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Khovanov–Rozansky homology for embedded graphs

by

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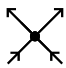
Abstract. For any positive integer n, Khovanov and Rozansky constructed a bigraded link homology from which you can recover the $\mathfrak{sl}_n$ link polynomial invariants. We generalize the Khovanov–Rozansky construction in the case of finite 4-valent graphs embedded in a ball $B^3 \subset \mathbb{R}^3$. More precisely, we prove that the homology associated to a diagram of a 4-valent graph embedded in $B^3 \subset \mathbb{R}^3$ is invariant under the graph moves introduced by Kauffman.

Introduction. We consider finite oriented 4-valent graphs embedded in a ball $B^3 \subset \mathbb{R}^3$. We fix a great circle on the boundary 2-sphere of B^3 and require that the boundary points of the embedded graph lie on this great circle and that the orientations around a vertex be as follows:



These graphs are called *open regular graphs*. A *diagram* Γ of an open regular graph is a generic projection of the graph onto the plane of the great circle. An isotopy of such a graph should not move its boundary points and should respect the cyclic order around the vertices. An embedded graph in B^3 without boundary points is called a *(closed) regular graph*. If an open regular graph can be embedded in the plane of the great circle, it is called *planar*, and we make no distinction between the graph and this generic projection (see Figure 1 for an example).

For any positive integer n, Khovanov and Rozansky categorified the $\mathfrak{sl}_n$ link polynomials [4] by associating to a link diagram a complex of matrix factorizations (see Section 1.1 for definition). The first step in their construction is to associate a matrix factorization to an open planar regular graph. It is then almost immediate to associate a complex $C_n(\Gamma)$ of matrix factorizations to a diagram Γ of an open regular graph. To prove that the homotopy

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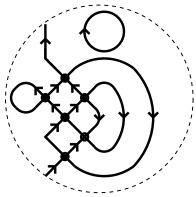


Fig. 1. Planar regular graph

type of this complex is an invariant of the open regular graph, we check that it is invariant under the graph moves, called RV-moves, introduced by Kauffman [2] (see Figure 4). We obtain the following theorem:

THEOREM 1. *Let $C_n(\Gamma_1)$ and $C_n(\Gamma_2)$ be complexes of matrix factorizations associated to diagrams Γ_1 and Γ_2 of open regular graphs in B^3. If there exists a sequence of RV-moves transforming Γ_1 into Γ_2 then $C_n(\Gamma_1)$ and $C_n(\Gamma_2)$ are homotopy equivalent.*

As pointed out by Kauffman and Vogel [3], link polynomial invariants give rise to graph invariants; the same is true for Khovanov–Rozansky link homology.

In Section 1, we recall the Khovanov–Rozansky construction and adapt it to the case of oriented 4-valent graphs embedded in $B^3 \subset \mathbb{R}^3$. In Section 2, we introduce Kauffman graph moves, and in Section 3, we prove the invariance up to homotopy of the complex of matrix factorizations under these moves.

1. Khovanov–Rozansky construction

1.1. Matrix factorizations. Let k be a positive integer, let $R = \mathbb{Q}[x_1, \ldots, x_k]$ be a commutative polynomial $\mathbb{Q}$-algebra, and let $w \in R$. An (R, w)-*matrix factorization* of the potential w over R consists of two free R-modules C^0, C^1 and two R-module maps

$$C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^0$$

such that $d^0 \circ d^1(m) = wm$ for all $m \in C^1$ and $d^1 \circ d^0(m) = wm$ for all $m \in C^0$.

A first example of matrix factorization is the following (R, ab)-matrix factorization:

$$R \xrightarrow{\times a} R \xrightarrow{\times b} R,$$

where $a, b \in R$. We denote this matrix factorization by $(a, b)_R$. We construct more matrix factorizations by tensoring over R such elementary factorizations (see [4] for the definition of tensor products). More precisely we denote by

$$\begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_k & b_k \end{pmatrix}_R$$

the tensor product over R of $(a_1, b_1)_R, \ldots, (a_k, b_k)_R$. It is a matrix factorization of the potential $w = a_1b_1 + \cdots + a_kb_k$. We consider the following $\mathbb{Z}$-grading on R: $\deg(x_i) = 2$ for $i = 1, \ldots, k$. A matrix factorization is *graded* if d^0 and d^1 are homogeneous and $\deg(d^0) = \deg(d^1)$. The grading on R induces a grading on C^0 and C^1: $C^0 = \bigoplus_{i\in\mathbb{Z}} C^{i,0}$, $C^1 = \bigoplus_{i\in\mathbb{Z}} C^{i,1}$, We denote by $\{\cdot\}$ the shift of the $\mathbb{Z}$-grading: for $i, k \in \mathbb{Z}$ and $j \in \mathbb{Z}/2\mathbb{Z}$, $C^{i,j}\{k\} = C^{i-k,j}$. For $k \in \mathbb{Z}$, let $\langle k\rangle$ be the shift of the $(\mathbb{Z}/2\mathbb{Z})$-grading by $k \pmod 2$. Given two graded (R, w)-matrix factorizations C and D, a *morphism* $f : C \to D$ is a pair of R-module homomorphisms $f^0 : C^0 \to D^0$ and $f^1 : C^1 \to D^1$ preserving the $\mathbb{Z}$-grading and such that the following diagram commutes:

$$\begin{array}{ccccc} C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & C^0 \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^0} \\ D^0 & \xrightarrow{\delta^0} & D^1 & \xrightarrow{\delta^1} & D^0 \end{array}$$

