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The solenoids are the only circle-like continua that admit expansive homeomorphisms

by

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Abstract. A homeomorphism $h : X \to X$ of a compactum X is *expansive* provided that for some fixed $c > 0$ and any distinct $x, y \in X$ there exists an integer n, dependent only on x and y, such that $\mathrm{d}(h^n(x), h^n(y)) > c$. It is shown that if X is a circle-like continuum that admits an expansive homeomorphism, then X is homeomorphic to a solenoid.

1. Introduction. The first known continuum to admit an expansive homeomorphism was the dyadic solenoid, as shown by R. F. Williams [12]. In this paper, the following will be shown: *If a circle-like continuum X admits an expansive homeomorphism, then X must be a solenoid.* A homeomorphism $h : X \to X$ of a compactum X is *expansive* provided that for some fixed $c > 0$ and any distinct $x, y \in X$ there exists an integer n, dependent only on x and y, such that $\mathrm{d}(h^n(x), h^n(y)) > c$. Here, c is called the *expansive constant*. Expansive homeomorphisms exhibit sensitive dependence on initial conditions in the strongest sense that no matter how close any two points are, either their images or pre-images will at some point be a certain distance apart.

A continuum X is *circle-like* if it is the inverse limit of simple closed curves. Equivalently, a continuum is circle-like if for every $\epsilon > 0$ there exists a circle-chain cover $\mathcal{U}$ of X with $\mathrm{mesh}(\mathcal{U}) < \epsilon$. A continuum is a *solenoid* if it is homeomorphic to $\varprojlim(S, z^{n(i)})_{i=1}^{\infty}$ where S is the unit circle in the complex plane and $n(i)$ is an integer greater than 1. It is well known that the *shift homeomorphism* of $\varprojlim(S, z^n)_{i=1}^{\infty}$ is expansive when $n \geq 2$. For more on inverse limits see [4]. Alex Clark showed in [3] that a solenoid must be composite to admit an expansive homeomorphism. A solenoid is *composite* if there exists a prime number p that divides an infinite number of $\{n(i)\}_{i=1}^{\infty}$.

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Also, it is known that if X is tree-like [9] or separates the plane into two complementary domains [10], then X does not admit an expansive homeomorphism. The following question remains open: *If X is a solenoid that admits an expansive homeomorphism, must X be homeomorphic to $\varprojlim(S, z^n)_{i=1}^{\infty}$ for some $n \geq 2$?*

2. Chains and circle-chains. In this section, properties of circle-like continua are developed by looking at open covers of the continua. This is used to show that nested refining circle-chains with very small folding of subchains can still limit to a solenoid. In later sections, it is shown that larger folding of subchains will prohibit the existence of an expansive homeomorphism.

Let $\mathcal{U}$ be a finite collection of open subsets of a continuum X. Then $\mathcal{U} = [U_0, \ldots, U_{n-1}]$ is a *chain* provided that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$, and $\mathcal{U} = [U_0, \ldots, U_{n-1}]_\circ$ is a *circle-chain* provided that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$ or $i, j \in \{0, n-1\}$. A cover $\mathcal{U}$ is *taut* provided that $U_i \cap U_j \neq \emptyset$ if and only if $\overline{U}_i \cap \overline{U}_j \neq \emptyset$. Given $\mathcal{U}$, define $\mathcal{U}^* = \bigcup_{U \in \mathcal{U}} U$. Then define the *core* of U_i by

$$\operatorname{core}(U_i) = U_i - \overline{(\mathcal{U} - \{U_i\})^*}.$$

Notice that a cover $\mathcal{U}$ of a continuum X is taut if and only if the core of each of its elements is nonempty.

Let H be a subset of X. Then $\mathcal{U}$ is a *proper cover* of H if $H \subset \mathcal{U}^*$ and for each $U \in \mathcal{U}$, $H \cap U \neq \emptyset$. Suppose that $\mathcal{V}$ is also a finite collection of open sets of X. Then $\mathcal{V}$ *refines* $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$; $\mathcal{V}$ is a *proper refinement* of $\mathcal{U}$ if $\mathcal{V}$ refines $\mathcal{U}$ and for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $V \subset U$; and $\mathcal{V}$ is an *n-refinement* of $\mathcal{U}$ if for any subchain $[V_{j_1}, \ldots, V_{j_n}]$ of $\mathcal{V}$ with n links there exists $U \in \mathcal{U}$ such that $[V_{j_1}, \ldots, V_{j_n}]^* \subset U$.

The following proposition will be useful later:

PROPOSITION 1. *Let $\mathcal{C}$ be a proper chain cover of a continuum X and $\mathcal{C}'$ a subchain of $\mathcal{C}$. Then there exists a subcontinuum $X' \subset X$ such that $\mathcal{C}'$ is a proper cover of X'.*

Proof. If no X' existed, then X would not be connected. This contradicts the fact that X is a continuum. ∎

Next define $\operatorname{mesh}(\mathcal{U}) = \sup\{\operatorname{diam}(U) \mid U \in \mathcal{U}\}$. A collection $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a *nested sequence of refining covers* if $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$ and $\lim_{i\to\infty} \operatorname{mesh}(\mathcal{U}_i) = 0$. A collection $\{\mathcal{U}_i\}_{i=1}^{\infty}$ *limits* to a space X if $X = \bigcap_{i=1}^{\infty} \mathcal{U}^*$. If $H \subset \mathcal{U}$, then define $\mathcal{U}(H) = \{U \in \mathcal{U} \mid U \cap H \neq \emptyset\}$. Likewise, if $\mathcal{V}$ refines $\mathcal{U}$ then define $\mathcal{U}(\mathcal{V}) = \{U \in \mathcal{U} \mid \text{there exists } V \in \mathcal{V} \text{ such that } V \subset U\}$.

In order to understand the topology of a circle-like continuum, it is important to measure how $\mathcal{U}_{i+1}$ winds in $\mathcal{U}_i$. To do this, Bing's definition of degree is needed [2]:

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be taut circle-chains such that

(1) $\mathcal{U}_2$ refines $\mathcal{U}_1$,
(2) $\mathcal{U}_1 = [U^1_0, \ldots, U^1_{n-1}]_\circ$,
(3) $\mathcal{U}_2 = [U^2_0, \ldots, U^2_{m-1}]_\circ$,
(4) U^2_0 and U^2_{m-1} both intersect the core of U^1_0.

For $U^2_i \in \mathcal{U}_2$, define $\Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_i) = j$ if there exists $U^1_j \in \mathcal{U}_1$ such that $U^2_i \subset U^1_j$. Note that there could be two choices for $\Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_i)$. In this case U^2_{i-1} and U^2_i are in the same element of $\mathcal{U}_1$. So we can inductively define $\Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_i) = \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_{i-1})$ in this special situation. Notice that since both U^2_0 and U^2_{m-1} intersect $\mathrm{core}(U^1_0)$, $\Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_0) = \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_{m-1}) = 0$. Next define

$$\Delta^{\mathcal{U}_2}_{\mathcal{U}_1} : \{0, 1, \ldots, m-1\} \to \mathbb{Z}$$

so that $\Delta^{\mathcal{U}_2}_{\mathcal{U}_1}(0) = 0$ and then continue inductively by

$$\Delta^{\mathcal{U}_2}_{\mathcal{U}_1}(i) = \begin{cases} \Delta^{\mathcal{U}_2}_{\mathcal{U}_1}(i-1) & \text{if } \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_i) = \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_{i-1}), \\ \Delta^{\mathcal{U}_2}_{\mathcal{U}_1}(i-1)+1 & \text{if } \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_i) = \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_{i-1})+1, \\ & \text{or } \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_i) = 0 \text{ and } \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_{i-1}) = n-1, \\ \Delta^{\mathcal{U}_2}_{\mathcal{U}_1}(i-1)-1 & \text{if } \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_i) = \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_{i-1})-1, \\ & \text{or } \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_i) = n-1 \text{ and } \Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_{i-1}) = 0. \end{cases}$$

Define the *degree* of $\mathcal{U}_2$ in $\mathcal{U}_1$ by

$$\deg_{\mathcal{U}_1}(\mathcal{U}_2) = \frac{|\Delta^{\mathcal{U}_2}_{\mathcal{U}_1}(m-1) - \Delta^{\mathcal{U}_2}_{\mathcal{U}_1}(0)|}{n}.$$

Notice that the degree of $\mathcal{U}_2$ in $\mathcal{U}_1$ is an integer that measures the number of times for which $\mathcal{U}_2$ "essentially circles" $\mathcal{U}_1$. Also, since both U^2_0 and U^2_{m-1} intersect the core of U^1_0, this value is independent of our choice for $\Gamma^{\mathcal{U}_2}_{\mathcal{U}_1}(U^2_i)$. The following theorem is also due to Bing:

Theorem 2 ([2]). *Suppose that $\mathcal{U}_0, \mathcal{U}_1$ and $\mathcal{U}_2$ are circle-chains such that $\mathcal{U}_2$ refines $\mathcal{U}_1$ and $\mathcal{U}_1$ refines $\mathcal{U}_0$. Then* $\deg_{\mathcal{U}_0}(\mathcal{U}_2) = \deg_{\mathcal{U}_0}(\mathcal{U}_1)\deg_{\mathcal{U}_1}(\mathcal{U}_2)$.

Similarly, if $\mathcal{V} = [V_0, \ldots, V_{p-1}]$ is a chain that refines $\mathcal{U}$, with $n = |\mathcal{U}|$, then we can define

$$\Gamma^{\mathcal{V}}_{\mathcal{U}} : \mathcal{V} \to \{0, \ldots, n-1\} \quad \text{and} \quad \Delta^{\mathcal{V}}_{\mathcal{U}} : \{0, \ldots, p-1\} \to \mathbb{Z}$$

in a similar way with the following additional requirement: If the endlinks V_0 and V_{p-1} are in the same element of $\mathcal{C}$, then $\Gamma^{\mathcal{V}}_{\mathcal{U}}(V_0) = \Gamma^{\mathcal{V}}_{\mathcal{U}}(V_{p-1})$. Notice that this is well defined for any i such that $V_0 \subset U_i$. Additionally, if V_0, V_{p-1} are in the same link of $\mathcal{U}$, then we can define

$$\deg_{\mathcal{U}}(\mathcal{V}) = \frac{|\Delta_{\mathcal{U}}^{\mathcal{V}}(p-1) - \Delta_{\mathcal{U}}^{\mathcal{V}}(0)|}{n}.$$

Proposition 3. *Suppose that $\Delta_{\mathcal{U}}^{\mathcal{V}}(i) < a < b < \Delta_{\mathcal{U}}^{\mathcal{V}}(j)$ where a, b are integers. Then there exists a set of consecutive integers $\{i', \ldots, j'\} \subset \{i, \ldots, j\}$ such that $\Delta_{\mathcal{U}}^{\mathcal{V}}(\{i', \ldots, j'\}) = \{a, \ldots, b\}$.*

Proof. This follows from the construction of $\Delta_{\mathcal{U}}^{\mathcal{V}}$. ■

Another aspect of determining the topology of X is by understanding how $\mathcal{U}_{i+1}$ is folded in $\mathcal{U}_i$. Let $\mathcal{V} = [V_0, \ldots, V_{p-1}]$ and $\mathcal{W} = [W_0, \ldots, W_{q-1}]$ be chains. Then $\mathcal{V}$ is *folded* in $\mathcal{W}$ if $\mathcal{V}$ refines $\mathcal{W}$ and there exists $j \in \{1, \ldots, p-2\}$ such that one of the following is true:

(1) $V_0, V_{p-1} \subset W_0$ and $V_j \subset W_{q-1}$,
(2) $V_0, V_{p-1} \subset W_{q-1}$ and $V_j \subset W_0$.

Next let $\mathcal{U}_\alpha$, $\mathcal{U}_\beta$ be circle-chains such that $\mathcal{U}_\beta$ refines $\mathcal{U}_\alpha$. A proper subchain $\widehat{\mathcal{U}}_\beta$ of $\mathcal{U}_\beta$ is *folded* in $\mathcal{U}_\alpha$ if there exists a chain $\mathcal{W}$ that refines $\mathcal{U}_\alpha$ such that $\widehat{\mathcal{U}}_\beta$ is folded in $\mathcal{W}$. (See Figure 1.) If $\widehat{\mathcal{U}}_\beta^*$ is not completely contained in any single element of $\mathcal{U}_\alpha$, then we say that $\widehat{\mathcal{U}}_\beta$ is *properly folded* in $\mathcal{U}_\alpha$.

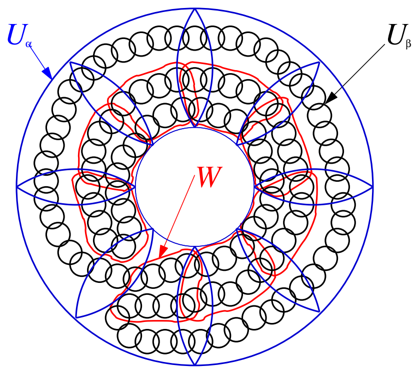

Fig. 1. Subchain $\widehat{\mathcal{U}}_\beta$ of $\mathcal{U}_\beta$ is folded in $\mathcal{U}_\alpha$.

Suppose that $\mathcal{U}$ is a finite open cover of X and let $\mathcal{U}' \subset \mathcal{U}$. A set W is an *amalgamation* of $\mathcal{U}'$ if $W = \mathcal{U}'^*$. A cover $\mathcal{W}$ is a *cover amalgamation* of $\mathcal{U}$ if

$$\mathcal{W} = \{W \mid W \text{ is an amalgamation of a subset of } \mathcal{U}\}.$$

Proposition 4. *Suppose that $\mathcal{U}_{j+1}$, $\mathcal{W}_j$ are circle-chains such that $\mathcal{U}_{j+1}$ refines $\mathcal{W}_j$ and no subchain of $\mathcal{U}_{j+1}$ is properly folded in $\mathcal{W}_j$. If $\mathcal{W}_{j+1}$ is a* *circle-chain that refines $\mathcal{W}_j$ such that each $W \in \mathcal{W}_{j+1}$ is an amalgamation of a subchain of $\mathcal{U}_{j+1}$, then no subchain of $\mathcal{W}_{j+1}$ is properly folded in $\mathcal{W}_j$.*

Proof. Suppose on the contrary that $\widehat{\mathcal{W}}_{j+1} = [W_0^{j+1}, \dots, W_{p-1}^{j+1}]$ is a subchain of $\mathcal{W}_{j+1}$ that is properly folded in $\mathcal{W}_j$. Then there exists a chain $\mathcal{C} = [C_0, \dots, C_{q-1}]$ that refines $\mathcal{W}_j$ such that $\widehat{\mathcal{W}}_{j+1}$ is folded in $\mathcal{C}$. Thus, without loss of generality, we may assume $W_0^{j+1} \cap \mathrm{core}(C_0)$, $W_{p-1}^{j+1} \cap \mathrm{core}(C_0)$ and $W_i^{j+1} \cap \mathrm{core}(C_{q-1})$ are all nonempty for some i. But since W_0^{j+1}, W_{p-1}^{j+1} and W_i^{j+1} are amalgamations of subchains of $\mathcal{U}_j$, there exist $U_{i_1}, U_{i_2}, U_{i_3} \in \mathcal{U}_j$ such that

(1) $[U_{i_1}, \dots, U_{i_2}, \dots, U_{i_3}]$ is a subchain of $\mathcal{U}_{j+1}$,
(2) $U_{i_1} \subset W_0^{j+1}$, $U_{i_2} \subset W_i^{j+1}$ and $U_{i_3} \subset W_{p-1}^{j+1}$,
(3) $U_{i_1} \cap \mathrm{core}(C_0)$, $U_{i_3} \cap \mathrm{core}(C_0)$ and $U_{i_2} \cap \mathrm{core}(C_{q-1})$ are all nonempty.

