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On unimodal maps with critical order $2 + \varepsilon$

by

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Abstract. It is proved that a smooth unimodal interval map with critical order $2+\varepsilon$ has no wild attractor if $\varepsilon > 0$ is small.

1. Introduction. In this paper, we continue the study of existence of wild attractors for unimodal interval maps. This problem, originated in [11], has been extensively studied. In particular, the original problem asked by Milnor, whether an S-unimodal interval map with a non-degenerate critical point has a wild attractor, was solved (see [9] and also [4, 5]). In [12], the second author of this paper extended the result to smooth unimodal maps with critical order not more than 2. In this paper, we extend the result further, and prove that smooth unimodal maps with critical order not much larger than 2 have no wild attractor.

Note that there exist unimodal maps with wild attractors (see [1]). For definition of wild attractors and more historical remarks, see the references above.

Unimodal maps. Let $N = [a, b]$ be a compact interval. A C^1 map $f : N \to N$ is called *unimodal* if there exists a unique $c \in (a, b)$ (called the *critical point*) such that $f'(c) = 0$ and f' has different signs on the components of $N \setminus \{c\}$. Let $\mathcal{U}$ be the collection of unimodal maps f which satisfy the following conditions:

- f is C^3 except at c;
- there exists $\ell > 1$ (called the *critical order*) and local C^3 diffeomorphisms ϕ and ψ with $\phi(c) = 0$ and $\psi(f(c)) = 0$ such that $\psi \circ f(x) = -|\phi(x)|^{\ell}$ in a neighborhood of c.

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NOTE. Without loss of generality, we shall assume that the critical point of f is located at 0 and $f(x) = -|x|^\ell + f(0)$ in a neighborhood of 0.

MAIN THEOREM. *There exists $\varepsilon > 0$ such that no $f \in \mathcal{U}$ with critical order $1 < \ell \leq 2 + \varepsilon$ has a wild attractor.*

A map $f \in \mathcal{U}$ is called *renormalizable* if there exist an interval I which contains the critical point c in its interior, and a positive integer $s > 1$, such that the intervals $I, f(I), \ldots, f^{s-1}(I)$ have pairwise disjoint interiors, $f^s(I) \subset I$, and $f^s(\partial I) \subset \partial I$. The unimodal map $f^s : I \to I$ is called a *renormalization* of f.

REMARK 1.1. It was already known to Milnor that if a map $f \in \mathcal{U}$ has a wild attractor, then f is at most finitely renormalizable and the critical point c is recurrent but not periodic. Since the property of having a wild attractor does not change under renormalization, *we may restrict ourselves to the case that f is non-renormalizable and has a recurrent and non-periodic critical point.* Let $\mathcal{F}$ be the collection of all maps in $\mathcal{U}$ which satisfy these properties.

Recall that an open interval T is called *nice* if $f^n(\partial T) \cap T = \emptyset$ for all $n \geq 0$. Let

$$\mathrm{D}(T) = \{x \in N : f^k(x) \in T \text{ for some } k \geq 1\}.$$

The *first entry map* $R_T : \mathrm{D}(T) \to T$ is defined as $x \mapsto f^{k(x)}(x)$, where $k(x)$ is the *entry time* of x into T, i.e., the minimal positive integer such that $f^{k(x)}(x) \in T$. It is well known that $k(x)$ is constant in any component of $\mathrm{D}(T)$. The map $R_T|_{\mathrm{D}(T)\cap T}$ is called the *first return map* of T. A component of $\mathrm{D}(T)$ is called an *entry domain*, and a component of $\mathrm{D}(T) \cap T$ is called a *return domain*.