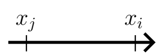
A *homotopy* h between morphisms $f, g : C \to D$ of matrix factorizations is a pair of morphisms $h^0 : C^0 \to D^1$ and $h^1 : C^1 \to D^0$ such that

$$f^0 - g^0 = h^1 \circ d^0 + \delta^1 \circ h^0 \quad \text{and} \quad f^1 - g^1 = h^0 \circ d^1 + \delta^0 \circ h^1.$$

Given $w \in R$, we denote by hmf^R_w the homotopy category of graded matrix factorizations of the potential w over R.

1.2. Planar regular graphs and matrix factorizations. Fix a positive integer n. We now recall the matrix factorizations that Khovanov and Rozansky [4] associate to an open planar regular graph Γ. Let k be a positive integer and $\{x_1, \ldots, x_k\}$ a set of marks on the edges of Γ such that every edge has at least one mark.

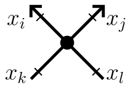
We consider the following piece of a planar regular graph:



The matrix factorization L^i_j associated to this piece is $(\pi_{ij}, x_i - x_j)_{\mathbb{Q}[x_i,x_j]}$:

$$\mathbb{Q}[x_i, x_j] \xrightarrow{\times\pi_{ij}} \mathbb{Q}[x_i, x_j]\{1 - n\} \xrightarrow{\times(x_i - x_j)} \mathbb{Q}[x_i, x_j],$$

where $\pi_{ij} = (x_i^{n+1} - x_j^{n+1})/(x_i - x_j)$. The grading shift makes L^i_j graded. More precisely, with this shift multiplications by π_{ij} and $x_i - x_j$ become homogeneous maps of degree $n+1$. We now consider the following piece Γ^1 of a planar regular graph:



Fix R to be the ring $\mathbb{Q}[x_i, x_j, x_k, x_l]$. We associate to such a piece a graded matrix factorization of the potential $w = x_i^{n+1} + x_j^{n+1} - x_k^{n+1} - x_l^{n+1}$ over R. We decompose w as

$$w = u_1(x_i, x_j, x_k, x_l)(x_i + x_j - x_k - x_l) + u_2(x_i, x_j, x_k, x_l)(x_i x_j - x_k x_l)$$

where

$$u_1 = \frac{g(x_i + x_j, x_i x_j) - g(x_k + x_l, x_i x_j)}{x_i + x_j - x_k - x_l},$$
$$u_2 = \frac{g(x_k + x_l, x_i x_j) - g(x_k + x_l, x_k x_l)}{x_i x_j - x_k x_l},$$

and g is the two-variable function satisfying $g(x+y, xy) = x^{n+1} + y^{n+1}$. We now define $C_n(\Gamma^1)$ to be the graded matrix factorization

$$\begin{pmatrix} u_1 & x_i + x_j - x_k - x_l \\ u_2 & x_i x_j - x_k x_l \end{pmatrix}_R \{-1\}.$$

In other words $C_n(\Gamma^1)$ is the tensor product over R of the graded matrix factorizations

$$R \xrightarrow{u_1} R\{1-n\} \xrightarrow{x_i + x_j - x_k - x_l} R$$

and

$$R \xrightarrow{u_2} R\{3-n\} \xrightarrow{x_i x_j - x_k x_l} R$$

with an additional shift by $\{-1\}$. As for L_j^i, the grading shift makes $C_n(\Gamma^1)$ graded.

We distinguish two kinds of tensor product of such elementary graded matrix factorizations: tensor product over $\mathbb{Q}$ corresponding topologically to a disjoint union of pieces and tensor product over some polynomial $\mathbb{Q}$-algebra corresponding to gluing of pieces along some endpoints (see [4] for a detailed treatment). The potential of graded matrix factorizations is additive with respect to both tensor products. We consider the two examples in Figure 2. The graded matrix factorization associated to the left diagram of Figure 2 is $L_k^i \otimes_{\mathbb{Q}} L_l^j$ and the graded matrix factorization associated to the right diagram in Figure 2 is $L_j^i \otimes_{\mathbb{Q}[x_j]} L_k^j$. In general, we can now associate to a planar regular graph Γ embedded in $\mathbb{R}^2$ the matrix factorization

$$C_n(\Gamma) = \Big(\bigotimes L_j^i\Big) \otimes \Big(\bigotimes C_n(\Gamma^1)\Big)$$