Hence $[U_{i_1}, \dots, U_{i_2}, \dots, U_{i_3}]$ is folded in $\mathcal{C}$ and thus in $\mathcal{W}_j$. By the third property, $[U_{i_1}, \dots, U_{i_2}, \dots, U_{i_3}]^*$ is not contained in any element of $\mathcal{W}_j$. Thus $[U_{i_1}, \dots, U_{i_2}, \dots, U_{i_3}]$ is properly folded in $\mathcal{W}_j$, which is a contradiction. ∎

LEMMA 5. *Let $\gamma > 0$ and $\mathcal{U}$ be a taut circle-chain cover of X such that $\max\{\gamma, \mathrm{mesh}(\mathcal{U})\} < (1/36)\,\mathrm{diam}(X)$. Then there exists a cover amalgamation $\mathcal{W}$ of $\mathcal{U}$ such that*

(1) *each $W \in \mathcal{W}$ is an amalgamation of a chain in $\mathcal{U}$,*
(2) *$\mathcal{W}$ is a taut circle-chain,*
(3) $\mathrm{mesh}(\mathcal{W}) \le 5\gamma + \mathrm{mesh}(\mathcal{U})$,
(4) $\mathrm{diam}(W) \ge 2\gamma$ *for each* $W \in \mathcal{W}$,
(5) $|\mathcal{W}| \ge 6$.

Proof. Let $\mathcal{U} = [U_0, \dots, U_{p-1}]_\circ$. Notice that if $[U_i, \dots, U_j]$ is a subchain of $\mathcal{U}$ then

$$\begin{aligned}\mathrm{diam}([U_i, \dots, U_j]^*) &\le \mathrm{diam}([U_i, \dots, U_j, U_{j+1}]^*)\\ &\le \mathrm{diam}([U_i, \dots, U_j]^*) + \mathrm{mesh}(\mathcal{U}).\end{aligned}$$

Thus, if $\mathrm{diam}([U_{i'}, \dots, U_{p-1}]^*) \ge 2\gamma$ then there exists $j' \in \{i', \dots, p-1\}$ such that

$$2\gamma \le \mathrm{diam}([U_{i'}, \dots, U_{j'}]^*) \le 3\gamma + \mathrm{mesh}(\mathcal{U}).$$

Hence there exists i_0 such that $2\gamma \le \mathrm{diam}([U_0, \dots, U_{i_0}]^*) \le 3\gamma + \mathrm{mesh}(\mathcal{U})$. Let $W_0 = \bigcup_{i \in \{0, \dots, i_0\}} U_i$. Suppose that $i_0, \dots, i_n$ and $W_0, \dots, W_n$ have been found. If $\mathrm{diam}([U_{i_n+1}, \dots, U_{p-1}]^*) \ge 2\gamma$ then there exists $i_{n+1} \in \{i_n+1, \dots, p-1\}$ such that $2\gamma \le \mathrm{diam}([U_{i_n+1}, \dots, U_{i_{n+1}}]^*) \le 3\gamma + \mathrm{mesh}(\mathcal{U})$. So let $W_{n+1} = \bigcup_{i \in \{i_n+1, \dots, i_{n+1}\}} U_i$ and continue the induction. On the other hand, if $\mathrm{diam}([U_{i_n+1},\dots,U_{p-1}]^*) < 2\gamma$, then replace

$$W_n = \bigcup_{i\in\{i_{n-1}+1,\dots,i_n\}} U_i \quad \text{with} \quad W_n = \bigcup_{i\in\{i_{n-1}+1,\dots,p-1\}} U_i$$

and stop the process. In this case, notice that

$$\begin{aligned}\mathrm{diam}&([U_{i_{n-1}+1},\dots,U_{p-1}]^*)\\ &\le \mathrm{diam}([U_{i_{n-1}+1},\dots,U_{i_n}]^*) + \mathrm{diam}([U_{i_n+1},\dots,U_{p-1}]^*)\\ &\le 3\gamma + \mathrm{mesh}(\mathcal{U}) + 2\gamma = 5\gamma + \mathrm{mesh}(\mathcal{U}).\end{aligned}$$

Since $\mathcal{U}$ is finite, this process must eventually stop, say at i_m. Let $\mathcal{W} = [W_0,\dots,W_m]$. Clearly $\mathcal{W}$ has properties (1)–(4) of the lemma. For property (5) notice that if $|\mathcal{W}| \le 5$ then $\mathrm{diam}(X) \le 5(5\gamma + \mathrm{mesh}(\mathcal{U})) \le 30\max\{\gamma, \mathrm{mesh}(\mathcal{U})\}$, which is a contradiction. ■

LEMMA 6. *Let $\gamma > 0$ and $\mathcal{U}$ be a taut circle-chain cover of X such that* $\max\{\gamma, \mathrm{mesh}(\mathcal{U})\} < (1/36)\,\mathrm{diam}(X)$. *Then there exists a cover amalgamation $\mathcal{W}$ of $\mathcal{U}$ such that*

(1) *$\mathcal{W}$ is a taut circle-chain,*
(2) *if $W, W' \in \mathcal{W}$ are such that $W \cap W' \neq \emptyset$ then $W \cap W'$ is an amalgamation of a chain in $\mathcal{U}$ such that* $\mathrm{diam}(W \cap W') \ge 2\gamma$,
(3) $\mathrm{mesh}(\mathcal{W}) \le 20\gamma + 4\,\mathrm{mesh}(\mathcal{U})$.

Proof. Let $\mathcal{W}' = [W'_0,\dots,W'_{m-1}]_\circ$ be a cover amalgamation of $\mathcal{U}$ that satisfies the conclusion of Lemma 5. Let

$$W_i = W'_{2i} \cup W'_{2i+1} \cup W'_{2(i+1)} \quad \text{for } 0 \le i \le \lfloor (m-4)/2 \rfloor$$

and

$$W_{\lfloor (m-4)/2\rfloor+1} = \begin{cases} W'_{m-2} \cup W'_{m-1} \cup W'_0 & \text{if } m \text{ is even,}\\ W'_{m-3} \cup W'_{m-2} \cup W'_{m-1} \cup W'_0 & \text{if } m \text{ is odd.}\end{cases}$$

Then $\mathcal{W} = [W_0,\dots,W_{\lfloor (m-4)/2\rfloor+1}]_\circ$ has the prescribed properties. ■

PROPOSITION 7. *Suppose $\mathcal{U}$, $\mathcal{W}$ are taut circle-chains and $\gamma > 0$ such that*

(1) *each $W \in \mathcal{W}$ is an amalgamation of some chain in $\mathcal{U}$,*
(2) *if $W, W' \in \mathcal{W}$ are such that $W \cap W' \neq \emptyset$, then $W \cap W'$ is an amalgamation of some chain in $\mathcal{U}$ and* $\mathrm{diam}(W \cap W') > 2\gamma$.

If $\widehat{\mathcal{U}}$ is a subchain of $\mathcal{U}$ such that $\mathrm{diam}(\widehat{\mathcal{U}}^*) \le 2\gamma$, *then there exists $W \in \mathcal{W}$ such that $\widehat{\mathcal{U}}^* \subset W$.*

Proof. Suppose on the contrary that no $W \in \mathcal{W}$ contains $\widehat{\mathcal{U}}^*$. Then there exist $W, W' \in \mathcal{W}$ such that $W \cap W' \neq \emptyset$ and $W \cap W' \subset \widehat{\mathcal{U}}^*$. However, this contradicts the fact that $\mathrm{diam}(W \cap W') > 2\gamma$. ■

THEOREM 8. *Let $\{\mathcal{U}_i\}_{i=1}^{\infty}$ be a nested refining sequence of taut circle-chain covers of X and $\xi \geq 2$ such that*

(1) *for each j, $\operatorname{mesh}(\mathcal{U}_{j+1}) \leq (1/25)\operatorname{mesh}(\mathcal{U}_j)$,*
(2) *there exists $n > 0$ such that if $\widehat{\mathcal{U}}_{j+1}$ is a subchain of $\mathcal{U}_{j+1}$ that is folded in $\mathcal{U}_j$ then $\operatorname{diam}(\widehat{\mathcal{U}}^*_{j+1})/\operatorname{mesh}(\mathcal{U}_j) \leq \xi$ for $j \geq n$.*

Then there exists a nested refining sequence $\{\mathcal{W}_i\}_{i=1}^{\infty}$ of taut circle-chain covers of X such that for each j, if $\widehat{\mathcal{W}}_{j+1}$ is a subchain of $\mathcal{W}_{j+1}$, then $\widehat{\mathcal{W}}_{j+1}$ is not properly folded in $\mathcal{W}_j$.

Proof. Let $\gamma_j = \xi \operatorname{mesh}(\mathcal{U}_j)$. There exists a $J > 0$ such that $\gamma_j < (1/36)\operatorname{diam}(X)$ for each $j \geq J$. Then for each $j > J$, let $\mathcal{W}_{j-J}$ be found from $\mathcal{U}_j$ as in Lemma 6 with $\gamma = \gamma_j$.

CLAIM 1. $\operatorname{mesh}(\mathcal{W}_i) \to 0$ *as* $i \to \infty$.

Notice that $\operatorname{mesh}(\mathcal{W}_i) < 25\xi \operatorname{mesh}(\mathcal{U}_{i+J})$. Since ξ is fixed, the claim follows from the fact that $\operatorname{mesh}(\mathcal{U}_i) \to 0$ as $i \to \infty$.

CLAIM 2. *$\mathcal{W}_{j+1}$ refines $\mathcal{W}_j$.*

Let $W' \in \mathcal{W}_{j+1}$. Then W' is the amalgamation of a chain $\mathcal{C}_{j+1+J}$ of $\mathcal{U}_{j+1+J}$ such that $\operatorname{diam}(\mathcal{C}^*_{j+1+J}) \leq 25\xi \operatorname{mesh}(\mathcal{U}_{j+1+J}) \leq \xi \operatorname{mesh}(\mathcal{U}_{j+J}) = \gamma_{j+J}$. Thus, $\mathcal{U}_{j+J}(\mathcal{C}_{j+1+J})$ is a chain of $\mathcal{U}_{j+J}$ such that $\operatorname{diam}(\mathcal{U}_{j+J}(\mathcal{C}_{j+1+J})^*) < 2\gamma_{j+J}$. However, this implies that $W' \subset \mathcal{U}_{j+J}(\mathcal{C}_{j+1+J})^* \subset W$ for some $W \in \mathcal{W}_j$ by the properties of Lemma 6 and Proposition 7.

CLAIM 3. *No subchain of $\mathcal{U}_{j+1}$ is properly folded in $\mathcal{W}_{j-J}$.*

Suppose on the contrary that $\widehat{\mathcal{U}}_{j+1} = [U_0^{j+1}, \ldots, U_{p-1}^{j+1}]$ is a subchain of $\mathcal{U}_{j+1}$ that is properly folded in $\mathcal{W}_{j-J}$. Then there exists a taut chain $\mathcal{C} = [C_0, \ldots, C_{q-1}]$ of minimal cardinality that refines $\mathcal{W}_{j-J}$ such that

(1) $U_0^{j+1}, U_{p-1}^{j+1} \subset C_0$,
(2) $U_i^{j+1} \subset C_{q-1}$,
(3) $U_0^{j+1} \not\subset C_{q-1}$ or $U_{p-1}^{j+1} \not\subset C_{q-1}$,
(4) $U_i^{j+1} \not\subset C_0$,

for some $i \in \{1, \ldots, p-2\}$. For each $W \in \mathcal{W}_{j-J}$, let $\widehat{\mathcal{U}}_j(W)$ be the subchain of $\mathcal{U}_j$ such that W is an amalgamation of $\widehat{\mathcal{U}}_j(W)$. Then define $W(k) \in \mathcal{W}_{j-J}$ such that $C_k \subset W(k)$. Next let

$$C_i^k = \bigcup \{U_\alpha^{j+1} \in \widehat{\mathcal{U}}_{j+1} \mid U_\alpha^{j+1} \subset C_k \cap U_i^j \text{ where } U_i^j \in \widehat{\mathcal{U}}_j(W(k))\}$$

and

$$\mathcal{C}_k = \{C_i^k \mid U_i^j \in \widehat{\mathcal{U}}_j(W(k)) \text{ and } C_i^k \neq \emptyset\}.$$

Then $\widetilde{\mathcal{C}} = \bigcup_{k=0}^{q-1} \mathcal{C}_k$ is a taut chain of minimal cardinality that refines $\mathcal{U}_j$ and is refined by $\widehat{\mathcal{U}}_{j+1}$. So there exist r, s, t such that $U_0^{j+1} \subset C_r^0$, $U_{p-1}^{j+1} \subset C_s^0$, $U_i^{j+1} \subset C_t^{q-1}$ and $U_i^{j+1} \not\subset C_r^0 \cup C_s^0$. Furthermore, there exist $\alpha \in \{0, \ldots, i\}$ or $\beta \in \{i, \ldots, p-1\}$ such that either $U_\alpha^{j+1} \subset C_s^0$ and $U_\alpha^{j+1} \not\subset C_t^{q-1}$, or $U_\beta^{j+1} \subset C_r^0$ and $U_\beta^{j+1} \not\subset C_t^{q-1}$. Thus either $[U_\alpha^{j+1}, \ldots, U_{p-1}^{j+1}]$ or $[U_0^{j+1}, \ldots, U_\beta^{j+1}]$ is a subchain of $\mathcal{U}_{j+1}$ that is properly folded in $\mathcal{U}_j$, which is a contradiction.