Principal nest. Consider $f \in \mathcal{F}$. Let q denote the orientation-reversing fixed point of f. Let $I^0 = (\hat{q}, q)$ where $f(\hat{q}) = f(q)$, and for all $n \geq 1$, let I^n be the return domain to I^{n-1} which contains the critical point. All these intervals I^n are nice. The sequence

$$I^0 \supset I^1 \supset I^2 \supset \cdots$$

is called the *principal nest*. Let g_n denote the first return map to I^n. Let $m(0) = 0$, and let $m(1) < m(2) < \cdots$ be all the *non-central return moments*, i.e., positive integers such that

$$g_{m(k)-1}(0) \notin I^{m(k)}.$$

Note that the fact that f is non-renormalizable implies that there are infinitely many non-central return moments.

Main Lemma. *For any $\eta > 0$ there exists $\varepsilon > 0$ such that for $f \in \mathcal{F}$ with critical order $1 < \ell \le 2+\varepsilon$, we have*

$$\limsup_{k\to\infty} \frac{|I^{m(k)+1}|}{|I^{m(k)}|} < \eta.$$

This result was claimed in [12] without a detailed proof. The proof given here is a modification of the method in that paper.

In [2], the authors assume this lemma, and prove that $f \in \mathcal{F}$ with critical order $2+\varepsilon$ has an acip provided that ε is small and f has only finitely many central returns, i.e., $m(k) - k$ is constant for all large k.

The main theorem and the main lemma also extend some results of [6].

Proposition 1.1. *For any $\ell_0 > 1$ there exists $\alpha = \alpha(\ell_0) > 0$ with the following property. Let $f \in \mathcal{F}$ be a map with critical order $1 < \ell \le \ell_0$ and assume that $\limsup |I^{m(k)+1}|/|I^{m(k)}| < \alpha$. Then f has no wild attractor.*

This proposition was (implicitly) proved in Section 7 of [9] in the case that f has negative Schwarzian. The last assumption becomes unnecessary due to [7]. One can also prove this proposition using the Theorem of [3] or the Theorem of [8].

Deduction of the Main Theorem from the Main Lemma. Let $\alpha = \alpha(3)$ be as in Proposition 1.1. By the Main Lemma, there exists $\varepsilon \in (0,1)$ such that if $f \in \mathcal{F}$ has critical order $\ell \in (1, 2+\varepsilon)$ then the assumption of Proposition 1.1 is satisfied, so f has no wild attractor. By the remark above, it follows that no map $f \in \mathcal{U}$ with critical order $\ell \in (1, 2+\varepsilon)$ has a wild attractor. ∎

1.1. *Preliminaries.* The following two lemmas were proved in [12].

Lemma 1.2. *Let $J \subset I^{m(k)-1} - I^{m(k)}$ be a return domain to $I^{m(k)-1}$ with return time s. Then there is an interval J' with $J \subset J' \subset I^{m(k)-1} - I^{m(k)}$ such that $f^s : J' \to I^{m(k-1)}$ is a diffeomorphism.*

Lemma 1.3. *Let s be the return time of 0 to $I^{m(k)}$. Then there is an interval $J \ni f(0)$ with $f^{-1}(J) \subset I^{m(k)}$ such that $f^{s-1} : J \to I^{m(k-1)}$ is a diffeomorphism.*

The following lemma on real bounds was proved in [10, 7].

Lemma 1.4. *For any $\ell_0 > 1$, there exists a constant $\varrho > 1$ such that if $f \in \mathcal{F}$ has critical order $\ell \in (1, \ell_0)$, then for all k sufficiently large,*

$$|I^{m(k)}| \ge \varrho |I^{m(k)+1}|. \tag{1.1}$$

Moreover, if $g_{m(k)}(I^{m(k)+1}) \not\ni 0$, then

$$|I^{m(k+1)-1}| \ge \varrho |I^{m(k+1)}|.$$

We shall use the following cross-ratio. For any two intervals $J \Subset T$, we define

$$C(T,J) = \frac{|T|\,|J|}{|L|\,|R|},$$

where L, R are the components of $T - J$. If $h : T \to \mathbb{R}$ is a homeomorphism onto its image, we write

