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Splitting stationary sets in $\mathcal{P}_\kappa\lambda$ for λ with small cofinality

by

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Abstract. For a regular uncountable cardinal κ and a cardinal λ with $\mathrm{cf}(\lambda) < \kappa < \lambda$, we investigate the consistency strength of the existence of a stationary set in $\mathcal{P}_\kappa\lambda$ which cannot be split into λ^+ many pairwise disjoint stationary subsets. To do this, we introduce a new notion for ideals, which is a variation of normality of ideals. We also prove that there is a stationary set S in $\mathcal{P}_\kappa\lambda$ such that every stationary subset of S can be split into λ^+ many pairwise disjoint stationary subsets.

1. Introduction. Let κ be a regular uncountable cardinal and λ be a cardinal with $\lambda \geq \kappa$. Splitting a stationary set in $\mathcal{P}_\kappa\lambda$ into pairwise disjoint stationary subsets is a classical problem of combinatorics on $\mathcal{P}_\kappa\lambda$.

Definition 1.1. For a stationary set S in $\mathcal{P}_\kappa\lambda$ and a cardinal μ, we say that $\mathrm{Sp}(S,\mu)$ *holds* if S can be split into μ many pairwise disjoint stationary subsets. $\mathrm{NSp}(S,\mu)$ is the negation of $\mathrm{Sp}(S,\mu)$.

More generally, we define the following:

Definition 1.2. Let I be an ideal over a set A, and let μ be a cardinal. We say that I is *weakly μ-saturated* if μ many pairwise disjoint I-positive sets do not exist.

$\mathrm{NSp}(S,\mu)$ is equivalent to the weak μ-saturation property of $\mathrm{NS}_{\kappa\lambda}|S$, the non-stationary ideal over $\mathcal{P}_\kappa\lambda$ restricted to S.

This problem is connected to the saturation property of $\mathrm{NS}_{\kappa\lambda}$, because for a stationary set S in $\mathcal{P}_\kappa\lambda$ and a cardinal $\mu \leq \lambda$, it is known that the following are equivalent:

(1) $\mathrm{NS}_{\kappa\lambda}|S$ is μ-saturated.
(2) $\mathrm{NSp}(S,\mu)$ holds.

Hence, if $\lambda^{<\kappa} = \lambda$, this problem can be translated into the local saturation property of $\mathrm{NS}_{\kappa\lambda}$. Furthermore, since every λ^+-saturated normal ideal over

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$\mathcal{P}_\kappa\lambda$ is precipitous, we can use the precipitousness to investigate a stationary set S with $\mathrm{NSp}(S,\lambda)$, and it turns out that the existence of such a stationary set has a very large consistency strength.

On the other hand, if $\mathrm{cf}(\lambda) < \kappa$, then every stationary set in $\mathcal{P}_\kappa\lambda$ has cardinality at least λ^+. Hence we can consider the possibility of splitting into λ^+ many stationary subsets. But we do not know if, for a stationary set S in $\mathcal{P}_\kappa\lambda$, λ^+-saturation of $\mathrm{NS}_{\kappa\lambda}|S$, or even precipitousness, follows from $\mathrm{NSp}(S,\lambda^+)$. Hence we cannot apply the saturation property and the precipitousness of $\mathrm{NS}_{\kappa\lambda}$ to investigate the properties of $\mathrm{NSp}(S,\lambda^+)$.

In this paper, where $\mathrm{cf}(\lambda)<\kappa$, we will investigate the consistency strength of the existence of a stationary set S in $\mathcal{P}_\kappa\lambda$ such that $\mathrm{NSp}(S,\lambda^+)$ holds, and of the existence of a weakly λ^+-saturated normal ideal over $\mathcal{P}_\kappa\lambda$. We will introduce a variation of normality of ideals, α-semi-weak normality, which will be the main tool used in this paper. This method was already used in Abe [1] and Burke [4] under some large cardinal assumptions. We prove that such assumptions can be dropped completely. Using this method, we prove the following:

Theorem 1.3. *Let* $\mathrm{cf}(\lambda) < \kappa$. *Suppose that there exists a weakly* λ^+-*saturated normal ideal over* $\mathcal{P}_\kappa\lambda$. *Then the following hold*:

1. *Every stationary subset of* $\{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \kappa\}$ *is reflecting*, *that is*, *for every stationary subset* E *of* $\{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \kappa\}$ *there exists* $\gamma < \lambda^+$ *such that* $E \cap \gamma$ *is stationary in* γ.
2. *There is no good scale for* λ.

The existence of a good scale is a very weak principle. The above theorem tells us that the existence of a stationary set S such that $\mathrm{NSp}(S,\lambda^+)$ holds is a very strong property, close to the λ^+-saturation of $\mathrm{NS}_{\kappa\lambda}|S$.

Foreman–Magidor [9] and Shioya [16] showed that $\mathrm{Sp}(\mathcal{P}_\kappa\lambda,\lambda^+)$ holds for λ with $\mathrm{cf}(\lambda) < \kappa$. Using the argument in the proof of Theorem 1.3, we will improve and refine this result to the following:

Theorem 1.4. *Let* $\mathrm{cf}(\lambda) < \kappa$. *Let* $S = \{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(|x|) \neq \mathrm{cf}(\lambda)\}$. *Then there is no weakly* λ^+-*saturated normal ideal* I *over* $\mathcal{P}_\kappa\lambda$ *with* $S \in I^*$. *In particular*, *the following holds*:

1. $\mathrm{Sp}(T,\lambda^+)$ *holds for every stationary subset* T *of* S.
2. *If* κ *is the successor cardinal of a cardinal* μ *with* $\mathrm{cf}(\mu) \neq \mathrm{cf}(\lambda)$ *then* $\mathrm{Sp}(T,\lambda^+)$ *holds for every stationary set* T *in* $\mathcal{P}_\kappa\lambda$.

Theorem 1.5. *Let* $\mathrm{cf}(\lambda) < \kappa$. *There exists a stationary set* S *in* $\mathcal{P}_\kappa\lambda$ *such that there is no weakly* λ^+-*saturated normal ideal* I *over* $\mathcal{P}_\kappa\lambda$ *with* $S \in I^*$. *In particular*, $\mathrm{Sp}(T,\lambda^+)$ *holds for every stationary subset* T *of* S.

Theorem 1.4 is also a refinement of the theorem of Burke and Cummings in [5]: They proved that there is no λ^+-saturated normal ideal I over $\mathcal{P}_\kappa\lambda$ such that $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(|x|) \neq \mathrm{cf}(\lambda)\} \in I^*$.

A rough outline of this paper is as follows: In Section 3, we introduce semi-weak normality of ideals. In Section 4, we consider basic properties of an ideal which is semi-weakly normal and has the weak saturation property. Using these observations, we prove Theorem 1.3. Section 5 brings the proof of Theorem 1.4, and Section 6, of Theorem 1.5. In Section 7, we show some results which are related to semi-weak normality and weak saturation of ideals.

2. Preliminaries. We refer the reader to Kanamori [11] for general background and basic notation.

Throughout this paper, κ denotes a regular uncountable cardinal, and λ denotes a cardinal with $\lambda \geq \kappa$. Except in Sections 3 and 7, λ will denote a singular cardinal with $\mathrm{cf}(\lambda) < \kappa$.

For ordinals $\alpha < \beta$, $[\alpha, \beta)$ denotes the interval $\{\gamma : \alpha \leq \gamma < \beta\}$.

For a set X of ordinals without the maximum element, let $\lim(X) = \{\alpha < \sup(X) : \sup(X \cap \alpha) = \alpha\}$.

For a regular cardinal μ and an ordinal δ with $\delta < \mu$, E^μ_δ (respectively $E^\mu_{<\delta}$) denotes $\{\alpha < \mu : \mathrm{cf}(\alpha) = \delta\}$ (respectively $\{\alpha < \mu : \mathrm{cf}(\alpha) < \delta\}$).

For an ordinal γ with uncountable cofinality and $S \subseteq \gamma$, S is *stationary in* γ if S intersects any club set in γ.

In this paper, an *ideal* means a non-principal proper ideal over an infinite set. An *ideal over* $\mathcal{P}_\kappa\lambda$ means a κ-complete fine proper ideal over $\mathcal{P}_\kappa\lambda$. For an ideal I over A, I^* denotes the dual filter of I and $I^+ = \mathcal{P}(A) \setminus I$. An element of I^+ is called an *I-positive set*. For an ideal I over A and $X \in I^+$, $I|X$ is the restriction of I to X, that is, $I|X = \{Y \in \mathcal{P}(A) : X \cap Y \in I\}$.

A set $C \subseteq \mathcal{P}_\kappa\lambda$ is *closed* if for every $\gamma < \kappa$ and $\subseteq$-increasing sequence $\langle x_\xi : \xi < \gamma\rangle$ in C, $\bigcup_{\xi<\gamma} x_\xi \in C$. A set $C \subseteq \mathcal{P}_\kappa\lambda$ is *unbounded* if $\forall x \in \mathcal{P}_\kappa\lambda$ $\exists y \in C$ $(x \subseteq y)$. A closed and unbounded set is called *club*. A set $S \subseteq \mathcal{P}_\kappa\lambda$ is *stationary* if it intersects every club set. The following is well-known:

Fact 2.1. *For $S \subseteq \mathcal{P}_\kappa\lambda$, the following are equivalent*:

(1) *S is stationary in $\mathcal{P}_\kappa\lambda$.*
(2) *For every $f : [\lambda]^{<\omega} \to \lambda$, there exists $x \in S$ such that $x \cap \kappa \in \kappa$ and $f``[x]^{<\omega} \subseteq x$.*

Moreover, *if $\kappa = \omega_1$ then* (1) *and* (2) *are equivalent to*

(3) *For every $f : [\lambda]^{<\omega} \to \lambda$, there exists $x \in S$ such that $f``[x]^{<\omega} \subseteq x$.*

$\mathrm{NS}_{\kappa\lambda}$ denotes the non-stationary ideal over $\mathcal{P}_\kappa\lambda$. That is, $\mathrm{NS}_{\kappa\lambda} = \{X \subseteq \mathcal{P}_\kappa\lambda : X \text{ is non-stationary}\}$.

Recall that, for an ideal I over A and a cardinal μ, we say that I is *weakly μ-saturated* if μ many pairwise disjoint I-positive sets do not exist. The items in the following note are easy to prove.

NOTE 2.2.

(1) *Every μ-saturated ideal is weakly μ-saturated.*
(2) *For ideals I and J over A, if I is weakly μ-saturated and $I \subseteq J$ then J is also weakly μ-saturated.*
(3) *If μ is a singular cardinal and I is weakly μ-saturated, then there exist a regular $\delta < \mu$ and $X \in I^+$ such that $I|X$ is weakly δ-saturated.*
(4) *If I is a normal ideal over $\mathcal{P}_\kappa\lambda$ and $\mu \leq \lambda$, then the μ-saturation of I is equivalent to I being weakly μ-saturated.*
(5) *For a stationary set S in $\mathcal{P}_\kappa\lambda$, $\mathrm{NS}_{\kappa\lambda}|S$ is weakly μ-saturated if and only if* $\mathrm{NSp}(S, \mu)$ *holds.*

We will need Shelah's pcf theory. Here we present some basic notations and facts from that theory. These can be found in Abraham–Magidor [2], Cummings [6], Eisworth [8], and Shelah [14].

Let λ be a singular cardinal, and let $\vec{\lambda} = \langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ be a strictly increasing sequence of regular cardinals with limit λ. We let $\Pi\vec{\lambda}$ denote the set

$$\{f : f \text{ is a function from } \mathrm{cf}(\lambda) \text{ to } \lambda \text{ and } f(i) \in \lambda_i \text{ for all } i < \mathrm{cf}(\lambda)\}.$$

We define binary relations $<^*$ and $\leq^*$ on $\Pi\vec{\lambda}$ by

$$f <^* g \Leftrightarrow \{i < \mathrm{cf}(\lambda) : f(i) < g(i)\} \text{ is cobounded,}$$
$$f \leq^* g \Leftrightarrow \{i < \mathrm{cf}(\lambda) : f(i) \leq g(i)\} \text{ is cobounded.}$$

A pair $\langle \vec{\lambda}, \vec{f}\rangle$ is a *scale for* λ if:

(1) $\vec{\lambda} = \langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ is a strictly increasing sequence of regular cardinals with limit λ.
(2) $\vec{f} = \langle f_\xi : \xi < \lambda^+\rangle$ is a $<^*$-increasing $<^*$-cofinal sequence in $\Pi\vec{\lambda}$.