It now follows from Proposition 4 that no subchain of $\mathcal{W}_{j+1}$ is properly folded in $\mathcal{W}_j$. Thus $\{\mathcal{W}_i\}_{i=1}^{\infty}$ has the properties of the theorem. ■

Next we need to show how "small folds" in refining covers do not prevent the nested intersection from being a solenoid. To do this we must first find a relationship between nested circle-chains and inverse limits of simple closed curves.

The *nerve* of a cover $\mathcal{U}$, denoted $N(\mathcal{U})$, is a geometric simplicial complex (a graph in this case) where each element $U_i \in \mathcal{U}$ is represented by a vertex $u_i \in N(\mathcal{U})$ and there exists an arc (edge) in $N(\mathcal{U})$ from u_i to u_j if and only if $U_i \cap U_j \neq \emptyset$. In a circle-chain, the nerve is always a simple closed curve. Furthermore, suppose that $[U_0, \ldots, U_{n-1}]_\circ$ is a circle-chain with nerve $[u_{n-1}, u_0] \cup \bigcup_{i=0}^{n-2} [u_i, u_{i+1}]$. For each $i \in \{0, \ldots, n-1\}$, let $f_i : [0,1] \to [u_i, u_{i+1}]$ be a homeomorphism such that $f_i(0) = u_i$ and $f_i(1) = u_{i+1}$, and $f_{n-1} : [0,1] \to [u_{n-1}, u_0]$ be a homeomorphism such that $f_{n-1}(0) = u_{n-1}$ and $f_i(1) = u_0$. Then for $r \in [0,1]$ and $i \in \{0, \ldots, n-2\}$, let $u_{i+r} = f_i(r)$.

We need the following well known theorem to prove the next lemma:

THEOREM 9 ([7]). *If $p : (Y, y_0) \to (X, x_0)$ and $p' : (Y', y_0') \to (X, x_0)$ are both simply connected covering spaces of X, then there exists a unique homeomorphism $\phi : (Y', y_0') \to (Y, y_0)$ such that $p \circ \phi = p'$.*

LEMMA 10. *Let $\mathcal{U}_i$ and $\mathcal{U}_{i+1}$ be taut circle-chains such that*

(1) *$\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$,*
(2) *no subchain of $\mathcal{U}_{i+1}$ is properly folded in $\mathcal{U}_i$.*

Suppose $h_i : N(\mathcal{U}_i) \to S$ is a homeomorphism such that $h_i(u_0^i) = e^0$. Then there exists a map $f_i : N(\mathcal{U}_{i+1}) \to N(\mathcal{U}_i)$ and a homeomorphism $h_{i+1} : N(\mathcal{U}_{i+1}) \to S$ such that $h_i \circ f_i = g_i \circ h_{i+1}$ where $g_i(z) = z^{\deg_{\mathcal{U}_i}(\mathcal{U}_{i+1})}$, $h_{i+1}(u_0^{i+1}) = e^0$ and $\operatorname{diam}(f_i^{-1}(x)) \le 3 \operatorname{mesh}(\mathcal{U}_i)$.

Proof. Since no subchain of $\mathcal{U}_{i+1}$ is properly folded in $\mathcal{U}_i$,

$$\Delta_{\mathcal{U}_i}^{\mathcal{U}_{i+1}} : \{0, \ldots, |\mathcal{U}_{i+1}| - 1\} \to \mathbb{Z}$$

can be defined such that $\Delta_{\mathcal{U}_i}^{\mathcal{U}_{i+1}}(j+1) \ge \Delta_{\mathcal{U}_i}^{\mathcal{U}_{i+1}}(j)$. Let $\mathcal{C}_j = [U_{i_j}^{i+1}, \ldots, U_{i_{j+1}-1}^{i+1}]$ be the maximal subchain of $\mathcal{U}_i$ such that $\Delta_{\mathcal{U}_i}^{\mathcal{U}_{i+1}}(k) = j$ for each $k \in$

$\{i_j, \ldots, i_{j+1} - 1\}$ where $i_0 = 0$. Let $n = |\mathcal{U}_i|$ and define f_i to map linearly such that

$$f_i([u^{i+1}_{i_p}, u^{i+1}_{i_{p+1}-1}]) = \begin{cases} [u^i_0, u^i_{1/4}] & \text{if } p = 0 \\ [u^i_{m-1/4}, u^i_{m+1/4}] & \text{if } m = p \bmod n \text{ for} \\ & p \in \{1, \ldots, n \deg_{\mathcal{U}_i}(\mathcal{U}_{i+1}) - 1\}, \end{cases}$$

$$f_i([u^{i+1}_{i_{p+1}-1}, u^{i+1}_{i_{p+1}}]) = [u^i_{m+1/4}, u^i_{m+3/4}] \text{ where } m = p \bmod n$$

and

$$f_i([u^{i+1}_{i_p}, u^{i+1}_{i_0}]) = [u^i_{n-1/4}, u^i_0] \quad \text{where } p = n \deg_{\mathcal{U}_i}(\mathcal{U}_{i+1}).$$

Under this construction, $\operatorname{diam}(f_i^{-1}(x)) \le 3 \operatorname{mesh}(\mathcal{U}_i)$ for each $x \in N(\mathcal{U}_i)$ and $N(\mathcal{U}_{i+1})$ is a covering space of S under $h_i \circ f_i$. Since S is a covering space of S under g_i and $|g_i^{-1}(s)| = |(h_i \circ f_i)^{-1}(s)|$, there exists a homeomorphism $h_{i+1} : N(\mathcal{U}_{i+1}) \to S$ such that $h_i \circ f_i = g_i \circ h_{i+1}$ by Theorem 9. ■

The following is the Anderson–Choquet embedding theorem:

THEOREM 11 ([1]). *Let the compact sets $\{M_i\}_{i=1}^{\infty}$ be subsets of a given compact metric space X, and let $f_i^j : M_j \to M_i$ be continuous surjections satisfying $f_i^k = f_i^j \circ f_j^k$ for each $i < j < k$. Suppose that*

1. *for every i and $\delta > 0$ there exists a $\delta' > 0$ such that if $i < j$, p and q are in M_j, and $\mathrm{d}(f_i^j(p), f_i^j(q)) < \delta$, then $\mathrm{d}(p, q) < \delta'$,*
2. *for every $\epsilon > 0$ there exists an integer k such that if $p \in M_k$ then*

$$\operatorname{diam}\Big(\bigcup_{k<j} (f_k^j)^{-1}(p)\Big) < \epsilon.$$

Then the inverse limit $M = \varprojlim\{M_i, f_i\}_{i=1}^{\infty}$ is homeomorphic to $Q = \bigcap_{i=1}^{\infty}(\overline{\bigcup_{i\le k} M_k})$, which is the sequential limiting set of the sequence $\{M_i\}_{i=1}^{\infty}$.

The following theorem is the main result of this section:

THEOREM 12. *Suppose that $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a nested refining sequence of taut circle-chain covers of X such that no subchain of $\mathcal{U}_{i+1}$ is properly folded in $\mathcal{U}_i$. Then X is homeomorphic to $\varprojlim(S, z^{n(i)})_{i=1}^{\infty}$ where $n(i) = \deg_{\mathcal{U}_i}(\mathcal{U}_{i+1})$.*

Proof. This follows directly from Lemma 10 and the Anderson–Choquet Embedding Theorem 11. ■

Hence, in order to prove the main theorem of this paper it suffices to show the following:

Suppose that $\{\mathcal{U}_j\}_{j=1}^{\infty}$ is a nested sequence of refining circle-chain covers that limits to X. If for every $\xi > 0$ there exists a subchain $\widehat{\mathcal{U}}_{j(\xi)+1}$ of $\mathcal{U}_{j(\xi)+1}$ that is properly folded in $\mathcal{U}_{j(\xi)}$ and $\operatorname{diam}(\widehat{\mathcal{U}}^{*}_{j(\xi)+1})/\operatorname{mesh}(\mathcal{U}_{j(\xi)}) \ge \xi$, then X does not admit an expansive homeomorphism.

3. Previous results and techniques. In this section we will examine previous results in the literature that will allow us to make assumptions on the behavior of an expansive homeomorphism.

A subcontinuum H of X is *stable* under a homeomorphism $h : X \to X$ if

$$\lim_{n\to\infty} \operatorname{diam}(h^n(H)) = 0.$$

Likewise, H is *unstable* under h if

$$\lim_{n\to-\infty} \operatorname{diam}(h^n(H)) = 0.$$

The following theorem is by Kato:

Theorem 13 ([8]). *If $h : X \to X$ is an expansive homeomorphism, then there exists either a stable or an unstable subcontinuum.*

Since h^{-1} is expansive if and only if h is expansive, we may assume $h : X \to X$ has an unstable subcontinuum. Also, if H is an unstable subcontinuum, then each subcontinuum of H is clearly unstable.

If $h : X \to X$ is a homeomorphism and $x, y \in X$ then define

$$\mathrm{d}_k^j(x,y) = \max\{\mathrm{d}(h^i(x),h^i(y)) \mid i \in \{k,\dots,j\}\},$$
$$\mathrm{d}_{-\infty}^j(x,y) = \sup\{\mathrm{d}(h^i(x),h^i(y)) \mid i \le j\}.$$

The next theorem is a version of Theorem 5.1 by Fathi [5]. Only the essential changes of the proof are included.

Theorem 14 ([5]). *If $h : X \to X$ is an expansive homeomorphism with expansive constant c, then there exists a metric $\mathrm{d} : X \times X \to [0,\infty)$ that preserves the topology on X with the following property: There exists $\alpha > 1$ such that*

(1) *if* $\operatorname{diam}(U) < c$ *then* $\operatorname{diam}(h^n(U)) < 4\alpha^{|n|+1} \operatorname{diam}(U)$,
(2) *if $x, y \in X$ such that if $\mathrm{d}_{-\infty}^n(x,y) \le c$, then*

$$\frac{\alpha^n}{4}\,\mathrm{d}(x,y) \le \mathrm{d}(h^n(x),h^n(y)) \le 4\alpha^{n+1}\,\mathrm{d}(x,y).$$

Proof. Let D be a metric on X defining its topology and let c be the expansive constant for h. For $x, y \in X$, define

$$n(x,y) = \begin{cases} \infty & \text{if } x = y,\\ \min\{n_0 \in \mathbb{N}\cup\{0\} \mid \max_{|i|\le n_0}\{\mathrm{D}(h^i(x),h^i(y))\} \ge c\} & \text{if } x \ne y. \end{cases}$$

Pick some $\alpha > 1$ and define $\rho : X\times X \to [0,\infty)$ by $\rho(x,y) = (4c\alpha)\alpha^{-n(x,y)}$. So if

$$\max_{|i|\le n-1} \rho(h^i(x),h^i(y)) \le 4c$$

then

$$\alpha^n\rho(x,y) \le \max\{\rho(h^n(x),h^n(y)),\rho(h^{-n}(x),h^{-n}(y))\} \le \alpha^{n+1}\rho(x,y).$$

Also, it can be shown that there exists $\alpha > 1$ such that ρ has the following properties:

(1) $\rho(x,y) = 0$ if and only if $x = y$,
(2) $\rho(x,y) = \rho(y,x)$,
(3) $\rho(x,y) \le 2\max\{\rho(x,z), \rho(z,y)\}$.

Therefore, by the Frink Metrization Theorem [6] there exists a metric d on X that preserves the topology and such that

$$\mathrm{d}(x,y) \le \rho(x,y) \le 4\,\mathrm{d}(x,y).$$

Thus, if $\mathrm{d}^n_{-\infty}(x,y) \le c$, then $\max_{|i|\le n-1} d(h^i(x), h^i(y)) \le c$. Hence

$$\frac{\alpha^n}{4}\,\mathrm{d}(x,y) \le \mathrm{d}(h^n(x), h^n(y)) \le 4\alpha^{n+1}\,\mathrm{d}(x,y). \ ■$$

In Theorem 14, α is called the *growth multiplier*.

Suppose that H is an unstable subcontinuum of h. Since X is circle-like and H is a proper subcontinuum, H must be chainable. Let $\mathcal{U}$ be a taut open cover for X. We say that the unstable subcontinuum H has *property* $E(\mathcal{U}, c)$ if there exist

(1) $n \in \mathbb{Z}$,
(2) $x, y \in h^n(H)$,
(3) a chain cover $\mathcal{C}$ of $h^n(H)$

such that

(1) $\mathcal{C}$ refines $\mathcal{U}$,
(2) there exists $C \in \mathcal{C}$ such that $x, y \in C$,
(3) $\mathrm{d}^0_{-\infty}(x,y) \ge c$.

For a homeomorphism h and a positive integer n, define $\mathcal{L}(h,n,\epsilon)$ to be a number greater than 0 such that

$$\mathrm{d}(x,y) < \mathcal{L}(h,n,\epsilon) \quad \text{implies} \quad \mathrm{d}(h^i(x), h^i(y)) < \epsilon \text{ for all } -n \le i \le n.$$

Since h is uniformly continuous, $\mathcal{L}(h,n,\epsilon)$ can always be found.

The following lemmas follow immediately from Lemmas 4 and 5 in [9]. Note that all chains are tree-covers.