$$\mathbf{C}(h;T,J) = \frac{C(h(T),h(J))}{C(T,J)}.$$

LEMMA 1.5 ([7, Theorem C]). *For each k sufficiently large, there is a positive number $\mathcal{O}_k$ with $\mathcal{O}_k \to 1$ as $k \to \infty$ and with the following property. Let $T \subset [-1,1]$ be an interval and let n be a positive integer. Assume that $f^n|_T$ is monotone and $f^n(T) \subset I^{m(k-1)}$. Then for any interval $J \Subset T$, we have*

$$\mathbf{C}(f^n;T,J) \geq \mathcal{O}_k.$$

We shall also use the following lemma which is implicit in the proof of the Main Theorem of [12, p. 390].

LEMMA 1.6. *For any $\ell_0 > 1$, there is a constant $C = C(\ell_0) > 0$ such that for any $f \in \mathcal{F}$ with critical order $1 < \ell \leq \ell_0$ and sufficiently large $k \geq 1$, we have*

$$\frac{|I^{m(k)}|}{|I^{m(k+1)}|} \geq \frac{|I^{m(k)}|}{|I^{m(k)+1}|} \geq C\left(\frac{|I^{m(k-1)}|}{|I^{m(k)}|}\right)^{1/\ell}.$$

2. Lower limit. For any $n \geq 0$, write $c_n = f^n(0)$. A *closest (critical) return time* is a positive integer s such that $c_k \notin (c_s, -c_s)$ for all $1 \leq k \leq s$. The point c_s will be called a *closest (critical) return.*

Let $s_1 < s_2 < \cdots$ be all the closest return times. Let n_0 be such that s_{n_0} is the return time of 0 to $I^{m(1)}$. For any $n \geq n_0$, let $k = k(n)$ be so that $c_{s_n} \in I^{m(k)} - I^{m(k+1)}$. Note that $c_{s_n} \in I^{m(k+1)-1} - I^{m(k+1)}$. Let $T_n \ni c_{s_n}$ be the maximal open interval such that:

- $f^{s_{n+1}-s_n}|_{T_n}$ is monotone,
- $f^{s_{n+1}-s_n}(T_n) \subset I^{m(k)-1}$.

Let x_n, y_n denote the endpoints of $f^{s_{n+1}-s_n}(T_n)$, with $|x_n| \leq |y_n|$. By Lemma 3.2 of [12], $y_n \in \partial I^{m(k)-1}$, $x_n \notin I^{m(k)}$, and $(x_n, y_n) \ni 0$. Let b_n be an endpoint of $I^{m(k+1)-1}$. Define

$$A_n = \frac{|b_n|^\ell - |c_{s_{n+1}}|^\ell}{|b_n|^\ell - |c_{s_n}|^\ell}, \qquad B_n = \left(\frac{|c_{s_n}|}{|c_{s_{n+1}}|}\right)^{\ell/2},$$

and

$$V_n = \frac{2|x_n|(|y_n| + |c_{s_n}|)}{(|y_n| + |x_n|)(|x_n| + |c_{s_n}|)}, \qquad W_n = \left(\frac{|x_n|}{|c_{s_{n-1}}|}\right)^{\ell/2}.$$

Moreover, define

$$\widetilde{A}_n = \frac{|b_n|^2 - |c_{s_{n+1}}|^2}{|b_n|^2 - |c_{s_n}|^2}, \qquad \widetilde{W}_n = \frac{|x_n|}{|c_{s_{n-1}}|}.$$