The following is an important fact of pcf theory:

FACT 2.3. *If λ is a singular cardinal then there exists a scale for λ.*

Let $\langle \vec{\lambda}, \vec{f}\rangle$ be a scale for λ. Let $\alpha < \lambda^+$ and let $\langle g_\xi : \xi < \alpha\rangle$ be a $<^*$-increasing sequence in $\Pi\vec{\lambda}$. Then $f \in \Pi\vec{\lambda}$ is an *exact upper bound* (*eub*) *for* $\langle g_\xi : \xi < \alpha\rangle$ if:

(1) $g_\xi <^* f$ for all $\xi < \alpha$.
(2) For every $g \in \Pi\vec{\mu}$, if $g <^* f$ then there exists $\xi < \alpha$ such that $g \leq^* g_\xi$.

An eub for $\langle g_\xi : \xi < \alpha\rangle$ is a least upper bound, and hence is unique modulo the bounded ideal, that is, if f and f' are eub for the sequence then $\{i < \mathrm{cf}(\lambda) : f(i) = f'(i)\}$ is cobounded.

For a limit ordinal $\alpha < \lambda^+$, we say that α is *good for* $\vec{f}$ if there exists an unbounded set a in α and $i^* < \mathrm{cf}(\lambda)$ such that $\langle f_\xi(j) : \xi \in a\rangle$ is strictly increasing for all j with $i^* < j < \mathrm{cf}(\lambda)$.

FACT 2.4. *For a limit ordinal* $\alpha < \lambda^+$ *with* $\mathrm{cf}(\alpha) > \mathrm{cf}(\lambda)$, *the following are equivalent*:

(1) α *is good for* $\vec{f}$.
(2) $\vec{f}|\alpha$ *has an eub* f *such that* $\{i<\mathrm{cf}(\lambda) : \mathrm{cf}(f(i))=\mathrm{cf}(\alpha)\}$ *is cobounded.*

The following fact follows from the combination of Lemmas 15 and 16 of Kojman [12].

FACT 2.5 (Kojman [12], Shelah). *Let* $\gamma < \lambda^+$ *be an ordinal*, *and let* ν *be a regular cardinal with* $\mathrm{cf}(\lambda) < \nu < \mathrm{cf}(\gamma)$. *If* $\{\alpha < \gamma : \mathrm{cf}(\alpha) = \nu,\ \alpha$ *is good for* $\vec{f}\}$ *is stationary in* γ, *then* $\vec{f}|\gamma$ *has an eub* f *such that* $\{i < \mathrm{cf}(\lambda) : \mathrm{cf}(f(i)) > \nu\}$ *is cobounded.*

A scale $\langle\vec{\lambda}, \vec{f}\rangle$ is called a *good scale* if there exists a club C in λ^+ such that every point of C with cofinality greater than $\mathrm{cf}(\lambda)$ is good for $\vec{f}$. It is known that the existence of a good scale is a very weak assumption.

For a strictly increasing sequence of regular cardinals $\vec{\lambda}$ and a set a, we define the function $\chi_a^{\vec{\lambda}} \in \Pi\vec{\lambda}$ by $\chi_a^{\vec{\lambda}}(i) = \sup(a \cap \lambda_i)$ if $\sup(a \cap \lambda_i) < \lambda_i$, and $\chi_a^{\vec{\lambda}}(i) = 0$ otherwise. When $\vec{\lambda}$ is clear from the context, we omit the superscript and write simply χ_a.

Let θ denote a sufficiently large regular cardinal. The following fact will be used:

FACT 2.6. *Let* $M \prec \langle H_\theta, \in\rangle$ *be such that* $\kappa \in M$ *and* $M \cap \kappa \in \kappa$. *For all* $a \in M$, *if* $|a| < \kappa$ *then* $a \subseteq M$.

For $M \prec \langle H_\theta, \in\rangle$ and a limit ordinal α, we say that M is *internally approachable of length* α if there exists an increasing sequence $\langle M_\xi : \xi < \alpha\rangle$ such that $\bigcup_{\xi<\alpha} M_\xi = M$, $M_\xi \prec \langle H_\theta, \in\rangle$ for all $\xi < \alpha$ and $\langle M_\xi : \xi \le \eta\rangle \in M_{\eta+1}$ for all $\eta < \alpha$.

FACT 2.7. *Let* $\mu < \kappa$ *be a regular cardinal and* $R \subseteq H_\theta$. *Then the set* $\{M\cap\lambda : |M| < \kappa,\ M \prec \langle H_\theta, \in, R\rangle,\ M$ *is internally approachable of length* $\mu\}$ *is stationary in* $\mathcal{P}_\kappa\lambda$.

3. Semi-weakly normal ideals. We introduce a variation of normality of ideals, α-semi-weak normality. This can be seen as a variation of the semi-weak normality of ideals in Abe [1]. Recall that an ideal I over $\mathcal{P}_\kappa\lambda$

is *semi-weakly normal* if for every $X \in I^+$ and every $f : X \to \lambda$ with $f(x) < \sup(x)$, there exists $\alpha < \lambda$ such that $\{x \in X : f(x) \leq \alpha\} \in I^+$. We extend this notion to any ideals.

DEFINITION 3.1. Let I be an ideal over A, and let α be an ordinal. A function f on A into the ordinals is called an *α-least function for I* if f fulfills the following conditions:

(1) $\{x \in A : \beta < f(x)\} \in I^*$ for all $\beta < \alpha$.
(2) For all functions g on A, if $\{x \in A : \beta < g(x)\} \in I^*$ for all $\beta < \alpha$, then $\{x \in A : f(x) \leq g(x)\} \in I^*$.

We say that I is *α-semi-weakly normal* (*α-s.w.n.* for short) if I has an α-least function.

Note that in the presence of (1), (2) is equivalent to the following:

(2)′ For all $X \in I^+$ and all functions g on X, if $\forall x \in X\ (g(x) < f(x))$ then there exists $\beta < \alpha$ such that $\{x \in X : g(x) \leq \beta\} \in I^+$.

Hence an ideal I over $\mathcal{P}_\kappa\lambda$ is semi-weakly normal if and only if the assignment $x \mapsto \sup(x)$ is a λ-least function for I.

NOTE 3.2.

(1) *If f is an α-least function for I, then $\{x \in A : f(x) \leq \alpha\} \in I^*$.*
(2) *An α-least function for I is unique modulo I, that is, if f and g are α-least functions for I, then $\{x \in A : f(x) = g(x)\} \in I^*$.*
(3) *If f is an α-least function for I and $X \in I^+$ then f is also an α-least function for $I|X$.*

Note that, under some assumptions, there exists a normal ideal over $\mathcal{P}_\kappa\lambda$ which is not λ^+-s.w.n. See Proposition 7.4. In spite of this, for any normal ideal I over $\mathcal{P}_\kappa\lambda$ and all α, we can find an α-s.w.n. normal ideal J such that $I \subseteq J$. The following proposition is the main result of this section.

PROPOSITION 3.3. *Let I be a normal ideal over $\mathcal{P}_\kappa\lambda$. Let $\mathcal{F}$ be a set of functions from $\mathcal{P}_\kappa\lambda$ to the ordinals. Then there exists a normal ideal J over $\mathcal{P}_\kappa\lambda$ extending I and a function f^* on $\mathcal{P}_\kappa\lambda$ satisfying the following*:

(1) *$\{x \in \mathcal{P}_\kappa\lambda : f(x) \leq f^*(x)\} \in J^*$ for all $f \in \mathcal{F}$.*
(2) *For all functions g on $\mathcal{P}_\kappa\lambda$ and $X \in J^+$, if $\forall x \in X\ (g(x) < f^*(x))$ then there exists $f \in \mathcal{F}$ such that $\{x \in X : g(x) \leq f(x)\} \in J^+$.*

In particular, *f^* is a least upper bound of $\mathcal{F}$ modulo J.*

Proof. This argument is inspired by Burke's proofs in [3] and [4].

Let $\alpha = \sup\{\sup(\text{range}(f)) : f \in \mathcal{F}\}$. Let $\gamma = |\alpha|^{\lambda^{<\kappa}}$. Fix an enumeration $\langle f_\xi : \xi < \gamma\rangle$ of $\mathcal{F}$. Let $\langle g_\xi : \xi < \gamma\rangle$ be an enumeration of all functions from $\mathcal{P}_\kappa\lambda$ to $\alpha + 1$. We assume that g_0 is the constant function on $\mathcal{P}_\kappa\lambda$ with value α. Let $\langle C_\xi : \xi < \gamma\rangle$ be an enumeration of I^*, and let $T = \{x \in \mathcal{P}_\kappa\gamma : 0 \in x \wedge \forall\xi \in x\ (x \cap \lambda \in C_\xi)\}$.

Claim 3.4. *T is stationary in $\mathcal{P}_\kappa\gamma$.*

Proof of Claim. It is enough to show that for every $f : [\gamma]^{<\omega} \to \gamma$ we can find $x \in T$ such that $x \cap \kappa \in x$ and $f``[x]^{<\omega} \subseteq x$. Fix a sufficiently large regular cardinal θ and choose $M \prec \langle H_\theta, \in, \kappa, \lambda, \gamma, f, I, \langle C_\xi : \xi < \gamma\rangle \ldots\rangle$ with $|M| = \lambda \subseteq M$. Fix a bijection $\pi : \lambda \to M \cap \gamma$. Then, because I is normal, we know the set $X = \{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \in \kappa,\ 0 \in x,\ \pi``x \cap \lambda = x,\ f``[\pi``x]^{<\omega} \subseteq \pi``x, \forall\xi \in \pi``x\ (x \in C_\xi)\}$ is in $I^*$. Take $x \in X$. Then $\pi``x \in \mathcal{P}_\kappa\gamma$ is the required set. $\blacksquare_{\text{Claim}}$

Take $x \in T$. Consider the set $a_x = \{g_\xi(x\cap\lambda) : \xi \in x, \forall\eta \in x\ (f_\eta(x\cap\lambda) \le g_\xi(x \cap \lambda))\}$. Then a_x is non-empty because $0 \in x$. Let $\xi_x \in x$ be such that $g_{\xi_x}(x \cap \lambda)$ is the least element of a_x. Since T is stationary in $\mathcal{P}_\kappa\gamma$, by the normality of $\mathrm{NS}_{\kappa\gamma}$, there exists $\xi^* < \gamma$ such that $\overline{T} = \{x \in T : \xi^* = \xi_x\}$ is stationary. Let J be the projection of $\mathrm{NS}_{\kappa\gamma}|\overline{T}$ to $\mathcal{P}_\kappa\lambda$, that is, $X \in J \Leftrightarrow \{x \in \overline{T} : x \cap \lambda \in X\}$ is non-stationary in $\mathcal{P}_\kappa\gamma$. It is easy to check that J is a normal ideal over $\mathcal{P}_\kappa\lambda$.