Lemma 15 ([9]). *Suppose that $h : X \to X$ is a homeomorphism of a continuum X and H is a subcontinuum of X. Suppose that there exist $a, b \in H$ and a tree-cover $\mathcal{T}$ of H such that a and b are in the same element of $\mathcal{T}$ and $\mathrm{d}^0_n(a,b) \ge \epsilon$ where $n \le 0$. Then there exist $x_\alpha, x_\beta \in H$ such that $\epsilon/3 \le \mathrm{d}^0_n(x_\alpha, x_\beta) < \epsilon$ and x_α, x_β are in the same element of $\mathcal{T}$.*

LEMMA 16 ([9]). *Let $h : X \to X$ be a homeomorphism of a compact space onto itself. Suppose that there are sequences $\{z_k\}_{k=1}^{\infty}$, $\{w_k\}_{k=1}^{\infty}$ such that $\mathrm{d}_{-\infty}^{k}(z_k, w_k) < \epsilon$. Then there exist a limit point z of $\{z_k\}_{k=1}^{\infty}$ and a limit point w of $\{w_k\}_{k=1}^{\infty}$ such that $\mathrm{d}(h^i(z), h^i(w)) < 2\epsilon$ for all i.*

The next theorem is the main result of this section:

THEOREM 17. *Let $h : X \to X$ be a homeomorphism. Suppose that for every $\delta > 0$ there exist an unstable subcontinuum H_δ and a taut open cover $\mathcal{U}_\delta$ with $\mathrm{mesh}(\mathcal{U}_\delta) < \delta$ such that H_δ has property $E(\mathcal{U}_\delta, c)$. Then c is not an expansive constant for h.*

Proof. Let $0 < \epsilon < c/3$. Suppose that H_k has property $E(\mathcal{U}_{\delta_k}, c)$ where $\delta_k < \mathcal{L}(h, k, \epsilon)$. Since H_k is unstable, we may assume that

$$(*) \qquad \mathrm{diam}(h^i(H_k)) < \epsilon \quad \text{for each } i \le 0.$$

Then there exist an integer n_k and points $x_k, y_k \in h^{n_k}(H_k)$ such that $\mathrm{d}(x_k, y_k) < \delta_k$ but $\mathrm{d}_{-\infty}^{0}(x_k, y_k) \ge c$. It follows from $(*)$ that $\mathrm{d}_{-n_k}^{0}(x_k, y_k) \ge c > \epsilon$. Thus from Lemma 15 there exist $x_k', y_k' \in h^{n_k}(H_k)$ such that $\epsilon/3 < \mathrm{d}_{-n_k}^{0}(x_k', y_k') < \epsilon$ and $\mathrm{d}(x_k', y_k') < \delta_k$. Let $n_k' \in \{-n_k, \dots, 0\}$ be such that $\mathrm{d}(h^{n_k'}(x_k'), h^{n_k'}(y_k')) > \epsilon/3$ and set $z_k = h^{n_k'}(x_k')$ and $w_k = h^{n_k'}(y_k')$. It now follows from $(*)$ and $\mathrm{d}(x_k', y_k') < \delta_k < \mathcal{L}(h, k, \epsilon)$ that $\mathrm{d}_{-\infty}^{k}(z_k, w_k) < \epsilon$. Thus by Lemma 16, there exist limit points z and w of $\{z_k\}_{k=1}^{\infty}$ and $\{w_k\}_{k=1}^{\infty}$ such that $\mathrm{d}(h^i(z), h^i(w)) < 2\epsilon < c$ for all i. Since $\mathrm{d}(w_k, z_k) > \epsilon/3$, z and w can be taken to be distinct. Thus c is not a constant of expansion for h. ∎

A circle-like continuum X has *degree* 1 if there exists a nested sequence $\{\mathcal{U}_j\}_{j=1}^{\infty}$ of refining covers that limit to X such that $\deg_{\mathcal{U}_{j+1}}(\mathcal{U}_j) \le 1$ for each j. The following theorem is an immediate corollary of Theorem 20 in [10].

THEOREM 18. *Degree* 1, *circle-like continua do not admit expansive homeomorphisms.*

From here on we will make the following **assumptions**:

(1) $h : X \to X$ is an expansive homeomorphism with expansive constant c and growth multiplier α,
(2) X is a circle-like continuum,
(3) $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a nested collection of taut refining circle-chain covers that limit to X,
(4) $\deg_{\mathcal{U}_{j+1}}(\mathcal{U}_j) \ge 2$ for each j,
(5) h has an unstable subcontinuum (usually denoted by H or K).

The primary focus of the proof of the main theorem will be to show either that c is not an expansive constant by finding an unstable subcontinuum H

that has property $E(\mathcal{U},c)$ where $\mathcal{U}$ has arbitrary mesh, or that Theorem 14 will be contradicted.

4. The chaining and wrapping of subcontinua. In this section we measure how subcontinua wrap in a circle-chain. If $\mathcal{V} = [V_0, \ldots, V_{p-1}]$ is a chain that refines a circle-chain $\mathcal{U}$, then define

$$C_i = \bigcup\{V_j \in \mathcal{V} \mid \Delta_{\mathcal{U}}^{\mathcal{V}}(j) - \min \Delta_{\mathcal{U}}^{\mathcal{V}} = i\}$$

and

$$\mathcal{C}(\mathcal{V},\mathcal{U}) = \{C_i \mid i \in \{0, \ldots, \max \Delta_{\mathcal{U}}^{\mathcal{V}} - \min \Delta_{\mathcal{U}}^{\mathcal{V}}\}\}.$$

Then notice that $\mathcal{C} = \mathcal{C}(\mathcal{V},\mathcal{U})$ is a chain that refines $\mathcal{U}$ such that $\Delta_{\mathcal{U}}^{\mathcal{C}}(i) = i$. (See Figure 2.)

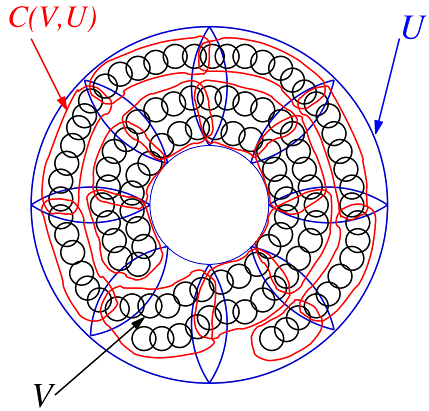

Fig. 2. $\mathcal{C}(\mathcal{V},\mathcal{U})$ is a chain that refines $\mathcal{U}$ such that $\Delta_{\mathcal{U}}^{\mathcal{C}}(i) = i$.

Likewise, if H is a proper subcontinuum of X then define $\mathcal{C}(H,\mathcal{U})$ to be some chain cover of H that refines $\mathcal{U}$ and has the property that $\Delta_{\mathcal{U}}^{\mathcal{C}(H,\mathcal{U})}(i) = i$. If $\mathcal{V}$ is a proper chain cover of H that refines $\mathcal{U}$, then we may take $\mathcal{C}(H,\mathcal{U}) = \mathcal{C}(\mathcal{V},\mathcal{U})$ when convenient. Notice that under this construction if $|\mathcal{U}| = k$ and $C_i, C_{i+kn} \in \mathcal{C}(H,\mathcal{U})$ then C_i and C_{i+kn} are in the same element of $\mathcal{U}$.

Next we are going to examine how the number of elements of $\mathcal{C}^2 = \mathcal{C}(H,\mathcal{U})$ is related to the number of elements of $\mathcal{C}^1 = \mathcal{C}(H,\mathcal{U}_0)$ when $\mathcal{U}$ refines $\mathcal{U}_0$. First create $C_j^1 \in \mathcal{C}^1$ by $C_j^1 = \bigcup\{C_i^2 \in \mathcal{C}^2 \mid j = \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i) - \min \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}\}$.

Proposition 19. *Let $\mathcal{U}$ and $\mathcal{U}_0$ be circle-chains such that $\mathcal{U}$ refines $\mathcal{U}_0$ and let $\mathcal{V}$ be a chain that refines $\mathcal{U}$. Let $\mathcal{C}^2 = \mathcal{C}(\mathcal{V},\mathcal{U})$ and p be such that $\Delta_{\mathcal{U}}^{\mathcal{V}}(p) = \min \Delta_{\mathcal{U}}^{\mathcal{V}}$. If $\Delta_{\mathcal{U}}^{\mathcal{V}}(k) = \beta + \min \Delta_{\mathcal{U}}^{\mathcal{V}}$, then $\Delta_{\mathcal{U}_0}^{\mathcal{V}}(k) = \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(\beta) + \Delta_{\mathcal{U}_0}^{\mathcal{V}}(p)$.*

Proof. It follows from the construction of $\mathcal{C}^2$ that if $\Delta_{\mathcal{U}}^{\mathcal{V}}(k) = \Delta_{\mathcal{U}}^{\mathcal{C}^2}(\beta) + \min \Delta_{\mathcal{U}}^{\mathcal{V}}$, then $V_k \subset C_\beta^2$. Thus $V_p \subset C_0^2$. Also, notice that if $V_i \subset C_\alpha^2$ and

$V_k \subset C^2_\beta$, then

$$\Delta^{\mathcal{V}}_{\mathcal{U}_0}(k) - \Delta^{\mathcal{V}}_{\mathcal{U}_0}(i) = \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(\beta) - \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(\alpha).$$

Since $\Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(0) = 0$, we have $\Delta^{\mathcal{V}}_{\mathcal{U}_0}(k) = \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(\beta) + \Delta^{\mathcal{V}}_{\mathcal{U}_0}(p)$. ∎

PROPOSITION 20. *$\Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(i+m) = \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(i) + k \deg_{\mathcal{U}_0}(\mathcal{U})$ where $|\mathcal{U}_0| = k$ and $|\mathcal{U}| = m$.*

Proof. Since C^2_i and C^2_{i+m} are in the same element of $\mathcal{U}$, they are in the same element of $\mathcal{U}_0$. Therefore

$$\deg_{\mathcal{U}_0}([C^2_i, \ldots, C^2_{i+m}]) = \frac{\Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(i+m) - \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(i)}{k}.$$

Also, by Theorem 2,

$$\deg_{\mathcal{U}_0}([C^2_i, \ldots, C^2_{i+m}]) = \deg_{\mathcal{U}_0}(\mathcal{U}) \deg_{\mathcal{U}}([C^2_i, \ldots, C^2_{i+m}]) = \deg_{\mathcal{U}_0}(\mathcal{U})$$

since

$$\deg_{\mathcal{U}}([C^2_i, \ldots, C^2_{i+m}]) = \frac{\Delta^{\mathcal{C}^2}_{\mathcal{U}}(i+m) - \Delta^{\mathcal{C}^2}_{\mathcal{U}}(i)}{m} = \frac{i+m-i}{m} = 1.$$

Thus, the proposition follows. ∎

PROPOSITION 21. *Let $\mathcal{C}$ be a chain that refines the circle-chain $\mathcal{U}_0$ and $\Delta^{\mathcal{C}}_{\mathcal{U}_0}(i+m) = \Delta^{\mathcal{C}}_{\mathcal{U}_0}(i) + \gamma$ for each i where $\gamma > 0$. If $j_2 \geq j_1 + m$ then there exists $\beta \in \{j_2 - m + 1, \ldots, j_2\}$ such that $\Delta^{\mathcal{C}}_{\mathcal{U}_0}(\beta) \geq \max_{i \leq j_1} \Delta^{\mathcal{C}}_{\mathcal{U}_0}(i) + \gamma$.*

Proof. Since $\Delta^{\mathcal{C}}_{\mathcal{U}_0}(i+m) = \Delta^{\mathcal{C}}_{\mathcal{U}_0}(i) + \gamma$, it follows that

$$\max_{i \leq j_1} \Delta^{\mathcal{C}}_{\mathcal{U}_0}(i) = \max_{j_1 - m + 1 \leq i \leq j_1} \Delta^{\mathcal{C}}_{\mathcal{U}_0}(i).$$

Let $\alpha \in \{j_1 - m + 1, \ldots, j_1\}$ be such that $\max_{i \leq j_1} \Delta^{\mathcal{C}}_{\mathcal{U}_0}(i) = \Delta^{\mathcal{C}}_{\mathcal{U}_0}(\alpha)$. There exists an integer s such that $\beta = \alpha + sm \in \{j_2 - m + 1, \ldots, j_2\}$. Since $j_2 - m \geq j_1 \geq \alpha$, we have $s \geq 1$. Hence,

$$\Delta^{\mathcal{C}}_{\mathcal{U}_0}(\beta) = \Delta^{\mathcal{C}}_{\mathcal{U}_0}(\alpha + sm) = \Delta^{\mathcal{C}}_{\mathcal{U}_0}(\alpha) + s\gamma \geq \max_{i \leq j_1} \Delta^{\mathcal{C}}_{\mathcal{U}_0}(i) + \gamma. \ ∎$$

Let

$$\max \Delta^{\mathcal{V}}_{\mathcal{U}}(\{i, \ldots, j\}) = \max_{k \in \{i, \ldots, j\}} \Delta^{\mathcal{V}}_{\mathcal{U}}(k), \quad \min \Delta^{\mathcal{V}}_{\mathcal{U}}(\{i, \ldots, j\}) = \min_{k \in \{i, \ldots, j\}} \Delta^{\mathcal{V}}_{\mathcal{U}}(k).$$

LEMMA 22. *Suppose that $\mathcal{U}$ and $\mathcal{U}_0$ are circle-chains, $\mathcal{V}$ is a chain and $m = |\mathcal{U}|$ such that*

(1) *$\mathcal{U}$ refines $\mathcal{U}_0$ with $\deg_{\mathcal{U}_0}(\mathcal{U}) \geq 1$,*
(2) *$\mathcal{V}$ refines $\mathcal{U}$,*
(3) *there exist subchains $\mathcal{V}_1 = [V_{i_1}, \ldots, V_{j_1}]$ and $\mathcal{V}_2 = [V_{i_2}, \ldots, V_{j_2}]$ of $\mathcal{V}$ such that*

$$\max \Delta^{\mathcal{V}}_{\mathcal{U}}(\{i_2, \ldots, j_2\}) - \max \Delta^{\mathcal{V}}_{\mathcal{U}}(\{i_1, \ldots, j_1\}) \geq m.$$

Then there exists $k \in \{i_2, \ldots, j_2\}$ *such that* $\Delta^{\mathcal{V}}_{\mathcal{U}_0}(k) \geq \max \Delta^{\mathcal{V}}_{\mathcal{U}_0}(\{i_1, \ldots, j_1\}) + |\mathcal{U}_0|$.