The argument in [12] shows the following

PROPOSITION 2.1. *There exists a constant $\sigma_0 > 1$ such that if $\ell < 3$ then for all n sufficiently large, we have*

$$\widetilde{A}_{n-1} V_n \widetilde{W}_n \geq \sigma_0.$$

Let

$$\mu_k = \frac{|I^{m(k+1)}|}{|I^{m(k)}|}.$$

PROPOSITION 2.2. *There is a constant $\sigma > 1$ such that for any $\alpha > 0$, there exists $\varepsilon = \varepsilon(\alpha) > 0$ which satisfies the following. If $f \in \mathcal{F}$ has critical order $1 < \ell \leq 2+\varepsilon$ and n is sufficiently large and $\mu_{k-1}, \mu_k, \mu_{k+1} \geq \alpha$, where $k = k(n)$, then*

$$\frac{|(f^{s_{n+1}})'(c_1)| B_n A_{n-1}}{|(f^{s_n})'(c_1)| B_{n-1} A_n} \geq \sigma, \tag{2.1}$$

$$\frac{|(f^{s_{n+1}})'(c_1)| B_n}{|(f^{s_n})'(c_1)| B_{n-1}} \geq \frac{1}{2} A_n V_n W_n. \tag{2.2}$$

Proof. By the Main Lemma of [12], we only need to consider the case $\ell > 2$. Given $\alpha > 0$. Suppose $f \in \mathcal{F}$ has critical order $\ell = 2 + \varepsilon$ for some $\varepsilon > 0$. By the definition, $c_{s_{n-1}} \in I^{m(k-1)} - I^{m(k+1)}$, $c_{s_{n+1}} \in I^{m(k)} - I^{m(k+2)}$ and $x_n, y_n \in \overline{I^{m(k)-1}} - I^{m(k)}$.

By the Lemma in [12], $\phi(x,\ell) = x^{1-\ell/2} \int_x^1 t^{\ell-1}\, dt$ is increasing in ℓ. Now the computation of [12, p. 397, first paragraph] yields

$$|(f^{s_{n+1}-s_n})'(f(c_{s_n}))| \frac{A_{n-1}B_n}{A_n B_{n-1}} \geq g\left(\frac{|c_{s_{n+1}}|}{|x_n|}, \varepsilon\right) \mathcal{O}_k A_{n-1} V_n W_n,$$

where

$$g(t,\varepsilon) = \frac{\phi(t,2)}{\phi(t, 2+\varepsilon)}.$$

Since

$$\frac{|c_{s_{n+1}}|}{|x_n|} \geq \frac{|I^{m(k+2)}|}{|I^{m(k-1)}|} \geq \alpha^3,$$

for any $0 < \sigma_1 < 1$, if ε is small enough,

$$g\left(\frac{|c_{s_{n+1}}|}{|x_n|}, \varepsilon\right) > \sigma_1.$$

In particular, (2.2) holds.

Let us prove (2.1). Since

$$\widetilde{\widetilde{W}}_n \geq \frac{|I^{m(k)}|}{|I^{m(k-1)}|} \geq \alpha,$$

for any $0 < \sigma_2 < 1$, provided that ε is small enough,

$$W_n = \widetilde{W}_n^{\varepsilon/2}\widetilde{\widetilde{W}}_n \geq \sigma_2 \widetilde{\widetilde{W}}_n.$$

Since $|c_{s_{n-1}}|/|y_n| \geq |I^{m(k+1)}|/|I^{m(k-1)}| \geq \alpha^2$ and $|b_{n-1}| = |y_n|$, we have

$$A_{n-1} = \frac{|y_n|^{2+\varepsilon} - |c_{s_n}|^{2+\varepsilon}}{|y_n|^{2+\varepsilon} - |c_{s_{n-1}}|^{2+\varepsilon}} \geq \frac{1 - \left(\frac{|c_{s_n}|}{|y_n|}\right)^2}{1 - \left(\frac{|c_{s_{n-1}}|}{|y_n|}\right)^2 \left(\frac{|c_{s_{n-1}}|}{|y_n|}\right)^\varepsilon} \geq \sigma_3 \widetilde{A}_{n-1}$$

where $\sigma_3 > 0$ can be arbitrarily close to 1 if ε is small.