Claim 3.5. *$I \subseteq J$.*

Proof of Claim. It is enough to show that $I^* \subseteq J^*$. Take $C_\xi \in I^*$. Then $\{x \in \overline{T} : \xi \notin x\}$ is non-stationary, hence $\{x \in \overline{T} : x \cap \lambda \notin C_\xi\}$ is non-stationary. This shows that $C_\xi \in J^*$. $\blacksquare_{\text{Claim}}$

Finally, we show that J and g_{ξ^*} are the required pair. Let $\eta < \gamma$ and we first check that $\{x \in \mathcal{P}_\kappa\lambda : g_{\xi^*}(x) \ge f_\eta(x)\} \in J^*$. For all $x \in \overline{T}$ with $\eta \in x$, we have $f_\eta(x \cap \lambda) \le g_{\xi_x}(x \cap \lambda) = g_{\xi^*}(x \cap \lambda)$. This shows that $\{x \in \overline{T} : g_{\xi^*}(x \cap \lambda) < f_\eta(x \cap \lambda)\}$ is non-stationary, hence we have $\{x \in \mathcal{P}_\kappa\lambda : g_{\xi^*}(x) \ge f_\eta(x)\} \in J^*$. Now take $X \in J^+$ and a function g such that $\forall x \in X\ (g(x) < g_{\xi^*}(x))$. We may assume that the range of g is contained in $\alpha + 1$, hence there exists $\zeta < \gamma$ such that $g_\zeta = g$. Since $X \in J^+$, $\{x \in \overline{T} : x\cap\lambda \in X, \zeta \in x\}$ is stationary. Let $x \in \overline{T}$ be such that $x\cap\lambda \in X$ and $\zeta \in x$. Then $g_\zeta(x \cap \lambda) = g(x \cap \lambda) < g_{\xi^*}(x \cap \lambda) = g_{\xi_x}(x \cap \lambda)$. By the minimality of $g_{\xi_x}(x\cap\lambda)$, $g(x\cap\lambda) < f_{\eta_x}(x\cap\lambda)$ for some $\eta_x \in x$. Thus, by the normality of $\mathrm{NS}_{\kappa\gamma}$, there exists $\eta < \gamma$ such that $\{x \in \overline{T} : x\cap\lambda \in X,\ g(x\cap\lambda) \le f_\eta(x\cap\lambda)\}$ is stationary. Then $\{x \in X : g(x) \le f_\eta(x)\} \in J^+$, as required. $\blacksquare$

Corollary 3.6. *For any normal ideal I over $\mathcal{P}_\kappa\lambda$ and any ordinal α, there exists an α-s.w.n. normal ideal over $\mathcal{P}_\kappa\lambda$ extending I.*

Proof. For $\beta < \alpha$, let $f_\beta : \mathcal{P}_\kappa\lambda \to \{\beta\}$ be the constant function with value β. By the previous proposition, we can find a normal ideal J over $\mathcal{P}_\kappa\lambda$ extending I and $f : \mathcal{P}_\kappa\lambda \to \mathrm{Ord}$ such that f is a least upper bound of the f_β's modulo J. Clearly f is an α-least function for J. $\blacksquare$

The previous proposition can be adapted for non-normal ideals in the following way. The following proposition will not be used until Section 7.

Proposition 3.7. *Let I be a κ-complete ideal over A. Let $\mathcal{F}$ be a set of functions from A to the ordinals. Then there exist an ideal J over A extending I and a function f^* on A satisfying the following*:

1. *J is κ-complete.*
2. *$\{a \in A : f(a) \leq f^*(a)\} \in J^*$ for all $f \in \mathcal{F}$.*
3. *For all functions g on A and $X \in J^+$, if $\forall a \in X\,(g(a) < f^*(a))$ then there exists $f \in \mathcal{F}$ such that $\{a \in X : g(a) \leq f(a)\} \in J^+$.*

Proof. Let $\alpha = \sup\{\sup(\mathrm{range}(f)) : f \in \mathcal{F}\}$ and $\gamma = |\alpha|^{|A|}$. Let $\langle f_\xi : \xi < \gamma\rangle$ be an enumeration of $\mathcal{F}$ and $\langle g_\xi : \xi < \gamma\rangle$ an enumeration of all functions from A to $\alpha + 1$. As before we assume that g_0 is the constant function from A to $\{\alpha\}$. Let $\langle C_\xi : \xi < \gamma\rangle$ be an enumeration of the members of I^*. Take $x \in \mathcal{P}_\kappa\gamma$ with $0 \in x$. Since I is κ-complete and $|x| < \kappa$, we have $\bigcap_{\xi\in x} C_\xi \in I^*$. Fix $s_x \in \bigcap_{\xi\in x} C_\xi$. Now let $b_x = \{g_\xi(s_x) : \xi \in x,\, \forall \eta \in x\ (f_\eta(s_x) \leq g_\xi(s_x))\}$. Let $\xi_x \in x$ be such that $g_{\xi_x}(s_x)$ is the least element of b_x. By Fodor's lemma, there exists $\xi^* < \gamma$ such that $T = \{x \in \mathcal{P}_\kappa\gamma : \xi_x = \xi^*\}$ is stationary in $\mathcal{P}_\kappa\gamma$. Define $J \subseteq \mathcal{P}(A)$ by $X \in J$ if and only if $\{x \in T : s_x \in X\}$ is non-stationary. It is easy to show that J is an ideal over A extending I, and J is κ-complete. Furthermore, g_{ξ^*} is the required function. ∎

Hence every maximal σ-complete ideal over A is α-s.w.n. for all α.

Corollary 3.8. *Let I be a κ-complete ideal over A and let α be an ordinal. Then there exists a κ-complete α-s.w.n. ideal J over A extending I.*

Next we introduce a strong form of α-semi-weak normality of ideals. This is an analogue of weak normality in Abe [1].

Definition 3.9. For an ideal I over A and an ordinal α, we say that I is *α-weakly normal* if I has an α-least function f satisfying the following: For all $X \in I^+$ and for all functions g on X, if $\forall x \in X\,(g(x) < f(x))$ then there exists $\beta < \alpha$ such that $\{x \in X : g(x) \leq \beta\} \in (I|X)^*$.

Note that, since an α-least function is unique modulo I, if I is α-weakly normal then any α-least function for I witnesses the α-weak normality of I.

The next proposition shows the connection between weak saturation and weak normality of ideals; it is an analogue of Lemma 1.1 in [1].

Proposition 3.10. *Let α be a limit ordinal and let I be an α-s.w.n. ideal over A. Then I is α-weakly normal if and only if I is weakly* $\mathrm{cf}(\alpha)$*-saturated.*

Proof. Suppose that I is weakly $\mathrm{cf}(\alpha)$-saturated. Let f be any α-least function for I. To show that I is α-weakly normal, take $X \in I^+$ and let $g : X \to \alpha + 1$ be such that $\forall x \in X\ (g(x) < f(x))$. Suppose that $\{x \in X :$

$g(x) > \beta\} \in I^+$ for all $\beta < \alpha$. For $\beta < \alpha$, since $\{x \in X : f(x) > g(x) > \beta\} \in I^+$, there exists $\gamma < \alpha$ such that $\beta < \gamma$ and $\{x \in X : \beta < g(x) \le \gamma\} \in I^+$. Using this observation, we can define a strictly increasing sequence $\langle \beta_\xi : \xi < \mathrm{cf}(\alpha)\rangle$ in α such that $X_\xi = \{x \in X : \beta_\xi < g(x) \le \beta_{\xi+1}\} \in I^+$. Then the X_ξ's are $\mathrm{cf}(\alpha)$ many pairwise disjoint I-positive sets, contradicting that I is weakly $\mathrm{cf}(\alpha)$-saturated.

For the converse, suppose that I is not $\mathrm{cf}(\alpha)$-saturated. Take $\mathrm{cf}(\alpha)$ many pairwise disjoint I-positive sets $\langle Y_\xi : \xi < \mathrm{cf}(\alpha)\rangle$. Let $\langle \gamma_\xi : \xi < \mathrm{cf}(\alpha)\rangle$ be a strictly increasing sequence of limit α. Since $\{x \in A : f(x) > \beta\} \in I^*$ for all $\beta < \alpha$, we may assume that $\forall x \in Y_\xi\ (f(x) > \gamma_\xi)$. Let $Y = \bigcup_{\xi<\mathrm{cf}(\alpha)} Y_\xi$ and define the function g on Y by $g(x) = \gamma_\xi \Leftrightarrow x \in Y_\xi$. Then it is easy to see that there is no $\beta < \alpha$ such that $\{x \in Y : g(x) \le \beta\} \in (I|Y)^*$, hence I is not α-weakly normal. ∎

4. Basic properties of λ^+-weakly normal ideals over $\mathcal{P}_\kappa\lambda$. In this section, we will observe some basic properties of λ^+-weakly normal ideals over $\mathcal{P}_\kappa\lambda$ with $\mathrm{cf}(\lambda) < \kappa$. Using them we will prove Theorem 1.3.

Throughout this section, we assume $\mathrm{cf}(\lambda) < \kappa$. Let I be a normal ideal over $\mathcal{P}_\kappa\lambda$ that is λ^+-weakly normal, and let $h_I : \mathcal{P}_\kappa\lambda \to \mathrm{Ord}$ be a witness.

PROPOSITION 4.1. $\{x \in \mathcal{P}_\kappa\lambda : h_I(x) < \lambda^+\} \in I^*$.

Proof. Suppose otherwise; then $X = \{x \in \mathcal{P}_\kappa\lambda : h_I(x) = \lambda^+\} \in I^+$ by Note 3.2(1). Fix a scale $\langle \vec{\lambda}, \vec{f}\,\rangle$ for λ. For each $x \in X$, choose $\xi_x < \lambda^+ = h_I(x)$ such that $\chi_x <^* f_{\xi_x}$. Then, by the λ^+-weak normality of I, there exists $\xi^* < \lambda^+$ such that $\{x \in X : \xi_x \le \xi^*\} \in I^+$. Since every I-positive set is unbounded, we can find $x \in X$ such that $\xi_x \le \xi^*$ and $f_{\xi^*} <^* \chi_x$. Then $f_{\xi^*} <^* \chi_x <^* f_{\xi_x}$, hence $\xi_x > \xi^*$. This is a contradiction. ∎

PROPOSITION 4.2. *For every club C in λ^+, $\{x \in \mathcal{P}_\kappa\lambda : h_I(x) \in C\} \in I^*$.*

Proof. Let $X = \{x \in \mathcal{P}_\kappa\lambda : h_I(x) \notin C\}$ and suppose $X \in I^+$. We define f on X by $f(x) = \sup(h_I(x) \cap C)$. Then $f(x) \in C$ and $f(x) < h_I(x)$. By the λ^+-weak normality of I, there exists $\alpha < \lambda^+$ such that $\{x \in X : f(x) \le \alpha\} \in I^+$. Choose $\beta \in C \setminus (\alpha + 1)$. Then $\{x \in \mathcal{P}_\kappa\lambda : h_I(x) > \beta\} \in I^*$, hence there exists $x \in X$ such that $\sup(h_I(x) \cap C) \le \alpha < \beta \in C \cap h_I(x)$. This is a contradiction. ∎

Note that we need only the λ^+-semi-weak normality of I to prove Propositions 4.1 and 4.2. From now on we will assume that $h_I(x)$ is a limit ordinal less than λ^+.

The proposition below will be used in Section 6 but not in Section 5.

PROPOSITION 4.3. *Let E be a stationary subset of $E^{\lambda^+}_{<\kappa}$. Then $\{x \in \mathcal{P}_\kappa\lambda : E \cap h_I(x)$ is stationary in $h_I(x)\} \in I^*$. Thus E is reflecting.*

Proof. Suppose otherwise, and let $X = \{x \in \mathcal{P}_\kappa\lambda : E \cap h_I(x)$ is nonstationary in $h_I(x)\} \in I^+$. For each $x \in X$, let c_x be a club in $h_I(x)$ such that $E \cap c_x = \emptyset$ and $\mathrm{ot}(c_x) = \mathrm{cf}(h_I(x))$. By induction on $\xi < \lambda^+$, we will define a strictly increasing sequence $\langle \alpha_\xi : \xi < \lambda^+\rangle$ so that $\{x \in X : [\alpha_\xi, \alpha_{\xi+1}) \cap c_x \neq \emptyset\} \in (I|X)^*$ for all $\xi < \lambda^+$. Suppose $\langle \alpha_\eta : \eta < \xi\rangle$ is defined and $\{x \in X : [\alpha_\eta, \alpha_{\eta+1}) \cap c_x \neq \emptyset\} \in (I|X)^*$ for all $\eta < \xi$ with $\eta + 1 < \xi$. If ξ is limit, then let $\alpha_\xi = \sup\{\alpha_\eta : \eta < \xi\}$. Suppose $\xi = \zeta + 1$. We define a function g on $\{x \in X : \alpha_\zeta < h_I(x)\}$ by $g(x) = \min(c_x \setminus \alpha_\zeta) + 1$. Note that $\{x \in X : \alpha_\zeta < h_I(x)\} \in (I|X)^*$. Hence, by the λ^+-weak normality of I, there exists $\alpha_\xi < \lambda^+$ such that $\{x \in X : g(x) \leq \alpha_\xi\} \in (I|X)^*$. Then clearly $\{x \in X : [\alpha_\zeta, \alpha_\xi) \cap c_x \neq \emptyset\} \in (I|X)^*$.