Proof. Let $\mathcal{C}^2 = \mathcal{C}(\mathcal{V},\mathcal{U})$, $\beta_i = \Delta^{\mathcal{V}}_{\mathcal{U}}(i) - \min \Delta^{\mathcal{V}}_{\mathcal{U}}$, and p be such that $\Delta^{\mathcal{V}}_{\mathcal{U}}(p) = \min \Delta^{\mathcal{V}}_{\mathcal{U}}$. Notice that since $\Delta^{\mathcal{C}^2}_{\mathcal{U}}(\beta_i) = \beta_i$,

$$\begin{aligned}\max \Delta^{\mathcal{C}^2}_{\mathcal{U}}(\{\beta_{i_2}, \ldots, \beta_{j_2}\}) &= \max\{\beta_{i_2}, \ldots, \beta_{j_2}\} \\ &= \max \Delta^{\mathcal{V}}_{\mathcal{U}}(\{i_2, \ldots, j_2\}) - \min \Delta^{\mathcal{V}}_{\mathcal{U}} \\ &\geq \max \Delta^{\mathcal{V}}_{\mathcal{U}}(\{i_1, \ldots, j_1\}) - \min \Delta^{\mathcal{V}}_{\mathcal{U}} + m \\ &= \max\{\beta_{i_1}, \ldots, \beta_{j_1}\} + m \\ &= \max \Delta^{\mathcal{C}^2}_{\mathcal{U}}(\{\beta_{i_1}, \ldots, \beta_{j_1}\}) + m.\end{aligned}$$

Also recall by Proposition 20, $\Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(i+m) = \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(i) + |\mathcal{U}_0| \deg_{\mathcal{U}_0}(\mathcal{U})$. So it follows from Proposition 21 that there exists

$$\beta \in \{\max \Delta^{\mathcal{C}^2}_{\mathcal{U}}(\{\beta_{i_2}, \ldots, \beta_{j_2}\}) - m + 1, \ldots, \max \Delta^{\mathcal{C}^2}_{\mathcal{U}}(\{\beta_{i_2}, \ldots, \beta_{j_2}\})$$

such that

$$\begin{aligned}\Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(\beta) &\geq \max_{\beta_i \leq \max \Delta^{\mathcal{C}^2}_{\mathcal{U}}(\{\beta_{i_1}, \ldots, \beta_{j_1}\})} \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(\beta_i) + |\mathcal{U}_0| \deg_{\mathcal{U}_0}(\mathcal{U}) \\ &\geq \max_{\beta_i \leq \max\{\beta_{i_1}, \ldots, \beta_{j_1}\}} \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(\beta_i) + |\mathcal{U}_0| \deg_{\mathcal{U}_0}(\mathcal{U}) \\ &\geq \max_{i \in \{i_1, \ldots, j_1\}} \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(\beta_i) + |\mathcal{U}_0|.\end{aligned}$$

Let $k \in \{i_2, \ldots, j_2\}$ be such that $\Delta^{\mathcal{V}}_{\mathcal{U}}(k) = \beta + \min \Delta^{\mathcal{V}}_{\mathcal{U}}$. So by Proposition 19,

$$\begin{aligned}\Delta^{\mathcal{V}}_{\mathcal{U}_0}(k) = \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(\beta) + \Delta^{\mathcal{V}}_{\mathcal{U}_0}(p) &\geq \max_{i \in \{i_1, \ldots, j_1\}} \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(\beta_i) + \Delta^{\mathcal{V}}_{\mathcal{U}_0}(p) + |\mathcal{U}_0| \\ &\geq \max_{i \in \{i_1, \ldots, j_1\}} \Delta^{\mathcal{V}}_{\mathcal{U}_0}(i) + |\mathcal{U}_0| = \max \Delta^{\mathcal{V}}_{\mathcal{U}_0}(\{i_1, \ldots, j_1\}) + |\mathcal{U}_0|. \quad \blacksquare\end{aligned}$$

Lemma 23.

$$|\mathcal{C}^2| \leq m\left(\frac{|\mathcal{C}^1| - (\max_{i \in \{0, \ldots, m-1\}} \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(i) - \min_{i \in \{0, \ldots, m-1\}} \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(i))}{k \deg_{\mathcal{U}_0}(\mathcal{U})} + 1\right).$$

Proof. The lemma follows from

$$\begin{aligned}|\mathcal{C}^1| &= \max \Delta^{\mathcal{C}^2}_{\mathcal{U}_0} - \min \Delta^{\mathcal{C}^2}_{\mathcal{U}_0} \\ &\geq \max_{i \in \{0, \ldots, m-1\}} \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(i) + k\left(\frac{|\mathcal{C}^2|}{m} \deg_{\mathcal{U}_0}(\mathcal{U}) - 1\right) - \min_{i \in \{0, \ldots, m-1\}} \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(i). \quad \blacksquare\end{aligned}$$

Proposition 24.

$$\max \Delta^{\mathcal{C}^2}_{\mathcal{U}_0} \leq \max_{i \in \{0, \ldots, m-1\}} \Delta^{\mathcal{C}^2}_{\mathcal{U}_0}(i) + k\,\frac{|\mathcal{C}^2|}{m} \deg_{\mathcal{U}_0}(\mathcal{U}).$$

Proof. Let $\mathcal{C}^2 = [C_0, \ldots, C_{q-1}]$. Since $\Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i+m) = \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i) + k\deg_{\mathcal{U}_0}(\mathcal{U})$, it follows that

$$\begin{aligned}\max \Delta_{\mathcal{U}_0}^{\mathcal{C}^2} &= \max_{i\in\{q-m,\ldots,q-1\}} \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i) \\ &= \max \{\Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i) + (r-1)k\deg_{\mathcal{U}_0}(\mathcal{U}) \mid i \in \{q-rm, \ldots, m-1\}\} \\ &\qquad \cup \{\Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i) + rk\deg_{\mathcal{U}_0}(\mathcal{U}) \mid i \in \{0, \ldots, q-rm-1\}\},\end{aligned}$$

where $r = \lfloor q/m \rfloor$. Thus

$$\max \Delta_{\mathcal{U}_0}^{\mathcal{C}^2} \le \max_{i\in\{0,\ldots,m-1\}} \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i) + k\,\frac{q}{m}\deg_{\mathcal{U}_0}(\mathcal{U}). \ \blacksquare$$

Proposition 25. $\min \Delta_{\mathcal{U}_0}^{\mathcal{C}^2} = \min_{i\in\{0,\ldots,m-1\}} \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i)$.

Proof. Follows from Proposition 20. ■

Theorem 26.

$$\begin{aligned} m&\left(\frac{|\mathcal{C}^1| - (\max_{i\in\{0,\ldots,m-1\}} \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i) - \min_{i\in\{0,\ldots,m-1\}} \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i))}{k\deg_{\mathcal{U}_0}(\mathcal{U})}\right) \\ &\le |\mathcal{C}^2| \\ &\le m\left(\frac{|\mathcal{C}^1| - (\max_{i\in\{0,\ldots,m-1\}} \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i) - \min_{i\in\{0,\ldots,m-1\}} \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}(i))}{k\deg_{\mathcal{U}_0}(\mathcal{U})} + 1\right).\end{aligned}$$

Proof. Follows from the fact that $|\mathcal{C}^1| = \max \Delta_{\mathcal{U}_0}^{\mathcal{C}^2} - \min \Delta_{\mathcal{U}_0}^{\mathcal{C}^2}$ and from Propositions 20, 24 and 25 and Lemma 23. ■

5. The behavior of unstable subcontinua. In this section we study the behavior of unstable subcontinua under h. In particular, if H is an unstable subcontinuum, then $h^n(H)$ is wrapped in any circle-chain at an approximately constant rate as n increases. Also, it is shown that for every chain there is an unstable subcontinuum properly covered by that chain.

The following theorem is the Mountain Climber Theorem [11]:

Theorem 27. *Suppose that $\phi : [a,b] \to [c,d]$ and $\psi : [a,b] \to [c,d]$ are continuous functions each with a finite number of monotone pieces and such that $\phi(a) = \psi(a) = c$ and $\phi(b) = \psi(b) = d$. Then there exist continuous functions $f, g : [0,1] \to [a,b]$ such that $f(0) = g(0) = a$, $f(1) = g(1) = b$ and $\phi \circ f = \psi \circ g$.*

Lemma 28. *Let Y be an arcwise connected plane continuum and $\pi_i : Y \to \mathbb{R}$ be the ith coordinate map for $i = 1, 2$. Suppose $q > 0$ is such that*

(1) *there exist $y_0, y_1 \in \pi_1^{-1}(0)$ such that $\mathrm{d}(y_0, y_1) > 2q$,*
(2) *if $w, z \in \pi_1^{-1}(0)$ then $\mathrm{d}(w, z) < q$ or $\mathrm{d}(w, z) > 2q$.*

Then there exist $x_1, x_2 \in \pi_1^{-1}(0)$ *and an arc* A *in* Y *from* x_1 *to* x_2 *such that*

(3) $\mathrm{d}(x_1, x_2) > 2q$,
(4) $\pi_1(A) \subset (-\infty, 0]$ *or* $\pi_1(A) \subset [0, \infty)$.

Proof. Since Y is arcwise connected, there exists an arc D in Y from y_0 to y_1. Let $p : [0,1] \to D$ be a homeomorphism such that $p(0) = y_0$ and $p(1) = y_1$. Let $D' = D \cap \pi_1^{-1}(0)$. Define $p(t)$, $p(t')$ to be *consecutive elements* of D' if there is no element $t'' \in p^{-1}(D')$ such that $t < t'' < t'$.

Claim. *There exist consecutive elements* $p(t)$, $p(t')$ *of* D' *such that*

$$\mathrm{d}(p(t), p(t')) > 2q.$$

Suppose on the contrary that if $p(t)$, $p(t')$ are consecutive elements of D' then $\mathrm{d}(p(t), p(t')) < q$. Then there exists a sequence $\{t_i\}_{i=0}^{n} \subset D'$ such that $0 = t_0 < t_1 < \cdots < t_n = 1$ and $\mathrm{d}(p(t_i), p(t_{i+1})) < q$ for each i. However since $\mathrm{d}(p(t_0), p(t_n)) = \mathrm{d}(y_0, y_1) > 2q$, it follows from the triangle inequality that there exist i, j such that $q \le \mathrm{d}(p(t_i), p(t_j)) \le 2q$, which contradicts hypothesis (2).

Next, let x_1, x_2 be consecutive elements of D' such that $\mathrm{d}(x_1, x_2) > 2q$ and A be the subarc of D from x_1 to x_2. Then $A \cap \pi_1^{-1}(0) = \{x_1, x_2\}$. Thus, $\pi_1(A) \subset (-\infty, 0]$ or $\pi_1(A) \subset [0, \infty)$. ∎

Lemma 29. *Suppose that* A *is a finitely piecewise linear arc in* $\mathbb{R}^2$ *with endpoints* $(0, y_1)$, $(0, y_2)$ *such that either* $\pi_1(A) \subset (-\infty, 0]$ *or* $\pi_1(A) \subset [0, \infty)$. *Then for every nonnegative* $r \le |y_1 - y_2|$ *there exist* $(x_r, p_1), (x_r, p_2) \in A$ *such that* $|p_1 - p_2| = r$.

Proof. Assume that $\pi_1(A) \subset [0, \infty)$ (the proof is similar for $\pi_1(A) \subset (-\infty, 0]$). Let $m = \max \pi_1(A)$ and $(m, y_m) \in A$. Let A_1 be the subarc of A from $(0, y_1)$ to (m, y_m) and A_2 be the subarc of A from $(0, y_2)$ to (m, y_m). Then by the Mountain Climber Theorem there exist continuous functions $f : [0,1] \to A_1$ and $g : [0,1] \to A_2$ such that $\pi_1 \circ f = \pi_1 \circ g$. Since $\mathrm{d}(f(0), g(0)) = |y_1 - y_2|$ and $\mathrm{d}(f(1), g(1)) = 0$, it follows from the intermediate value theorem that there exists $t_r \in [0,1]$ such that $\mathrm{d}(f(t_r), g(t_r)) = r$. Letting $(x_r, p_1) = f(t_r)$ and $(x_r, p_2) = g(t_r)$ completes the proof. ∎

Lemma 30. *Suppose that* $h : X \to X$ *is a homeomorphism*, $\mathcal{U}_0, \mathcal{U}_1$ *are circle-chains*, $\mathcal{V} = [V_0, \ldots, V_{p-1}]$ *is a chain and* $i, j \in \{0, \ldots, p-1\}$ *are such that*

(1) $\mathcal{V}$ *refines* $\mathcal{U}_1$ *and* $h(\mathcal{V})$ *refines* $\mathcal{U}_0$,
(2) $\Delta_{\mathcal{U}_1}^{\mathcal{V}}(i) = \Delta_{\mathcal{U}_1}^{\mathcal{V}}(j)$,
(3) $|\Delta_{\mathcal{U}_0}^{h(\mathcal{V})}(i) - \Delta_{\mathcal{U}_0}^{h(\mathcal{V})}(j)| \ge 6$.

Then there exists $i', j' \in \{i, \dots, j\}$ *such that* $\Delta^{\mathcal{V}}_{\mathcal{U}_1}(i') = \Delta^{\mathcal{V}}_{\mathcal{U}_1}(j')$ *and*

$$2 \le |\Delta^{h(\mathcal{V})}_{\mathcal{U}_0}(i') - \Delta^{h(\mathcal{V})}_{\mathcal{U}_0}(j')| \le 4.$$

Proof. Let $Y =$ the union of straight line segments in $\mathbb{R}$ from

$$(\Delta^{\mathcal{V}}_{\mathcal{U}_1}(k), \Delta^{h(\mathcal{V})}_{\mathcal{U}_0}(k)) \text{ to } (\Delta^{\mathcal{V}}_{\mathcal{U}_1}(k+1), \Delta^{h(\mathcal{V})}_{\mathcal{U}_0}(k+1))$$

for $k \in \{i, \dots, j\}$ where i and j are defined in the statement of the lemma. Then let $Y_i = \{(a - \Delta^{\mathcal{V}}_{\mathcal{U}_1}(i), b) \mid (a, b) \in Y\}$. Notice that

(1) Y_i is arcwise connected,
(2) $(0, \Delta^{h(\mathcal{V})}_{\mathcal{U}_0}(i)), (0, \Delta^{h(\mathcal{V})}_{\mathcal{U}_0}(j)) \in Y_i$

Thus by Lemmas 28 and 29 there exist $(x_3, p_1), (x_3, p_2) \in Y_i$ such that $|p_1 - p_2| = 3$. Since Y_i is composed of line segments of the form $[(\alpha, \beta), (\alpha+1, \beta)]$, $[(\alpha, \beta), (\alpha+1, \beta+1)]$, $[(\alpha, \beta), (\alpha+1, \beta-1)]$ and $[(\alpha, \beta), (\alpha, \beta+1)]$ where $\alpha, \beta \in \mathbb{Z}$, there exist $y_1 \in \{\lceil p_1 \rceil, \lfloor p_1 \rfloor\}$ and $y_2 \in \{\lceil p_2 \rceil, \lfloor p_2 \rfloor\}$ such that $(\lceil x_3 \rceil, y_1), (\lceil x_3 \rceil, y_2) \in Y_i$. Hence $|y_1 - y_2| \in \{2, 3, 4\}$. Since $\lceil x_3 \rceil$, y_1 and y_2 are all integers, there exist integers i', j' such that $\Delta^{\mathcal{V}}_{\mathcal{U}_1}(i') = \Delta^{\mathcal{V}}_{\mathcal{U}_1}(j') = \lceil x_3 \rceil$, $\Delta^{h(\mathcal{V})}_{\mathcal{U}_0}(i') = y_1$ and $\Delta^{h(\mathcal{V})}_{\mathcal{U}_0}(j') = y_2$. Thus, the theorem follows. ∎

If $\mathcal{U}$ is a taut finite open cover then define

$$\mathrm{d}(\mathcal{U}) = \min\{\mathrm{d}(U_i, U_j) \mid U_i, U_j \in \mathcal{U} \text{ and } U_i \cap U_j = \emptyset\}.$$

Notice that since $\mathcal{U}$ is taut, $\mathrm{d}(\mathcal{U}) > 0$. Also let $\mathrm{Leb}(\mathcal{U})$ be the Lebesgue number for $\mathcal{U}$.