Therefore we get

$$|(f^{s_{n+1}-s_n})'(f(c_{s_n}))| \, \frac{A_{n-1}B_n}{A_n B_{n-1}} \geq \sigma_1\sigma_2\sigma_3 \mathcal{O}_k \widetilde{A}_{n-1} V_n \widetilde{\widetilde{W}}_n$$

where $\sigma_1, \sigma_2, \sigma_3$ can be arbitrarily close to 1 if ε is sufficiently small. By Proposition 2.1, (2.1) follows. ∎

COROLLARY 2.3. *For any $\alpha > 0$, there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that for $f \in \mathcal{F}$ with critical order $1 < \ell \leq 2 + \varepsilon$, we have*

$$\liminf_{k\to\infty} \frac{|I^{m(k+1)}|}{|I^{m(k)}|} < \alpha.$$

Proof. Suppose the statement is not true. Then there exists $\alpha > 0$, and for any $\varepsilon > 0$ there exists a map $f \in \mathcal{F}$ with critical order $1 < \ell \leq 2 + \varepsilon$ such that

$$\liminf_{k\to\infty} \frac{|I^{m(k+1)}|}{|I^{m(k)}|} \geq \alpha.$$

Therefore for k large enough, $|I^{m(k+1)}|/|I^{m(k)}| \geq \alpha/2$. Provided that ε is small enough, Proposition 2.2 implies that

$$|(f^{s_{n+1}-s_n})'(f(c_{s_n}))| \, \frac{A_{n-1}B_n}{A_n B_{n-1}} > \sigma$$

for all n large, where $\sigma > 1$ is a constant. But as shown in the proof of the Main Theorem in [12], this implies that f has decay of geometry and thus $\lim |I^{m(k+1)}|/|I^{m(k)}| = 0$, which is a contradiction. ∎

3. Proof of the Main Lemma

PROPOSITION 3.1. *For any $\ell_0 > 1$, there exists a constant $C = C(\ell_0) > 0$ satisfying the following. Let $f \in \mathcal{F}$ be a map with critical order $1 < \ell < \ell_0$. Assume that k is sufficiently large and $\mu_k, \mu_{k-1} < \alpha$. Let n be such that s_n*

is the return time of 0 *into* $I^{m(k)}$. *Then*

$$|(f^{s_n})'(c_1)|B_{n-1} > C/\sqrt{\alpha}.$$

Proof. Notice that $c_{s_n} \in I^{m(k+1)-1} - I^{m(k+1)}$ is also the first return of 0 to $I^{m(k+1)-1}$. By Lemma 1.3, there is an interval $J \ni c_1$ such that $f^{-1}(J) \subset I^{m(k)}$ and $f^{s_n-1} : J \to I^{m(k-1)}$ is a diffeomorphism. We will consider two cases.

Case 1: $|f^{s_n}(I^{m(k+1)})| \geq 0.1|I^{m(k+1)-1}|$. Since $f(I^{m(k+1)}) \subset J$ and $f^{s_n}(I^{m(k+1)}) \subset I^{m(k)}$ is well-inside $I^{m(k-1)}$, $f^{s_n-1}|_{f(I^{m(k+1)})}$ has uniformly bounded distortion. Therefore

$$|(f^{s_n-1})'(c_1)| \geq C\,\frac{|f^{s_n}(I^{m(k+1)})|}{|f(I^{m(k+1)})|} \geq C\,\frac{|I^{m(k+1)-1}|}{|I^{m(k+1)}|^{\ell}}.$$