Let $C = \mathrm{Lim}\{\alpha_\xi : \xi < \lambda^+\}$. Then C is a club set in λ^+. Since E is stationary, we have $E \cap C \neq \emptyset$. Let $\alpha \in E \cap C$. Since $E \subseteq E^{\lambda^+}_{<\kappa}$, we can take $a \subseteq \lambda^+$ such that $\sup\{\alpha_\xi : \xi \in a\} = \alpha$ and $\mathrm{ot}(a) = \mathrm{cf}(\alpha) < \kappa$. For each $\xi \in a$, let $X_\xi = \{x \in X : [\alpha_\xi, \alpha_{\xi+1}) \cap c_x \neq \emptyset\} \in (I|X)^*$. Since I is κ-complete, we have $\bigcap_{\xi\in a} X_\xi \in (I|X)^*$. Take $x \in \bigcap_{\xi\in a} X_\xi$ with $h_I(x) > \alpha$. Then, since $[\alpha_\xi, \alpha_{\xi+1}) \cap c_x \neq \emptyset$ for all $\xi \in a$, c_x is a club in $h_I(x)$, and $h_I(x) > \alpha$, we have $\alpha = \sup\{\alpha_\xi : \xi \in a\} \in c_x$. Then $\alpha \in E \cap c_x$, which is a contradiction. ∎

We do not need the normality of ideals to prove the previous proposition, but we need it in the next one.

Proposition 4.4. $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(h_I(x)) > \mathrm{ot}(x)\} \in I^*$.

Proof. Notice that $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{ot}(x)$ is not regular$\} \in I^*$ because $\{x \in \mathcal{P}_\kappa\lambda : \sup(x) = \lambda\}$ is cobounded.

Suppose to the contrary $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(h_I(x)) < \mathrm{ot}(x)\} \in I^+$. Then, by the normality of I, there exists $\gamma < \lambda$ such that $X = \{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(h_I(x)) = \mathrm{ot}(x \cap \gamma)\} \in I^+$. For each $x \in X$, fix an unbounded set $b_x \subseteq h_I(x)$ with $\mathrm{ot}(b_x) = \mathrm{cf}(h_I(x)) = \mathrm{ot}(x \cap \gamma)$. As in the proof of Proposition 4.3, we can construct a strictly increasing sequence $\langle \alpha_\xi : \xi < \gamma\rangle$ such that $\{x \in X : [\alpha_\xi, \alpha_{\xi+1}) \cap b_x \neq \emptyset\} \in (I|X)^*$ for all $\xi < \gamma$. Let $\alpha = \sup\{\alpha_\xi : \xi < \gamma\} < \lambda^+$.

Since I is normal and $\gamma < \lambda$, $Y = \{x \in X : h_I(x) > \alpha, \forall \xi \in x \cap \gamma\,([\alpha_\xi, \alpha_{\xi+1}) \cap b_x \neq \emptyset)\}$ is in $(I|X)^*$. Take $x \in Y$. Then, since $\mathrm{ot}(b_x) = \mathrm{ot}(x \cap \gamma)$, we have $\sup\{\alpha_\xi : \xi \in x \cap \gamma\} = \sup(b_x) = h_I(x)$. However, $\sup\{\alpha_\xi : \xi \in x \cap \gamma\} \leq \alpha < h_I(x)$. This is a contradiction. ∎

Lemma 4.5. *Let $\langle \vec{\lambda}, \vec{f}\rangle$ be a scale for λ. Let X be the set of all $x \in \mathcal{P}_\kappa\lambda$ such that for all $f \in \Pi\vec{\lambda}$, if f is an eub for $\vec{f}|h_I(x)$ then $\{i < \mathrm{cf}(\lambda) : f(i) \cap x$ is unbounded in $f(i)\}$ is cobounded in $\mathrm{cf}(\lambda)$. Then $X \in I^*$.*

Proof. Suppose otherwise, and let $Y = \mathcal{P}_\kappa\lambda \setminus X \in I^+$. For each $x \in Y$, let f_x be an eub for $\vec{f}|h_I(x)$ such that $a_x = \{i < \mathrm{cf}(\lambda) : f_x(i) \cap x$ is bounded in $f_x(i)\}$ is unbounded. Define $g_x \in \Pi\vec{\lambda}$ by $g_x(i) = \sup(x \cap f_x(i))$ if $f_x(i) \cap x$ is bounded in $f_x(i)$, and $g_x(i) = 0$ otherwise. Then $g_x <^* f_x$, hence there exists $\xi_x < h_I(x)$ such that $g_x \leq^* f_{\xi_x}$.

By the λ^+-weak normality of I, there exists $\xi^* < \lambda^+$ such that $\{x \in Y : \xi_x \leq \xi^*\} \in I^+$. Then there exists $x \in Y$ such that $\xi_x \leq \xi^* < h_I(x)$ and $\mathrm{range}(f_{\xi^*}) \subseteq x$. Note that $f_{\xi^*} <^* f_x$ because $h_I(x)$ is limit. Choose $i < \mathrm{cf}(\lambda)$ such that $\sup(x \cap f_x(i)) = g_x(i) < f_{\xi_x}(i) \leq f_{\xi^*}(i) < f_x(i)$. Since $f_{\xi^*}(i) \in x$, we have $f_{\xi^*}(i) \leq \sup(x \cap f_x(i)) < f_{\xi^*}(i)$. This is a contradiction. ■

Proposition 4.6. *Let $\langle\vec{\lambda}, \vec{f}\rangle$ be a scale for λ. Then $\{x \in \mathcal{P}_\kappa\lambda : h_I(x)$ is not good for $\vec{f}\} \in I^*$.*

Proof. Let X be as in the last lemma. Let $Z = \{x \in X : \mathrm{cf}(h_I(x)) > \mathrm{ot}(x) > \mathrm{cf}(\lambda)\}$. Then $Z \in I^*$ by Proposition 4.4. Let $x \in Z$. We claim that $h_I(x)$ is not good. Suppose otherwise. Then by Fact 2.4 and $\mathrm{cf}(h_I(x)) > \mathrm{ot}(x) > \mathrm{cf}(\lambda)$, there exists an eub f for $\vec{f}|h_I(x)$ such that $\{i < \mathrm{cf}(\lambda) : \mathrm{cf}(f(i)) = \mathrm{cf}(h_I(x))\}$ is cobounded. On the other hand, $\{i < \mathrm{cf}(\lambda) : \mathrm{cf}(f(i)) \leq \mathrm{ot}(x)\}$ is cobounded by $x \in X$. Since $\mathrm{ot}(x) < \mathrm{cf}(h_I(x))$, this is a contradiction. ■

Proposition 4.6 will not be used in later sections. Combining Propositions 4.4 and 4.6, we have the following:

Proposition 4.7. *Suppose that there exists a normal λ^+-weakly normal ideal over $\mathcal{P}_\kappa\lambda$. Then there exists no good scale for λ.*

Corollary 4.8. *Suppose that there exists a weakly λ^+-saturated normal ideal over $\mathcal{P}_\kappa\lambda$. Then every stationary subset of $E^{\lambda^+}_{<\kappa}$ is reflecting, and there is no good scale for λ.*

Proof. Let J be a weakly λ^+-saturated normal ideal over $\mathcal{P}_\kappa\lambda$. By Corollary 3.6, there exists a λ^+-s.w.n. normal ideal I extending J. By Note 2.2(2), I is weakly λ^+-saturated. Thus I is a λ^+-weakly normal ideal by Proposition 3.10, and the assertion follows from Proposition 4.3 and 4.7. ■

5. Splitting stationary subsets of $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(|x|) \neq \mathrm{cf}(\lambda)\}$. In this section, we will prove Theorem 1.4. As in Section 4, throughout this section we assume that $\mathrm{cf}(\lambda) < \kappa$.

Proposition 5.1. *Let $S = \{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(|x|) \neq \mathrm{cf}(\lambda)\}$. Then there is no weakly λ^+-saturated normal ideal I over $\mathcal{P}_\kappa\lambda$ such that $S \in I^*$.*

Proof. Suppose that such an ideal exists. Then, by Corollary 3.6, there exists a λ^+-s.w.n. normal ideal I extending our weakly λ^+-saturated normal ideal over $\mathcal{P}_\kappa\lambda$. Therefore, by Note 2.2(2) and Proposition 3.10, I is a normal λ^+-weakly normal ideal with $S \in I^*$. Note that $\mathrm{cf}(\lambda)^+ < \kappa$: If $\mathrm{cf}(\lambda)^+ = \kappa$ then $\{x \in \mathcal{P}_\kappa\lambda : |x| = \mathrm{cf}(\lambda)\}$ would be cobounded, which contradicts $S \in I^*$.

Since $\{x \in \mathcal{P}_\kappa\lambda : \sup(x) = \lambda\}$ is cobounded, the set $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(\mathrm{ot}(x)) = \mathrm{cf}(\lambda)\}$ is cobounded. Thus $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{ot}(x) > |x|\} \in I^*$. Fix a scale $\langle \vec{\lambda}, \vec{f} \rangle$ for λ. Define $X = \{x \in S : \mathrm{cf}(h_I(x)) > \mathrm{ot}(x) > |x| > \mathrm{cf}(\lambda), \forall f$ (f is an eub for $\vec{f}|h_I(x) \Rightarrow \{i < \mathrm{cf}(\lambda) : f(i) \cap x$ is unbounded in $f(i)\}$ is cobounded)$\}$. Then $X \in I^*$ by Proposition 4.4 and Lemma 4.5.

Define $Y = \{x \in X : \forall \alpha \in x$ ($\mathrm{ot}(x \cap \alpha) > \mathrm{cf}(\lambda)$ is regular $\Rightarrow \{\beta < h_I(x) : \mathrm{cf}(\beta) = \mathrm{ot}(x \cap \alpha)$, β is good$\}$ is stationary in $h_I(x)\}$.

Claim 5.2. *$Y \in I^*$.*

Proof of Claim. Suppose not. Using the normality of I, there exists $\alpha^* < \lambda$ such that $Z = \{x \in X : \mathrm{ot}(x \cap \alpha^*)$ is regular, $\mathrm{ot}(x \cap \alpha^*) > \mathrm{cf}(\lambda)$, $\{\beta < h_I(x) : \mathrm{cf}(\beta) = \mathrm{ot}(x \cap \alpha^*)$, β is good$\}$ is not stationary in $h_I(x)\} \in I^+$.

Subclaim 5.3. *α^* is regular with* $\mathrm{cf}(\lambda) < \alpha^* < \lambda$.

Proof of Subclaim. It is easy to check that $\mathrm{cf}(\lambda) < \alpha^* < \lambda$. Suppose that $\mathrm{cf}(\alpha^*) < \alpha^*$. Fix a cofinal map $\pi : \mathrm{cf}(\alpha^*) \to \alpha^*$. Then $\{x \in \mathcal{P}_\kappa\lambda : \pi``(x \cap \mathrm{cf}(\alpha^*))$ is cofinal in $\sup(x \cap \alpha^*)\}$ contains a club. Hence there is $x \in Z$ such that $\mathrm{ot}(x \cap \alpha^*)$ is not regular. This is a contradiction. ■$_{\text{Subclaim}}$

For each $x \in Z$, fix a club c_x in $h_I(x)$ such that $c_x \cap \{\beta < h_I(x) : \mathrm{cf}(\beta) = \mathrm{ot}(x \cap \alpha^*)$, β is good$\} = \emptyset$ and $\mathrm{ot}(c_x) = \mathrm{cf}(h_I(x))$. We will find $x \in Z$ and $\beta \in c_x$ such that $\mathrm{cf}(\beta) = \mathrm{ot}(x \cap \alpha^*)$ and β is good, which is a contradiction. Let θ be a sufficiently large regular cardinal. By the λ^+-weak normality of I, we can build an increasing continuous sequence $\langle M_\xi : \xi < \alpha^* \rangle$ satisfying the following. For all $\xi < \alpha^*$:

(1) $|M_\xi| < \alpha^*$, $M_\xi \cap \alpha^* \in \alpha^*$, $M_\xi \prec \langle H_\theta, \in \rangle$, and $\alpha^*, \langle \vec{\lambda}, \vec{f} \rangle \in M_\xi$.
(2) $\langle M_\eta : \eta \leq \xi \rangle \in M_{\xi+1}$.
(3) $Z_\xi = \{x \in Z : [\sup(M_\xi \cap \lambda^+), \sup(M_{\xi+1} \cap \lambda^+)) \cap c_x \neq \emptyset\} \in (I|Z)^*$.