Lemma 31. *Let $h : X \to X$ be an expansive homeomorphism with expansive constant c and growth multiplier $\alpha > 1$. Let $\mathcal{U}_0$, $\mathcal{U}_1$ and $\mathcal{U}_2$ be circle-chains, $\mathcal{C} = [C_0, \dots, C_{k-1}]$ be a chain and $i, j \in \{0, \dots, k-1\}$ be such that*

(1) $|\mathcal{U}_0| \ge 6$,
(2) $[C_i, \dots, C_j]$ *refines* $\mathcal{U}_2$, $\mathcal{U}_2$ *refines* $\mathcal{U}_1$ *and* $\mathcal{U}_1$ *refines* $\mathcal{U}_0$,
(3) $\mathrm{mesh}(\mathcal{U}_2) < (1/4)\,\mathrm{Leb}(\mathcal{U}_1)$, $\mathrm{mesh}(\mathcal{U}_1) < \mathrm{d}(\mathcal{U}_0)$ *and* $\mathrm{mesh}(h(\mathcal{U}_1)) < \mathrm{d}(\mathcal{U}_0)$,
(4) $\deg_{\mathcal{U}_1}(\mathcal{U}_2) \ge 1$ *and* $\deg_{\mathcal{U}_0}(\mathcal{U}_1) \ge 1$,
(5) C_i *and* C_j *are in the same element of* $\mathcal{U}_2$,
(6) $\deg_{\mathcal{U}_2}([C_i, \dots, C_j]) \ge 1$,
(7) $\mathcal{C}$ *properly covers an unstable subcontinuum* H.

Then $\mathrm{d}^0_{-\infty}(x, y) \ge c$ *for all* $x \in H \cap C_i$ *and* $y \in H \cap C_j$.

Proof. Pick $x \in H \cap C_i$ and $y \in H \cap C_j$ and let $N_H < 0$ be such that $\mathrm{diam}(h^{N_H}(H)) < \frac{1}{12\alpha}\,\mathrm{Leb}(\mathcal{U}_1)$. Then let $\mathcal{C}' = [C'_0, \dots, C'_{p-1}]$ be an open chain from x to y that refines $\mathcal{C}$ and such that $\mathrm{mesh}(h^n(\mathcal{C}')) < \frac{1}{12\alpha}\,\mathrm{Leb}(\mathcal{U}_1)$ for each $n \in \{N_H, \dots, 0\}$. It follows that $\deg_{\mathcal{U}_2}(\mathcal{C}') \ge 1$ and hence $\deg_{\mathcal{U}_1}(\mathcal{C}') \ge 1$ and $\deg_{\mathcal{U}_0}(\mathcal{C}') \ge 1$. Also, $\mathrm{diam}(h^{N_H}(\mathcal{C}')^*) < \mathrm{Leb}(\mathcal{U}_1)$. Therefore there exists $U_i^1 \in \mathcal{U}_1$ that contains $h^{N_H}(\mathcal{C}')$. Thus, $\deg_{\mathcal{U}_1}(h^{N_H}(\mathcal{C}')) = 0$ and $\deg_{\mathcal{U}_0}(h^{N_H}(\mathcal{C}')) = 0$.

Suppose on the contrary that $\mathrm{d}^0_{-\infty}(x,y) < c$. Then by Theorem 14,

$$\mathrm{d}(h^n(x), h^n(y)) < (4\alpha/\alpha^{|n|})\,\mathrm{d}(x,y) < (1/3)\,\mathrm{Leb}(\mathcal{U}_1)$$

for each $n \le 0$. Thus the endlinks of $h^n(\mathcal{C}')$ are always in the same element of $\mathcal{U}_1$ for each $n \in \{N_H, \dots, 0\}$ by Theorem 14 and the triangle inequality. Since $\deg_{\mathcal{U}_1}(h^{N_H}(\mathcal{C}')) = 0$, $\deg_{\mathcal{U}_0}(h^{N_H}(\mathcal{C}')) = 0$ and $\deg_{\mathcal{U}_0}(\mathcal{C}') \ge 1$, there exists an $N' \in \{N_H, \dots, -1\}$ such that $\deg_{\mathcal{U}_1}(h^{N'}(\mathcal{C}')) = 0$ and $\deg_{\mathcal{U}_0}(h^{N'+1}(\mathcal{C}')) \ge 1$. Hence $\Delta_{\mathcal{U}_1}^{h^{N'}(\mathcal{C}')}(0) = \Delta_{\mathcal{U}_1}^{h^{N'}(\mathcal{C}')}(p-1)$ and

$$|\Delta_{\mathcal{U}_0}^{h^{N'+1}(\mathcal{C}')}(0) - \Delta_{\mathcal{U}_0}^{h^{N'+1}(\mathcal{C}')}(p-1)| \ge |\mathcal{U}_0| \ge 6.$$

Therefore, by Lemma 30, there exist $C'_{i'}, C'_{j'} \in \mathcal{C}'$, $U^1_\alpha \in \mathcal{U}_1$ and $U^0_\beta, U^0_\gamma \in \mathcal{U}_0$ with $2 \le |\beta - \gamma| \le 4$ such that $h^{N'}(C'_{i'}), h^{N'}(C'_{j'}) \subset U^1_\alpha$ but $h^{N'+1}(C'_{i'}) \subset U^0_\beta$ and $h^{N'+1}(C'_{j'}) \subset U^0_\gamma$. However, that implies that $\mathrm{diam}(h(U^1_\alpha)) > \mathrm{d}(\mathcal{U}_0)$, which is a contradiction. ∎

LEMMA 32. *Given $\mathcal{U}_2$ as defined in Lemma* 31, *there exists $N > 0$ such that if H is an unstable subcontinuum such that $|\mathcal{C}(H,\mathcal{U}_2)| \ge |\mathcal{U}_2|$, then $|\mathcal{C}(h^n(H),\mathcal{U}_2)| \ge (3/2)|\mathcal{C}(H,\mathcal{U}_2)|$ for all $n \ge N$.*

Proof. There exists N such that $\alpha^n\,\mathrm{d}(\mathcal{U}_2) > 8\,\mathrm{mesh}(\mathcal{U}_2)$ for all $n \ge N$. Let $\mathcal{C}(H,\mathcal{U}_2) = [C_0, \dots, C_{p-1}]$. For each $i \in \{0, \dots, p-1\}$ pick $x_i \in C_i \cap C_{i+1} \cap H$ and $\widehat{x}_i \in \mathrm{core}(C_i) \cap H$ such that $\mathrm{d}(\widehat{x}_i, \mathrm{Bd}(\mathrm{core}(C_i))) \ge (1/2)\,\mathrm{d}(\mathcal{U}_2)$. Let $E = \{x_i\}_{i=0}^{p-2} \cup \{\widehat{x}_i\}_{i=0}^{p-1}$. Then E has the property that if x, y are distinct elements of E then one of the following must be true:

(1) $\mathrm{d}(x,y) \ge (1/2)\,\mathrm{d}(\mathcal{U}_2)$,
(2) $\mathrm{d}^0_{-\infty}(x,y) \ge c$.

This follows from the fact that if (1) is false, then x and y are in the same element of $\mathcal{U}_2$ but different elements of $\mathcal{C}(H,\mathcal{U}_2)$. Hence, it follows from Lemma 31 that $\mathrm{d}^0_{-\infty}(x,y) \ge c$.

Now if $|\mathcal{C}(h^n(H),\mathcal{U}_2)| < (3/2)|\mathcal{C}(H,\mathcal{U}_2)|$, then it follows from the pigeonhole principle that there exist distinct $x, y \in E$ such that $h^n(x)$, $h^n(y)$ are in the same element of $\mathcal{C}(h^n(H),\mathcal{U}_2)$. Thus, if $\mathrm{d}^0_{-\infty}(h^n(x), h^n(y)) \ge c$ then H has property $E(\mathcal{U}_2, c)$. Since the mesh of $\mathcal{U}_2$ can be chosen arbitrarily, c cannot be the expansive constant by Theorem 17.

On the other hand, if $\mathrm{d}^0_{-\infty}(h^n(x), h^n(y)) < c$, then

$$\mathrm{d}(h^n(x), h^n(y)) < \mathrm{mesh}(\mathcal{U}_2) < (1/8)\alpha^n\,\mathrm{d}(\mathcal{U}_2) \le (1/4)\alpha^n\,\mathrm{d}(x,y)$$

whenever $n \ge N$. However, this contradicts Theorem 14. ∎

The next theorem shows that $h^n(H)$ wraps any circle-chain at an approximately constant rate.

THEOREM 33. *There exists an integer N' such that if $\mathcal{U}$ is any circle-chain that refines $\mathcal{U}_2$ (as defined in Lemma 31) and H is any unstable subcontinuum such that $|\mathcal{C}(H,\mathcal{U})| \geq |\mathcal{U}|$ then $|\mathcal{C}(h^n(H),\mathcal{U})| \geq 3|\mathcal{C}(H,\mathcal{U})|$ for all $n \geq N'$.*

Proof. Let $\widehat{\mathcal{C}} = \mathcal{C}(H,\mathcal{U}_2)$, $\widehat{\mathcal{C}}_n = \mathcal{C}(h^n(H),\mathcal{U}_2)$, $\mathcal{C} = \mathcal{C}(H,\mathcal{U})$ and $\mathcal{C}_n = \mathcal{C}(h^n(H),\mathcal{U})$. Also let $k = |\mathcal{U}_2|$, $m = |\mathcal{U}|$ and $N' > N\log(6)/\log(3/2)$ where N is defined from Lemma 32. Then by Lemma 32, $|\widehat{\mathcal{C}}_n| \geq 6|\widehat{\mathcal{C}}|$ for all $n \geq N'$. Since $|\mathcal{C}| \geq |\mathcal{U}|$,

$$\begin{aligned}
|\widehat{\mathcal{C}}| &\geq \max_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}}_{\mathcal{U}_2}(i) - \min_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}}_{\mathcal{U}_2}(i) \\
&= \frac{k}{k}\Big(\max_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}}_{\mathcal{U}_2}(i) - \min_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}}_{\mathcal{U}_2}(i)\Big) \\
&\geq k\,\frac{\Delta^{\mathcal{U}}_{\mathcal{U}_2}(m-1) - \Delta^{\mathcal{U}}_{\mathcal{U}_2}(0)}{k} \geq k \deg_{\mathcal{U}_2}(\mathcal{U}).
\end{aligned}$$

Thus, $3 \leq 3|\widehat{\mathcal{C}}|/(k\deg_{\mathcal{U}_2}(\mathcal{U}))$. Also, since $|\mathcal{U}| = m$, $\Delta^{\mathcal{C}}_{\mathcal{U}}(i) = i$ and $\Delta^{\mathcal{C}_n}_{\mathcal{U}}(i) = i$,

$$\begin{aligned}
&\max_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}}_{\mathcal{U}_2}(i) - \min_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}}_{\mathcal{U}_2}(i) \\
&\qquad= \max_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{U}}_{\mathcal{U}_2}(i) - \min_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{U}}_{\mathcal{U}_2}(i) \\
&\qquad= \max_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}_n}_{\mathcal{U}_2}(i) - \min_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}_n}_{\mathcal{U}_2}(i).
\end{aligned}$$

So, by two applications of Theorem 26 (set $\widehat{\mathcal{C}} = \mathcal{C}^1$ and $\mathcal{C} = \mathcal{C}^2$ in the first application and $\widehat{\mathcal{C}}_n = \mathcal{C}^1$ and $\mathcal{C}_n = \mathcal{C}^2$ in the second)

$$\begin{aligned}
3|\mathcal{C}| &\leq m\bigg(\frac{3|\widehat{\mathcal{C}}| - 3(\max_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}}_{\mathcal{U}_2}(i) - \min_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}}_{\mathcal{U}_2}(i))}{k\deg_{\mathcal{U}_2}(\mathcal{U})} + 3\bigg) \\
&\leq m\bigg(\frac{6|\widehat{\mathcal{C}}| - 3(\max_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}}_{\mathcal{U}_2}(i) - \min_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}}_{\mathcal{U}_2}(i))}{k\deg_{\mathcal{U}_2}(\mathcal{U})}\bigg) \\
&\leq m\bigg(\frac{|\widehat{\mathcal{C}}_n| - (\max_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}_n}_{\mathcal{U}_2}(i) - \min_{i\in\{0,\dots,m-1\}} \Delta^{\mathcal{C}_n}_{\mathcal{U}_2}(i))}{k\deg_{\mathcal{U}_2}(\mathcal{U})}\bigg) \leq |\mathcal{C}_n|. \ \blacksquare
\end{aligned}$$

The next set of results show that for every subchain of a circle-chain cover, there is an unstable subcontinuum properly covered by that chain.

LEMMA 34. *Suppose that H is an unstable subcontinuum and $\mathcal{U}$ is a circle-chain cover such that $\mathcal{U}(h^n(H))$ is a proper subchain of $\mathcal{U}$ for each $n \geq 0$. Then H has property $E(\mathcal{U},c)$.*

Proof. Let H be an unstable subcontinuum and $k = |\mathcal{U}|$. Then there exist distinct points $\{x_1, \dots, x_k\} \subset H$ such that $\mathrm{d}(x_i, x_j) \geq \mathrm{diam}(H)/k$ whenever $i \neq j$. Pick M such that $\alpha^M \mathrm{diam}(H)/k > c$. Then $\{h^M(x_1), \dots, h^M(x_k)\} \subset h^M(H)$ has the property that $\mathrm{d}^0_{-\infty}(h^M(x_i), h^M(x_j)) > c$ for $i \neq j$. By hypothesis, $\mathcal{U}(h^M(H))$ is a proper subchain of H. Thus, $|\mathcal{U}(h^M(H))| < k$. So by the pigeon-hole principle, there exist $U \in \mathcal{U}(h^M(H))$ and distinct $i', j' \in \{1, \dots, k\}$ such that $h^M(x_i), h^M(x_j) \in U$. Hence, H has property $E(\mathcal{U}, c)$. ∎

LEMMA 35. *Suppose that $h : X \to X$ is an expansive homeomorphism of a circle-like continuum with an unstable subcontinuum. If $\mathcal{U}$ is a taut circle-chain cover of X, then there exists an unstable subcontinuum $H_{\mathcal{U}}$ such that $\mathcal{U}(H_{\mathcal{U}}) = \mathcal{U}$.*

Proof. Let $\{\mathcal{U}_i\}_{i=1}^{\infty}$ be a nested sequence of refining circle-chains that limit to X. Then there exists $M > 0$ such that for each $i \geq M$, $\mathcal{U}_i$ refines $\mathcal{U}$. Thus, if H is a continuum such that $\mathcal{U}_i(H) = \mathcal{U}_i$, then $\mathcal{U}(H) = \mathcal{U}$. Suppose that every unstable subcontinuum H of X has the property that $\mathcal{U}(H)$ is a proper subchain of $\mathcal{U}$. Then $\mathcal{U}_i(H)$ is a proper subchain of $\mathcal{U}_i$ for each $i \geq M$. Thus, every unstable subcontinuum has property $E(\mathcal{U}_i, c)$ for each i by Lemma 34. However, this contradicts the fact c is the expansive constant by Theorem 17. ∎

LEMMA 36. *Suppose that $|\mathcal{C}(H,\mathcal{U})| \geq 2|\mathcal{U}|$. Then for every proper subchain $\widehat{\mathcal{U}}$ of $\mathcal{U}$ there exists a subcontinuum K of H such that $\widehat{\mathcal{U}}$ is a proper cover of K.*

Proof. Let $\mathcal{U} = [U_0, \dots, U_{n-1}]_\circ$ and $\widehat{\mathcal{U}}$ be a subchain of $\mathcal{U}$. Then $\widehat{\mathcal{U}}$ is of the form $[U_i, \dots, U_j]$ where $i < j$, or of the form $[U_i, \dots, U_{n-1}, U_0, \dots, U_j]$ where $j < i-1$. Then there exists a subchain $\mathcal{C} = [C_{n+i}, \dots, C_{n+j}]$ of $\mathcal{C}(H,\mathcal{U})$ (or similarly a subchain $\mathcal{C} = [C_i, \dots, C_{n-1}, C_n, \dots, C_{n+j}]$ of $\mathcal{C}(H,\mathcal{U})$) where $C_{k'} \subset U_k$ and $k = k' \bmod n$. Then by Proposition 1, there exists a subchain K of H that is properly covered by $\mathcal{C}$ and hence properly covered by $\widehat{\mathcal{U}}$. ∎

THEOREM 37. *If $\widehat{\mathcal{U}}$ is a proper subchain of a circle-chain cover $\mathcal{U}$, then there exists an unstable subcontinuum H that is properly covered by $\widehat{\mathcal{U}}$.*

Proof. This follows directly from Theorem 33 and Lemmas 35 and 36. ∎

6. Growth of folds in circle chains. In this section we show that under certain conditions, small folds in circle-chains grow to large folds under h.