So

$$\begin{aligned}|(f^{s_n})'(c_1)|B_{n-1} &= |(f^{s_n-1})'(c_1)|\,|f'(c_{s_n})|\left(\frac{|c_{s_{n-1}}|}{|c_{s_n}|}\right)^{\ell/2}\\ &\geq C\,\frac{|I^{m(k+1)-1}|}{|I^{m(k+1)}|^{\ell}}\,\ell|I^{m(k+1)}|^{\ell-1}\left(\frac{|I^{m(k)}|}{|I^{m(k+1)-1}|}\right)^{\ell/2}\\ &\geq C\,\frac{|I^{m(k+1)-1}|}{|I^{m(k+1)}|}\left(\frac{|I^{m(k)}|}{|I^{m(k+1)-1}|}\right)^{\ell/2}.\end{aligned}$$

If $m(k+1) = m(k)+1$, then

$$|(f^{s_n})'(c_1)|B_{n-1} \geq C\,\frac{|I^{m(k)}|}{|I^{m(k+1)}|} \geq \frac{C}{\alpha}.$$

If $m(k+1) \geq m(k)+2$, then

$$|(f^{s_n})'(c_1)|B_{n-1} \geq C\left(\frac{|I^{m(k)}|}{|I^{m(k+1)-1}|}\right)^{\ell/2} \geq C\left(\frac{|I^{m(k)}|}{|I^{m(k)+1}|}\right)^{\ell/2} \geq \frac{C}{\sqrt{\alpha}}$$

by Lemma 1.6.

Case 2: $|f^{s_n}(I^{m(k+1)})| < 0.1|I^{m(k+1)-1}|$. First we assume that $m(k+1) = m(k)+1$. By Lemma 1.4, $\varrho' = |I^{m(k-1)}|/|I^{m(k)}| > \varrho > 1$. Let $J' \subset J$ be such that $f^{s_n-1}(J') = \frac{\varrho'+1}{2}I^{m(k)}$. Let $K' = f^{-1}(J') \subset I^{m(k)}$ and K be a component of $K' - \{0\}$.

Since $\frac{\varrho'+1}{2}I^{m(k)}$ is well-inside $I^{m(k+1)}$, $f^{s_n-1}|_{J'}$ has uniformly bounded distortion. Thus

$$|(f^{s_n-1})'(c_1)| \geq C\,\frac{|f^{s_n}K|}{|f(K)|} \geq C\,\frac{|I^{m(k-1)}|}{|I^{m(k)}|^{\ell}}.$$

By assumption, c_{s_n} is close to the endpoint of $I^{m(k)}$, and it follows that

$$|(f^{s_n})'(c_1)|B_{n-1} \geq |(f^{s_n})'(c_1)| = |(f^{s_n-1})'(c_1)|\,|f'(c_{s_n})| \geq C\,\frac{|I^{m(k-1)}|}{|I^{m(k)}|^{\ell}}\,|I^{m(k)}|^{\ell-1} \geq \frac{C}{\alpha}.$$

Now we assume that $m(k+1)-m(k) \geq 2$. By Lemma 1.4, $\varrho' := |I^{m(k)}|/|I^{m(k+1)-1}| \geq |I^{m(k)}|/|I^{m(k)+1}| > \varrho$. Let $J' \subset J$ be such that $f^{s_n-1}(J') = T := \frac{\varrho'+1}{2} I^{m(k+1)-1}$. Let $K' = f^{-1}(J')$. Then $K' \subset T$, since otherwise f^{s_n} will have an attracting periodic point in K'. Let K be a component of $K' - \{0\}$. Since $f^{s_n-1}|_{J'}$ has uniformly bounded distortion, we have

$$|(f^{s_n-1})'(c_1)| \geq C\,\frac{|f^{s_n}(K)|}{|f(K)|} \geq C\,\frac{|I^{m(k+1)-1}|}{|I^{m(k+1)-1}|^{\ell}}.$$