Finally, let $M_{\alpha^*} = \bigcup_{\xi<\alpha^*} M_\xi$. Note that, by a standard argument, for each $\xi \leq \alpha^*$ with $\mathrm{cf}(\xi) > \mathrm{cf}(\lambda)$, $\sup(M_\xi \cap \lambda^+)$ is good for $\vec{f}$ (see Cummings [5]). Since I is normal and $\alpha^* < \lambda$, we have $\triangle_{\xi<\alpha^*} Z_\xi = \{x \in \mathcal{P}_\kappa\lambda : \forall \xi \in x \cap \alpha^* \,(x \in Z_\xi)\} \in (I|Z)^*$. Take $x \in \triangle_{\xi<\alpha^*} Z_\xi$. Let $\alpha = \sup(x \cap \alpha^*)$. Clearly α is limit. Then, since $[\sup(M_\xi \cap \lambda^+), \sup(M_{\xi+1} \cap \lambda^+)) \cap c_x \neq \emptyset$ for all $\xi \in x \cap \alpha^*$ and $\mathrm{ot}(x \cap \alpha^*) \leq \mathrm{ot}(x) < \mathrm{cf}(h_I(x)) = \mathrm{ot}(c_x)$, we have $\beta = \sup(M^*_\alpha \cap \lambda^+) = \sup\{\sup(M_\xi \cap \lambda^+) : \xi \in x \cap \alpha^*\} \in c_x$. Moreover $\mathrm{cf}(\beta) = \mathrm{ot}(x \cap \alpha^*) > \mathrm{cf}(\lambda)$, and hence β is good for $\vec{f}$, as required. ■$_{\text{Claim}}$

Fix $x \in Y$. Since $\mathrm{ot}(x) > |x| > \mathrm{cf}(\lambda)$, there is $\alpha \in x$ with $\mathrm{ot}(x \cap \alpha) = \mathrm{cf}(\lambda)^+$. Hence $\{\beta < h_I(x) : \mathrm{cf}(\beta) = \mathrm{cf}(\lambda)^+$, β is good for $\vec{f}\}$ is stationary as $x \in Y$. Therefore by Fact 2.5, $\vec{f}|h_I(x)$ has an eub f, say. Since $x \in Y$, $\{\beta < h_I(x) : \mathrm{cf}(\beta) = \nu$, β is good for $\vec{f}\}$ is stationary for all regular $\nu \leq |x|$ with $\nu > \mathrm{cf}(\lambda)$. By Fact 2.5 and the uniqueness of an eub, $\{i < \mathrm{cf}(\lambda) :$

$\mathrm{cf}(f(i)) > \nu\}$ is cobounded for all regular $\nu \le |x|$. On the other hand, $\{i < \mathrm{cf}(\lambda) : \mathrm{cf}(f(i)) \le |x|\}$ is cobounded as $x \in X$. Therefore $|x|$ is singular, and hence $\{i < \mathrm{cf}(\lambda) : \mathrm{cf}(f(i)) < |x|\}$ is cobounded. Since $\mathrm{cf}(|x|) \neq \mathrm{cf}(\lambda)$, there is $\nu < |x|$ such that $\{i < \mathrm{cf}(\lambda) : \mathrm{cf}(f(i)) < \nu\}$ is unbounded. This is a contradiction. ∎

COROLLARY 5.4. *Let* $S = \{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(|x|) \neq \mathrm{cf}(\lambda)\}$. *Then* $\mathrm{Sp}(T, \lambda^+)$ *for every stationary subset* T *of* S. *In particular*, *if* $\kappa = \mu^+$ *with* $\mathrm{cf}(\mu) \neq \mathrm{cf}(\lambda)$, *then* $\mathrm{Sp}(T, \lambda^+)$ *for every stationary subset* T *of* $\mathcal{P}_\kappa\lambda$.

6. Splitting stationary subsets of some definable set. In this section we prove Theorem 1.5. As in the previous section, we assume that $\mathrm{cf}(\lambda) < \kappa$. Let θ be a sufficiently large regular cardinal, and let $\varDelta$ be a well-ordering on H_θ. Let $\mathcal{M} = \langle H_\theta, \in, \varDelta, \kappa, \lambda\rangle$.

We define a canonical function that is λ^+-least for all λ^+-weakly normal ideals. Using this function, we prove Theorem 1.5.

DEFINITION 6.1. Define $h^* : \mathcal{P}_\kappa\lambda \to \lambda^+ + 1$ by $h^*(x) = \sup\{\sup(M \cap \lambda^+) : M \prec \mathcal{M},\ M \cap \lambda = x\}$.

NOTE 6.2.

(1) *$h^*(x)$ is* 0 *or a limit ordinal.*
(2) $\{x \in \mathcal{P}_\kappa\lambda : h^*(x)$ *is a limit ordinal* $> \alpha\}$ *contains a club for all* $\alpha < \lambda^+$.

We prove that $h^*(x) < \lambda^+$ for all $x \in \mathcal{P}_\kappa\lambda$ such that $x \cap \kappa \in \kappa$.

LEMMA 6.3. *Let* $x \in \mathcal{P}_\kappa\lambda$ *be such that* $x \cap \kappa \in \kappa$ *and let* $\langle \vec{\lambda}, \vec{f}\rangle$ *be the* $\varDelta$*-least scale for* λ. *Then* $h^*(x) \le \min\{\xi : \chi_x \le^* f_\xi\}$. *In particular*, $\{x \in \mathcal{P}_\kappa\lambda : h^*(x) < \lambda^+\}$ *contains a club.*

Proof. Suppose to the contrary $\xi < h^*(x)$ and $\chi_x \le^* f_\xi$ for some $\xi < \lambda^+$. Then there exists $M \prec \mathcal{M}$ such that $M \cap \lambda = x$ and $\sup(M \cap \lambda^+) > \xi$. Take $\eta \in M \cap \lambda^+ \setminus \xi$. Then $\chi_x \le^* f_\xi \le^* f_\eta$, hence there exists $i < \mathrm{cf}(\lambda)$ such that $\sup(x \cap \lambda_i) \le f_\xi(i) \le f_\eta(i)$. But since $\eta \in M$ and $M \cap \kappa \in \kappa$, we have $f_\eta(i) \in M \cap \lambda_i$. Therefore $f_\eta(i) < \sup(M \cap \lambda_i) = \sup(x \cap \lambda_i)$, which is a contradiction. ∎

PROPOSITION 6.4. *Let I be a normal λ^+-weakly normal ideal*, *and let h_I be a λ^+-least function for I. Then* $\{x \in \mathcal{P}_\kappa\lambda : h_I(x) = h^*(x)\} \in I^*$, *that is*, *$h^*$ is a λ^+-least function for I.*

Proof. We have $\{x \in \mathcal{P}_\kappa\lambda : h_I(x) \le h^*(x)\} \in I^*$ by Note 6.2(2). It remains to show that $\{x \in \mathcal{P}_\kappa\lambda : h^*(x) \le h_I(x)\} \in I^*$.

Let $\langle \vec{\lambda}, \vec{f}\rangle$ be the $\varDelta$-least scale for λ. Let $\langle E_\xi : \xi < \lambda\rangle$ be the $\varDelta$-least sequence of pairwise disjoint stationary subsets of $E^{\lambda^+}_\omega$. By Proposition 4.3 and the normality of I, $S = \{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa \in \kappa,\ \forall \xi \in x\ (E_\xi \cap h_I(x)$ is stationary in $h_I(x))\} \in I^*$. We claim that $S \subseteq \{x \in \mathcal{P}_\kappa\lambda : h^*(x) \le h_I(x)\}$. Suppose to the contrary $h_I(x) < h^*(x)$ for some $x \in S$. Fix $M \prec \mathcal{M}$ such that $M \cap \lambda = x$ and $h_I(x) < \sup(M \cap \lambda^+) \le h^*(x)$. We know that $\langle E_\xi : \xi < \lambda\rangle, \langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle \in M$. Fix $\alpha \in M \cap \lambda^+$ with $h_I(x) < \alpha$. For each $\beta < \alpha$, let $a_\beta = \{\xi < \lambda : E_\xi \cap \beta$ is stationary in $\beta\}$. Note that, since $|a_\beta| \le \mathrm{cf}(\beta) < \lambda$, there exists $i < \mathrm{cf}(\lambda)$ such that $\sup(a_\beta \cap \lambda_j) < \lambda_j$ for all $j > i$. Let $\xi_\beta < \lambda^+$ be the minimal $\xi < \lambda^+$ such that $\chi_{a_\beta} <^* f_\xi$. Since the sequence $\langle \xi_\beta : \beta < \alpha\rangle$ is definable in M, we have $\langle \xi_\beta : \beta < \alpha\rangle \in M$. Let $\xi^* = \sup\{\xi_\beta : \beta < \alpha\} \in M$. Since $\alpha < \lambda^+$, we know $\xi^* < \lambda^+$. Then, because $h_I(x) < \alpha$, we have $\chi_{a_{h_I(x)}} <^* f_{\xi_{h_I(x)}} \le^* f_{\xi^*}$. Thus there exists $i < \mathrm{cf}(\lambda)$ such that $\sup(a_{h_I(x)} \cap \lambda_i) < f_{\xi^*}(i)$. Since $f_{\xi^*} \in M$, we have $f_{\xi^*}(i) \in M$ and $f_{\xi^*}(i) < \sup(M \cap \lambda_i) = \sup(x \cap \lambda_i)$. Hence $\sup(a_{h_I(x)} \cap \lambda_i) < \sup(x \cap \lambda_i)$. On the other hand, $x \subseteq a_{h_I(x)}$ by the choice of x. In particular, $\sup(x \cap \lambda_i) \le \sup(a_{h_I(x)} \cap \lambda_i)$. This is a contradiction. ∎

We can show the following, which obviously implies Theorem 1.5.

PROPOSITION 6.5. *Let* $S = \{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(h^*(x)) < \mathrm{ot}(x)\}$.

(1) S *is stationary in* $\mathcal{P}_\kappa\lambda$.
(2) *There is no weakly* λ^+*-saturated normal ideal* I *with* $S \in I^*$.
(3) $\mathrm{Sp}(T, \lambda^+)$ *holds for any stationary subset* T *of* S.

(3) follows from (2), which in turn follows from Propositions 4.4, 6.4, and the proof of Proposition 5.1. (1) follows from Fact 2.7 and the propositions and fact below.

PROPOSITION 6.6. *Let* $M \prec \mathcal{M}$ *be such that* $|M| < \kappa$ *and* $M \cap \kappa \in \kappa$. *Let* $\langle \vec{\lambda}, \vec{f}\rangle \in M$ *be the* Δ*-least scale for* λ. *If* $\chi_{M\cap\lambda} \le^* f_{\sup(M\cap\lambda^+)}$, *then* $h^*(M \cap \lambda) = \sup(M \cap \lambda^+)$.

Proof. It follows from the definition of h^* that $\sup(M\cap\lambda^+) \le h^*(M\cap\lambda)$, and $\sup(M \cap \lambda^+) \ge h^*(M \cap \lambda)$ follows from Lemma 6.3. ∎

The following fact is just Lemma 12 in Foreman–Magidor [9].