LEMMA 38. *Let $\mathcal{U}$ be a circle-chain and $\epsilon > 0$. Then there exists a positive integer N_ϵ such that if H is an unstable subcontinuum with* $\mathrm{diam}(H) > \epsilon$ *then $\mathcal{U}(h^n(H)) = \mathcal{U}$ for all $n \geq N_\epsilon$.*

Proof. Suppose on the contrary that there exists a sequence $\{H_i\}_{i=1}^{\infty}$ of unstable subcontinua and an increasing sequence $\{N_i\}_{i=1}^{\infty}$ of positive integers such that $\mathrm{diam}(H_i) > \epsilon$ and $\mathcal{U}(h^{N_i}(H_i))$ is a proper subchain of $\mathcal{U}$. Let $k = |\mathcal{U}|$ and M be such that $\alpha^n\epsilon/k > \mathrm{mesh}(\mathcal{U})$ for all $n \geq M$. Choose i such that $N_i > M$. Since $\mathrm{diam}(H_i) > \epsilon$, there exist points $\{x_1, \ldots, x_k\} \subset H_i$ such that $\mathrm{d}(x_\alpha, x_\beta) > \epsilon/k$ for all $\alpha \neq \beta$. Then since $|\mathcal{U}(h^{N_i}(H_i))| < k$, it follows from the pigeon-hole principle that there exist distinct $\alpha', \beta' \in \{1, \ldots, k\}$ such that $h^{N_i}(x_{\alpha'})$, $h^{N_i}(x_{\beta'})$ are in the same element of $\mathcal{U}$. Thus,

$$\alpha^{N_i}\,\mathrm{d}(x_{\alpha'}, x_{\beta'}) > \alpha^M\epsilon/k > \mathrm{mesh}(\mathcal{U}) > \mathrm{d}(h^{N_i}(x_{\alpha'}), h^{N_i}(x_{\beta'})).$$

Therefore by Theorem 14, $\mathrm{d}_{-\infty}^{0}(h^{N_i}(x_{\alpha'}), h^{N_i}(x_{\beta'})) > c$. Thus, H_i has property $E(\mathcal{U}, c)$. Since $\mathcal{U}$ can be taken with arbitrarily small mesh, c is not the expansive constant for h, which is a contradiction. ∎

LEMMA 39. *Let $\{\mathcal{U}_i\}_{i=1}^{\infty}$ be a nested sequence of refining covers that limits to a circle-like continuum X such that $\mathcal{U}_0$, $\mathcal{U}_1$ and $\mathcal{U}_2$ have the properties of Lemma* 31 *and let $\epsilon > 0$. Suppose that for each $i > 0$ there exists $j > i$ such that there is a subchain $\widehat{\mathcal{U}}_j$ of $\mathcal{U}_j$ that is folded in $\mathcal{U}_i$ and with $\mathrm{diam}(\widehat{\mathcal{U}}_j^*) > \epsilon$. Then there exists $i' > i$ such that if $j' > i'$ then there is a subchain $\widehat{\mathcal{U}}_{j'}$ of $\mathcal{U}_{j'}$ that is folded in $\mathcal{U}_i$ and $\mathcal{U}_i(\widehat{\mathcal{U}}_{j'}) = \mathcal{U}_i$.*

Proof. By Lemma 38, if H is an unstable subcontinuum such that $\mathrm{diam}(H) > \epsilon/2$ then there exists $M = N_{\epsilon/2}$ such that $\mathcal{U}_i(h^n(H)) = \mathcal{U}_i$ for each $n \geq M$. Let $\widehat{i} > i$ be such that if $j \geq \widehat{i}$ then $h^n(\mathcal{U}_j)$ refines $\mathcal{U}_i$ for each $n \in \{0, \ldots, M\}$ and $\mathrm{mesh}(\mathcal{U}_j) < \epsilon/6$. Let $\widehat{\mathcal{U}}_{\widehat{i}+1}$ be a subchain of $\mathcal{U}_{\widehat{i}+1}$ such that $\mathrm{diam}(\widehat{\mathcal{U}}_{\widehat{i}+1}^*) > \epsilon$ and $\widehat{\mathcal{U}}_{\widehat{i}+1}$ is folded in $\mathcal{U}_{\widehat{i}}$. Then there exists an unstable subcontinuum H that is properly covered by $\widehat{\mathcal{U}}_{\widehat{i}+1}$ with $\mathrm{diam}(H) > \epsilon/2$ by Theorem 37. Let $\mathcal{W}$ be a chain such that $\widehat{\mathcal{U}}_{\widehat{i}+1}$ is folded in $\mathcal{W}$ and $\mathcal{W}$ refines $\mathcal{U}_{\widehat{i}}$. Then $h^M(\mathcal{W})$ refines $\mathcal{U}_i$. So $h^M(\widehat{\mathcal{U}}_{\widehat{i}+1})$ is a proper subchain of $h^M(\mathcal{U}_{\widehat{i}+1})$ that is folded in $h^M(\mathcal{W})$ and hence folded in $\mathcal{U}_i$. Also, $h^M(\widehat{\mathcal{U}}_{\widehat{i}+1})$ covers $h^M(H)$. Therefore, $\mathcal{U}_i(h^M(\widehat{\mathcal{U}}_{\widehat{i}+1})) = \mathcal{U}_i(h^M(H)) = \mathcal{U}_i$.

Let $i' > \widehat{i} + 1$ be such that if $j' \geq i'$, then $\mathcal{U}_j'$ refines $h^M(\widehat{\mathcal{U}}_{\widehat{i}+1})$. Then for each $j' \geq i'$ there exists a proper subchain $\widehat{\mathcal{U}}_{j'}$ such that $h^M(\widehat{\mathcal{U}}_{\widehat{i}+1})(\widehat{\mathcal{U}}_{j'}) = h^M(\widehat{\mathcal{U}}_{\widehat{i}+1})$. It is easily checked that $\widehat{\mathcal{U}}_{j'}$ has the prescribed properties of the theorem. ∎

THEOREM 40. *Let $\{\mathcal{U}_i\}_{i=1}^{\infty}$ be a nested sequence of refining covers that limit to a circle-chain X such that $\mathcal{U}_0$, $\mathcal{U}_1$ and $\mathcal{U}_2$ have the properties of Lemma* 31. *Suppose that for each $i > 0$ and $\xi > 0$ there exists $j > i$ such that there is a subchain $\widehat{\mathcal{U}}_{j+1}$ of $\mathcal{U}_{j+1}$ that is folded in $\mathcal{U}_j$ and*

$\operatorname{diam}(\widehat{\mathcal{U}}^*_{j+1})/\operatorname{mesh}(\mathcal{U}_j) > \xi$. *Then there exists* $i' > i$ *such that if* $j' > i'$ *then there exists a subchain* $\widehat{\mathcal{U}}_{j'}$ *of* $\mathcal{U}_{j'}$ *that is folded in* $\mathcal{U}_i$ *and* $\mathcal{U}_i(\widehat{\mathcal{U}}_{j'}) = \mathcal{U}_i$.

Proof. Let $0 < \epsilon < c$ and $i > 0$. Then there exists a $j > i$ such that

$$\frac{\epsilon}{\operatorname{mesh}(\mathcal{U}_j)} > \frac{\operatorname{diam}(\widehat{\mathcal{U}}^*_{j+1})}{\operatorname{mesh}(\mathcal{U}_j)} > \max\left\{32\alpha^2 \frac{\epsilon}{\operatorname{Leb}(\mathcal{U}_i)}, 6\right\}.$$

Thus, by Theorem 37, there exists an unstable subcontinuum H that is properly covered by $\widehat{\mathcal{U}}_{j+1}$ with

$$(1/2)\operatorname{diam}(\widehat{\mathcal{U}}^*_{j+1}) < \operatorname{diam}(H) < \operatorname{diam}(\widehat{\mathcal{U}}^*_{j+1}) < \epsilon.$$

Let M be the smallest positive integer such that $(\alpha^M/4)\operatorname{diam}(H) > \epsilon$. Then $\operatorname{diam}(h^M(\widehat{\mathcal{U}}^*_{j+1})) > \epsilon$. However, since $(\alpha^{M-1}/4)\operatorname{diam}(H) \le \epsilon$, it follows from Theorem 14 that

$$\begin{aligned}\operatorname{mesh}(h^M(\mathcal{U}_j)) &\le 4\alpha^{M+1}\operatorname{mesh}(\mathcal{U}_j) \le 8\alpha^{M+1}\frac{\operatorname{diam}(H)}{\operatorname{diam}(\widehat{\mathcal{U}}^*_{j+1})}\operatorname{mesh}(\mathcal{U}_j)\\ &\le 32\alpha^2\frac{\epsilon}{\operatorname{diam}(\widehat{\mathcal{U}}^*_{j+1})}\operatorname{mesh}(\mathcal{U}_j) < \operatorname{Leb}(\mathcal{U}_i).\end{aligned}$$

Thus, $h^M(\mathcal{U}_j)$ refines $\mathcal{U}_i$. Since h^M is a homeomorphism, $h^M(\widehat{\mathcal{U}}_{j+1})$ is folded in $h^M(\mathcal{U}_j)$ and thus folded in $\mathcal{U}_i$. The theorem now follows from Lemma 39. ∎

7. Main result. When an unstable subcontinuum is folded in a circle-chain, it creates "parallel" subcontinua. The understanding of the behavior of these parallel unstable subcontinua under h is crucial in proving the main result. The next several results examine this behavior.

Subcontinua H_1, K_1 are $(\mathcal{U}, H)$-*parallel* if H_1, K_1 are subcontinua of H such that there exists a chain $\mathcal{W}$ that properly covers H and refines $\mathcal{U}$ with the property that $\mathcal{W}(H_1) = \mathcal{W}(K_1)$.

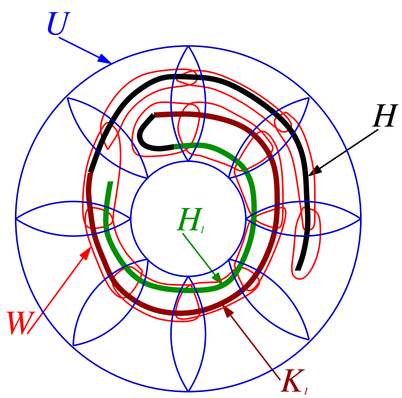

Fig. 3. H_1, K_1 are $(\mathcal{U}, H)$-parallel.

Lemma 41. *If $\widehat{\mathcal{U}}_k$ is a subchain of $\mathcal{U}_k$ that is folded in a circle-chain $\mathcal{U}$ such that $\mathcal{U}(\widehat{\mathcal{U}}_k) = \mathcal{U}$, then there exists an unstable subcontinuum H and disjoint subcontinua $H_1, K_1 \subset H$ that are $(\mathcal{U}, H)$-parallel with $\mathcal{U}(H_1) = \mathcal{U}(K_1) = \mathcal{U}$.*

Proof. Let $\widehat{\mathcal{U}}_k = [U_0, \dots, U_{p-1}]$. Then there exists a chain $\mathcal{W} = [W_0, \dots, W_{q-1}]$ that refines $\mathcal{U}$ such that $\widehat{\mathcal{U}}_k$ is folded in $\mathcal{W}$. Notice that $\mathcal{U}(\mathcal{W}) = \mathcal{U}$. Without loss of generality, assume that $U_0, U_{p-1} \subset W_0$ and $U_j \subset W_{q-1}$ for some j. Then by Theorem 37, there exists an unstable subcontinuum H that is properly covered by $\widehat{\mathcal{U}}_k$ and hence $\mathcal{W}$. Let $\mathcal{V} = [V_0, \dots, V_{s-1}]$ be a chain cover of H that is a 3-refinement of $\widehat{\mathcal{U}}_k$. Then there exist α, β such that $V_\alpha \subset U_0 \subset W_0$ and $V_\beta \subset U_{p-1} \subset W_0$. We may assume that $\alpha < \beta$. Then since $\mathcal{V}$ is a 3-refinement of $\widehat{\mathcal{U}}_k$, there exists an m between α and β such that $V_{m-1} \cup V_m \cup V_{m+1} \subset U_j \subset W_{q-1}$. Let H_1 be a subcontinuum of H that is properly covered by $[V_\alpha, \dots, V_{m-1}]$, and K_1 be a subcontinuum of H that is properly covered by $[V_{m+1}, \dots, V_\beta]$. Then H_1 and K_1 are disjoint. Also, $\mathcal{W}(H_1) = \mathcal{W}(K_1) = \mathcal{W}$. So H_1, K_1 are $(\mathcal{U}, H)$-parallel. Furthermore, it follows from $\mathcal{U}(\mathcal{W}) = \mathcal{U}$ that $\mathcal{U}(H_1) = \mathcal{U}(K_1) = \mathcal{U}$. ∎

Proposition 42. *Suppose that H_1 and K_1 are subcontinua of H, $\mathcal{U}$ is a circle-chain and $\mathcal{V}$ is a chain cover of H such that*

(1) *$\mathcal{V}$ refines $\mathcal{U}$,*
(2) *$\mathcal{V}(H_1) = [V_{i_1}, \dots, V_{i_2}]$ and $\mathcal{V}(K_1) = [V_{j_1}, \dots, V_{j_2}]$,*
(3) *$\Delta^{\mathcal{V}}_{\mathcal{U}}(\{i_1, \dots, i_2\}) = \Delta^{\mathcal{V}}_{\mathcal{U}}(\{j_1, \dots, j_2\})$.*

Then H_1, K_1 are $(\mathcal{U}, H)$-parallel.