By assumption, c_{s_n} is close to $\partial I^{m(k+1)-1}$, and it follows that $|(f^{s_n})'(c_1)| = |(f^{s_n-1})'(c_1)|\,|f'(c_{s_n})|$ is bounded away from 0. Therefore

$$|(f^{s_n})'(c_1)|B_{n-1} \geq C\left(\frac{|c_{s_{n-1}}|}{|c_{s_n}|}\right)^{\ell/2} \geq C\left(\frac{|I^{m(k)}|}{|I^{m(k+1)-1}|}\right)^{\ell/2} \geq \frac{C}{\sqrt{\alpha}},$$

where we use Lemma 1.6. ∎

LEMMA 3.2. *There is a constant $\beta > 0$ such that $W_n \geq \beta$ for each $n > 0$.*

Proof. For each $n > 0$ let $k > 0$ be such that $c_{s_n} \in I^{m(k)} - I^{m(k+1)}$ and let $p = m(k+1) - m(k)$. We may assume that $W_n < 1$, so by Lemma 3.2 of [12], we have $p \geq 2$, $0 \notin g_{m(k)}(I^{m(k)+1})$ and c_{s_n} is the first return of 0 to $I^{m(k)}$. Moreover, let q be such that $c_{s_n} = g^q_{m(k)-1}(0)$. Then there exist $1 \leq q' < q$ and $1 \leq p' < p$ such that $x_n = g^{q'}_{m(k)-1}(g^{p'}_{m(k)}(0))$ and $c_{s_{n-1}} = g^{q'}_{m(k)-1}(0)$.

Let J_1 be the entry domain to $I^{m(k)}$ which contains $c_{s_{n-1}} = g^{q'}_{m(k)-1}(0)$. Then $g^{q-q'}_{m(k)-1}|_{J_1} : J_1 \to I^{m(k)}$ is a diffeomorphism and

$$I^{m(k)-1} - I^{m(k)} \supset J_1 \supset g^{q'}_{m(k)-1}(I^{m(k)+1}) \ni g^{q'}_{m(k)-1}(g^{p'}_{m(k)}(0)) = x_n.$$

Therefore $g^{q-q'}_{m(k)-1}((x_n, c_{s_{n-1}})) \subset (c_{s_n}, w_n)$, where $w_n = g^{p}_{m(k)}(0) \in I^{m(k)} - I^{m(k)+1}$.

Let $J \subset I^{m(k)-1} - I^{m(k)}$ be the entry domain to $I^{m(k)-1}$ which contains $c_{s_{n-1}}$. Then $J \supset J_1$. By Lemma 1.2, there is an interval J' with $J \subset J' \subset I^{m(k)-1} - I^{m(k)}$ such that $g^{q-q'}_{m(k)-1} : J' \to I^{m(k-1)}$ is a diffeomorphism. Since (c_{s_n}, w_n) is well-inside $I^{m(k-1)}$, $(x_n, c_{s_{n-1}})$ is well-inside J'. Since $0 \notin J'$, it follows that W_n is bounded away from zero. ∎

Before proving the Main Lemma we need a lemma which is implicit in the proof of the Main Theorem in [12].

LEMMA 3.3. *For any* $\ell_0 > 1$ *and* $\eta > 0$, *there exists* $\xi > 0$ *satisfying the following. Let* $f \in \mathcal{F}$ *be a map with critical order* $1 < \ell \le \ell_0$, *and assume that* $|(f^{s_n})'(c_1)|B_{n-1} > \xi$ *for all sufficiently large* n. *Then*

$$\frac{|I^{m(k)+1}|}{|I^{m(k)}|} < \eta$$

for all sufficiently large k.

Proof of the Main Lemma. For any $\alpha > 0$, by Lemma 1.6, there exists $\alpha_1 > 0$ (we may assume $\alpha_1 < \alpha$) such that if $\mu_{k-1} < \alpha_1$, then $\mu_k < \alpha$. By Corollary 2.3, there exists $\varepsilon > 0$ such that if f is a non-renormalizable C^3 unimodal map with critical order $\ell \le 2 + \varepsilon$, then

$$\liminf_{k\to\infty} \frac{|I^{m(k+1)}|}{|I^{m(k)}|} < \alpha_1.$$

Let $k_1 < k_2 < \cdots$ be all the integers such that $\mu_{k_i-1} < \alpha_1$. Then $\mu_{k_i} < \alpha$ and $\mu_k \ge \alpha_1$ if $k \ne k_i - 1$ $(i = 1, 2, \ldots)$.