FACT 6.7. *Let* $\langle \vec{\lambda}, \vec{f}\rangle$ *be a scale for* λ. *Suppose* $\kappa > \omega_1$. *Let* $M \prec \mathcal{M}$ *be such that* $\langle \vec{\lambda}, \vec{f}\rangle \in M$, $|M| < \kappa$, $M \cap \kappa \in \kappa$ *and* M *is internally approachable of length* α *with* $\mathrm{cf}(\alpha) \ne \mathrm{cf}(\lambda)$. *Then* $\chi_{M\cap\lambda} \le^* f_{\sup(M\cap\lambda^+)}$.

PROPOSITION 6.8. *Suppose* $\mathrm{cf}(\lambda) = \omega$. *Let* E *be a stationary subset of* $E^{\lambda^+}_\omega$. *Let* $\langle \vec{\lambda}, \vec{f}\rangle$ *be a scale for* λ. *Then* $\{x \in \mathcal{P}_{\omega_1}\lambda^+ : \sup(x) \in E, \chi_x \le^* f_{\sup(x)}\}$ *is stationary in* $\mathcal{P}_{\omega_1}\lambda^+$.

Note that Shelah [15] proved a strong version of the above proposition under additional assumptions. Now we will give a proof of Proposition 6.8, which is based on Shelah's argument in [15].

To show Proposition 6.8, it is enough to show that for every $f : [\lambda^+]^{<\omega} \to \lambda^+$ there exists $x \in \mathcal{P}_{\omega_1}\lambda^+$ such that x is closed under f, $\sup(x) \in E$, and $\chi_x \leq^* f_{\sup(x)}$. Now fix a function $f : [\lambda^+]^{<\omega} \to \lambda^+$.

We recall some well-known notions. A *tree* is a poset $\langle T, \subseteq\rangle$ such that $T \subseteq {}^{<\omega}\lambda^+$ is closed under initial segments. For a tree T and $s \in T$, let $\mathrm{Suc}_T(s)$ be the set of immediate successors of s in T. An element $s \in T$ is called the *stem* of T, denoted by $\mathrm{Stm}(T)$, if $\forall s' \in T\ (s \subseteq s' \vee s' \subseteq s)$ and $|\mathrm{Suc}_T(s)| \neq 1$. A tree T is *perfect* if $|\mathrm{Suc}_T(s)| = \lambda^+$ for all $s \in T$ with $\mathrm{Stm}(T) \subseteq s$.

Definition 6.9. For a tree T and a regular uncountable cardinal $\mu < \lambda$, we say that T is *bounded in* μ if there exists $\gamma < \mu$ such that $\sup(\mathrm{Cl}_f(s) \cap \mu) < \gamma$ for every $s \in T$. Here $\mathrm{Cl}_f(s)$ is the closure of range(s) under f.

First we prove the following.

Lemma 6.10. *For every perfect tree T and every regular uncountable cardinal $\mu < \lambda$, there exists a perfect subtree $T' \subseteq T$ such that T' is bounded in μ and* $\mathrm{Stm}(T) = \mathrm{Stm}(T')$.

Proof. Let T and μ be as above. First we define a two-player game $G(\gamma)$ for $\gamma < \mu$.

Player I :	α_0		α_1		$\cdots$	α_i		$\cdots$
Player II :		β_0		β_1	$\cdots$		β_i	$\cdots$

Players choose ordinals less than λ^+ alternately with $\alpha_i \leq \beta_i$ for all $i < \omega$. Player II wins if $\mathrm{Stm}(T)^\frown\langle\beta_i : i < n\rangle \in T$ and $\sup(\mathrm{Cl}_f(\mathrm{Stm}(T)^\frown\langle\beta_i : i < n\rangle) \cap \mu) \leq \gamma$ for all $n < \omega$. Otherwise I wins.

Clearly the game $G(\gamma)$ is open, hence Player I has a winning strategy, or else Player II has.

Claim 6.11. *There exists $\gamma < \mu$ such that Player II has a winning strategy in $G(\gamma)$.*

Proof of Claim. Fix a large regular cardinal θ. Choose $N \prec \langle H_\theta, \in, T, \lambda^+, \mu, f\rangle$ such that $N \cap \lambda^+ \in E^{\lambda^+}_\omega$. Fix $a \in [N \cap \lambda^+]^\omega$ such that $\sup(a) = \sup(N \cap \lambda^+)$. Let M_0 be the Skolem hull of a under $\langle H_\theta, \in, T, \lambda^+, \mu, f\rangle$, and M_1 be that of $a \cup \{\sup(M_0 \cap \mu)\}$. Then M_0 and M_1 are countable elementary submodels of $\langle H_\theta, \in, T, \lambda^+, \mu, f\rangle$ such that $M_0 \subseteq M_1$, $\sup(M_0 \cap \mu) \in M_1$, and $\sup(M_0 \cap \lambda^+) = \sup(M_1 \cap \lambda^+)$.

Let $\gamma = \sup(M_0 \cap \mu) \in M_1$. We show that Player II in $G(\gamma)$ has a winning strategy. Suppose not. Then Player I has a winning strategy $\sigma : {}^{<\omega}\lambda^+ \to \lambda^+$ in the game $G(\gamma)$ with $\sigma \in M_1$. Define $\langle\alpha_i, \beta_i : i < \omega\rangle$ by induction on $i < \omega$ such that:

(1) $\alpha_i \in M_1 \cap \lambda^+$ and $\beta_i \in M_0 \cap \lambda^+$.
(2) $\alpha_i \leq \beta_i$ and $\alpha_i = \sigma(\langle \beta_j : j < i \rangle)$.
(3) $\mathrm{Stm}(T)^\frown \langle \beta_j : j < i \rangle \in T$.

Suppose α_j and β_j are defined for all $j < i$. Since $\beta_0, \ldots, \beta_{i-1} \in M_0 \subseteq M_1$, we know $\alpha_i = \sigma(\langle \beta_0, \ldots, \beta_{i-1} \rangle) \in M_1$. Now choose $\beta_i \in M_0 \cap \lambda^+$ such that $\beta_i \geq \alpha_i$ and $\beta_i \in \mathrm{Suc}_T(\mathrm{Stm}(T)^\frown \langle \beta_0 \ldots, \beta_{i-1} \rangle)$. This is possible because $T, \langle \beta_0 \ldots, \beta_{i-1} \rangle \in M_0$, $\mathrm{Suc}_T(\mathrm{Stm}(T)^\frown \langle \beta_0 \ldots, \beta_{i-1} \rangle)$ is unbounded in λ^+, and $\sup(M_0 \cap \lambda^+) = \sup(M_1 \cap \lambda^+)$.

Then $\mathrm{Stm}(T)^\frown \langle \beta_i : i < n \rangle \in T$ for all $n \in \omega$, but since the β_i's are in M_0 and $M_0 \cap \lambda^+$ is closed under f, we have $\mathrm{Cl}_f(\mathrm{Stm}(T)^\frown \langle \beta_i : i < n \rangle) \cap \mu \subseteq M_0 \cap \mu$ and $\sup(\mathrm{Cl}_f(\mathrm{Stm}(T)^\frown \langle \beta_i : i < n \rangle) \cap \mu) < \sup(M_0 \cap \mu) = \gamma$ for all $n < \omega$. Thus Player I has followed the strategy σ but loses in this game. This is a contradiction. $\blacksquare_{\text{Claim}}$

Fix $\gamma < \mu$ such that Player II has a winning strategy in $G(\gamma)$. Let $\sigma : {}^{<\omega}\lambda^+ \to \lambda^+$ be such a strategy. Let T' be the set of all $s \in T$ such that $s \subseteq \mathrm{Stm}(T)$ or $s = \mathrm{Stm}(T)^\frown \langle \sigma(s'|i) : i < \mathrm{length}(s') \rangle$ for some $s' \in {}^{<\omega}\lambda^+$. It is easy to check that T' is a perfect subtree of T with $\mathrm{Stm}(T) = \mathrm{Stm}(T')$ and $\sup(\mathrm{Cl}_f(s) \cap \mu) \leq \gamma$ for all $s \in T'$. Thus T' is bounded in μ. $\blacksquare$

Now we start the proof of Proposition 6.8.

Proof of Proposition 6.8. Take a large regular cardinal θ. Since E is a stationary subset of $E^{\lambda^+}_\omega$, we can find a sequence of countable models $\langle M_i : i < \omega \rangle$ such that $M_i \prec \langle H_\theta, \in, \lambda^+, f, \langle \vec{\lambda}, \vec{f} \rangle \ldots \rangle$, $M_i \in M_{i+1}$ and $\sup\{\sup(M_i \cap \lambda^+) : i < \omega\} \in E$. Because each M_i is countable and $\mathrm{cf}(\lambda) = \omega$, we know that $\vec{\lambda} \subseteq M_i$ and $\sup(M_i \cap \lambda_n) < \lambda_n$ for all $n < \omega$. Let $M = \bigcup_{i<\omega} M_i$ and $\xi = \sup(M \cap \lambda^+) \in E$. We will find $x \in \mathcal{P}_{\omega_1}\lambda^+$ such that $x \subseteq M \cap \lambda^+$, x is closed under f, $\sup(x) = \xi$, and $\chi_x \leq^* f_\xi$. For each $i < \omega$, because $M_i \in M_{i+1}$, there exists $\eta \in M_{i+1} \cap \lambda^+$ such that $\chi_{M_i} \leq^* f_\eta$. Since $\sup(M_{i+1} \cap \lambda^+) < \xi$, we know that $\chi_{M_i} <^* f_\xi$. Let $n_i < \omega$ be the minimal with $\chi_{M_i}(k) < f_\xi(k)$ for all $k \geq n_i$. Since $M_0 \subseteq M_1 \subseteq \cdots$, the sequence $\langle n_i : i < \omega \rangle$ is increasing. If $\{n_i : i < \omega\}$ is finite, then $\chi_M \leq^* f_\xi$. Hence $M \cap \lambda^+$ is the required set. So we may assume that $\langle n_i : i < \omega \rangle$ is a strictly increasing sequence.

Lemma 6.10 allows us to define an ordinal α_i and a perfect tree $T_i \in M_i$ by induction on $i < \omega$ so that:

(1) $\sup(M_i \cap \lambda^+) < \alpha_i \in M_{i+1} \cap \lambda^+$.
(2) $T_{i+1} \subseteq T_i$.
(3) T_i is bounded in λ_k for all $k < n_{i+1}$.
(4) $\langle \alpha_j : j < i \rangle = \mathrm{Stm}(T_i)$.

Let x be the closure of $\{\alpha_i : i < \omega\}$ under f. Since $\{\alpha_i : i < \omega\} \subseteq M$ and $M \cap \lambda^+$ is closed under f, we have $x \subseteq M \cap \lambda^+$. This implies that $\sup(x) = \xi$ because $\sup_{i<\omega} \alpha_i = \sup_{i<\omega} \sup(M_i \cap \lambda^+) = \xi$ by (1). It remains to prove $\chi_x \leq^* f_\xi$. Fix $k > n_0$. It is enough to show that $\chi_x(k) \leq f_\xi(k)$. Take $i < \omega$ with $n_i \leq k < n_{i+1}$. For $l < \omega$, let $x_l = \mathrm{Cl}_f(\langle \alpha_j : j < l \rangle)$. It is easy to see that $x = \bigcup_{l<\omega} x_l$. Note that $\chi_{x_l}(k) < \chi_{M_i}(k)$ for all $l < \omega$ because (3) holds in M_i and $\langle \alpha_j : j < l \rangle \in T_i$ (even though $\langle \alpha_j : j < l \rangle \notin M_i$ for all $i < l$). Therefore $\chi_x(k) = \sup_{l<\omega} \chi_{x_l}(k) < \chi_{M_i}(k) \leq f_\xi(k)$. ∎

To conclude this section, using the function h^* and Proposition 6.8, we show that $\mathrm{NS}_{\omega_1\lambda}$ is not λ^{++}-saturated if $\mathrm{cf}(\lambda) = \omega$. This supplements Foreman–Magidor's theorem [9] that $\mathrm{NS}_{\kappa\lambda}$ is not λ^{++}-saturated if $\omega_2 \leq \kappa$ and $\mathrm{cf}(\lambda) < \kappa$.