Proof. Let $\mathcal{C} = \mathcal{C}(\mathcal{V}, \mathcal{U})$. Then $\mathcal{C}$ is a chain cover of H that refines $\mathcal{U}$. Also,

$$\mathcal{C}(H_1) = \{C_{\alpha - \min \Delta^{\mathcal{V}}_{\mathcal{U}}}\}_{\alpha \in \Delta^{\mathcal{V}}_{\mathcal{U}}(\{i_1, \dots, i_2\})} = \{C_{\alpha - \min \Delta^{\mathcal{V}}_{\mathcal{U}}}\}_{\alpha \in \Delta^{\mathcal{V}}_{\mathcal{U}}(\{j_1, \dots, j_2\})} = \mathcal{C}(K_1).$$

Hence, H_1, K_1 are $(\mathcal{U}, H)$-parallel. ∎

Notice that Lemma 22 shows that if subcontinua are not close to being parallel in a circle-chain $\mathcal{U}$ then they are not close to being parallel in a circle-chain $\mathcal{U}_0$ that is refined by $\mathcal{U}$. The next lemma will show that if H and K are "large" parallel subcontinua, then there will exist "large" parallel subcontinua of $h(H)$ and $h(K)$. This will be used to build an inductive argument in the main theorem.

Lemma 43. *Suppose*

(1) *$\mathcal{U}_0, \mathcal{U}_1$ and $\mathcal{U}_2$ have the properties of Lemma* 31,
(2) *H is an unstable subcontinuum,*
(3) *N' is found from Theorem* 33,

(4) *$\mathcal{U}$ is a circle-chain such that $h^n(\mathcal{U})$ refines $\mathcal{U}_2$ for all $n \in \{0,\dots,N'\}$,*
(5) *H_1, K_1 are $(\mathcal{U}, H)$-parallel subcontinua with $\mathcal{U}(H_1) = \mathcal{U}(K_1) = \mathcal{U}$.*

Then there exist subcontinua $H_2 \subset h^{N'}(H_1)$ and $K_2 \subset h^{N'}(K_1)$ that are $(\mathcal{U}, h^N(H))$-parallel and such that $\mathcal{U}(H_2) = \mathcal{U}(K_2) = \mathcal{U}$.

Proof. Let $\mathcal{V}$ be a chain cover of H such that $h^n(\mathcal{V})$ refines $\mathcal{U}$ for each $n \in \{0,\dots,N'\}$, and let $\mathcal{V}(H_1) = [V_{i_1},\dots,V_{j_1}]$ and $\mathcal{V}(K_1) = [V_{i_2},\dots,V_{j_2}]$. Let

$$\begin{aligned}
&m = |\mathcal{U}|,\\
&\mathcal{C}^2 = \mathcal{C}(H,\mathcal{U}) = \mathcal{C}(\mathcal{V},\mathcal{U}) = [C^2_0,\dots,C^2_{p-1}],\\
&\mathcal{C}^1 = \mathcal{C}(h^{N'}(H),\mathcal{U}) = \mathcal{C}(h^{N'}(\mathcal{V}),\mathcal{U}) = [C^1_0,\dots,C^1_{q-1}].
\end{aligned}$$

Claim 1. *If $\Delta^{\mathcal{V}}_{\mathcal{U}}(i) = \Delta^{\mathcal{V}}_{\mathcal{U}}(j)$, then $h^{N'}(V_i), h^{N'}(V_j)$ are in the same element of $\mathcal{U}_0$.*

Since $\Delta^{\mathcal{V}}_{\mathcal{U}}(i) = \Delta^{\mathcal{V}}_{\mathcal{U}}(j)$, there exists $U \in \mathcal{U}$ such that $V_i, V_j \subset U$. Since $h^{N'}(\mathcal{U})$ refines $\mathcal{U}_2$ and hence $\mathcal{U}_0$, it follows that there exists $U^0 \in \mathcal{U}_0$ such that $h^{N'}(V_i), h^{N'}(V_j) \subset h^{N'}(U) \subset U^0$.

Let $i'_1, j'_1 \in \{i_1,\dots,j_1\}$ and $i'_2, j'_2 \in \{i_2,\dots,j_2\}$ be such that

$$\begin{aligned}
\Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(i'_1) &= \min \Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(\{i_1,\dots,j_1\}),\\
\Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(j'_1) &= \max \Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(\{i_1,\dots,j_1\}),\\
\Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(i'_2) &= \min \Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(\{i_2,\dots,j_2\}),\\
\Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(j'_2) &= \max \Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(\{i_2,\dots,j_2\}).
\end{aligned}$$

Claim 2. *$\Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(j'_1) - \Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(i'_1) \ge 3m$.*

By Theorem 33,

$$\Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(j'_1) - \Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(i'_1) = |\mathcal{C}^1(h^{N'}(H_1))| \ge 3|\mathcal{C}^2(H_1)| \ge 3|\mathcal{U}(H_1)| = 3m.$$

Claim 3. *$\Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(j'_2) - \Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(i'_2) \ge 3m$.*

The proof is similar to that of Claim 2.

Claim 4. *$\Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(j'_1) - \Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(i'_2) \ge m$.*

Suppose on the contrary that $\Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(j'_1) - \Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(i'_2) < m$. It then follows from Claim 3 that $\Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(j'_2) - \Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}}(j'_1) \ge 2m$. Hence it follows from Lemma 22 that there exists $k_2 \in \{i_2,\dots,j_2\}$ such that $\Delta^{h^{N'}(\mathcal{V})}_{\mathcal{U}_0}(k_2) \ge$

$\max \Delta_{\mathcal{U}_0}^{h^{N'}(\mathcal{V})}(\{i_1,\dots,j_1\})+|\mathcal{U}_0|$. Since H_1, K_1 are $(\mathcal{U},H)$-parallel, there exists $k_1\in\{i_1,\dots,j_1\}$ such that $\Delta_{\mathcal{U}}^{\mathcal{V}}(k_1)=\Delta_{\mathcal{U}}^{\mathcal{V}}(k_2)$. However, $|\Delta_{\mathcal{U}_0}^{h^{N'}(\mathcal{V})}(k_2)-\Delta_{\mathcal{U}_0}^{h^{N'}(\mathcal{V})}(k_1)|\ge|\mathcal{U}_0|$. Thus by Lemma 30, there exist $i',j'\in\{0,\dots,p-1\}$ such that $\Delta_{\mathcal{U}}^{\mathcal{V}}(i')=\Delta_{\mathcal{U}}^{\mathcal{V}}(j')$ but $2\le|\Delta_{\mathcal{U}_0}^{h^{N'}(\mathcal{V})}(i')-\Delta_{\mathcal{U}_0}^{h^{N'}(\mathcal{V})}(j')|\le 4$. This implies that $h^{N'}(V_{i'})$ and $h^{N'}(V_{j'})$ are not in the same element of $\mathcal{U}_0$, which contradicts Claim 1.

CLAIM 5. $\Delta_{\mathcal{U}}^{h^{N'}(\mathcal{V})}(j_2')-\Delta_{\mathcal{U}}^{h^{N'}(\mathcal{V})}(i_1')\ge m$.

The proof is similar to that of Claim 4.

Let

$$M_2=\min\{\Delta_{\mathcal{U}_0}^{h^{N'}(\mathcal{V})}(j_1'),\Delta_{\mathcal{U}}^{h^{N'}(\mathcal{V})}(j_2')\},\quad M_1=\max\{\Delta_{\mathcal{U}}^{h^{N'}(\mathcal{V})}(i_1'),\Delta_{\mathcal{U}}^{h^{N'}(\mathcal{V})}(i_2')\}.$$

Then $M_2-M_1\ge m$ by Claims 2–5. Thus, there exist $\widehat{i}_1,\widehat{j}_1\in\{i_1',\dots,j_1'\}$ and $\widehat{i}_2,\widehat{j}_2\in\{i_2',\dots,j_2'\}$ such that $\Delta_{\mathcal{U}_0}^{h^{N'}(\mathcal{V})}(\{\widehat{i}_1,\dots,\widehat{j}_1\})=\{M_1,\dots,M_2\}=\Delta_{\mathcal{U}_0}^{h^{N'}(\mathcal{V})}(\{\widehat{i}_2,\dots,\widehat{j}_2\})$. Therefore by Proposition 1, there exist subcontinua $H_2\subset h^{N'}(H_1)$ and $K_2\subset h^{N'}(K_1)$ that are properly covered by

$$[C^1_{M_1-\min\Delta_{\mathcal{U}_0}^{h^{N'}(\mathcal{V})}},\dots,C^1_{M_2-\min\Delta_{\mathcal{U}_0}^{h^{N'}(\mathcal{V})}}]\subset\mathcal{C}^1.$$

Hence H_2,K_2 are $(\mathcal{U},h^{N'}(H))$-parallel and $\mathcal{U}(H_2)=\mathcal{U}(K_2)=\mathcal{U}$. ∎

The following theorem and corollary are the main results of this paper:

THEOREM 44. *Let $\{\mathcal{U}_i\}_{i=1}^{\infty}$ be a nested sequence of refining covers that limit to a circle-chain X such that $\mathcal{U}_0$, $\mathcal{U}_1$ and $\mathcal{U}_2$ have the properties of Lemma* 31*. Suppose that for each $i>0$ and $\xi>0$ there exists $j>i$ such that there is a subchain $\widehat{\mathcal{U}}_{j+1}$ of $\mathcal{U}_{j+1}$ that is folded in $\mathcal{U}_j$ and $\operatorname{diam}(\widehat{\mathcal{U}}^*_{j+1})/\operatorname{mesh}(\mathcal{U}_j)>\xi$. Then X does not admit an expansive homeomorphism.*

Proof. Suppose that $h:X\to X$ is an expansive homeomorphism with expansive constant c and growth multiplier α. Let $i>2$ and N' be defined as in Theorem 33. Then by Theorem 40, there exists a $j>i$ such that $h^n(\mathcal{U}_j)$ refines $\mathcal{U}_i$ for each $n\in\{0,\dots,N'\}$ and there exists a subchain $\widehat{\mathcal{U}}_j=[U_0^j,\dots,U_{p-1}^j]$ of $\mathcal{U}_j$ that is folded in $\mathcal{U}_i$ with $\mathcal{U}_i(\widehat{\mathcal{U}}_j)=\mathcal{U}_i$. Let

$$\mathcal{C}^0=\mathcal{C}(\widehat{\mathcal{U}}_j,\mathcal{U}_i)=[C_0^0,\dots,C_{q-1}^0].$$

Without loss of generality, we may assume that $U_0^j,U_{p-1}^j\subset C_0^0$ and $U_t^j\subset C_{q-1}^0$. By Theorem 37 there exists an unstable subcontinuum H of X that is properly covered by $\widehat{\mathcal{U}}_j$. Let $\mathcal{V}=[V_0,\dots,V_{s-1}]$ be a chain cover of H that 3-

refines $\widehat{\mathcal{U}}_j$. Then there exists $\beta \in \{0, \ldots, s-1\}$ such that $V_{\beta-1}, V_\beta, V_{\beta+1} \subset U_t^j \subset C_{q-1}^0$. By Proposition 1 there exist disjoint subcontinua H_1 and K_1 of H that are properly covered by $[V_0, \ldots, V_{\beta-1}]$ and $[V_{\beta+1}, \ldots, V_{s-1}]$ respectively. Thus, $\mathcal{C}^0(H_1) = \mathcal{C}^0(K_1) = \mathcal{C}^0$. It follows that H_1, K_1 are $(\mathcal{U}_i, H)$-parallel and that $\mathcal{U}_i(H_1) = \mathcal{U}_i(K_1) = \mathcal{U}_i$. Let $\gamma = \mathrm{d}(H_1, K_1)$.

Then by Lemma 43 there exists subcontinua $H_2 \subset h^{N'}(H_1)$ and $K_2 \subset h^{N'}(K_1)$ that are $(\mathcal{U}_i, h^{N'}(H))$-parallel and $\mathcal{U}_i(H_2) = \mathcal{U}_i(K_2) = \mathcal{U}_i$. Continuing inductively by Lemma 43 there exist subcontinua $H_m \subset h^{N'}(H_{m-1})$ and $K_m \subset h^{N'}(K_{m-1})$ that are $(\mathcal{U}_i, h^{(m-1)N'}(H))$-parallel and $\mathcal{U}_i(H_m) = \mathcal{U}_i(K_m) = \mathcal{U}_i$. Let k be an integer such that $\alpha^{kN'}\gamma/4 > \mathrm{mesh}(\mathcal{U}_i)$. Since H_k, K_k are $(\mathcal{U}_i, h^{(m-1)N'}(H))$-parallel, there exist $x \in H_k$ and $y \in K_k$ that are in the same element of $\mathcal{C}(h^{kN'}(H), \mathcal{U}_i)$. Also, $h^{-kN'}(x) \in H_1$ and $h^{-kN'}(y) \in K_1$. Thus

$$\begin{aligned}\mathrm{d}(h^{kN'}(h^{-kN'}(x)), h^{kN'}(h^{-kN'}(x))) &= \mathrm{d}(x,y) < \mathrm{mesh}(\mathcal{C}(h^{kN'}(H), \mathcal{U}_i)) \\ &\le \mathrm{mesh}(\mathcal{U}_i) < \alpha^{kN'}\gamma/4 < \ \alpha^{kN'}/4\, \mathrm{d}(h^{-kN'}(x), h^{-kN'}(y)).\end{aligned}$$

So by Theorem 14, $\mathrm{d}_{-\infty}^0(x,y) = \mathrm{d}_{-\infty}^{kN'}(h^{-kN'}(x), h^{-kN'}(y)) \ge c$. Thus H has property $E(\mathcal{U}_i, c)$. Since $\mathcal{U}_i$ was arbitrarily chosen, h cannot be an expansive homeomorphism. ∎

COROLLARY 45. *If X is a circle-like continuum that admits an expansive homeomorphism, then X is a solenoid.*

Proof. This follows from Theorems 8, 12 and 44. ∎

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