To complete the proof, by Lemma 3.3, it is enough to prove that the quantity $|(f^{s_m})'(c_1)|B_{m-1}$ is large for m sufficiently large.

Fix a large integer $i \ge 1$. Let $n = n(i)$ be such that s_n is the return time of 0 into $I^{m(k_i)}$. Since $\mu_{k_i-1}, \mu_{k_i} < \alpha$, by Proposition 3.1,

$$|(f^{s_n})'(c_1)|B_{n-1} > C/\sqrt{\alpha}$$

for some constant $C > 0$.

Certainly we may assume that $k_{i+1} \ge k_i + 2$, so that $\mu_k \ge \alpha_1$ for all $k_i \le k \le k_{i+1} - 2$. By Lemma 1.6, there exists $\alpha_2 \in (0, \alpha_1)$ such that $\mu_{k_i-1} \ge \alpha_2$. Let $N = N(i)$ be such that $c_{s_{n+N}}$ is the first return of 0 to $I^{m(k_{i+1}-1)}$. By Proposition 2.2, for any $n \le m \le n + N - 1$,

$$\frac{|(f^{s_{m+1}})'(c_1)|B_m}{|(f^{s_m})'(c_1)|B_{m-1}} \ge \frac{1}{2} A_m V_m W_m$$

If $|(f^{s_{m+1}})'(c_1)|B_m \ge |(f^{s_m})'(c_1)|B_{m-1}$ for all $n \le m \le n + N - 1$, then $|(f^{s_m})'(c_1)|B_{m-1} > C/\sqrt{\alpha}$ for all $n \le m \le n+N$. Otherwise, let $n+N-1 \ge \widetilde{n} \ge n$ be minimal such that

$$|(f^{s_{\widetilde{n}+1}})'(c_1)|B_{\widetilde{n}} < |(f^{s_{\widetilde{n}}})'(c_1)|B_{\widetilde{n}-1}.$$

Then $A_{\widetilde{n}} V_{\widetilde{n}} W_{\widetilde{n}} \le 2$. Since $V_{\widetilde{n}} > 1$ and $W_{\widetilde{n}} \ge \beta$ (Lemma 3.2), we obtain $A_{\widetilde{n}} \le 1/2\beta$. Since

$$|(f^{s_{\widetilde{n}}})'(c_1)|B_{\widetilde{n}-1} \ge \cdots \ge |(f^{s_n})'(c_1)|B_{n-1} \ge C/\sqrt{\alpha},$$

and

$$\frac{|(f^{s_{\widetilde{n}+1}})'(c_1)|B_{\widetilde{n}}}{|(f^{s_{\widetilde{n}}})'(c_1)|B_{\widetilde{n}-1}} \geq \frac{1}{2} A_{\widetilde{n}} V_{\widetilde{n}} W_{\widetilde{n}} \geq \frac{\beta}{2},$$

we have

$$|(f^{s_{\widetilde{n}+1}})'(c_1)| \frac{B_{\widetilde{n}}}{A_{\widetilde{n}}} \geq \frac{\beta^2 C}{\sqrt{\alpha}}.$$

Now by Proposition 2.2, for any $\widetilde{n}+1 \leq m \leq n+N$,

$$|(f^{s_m})'(c_1)|B_{m-1} \geq |(f^{s_m})'(c_1)| \frac{B_{m-1}}{A_{m-1}} > \frac{\beta^2 C}{\sqrt{\alpha}}.$$

Since $\alpha > 0$ can be arbitrarily small, the proof is finished. ∎

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