PROPOSITION 6.12. *Suppose* $\mathrm{cf}(\lambda) = \omega$. *Then* $\mathrm{NS}_{\omega_1\lambda}$ *is not* λ^{++}-*saturated, in fact we can find a family of stationary sets* $\langle X_\xi : \xi < \lambda^{++} \rangle$ *in* $\mathrm{NS}_{\omega_1\lambda}$ *such that* $X_\xi \cap X_\eta$ *is not unbounded for* $\xi \neq \eta$.

Proof. We use the argument in the proof of Theorem 13 in [9] with Proposition 6.8.

By a theorem of Gitik and Shelah [10], $\mathrm{NS}_{\lambda^+} | E^{\lambda^+}_\omega$ is not λ^{++}-saturated, where NS_{λ^+} is the non-stationary ideal over λ^+. So there exists a family of stationary subsets of $E^{\lambda^+}_\omega$, $\langle E_\alpha : \alpha < \lambda^{++} \rangle$, such that $E_\alpha \cap E_\beta$ is non-stationary for $\alpha \neq \beta$. By using a diagonal union, we may assume that $E_\alpha \cap E_\beta$ is bounded in λ^+ for each $\alpha \neq \beta$. Let $\langle \vec{\lambda}, \vec{f} \rangle$ be a scale for λ. Let $X_\alpha = \{M \cap \lambda : M \prec \mathcal{M}, |M| = \omega, \chi_M \leq^* f_{\sup(M\cap\lambda^+)}, \sup(M \cap \lambda^+) \in E_\alpha\}$. By using Proposition 6.8, we know that X_α is stationary in $\mathcal{P}_{\omega_1}\lambda^+$. We claim that $\langle X_\alpha : \alpha < \lambda^{++} \rangle$ is the required family. Take $\alpha < \beta < \lambda^{++}$. Then there exists $\gamma < \lambda^+$ such that $E_\alpha \cap E_\beta \subseteq \gamma$. Fix $x \in X_\alpha \cap X_\beta$. It is enough to show that $\chi_x <^* f_\gamma$. Take M and N that witness $x \in X_\alpha$ and $x \in X_\beta$ respectively. Then $\sup(M \cap \lambda^+) = h^*(x) = \sup(N \cap \lambda^+) \in E_\alpha \cap E_\beta \subseteq \gamma$ by Proposition 6.6. Therefore $\chi_x = \chi_M \leq f_{\sup(M\cap\lambda^+)} <^* f_\gamma$. ∎

7. Some related results. In this section, we will prove some results which are related to semi-weak normality and weak normality.

The α-semi-weak normality of ideals can be characterized in the context of *generic ultrapowers*. For an ideal I over A, let $\mathbb{P}_I$ be the generic ultrapower poset $\langle I^+, \subseteq_I \rangle$ associated to I, where $X \subseteq_I Y$ if $X \setminus Y \in I$. For a $(V, \mathbb{P}_I)$-generic G, let $\mathrm{Ult}(V, G) = \langle \overline{V}, \in^* \rangle$ be the generic ultrapower of V by G, and $j_G : V \to \overline{V}$ be the generic elementary embedding induced by G. Recall that I is *precipitous* if $\mathrm{Ult}(V, G)$ is well-founded for all $(V, \mathbb{P}_I)$-generic G, and I is *nowhere precipitous* if $I|X$ is precipitous for no $X \in I^+$.

Lemma 7.1. *For an ideal I over A and an ordinal α, the following are equivalent*:

(1) *I is α-s.w.n.*
(2) *There exists a function $f \in V$ on A such that for any $(V, \mathbb{P}_I)$-generic G, $[f]_G$ is the supremum of $\{j_G(\beta) : \beta < \alpha\}$ in the order $\in^*$, where $[f]_G$ is the equivalence class of f modulo G.*

The proof is straightforward. Note that we do not require that $\overline{V}$ is well-founded, and that if I is maximal, then the $(V, \mathbb{P})$-generic filter is just the dual filter of I.

In the above lemma, if f is the function that witnesses (2), then f is an α-least function for I. The lemma below immediately follows from Lemma 7.1.

Lemma 7.2. *If I is a precipitous ideal then for every $X \in I^+$ and every ordinal α there exists $Y \in (I|X)^+$ such that $I|Y$ is α-s.w.n.*

The next proposition implies that if $\lambda^{<\kappa} = \lambda$, then every normal ideal over $\mathcal{P}_\kappa\lambda$ is α-s.w.n. for any ordinal α.

Proposition 7.3. *If α is an ordinal with* $\mathrm{cf}(\alpha) < \lambda^+$ *or* $\mathrm{cf}(\alpha) > \lambda^{<\kappa}$, *then every normal ideal over $\mathcal{P}_\kappa\lambda$ is α-s.w.n.*

Proof. If $\mathrm{cf}(\alpha) < \kappa$ or $\mathrm{cf}(\alpha) > \lambda^{<\kappa}$, then it is easy to see that the constant function with value α is an α-least function for any normal ideal over $\mathcal{P}_\kappa\lambda$. Therefore we may assume that $\kappa \le \mathrm{cf}(\alpha) < \lambda^+$. Fix a function $g : \lambda \to \alpha$ such that $g``\mathrm{cf}(\alpha)$ is cofinal in $\alpha$. Define $f : \mathcal{P}_\kappa\lambda \to \alpha$ by $f(x) = \sup(g``x)$. It is easy to check that f is an α-least function for any normal ideal over $\mathcal{P}_\kappa\lambda$. ■

From the last proposition, we know that $\mathrm{NS}_{\kappa\lambda}$ is α-s.w.n. for every ordinal α if $\lambda^{<\kappa} = \lambda$. On the other hand, if $\lambda^{<\kappa} > \lambda$ then $\mathrm{NS}_{\kappa\lambda}$ is not $\lambda^{<\kappa}$-s.w.n. in general.

For an ideal I over A, the cardinals $\mathrm{cof}(I)$ and $\mathrm{non}(I)$ are defined by

$$\mathrm{cof}(I) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq I,\ \forall X \in I\ \exists Y \in \mathcal{F}\ (X \subseteq Y)\}, \tag{1}$$

$$\mathrm{non}(I) = \min\{|X| : X \in I^+\}. \tag{2}$$

In Matsubara–Shioya [13] it was proved that an ideal I is nowhere precipitous if $\mathrm{non}(I) = \mathrm{cof}(I)$. We improve their result in terms of semi-weak normality.

Proposition 7.4. *Let I be an ideal over A. Suppose* $\mathrm{cof}(I) = \mathrm{non}(I)$. *Then for every $X \in I^+$, $I|X$ is not* $\mathrm{cof}(I)$*-s.w.n. In particular, I is nowhere precipitous.*

Proof. Let $\mu = \mathrm{cof}(I) = \mathrm{non}(I)$. Note that if $Y \subseteq A$ and $|Y| < \mu$ then $Y \in I$.

Let $X \in I^+$ and let $f : A \to \mu + 1$ be such that $\{x \in A : \alpha < f(x)\} \in (I|X)^*$ for all $\alpha < \mu$. We will find $Y \in (I|X)^+$ and a function g on Y such that $\forall x \in Y\ (g(x) < f(x))$ but $\{x \in Y : g(x) \le \alpha\} \in I$ for all $\alpha < \mu$, hence f is not a μ-least function.

Fix $\mathcal{F} \subseteq I$ such that $|\mathcal{F}| = \mu = \mathrm{cof}(I)$ and $\forall Y \in I\ \exists Z \in \mathcal{F}\ (Y \subseteq Z)$. Let $\langle X_\xi : \xi < \mu\rangle$ be an enumeration of $\mathcal{F}$. By induction on $\xi < \mu$ we define a sequence $\langle x_\xi : \xi < \mu\rangle$ satisfying the following, for all $\xi < \mu$:

(1) $x_\xi \in X \setminus X_\xi$.
(2) $f(x_\xi) > \xi$.
(3) $\forall \eta < \xi\ (x_\eta \ne x_\xi)$.

Suppose $\xi < \mu$ and $\langle x_\eta : \eta < \xi\rangle$ are defined. Since $\{x \in X : f(x) > \xi\} \in (I|X)^*$ and $X_\xi \cup \{x_\eta : \eta < \xi\} \in I$, there exists $x_\xi \in X \setminus X_\xi$ such that $f(x_\xi) > \xi$ and $\forall \eta < \xi\ (x_\eta \ne x_\xi)$.

Let $Y = \{x_\xi : \xi < \mu\}$. By the induction hypothesis (1), Y is an I-positive subset of X. Define the function g on Y by $g(x_\xi) = \xi$ for all $\xi < \mu$. Then $g(x_\xi) = \xi < f(x_\xi)$ for all $\xi < \mu$. Let $\alpha < \mu$. Then $\{x_\xi : g(x_\xi) \le \alpha\} = \{x_\xi : \xi \le \alpha\}$, hence $|\{x_\xi : g(x_\xi) \le \alpha\}| = |\alpha| < \mu$. Therefore $\{x_\xi : g(x_\xi) \le \alpha\} \in I$. ■

For an ordinal $\gamma \ge \kappa$, let $\mathrm{cf}(\kappa, \gamma)$ denote the minimal size of unbounded sets in $\mathcal{P}_\kappa\gamma$. Notice that $|\mathcal{P}_\kappa\lambda| = \lambda^{<\kappa} = \mathrm{cf}(\kappa,\lambda) + 2^{<\kappa}$. Solovay [17] showed that $\mathrm{cf}(\kappa,\lambda) = \lambda$ if $\mathrm{cf}(\lambda) \ge \kappa$ and $\mathcal{P}_\kappa\lambda$ carries a maximal ideal (that is, κ is λ-compact). Abe [1] obtained the same result assuming that $\mathcal{P}_\kappa\lambda$ carries a weakly normal ideal and λ is regular. Here an ideal I over $\mathcal{P}_\kappa\lambda$ is called *weakly normal* if for every function $f : \mathcal{P}_\kappa\lambda \to \lambda$ such that $f(x) < \sup(x)$ for all $x \in \mathcal{P}_\kappa\lambda$, there exists $\alpha < \lambda$ such that $\{x \in \mathcal{P}_\kappa\lambda : f(x) \le \alpha\} \in I^*$. Every weakly normal ideal over $\mathcal{P}_\kappa\lambda$ is weakly $\mathrm{cf}(\lambda)$-saturated, and he also showed that if I is λ-saturated normal ideal over $\mathcal{P}_\kappa\lambda$ and $\mathrm{cf}(\lambda) \ge \kappa$, then I is weakly normal. Relevantly, Burke [4] showed that if $\mathrm{cf}(\lambda) \ge \kappa$, $\mathcal{P}_\kappa\lambda$ carries a weakly λ-saturated ideal, and there exists a large cardinal greater than λ then $\mathrm{cf}(\kappa,\lambda) = \lambda$. We prove that $\mathrm{cf}(\kappa,\lambda) = \lambda$ is implied by the existence of a weakly λ-saturated ideal.

PROPOSITION 7.5. *Suppose that* $\mathrm{cf}(\lambda) \ge \kappa$. *If* $\mathcal{P}_\kappa\lambda$ *carries a weakly* λ-*saturated ideal*, *then* $\mathrm{cf}(\kappa,\lambda) = \lambda$. *In particular*, $\lambda^{<\kappa} = \lambda + 2^{<\kappa}$.

Proof. Case 1: $\mathrm{cf}(\lambda) = \lambda$. By [1], it is enough to show the existence of a weakly normal ideal over $\mathcal{P}_\kappa\lambda$.

Since we are assuming the existence of a weakly λ-saturated ideal, $\mathcal{P}_\kappa\lambda$ carries a λ-weakly normal ideal I by Corollary 3.8 and Proposition 3.10. Let h_I be a λ-least function for I. Then $\{x \in \mathcal{P}_\kappa\lambda : h_I(x) \le \sup(x)\} \in I^*$, since if not, then there exists $\alpha < \lambda$ such that $\{x \in \mathcal{P}_\kappa\lambda : \sup(x) \le \alpha\} \in I^+$, which is impossible. Now define an ideal J over $\mathcal{P}_\kappa\lambda$ by $X \in J$ if and only if $\{x \in \mathcal{P}_\kappa\lambda : x \cap h_I(x) \in X\} \in I$. It is easy to check that J is a weakly

λ-saturated ideal, and the function $x \mapsto \sup(x)$ is a λ-least function of J. Hence J is weakly normal.

Case 2: $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. Let I be a weakly λ-saturated ideal over $\mathcal{P}_\kappa\lambda$. Since λ is singular, by Note 2.2(3) there exist $X \in I^+$ and $\mu < \lambda$ such that $I|X$ is weakly μ-saturated. Then for each regular γ with $\mu < \gamma < \lambda$, the ideal I_γ over $\mathcal{P}_\kappa\gamma$ defined by $Y \in I_\gamma \Leftrightarrow \{x \in X : x \cap \gamma \in Y\} \in (I|X)$ is weakly γ-saturated. Hence $\mathrm{cf}(\kappa, \gamma) = \gamma$ by Case 1. Then, because $\mathcal{P}_\kappa\lambda = \bigcup_{\delta<\lambda} \mathcal{P}_\kappa\delta$, we have $\mathrm{cf}(\kappa, \lambda) = \lambda$. ∎

A similar result holds when $\mathrm{cf}(\lambda) < \kappa$.

Proposition 7.6. *Suppose that* $\mathrm{cf}(\lambda) < \kappa$ *and there exists a weakly* λ^+*-saturated ideal over* $\mathcal{P}_\kappa\lambda$. *Then there exists* $\delta < \lambda$ *such that* $\delta \leq \kappa^{\mathrm{cf}(\lambda)}$ *and* $\mathrm{cf}(\kappa, \lambda) = \lambda^+ + \mathrm{cf}(\kappa, \delta)$. *In particular*, $\lambda^{<\kappa} = \lambda^+ + 2^{<\kappa}$.

Proof. By Corollary 3.8 and Proposition 3.10, there is a λ^+-weakly normal ideal I over $\mathcal{P}_\kappa\lambda$. First we claim that $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(h_I(x)) \leq \kappa^{\mathrm{cf}(\lambda)}\} \in I^*$. Take $F \subseteq [\lambda]^{\mathrm{cf}(\lambda)}$ such that $|F| = \lambda^+$. Take a 1-1 enumeration $\langle d_\xi : \xi < \lambda^+\rangle$ of F. For each $x \in \mathcal{P}_\kappa\lambda$, let $e_x = \{\xi < \lambda^+ : d_\xi \subsetneq x\}$. Then clearly $|e_x| \leq \kappa^{\mathrm{cf}(\lambda)}$. Thus it is enough to show that $\{x \in \mathcal{P}_\kappa\lambda : e_x \cap h_I(x)$ is unbounded in $h_I(x)\} \in I^*$. If not, then by the λ^+-weak normality of I, there exists $\xi^* < \lambda^+$ such that $\{x \in \mathcal{P}_\kappa\lambda : e_x \cap h_I(x) \subseteq \xi^*\} \in I^+$. But then we can pick $x \in \mathcal{P}_\kappa\lambda$ such that $e_x \cap h_I(x) \subseteq \xi^*$ and $d_{\xi^*} \subsetneq x$, which is impossible.

Since $\{x \in \mathcal{P}_\kappa\lambda : h_I(x) < \lambda^+\} \in I^*$ and $\mathrm{cf}(\lambda) < \kappa$, there exists a regular $\delta < \lambda$ such that $X = \{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(h_I(x)) \leq \delta\} \in I^+$. By the previous claim, we may assume that $\delta \leq \kappa^{\mathrm{cf}(\lambda)}$.

We will construct a sequence $\langle c_\xi : \xi \in E^{\lambda^+}_{\leq\delta}\rangle$ such that $c_\xi \subseteq \lambda$, $\mathrm{ot}(c_\xi) \leq \mathrm{cf}(\xi)$, and $\{x \in X : \eta \in c_{h_I(x)}\} \in (I|X)^*$ for all $\eta < \lambda$. When such a sequence is constructed, then for each $\xi \in E^{\lambda^+}_{\leq\delta}$ fix $Y_\xi \subseteq \mathcal{P}_\kappa c_\xi$ such that Y_ξ is unbounded in $\mathcal{P}_\kappa c_\xi$ and $|Y_\xi| \leq \mathrm{cf}(\kappa, \delta)$. Let $Y = \bigcup\{Y_\xi : \xi \in E^{\lambda^+}_{\leq\delta}\}$. Then Y is an unbounded set in $\mathcal{P}_\kappa\lambda$ and $|Y| = \lambda^+ + \mathrm{cf}(\kappa, \delta)$, which completes the proof.

Fix $\langle a_\xi : \xi < \lambda^+\rangle$ such that $a_\xi \subseteq \xi$ is an unbounded set in ξ with $\mathrm{ot}(a_\xi) = \mathrm{cf}(\xi)$. As in the proof of Proposition 4.3 we can construct a strictly increasing sequence $\langle \alpha_\eta : \eta < \lambda\rangle$ such that $\{x \in X : [\alpha_\eta, \alpha_{\eta+1}) \cap a_{h_I(x)} \neq \emptyset\} \in (I|X)^*$ for all $\eta < \lambda$. Now define $\langle c_\xi : \xi \in E^{\lambda^+}_{\leq\delta}\rangle$ by $c_\xi = \{\eta < \lambda : [\alpha_\eta, \alpha_{\eta+1}) \cap a_\xi \neq \emptyset\}$. Since $\mathrm{ot}(a_\xi) = \mathrm{cf}(\xi)$ and the α_η's are strictly increasing, we have $\mathrm{ot}(c_\xi) \leq \mathrm{cf}(\xi)$. Then clearly $\{x \in X : \eta \in c_{h_I(x)}\} \in (I|X)^*$ for all $\eta < \lambda$. ∎

We do not know whether $\mathrm{cf}(\kappa, \lambda)$ must be λ^+ even if $\mathcal{P}_\kappa\lambda$ carries a weakly λ^+-saturated ideal. On the other hand, $\mathrm{cf}(\kappa, \lambda) = \lambda^+$ must hold if $\mathcal{P}_\kappa\lambda$ carries a weakly λ-saturated ideal:

If $\mathcal{P}_\kappa\lambda$ carries a weakly λ-saturated ideal, then, since λ is singular, there exists $\delta < \lambda$ such that, for every regular μ with $\delta < \mu < \lambda$, $\mathcal{P}_\kappa\mu$ carries a weakly μ-saturated ideal. Then $\mathrm{cf}(\kappa,\mu)=\mu$ by Proposition 7.5. Hence $\mathrm{cf}(\kappa,\lambda)=\lambda^+$ in view of Proposition 7.6.

COROLLARY 7.7. *Suppose that $\mathcal{P}_\kappa\lambda$ carries a weakly λ-saturated ideal. Then*

$$\mathrm{cf}(\kappa,\lambda)=\begin{cases}\lambda & \text{if } \mathrm{cf}(\lambda)\geq\kappa,\\ \lambda^+ & \text{if } \mathrm{cf}(\lambda)<\kappa.\end{cases}$$

Finally, we give other characterizations of weak λ^+-saturation of ideals, which shows that λ^+-weak normality of a normal ideal follows from weak λ^+-saturation.

PROPOSITION 7.8. *Suppose* $\mathrm{cf}(\lambda)<\kappa$. *Let I be a normal ideal over $\mathcal{P}_\kappa\lambda$. Then the following are equivalent*:

(1) *I is weakly λ^+-saturated.*
(2) *The function h^* in Definition* 6.1 *is a λ^+-least function for any normal ideal over $\mathcal{P}_\kappa\lambda$ extending I.*
(3) *There exists a function $f:\mathcal{P}_\kappa\lambda\to\lambda^+$ such that f is a λ^+-least function for any normal ideal over $\mathcal{P}_\kappa\lambda$ extending I.*

In particular, every weakly λ^+-saturated normal ideal over $\mathcal{P}_\kappa\lambda$ is λ^+-weakly normal.

Proof. We prove (1)⇒(2)⇒(3)⇒(1).

(1)⇒(2). Let J be a normal ideal extending I. To show that h^* is a λ^+-least function for J, it is enough to show that for all $X\in J^+$ and all functions g on X, if $\forall x\in X\ (f(x)<h^*(x))$ then $\{x\in X: f(x)\leq\beta\}\in J^+$ for some $\beta<\lambda^+$. Since I is weakly λ^+-saturated and $I\subseteq J\subseteq J|X$, $J|X$ is a weakly λ^+-saturated normal ideal. Take a normal λ^+-weakly normal ideal $\overline{J}$ extending $J|X$. Then h^* is a λ^+-least function for $\overline{J}$. Since $X\in\overline{J}^*$, there exists $\beta<\lambda^+$ such that $\{x\in X: f(x)\leq\beta\}\in\overline{J}^+$. Then clearly $\{x\in X: f(x)\leq\beta\}\in J^+$.

(2)⇒(3) is trivial.

(3)⇒(1). Let f be a function witnessing (3). Then f is a λ^+-least function for I. To see that I is weakly λ^+-saturated, since I is λ^+-s.w.n., we prove that for all $X\in I^+$ and all functions g on X, if $\forall x\in X\ (g(x)<f(x))$ then there exists $\alpha<\lambda^+$ such that $\{x\in X: g(x)\leq\alpha\}\in(I|X)^*$. Suppose not. Then $X_\alpha=\{x\in X: g(x)>\alpha\}\in I^+$ for all $\alpha<\lambda^*$. Notice that $X_\alpha\supseteq X_\beta$ for all $\alpha<\beta<\lambda^+$. Now we consider the filter F over $\mathcal{P}_\kappa\lambda$ generated by $I^*\cup\{X_\alpha:\alpha<\lambda^+\}$, that is, for $X\subseteq\mathcal{P}_\kappa\lambda$, $X\in F$ if and only if $Y_0\cap\cdots\cap Y_n\cap X_{\alpha_0}\cap\cdots\cap X_{\alpha_m}\subseteq X$ for some $Y_0,\ldots,Y_n\in I^*$ and $\alpha_0,\ldots,\alpha_m<\lambda^+$. We claim that F is a normal filter. Notice that I^* is a filter and the X_α's are $\supseteq$-decreasing, so $X\in F$ if and only if $Y\cap X_\alpha\subseteq X$ for some $Y\in I^*$ and $\alpha<\lambda^+$. Hence it is easy to check that F is a proper filter. Because $I^*\subseteq F$, F is fine. To check that F is normal, take $Z_\xi\in F$

$(\xi < \lambda)$. Then for each $\xi < \lambda$, there exist $Y_\xi \in I^*$ and $\alpha_\xi < \lambda^+$ such that $Y_\xi \cap X_{\alpha_\xi} \subseteq Z_\xi$. Let $\alpha^* = \sup\{\alpha_\xi : \xi < \lambda\} < \lambda^+$. Then $Y_\xi \cap X_{\alpha^*} \subseteq Z_\xi$ for all $\xi < \lambda$. Since I^* is normal, we have $\triangle_{\xi<\lambda} Y_\xi \in I^*$. Because $X_{\alpha^*} \in F$, we have $(\triangle_{\xi<\lambda} Y_\xi) \cap X_{\alpha^*} \in F$ and $(\triangle_{\xi<\lambda} Y_\xi) \cap X_{\alpha^*} = \triangle_{\xi<\lambda}(Y_\xi \cap X_{\alpha^*}) \subseteq \triangle_{\xi<\lambda} Z_\xi \in F$. Therefore F is normal.

Let J be the dual ideal of F. Then J is normal, $X \in J^*$, and, by assumption (3), f is a λ^+-least function for J. However, since $X_\alpha = \{x \in X : g(x) > \alpha\} \in J^*$, there is no $\alpha < \lambda^+$ such that $\{x \in X : g(x) < \alpha\} \in J^+$. This is a contradiction. ■

8. Questions. There are some open questions. Let $\mathrm{cf}(\lambda) < \kappa$.

(1) Is it consistent that $\mathcal{P}_\kappa\lambda$ carries a normal ideal which is weakly λ^+-saturated but not λ^+-saturated? Or not precipitous?

(2) Is it consistent that κ is a successor cardinal and $\mathcal{P}_\kappa\lambda$ carries a weakly λ^+-saturated ideal?

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