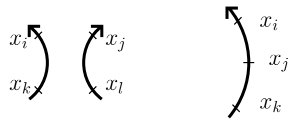
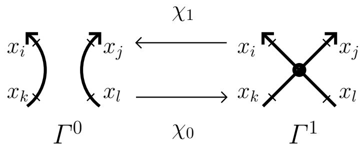


Fig. 2. Two examples

where the first tensor product runs through all the oriented arcs starting and ending at marks and with no interior mark, and where the second runs through all 4-valent vertices. The tensor products are over suitable polynomial $\mathbb{Q}$-algebras (see [4]). The homotopy type of this matrix factorization does not depend on the choice of marks [4].

1.3. Regular graph embedded in $\mathbb{R}^3$ and the complex of graded matrix factorizations. We define two morphisms χ_0 and χ_1 of graded matrix factorizations between elementary matrix factorizations as depicted in the following diagram:



The matrix factorization $C_n(\Gamma^0)$ is the tensor product over $\mathbb{Q}$ of L_k^i and L_l^j and is given by

$$\begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R\{1-n\} \\ R\{1-n\} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix}$$

where

$$P_0 = \begin{pmatrix} \pi_{ik} & x_j - x_l \\ \pi_{jl} & x_k - x_i \end{pmatrix}, \qquad P_1 = \begin{pmatrix} x_i - x_k & x_j - x_l \\ \pi_{jl} & -\pi_{ik} \end{pmatrix}.$$

The matrix factorization $C_n(\Gamma^1)$ is

$$\begin{pmatrix} R\{-1\} \\ R\{3-2n\} \end{pmatrix} \xrightarrow{Q_0} \begin{pmatrix} R\{-n\} \\ R\{2-n\} \end{pmatrix} \xrightarrow{Q_1} \begin{pmatrix} R\{-1\} \\ R\{3-2n\} \end{pmatrix}$$

where

$$Q_0 = \begin{pmatrix} u_1 & x_i x_j - x_k x_l \\ u_2 & -x_i - x - j + x_k + x_l \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} x_i + x_j - x_k - x_l & x_i x_j - x_k x_l \\ u_2 & -u_1 \end{pmatrix}.$$

We define $\chi_0 : C_n(\Gamma^0) \to C_n(\Gamma^1)$ by the pair of matrices

$$U_0 = \begin{pmatrix} x_k - x_j & 0 \\ a & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} x_k & -x_j \\ -1 & 1 \end{pmatrix}$$

acting on $C_n^0(\Gamma^0)$ and $C_n^1(\Gamma^0)$ respectively. Let a be equal to $-u_2 + (u_1 + x_i u_2 - \pi_{jl})/(x_i - x_k)$, and define the morphism $\chi_1 : C_n(\Gamma^1) \to C_n(\Gamma^0)$ by the pair of matrices

$$V_0 = \begin{pmatrix} 1 & 0 \\ -a & x_k - x_j \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 & x_j \\ 1 & x_k \end{pmatrix}$$

acting on $C_n^0(\Gamma^1)$ and $C_n^1(\Gamma^1)$ respectively. The maps χ_0 and χ_1 are morphisms of graded matrix factorizations and are of degree 1 (for the grading $\{\cdot\}$).

We now consider a regular graph embedded in $\mathbb{R}^3$. We denote by D a diagram for this graph. It has three different types of crossing: positive, negative and singular (see Figure 3).

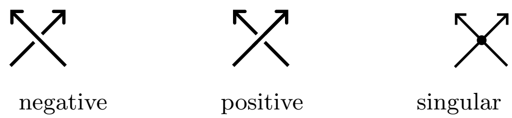


Fig. 3. Crossings

Let k, r, and s be positive integers. We put marks $\{x_1, \ldots, x_k\}$ on D such that every arc between two crossings has at least one mark. We also put marks $\{p_1, \ldots, p_r\}$ on every positive or negative crossing and $\{q_1, \ldots, q_s\}$ on every singular crossing. As for planar regular graphs, we want to associate to every elementary piece of a regular graph diagram an algebraic object, and in this case it is a complex of graded matrix factorizations.

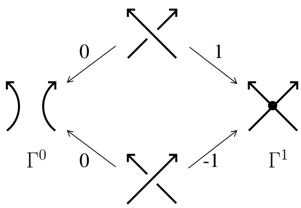
For an arc that contains no crossings and no other marks, define L_j^i as above and consider it as the chain complex

$$0 \to L_j^i \to 0,$$

where L_j^i is in cohomological degree 0. For a singular crossing q, define $C_n(q)$ as $C_n(\Gamma^1)$ and consider it as the chain complex

$$0 \to C_n(\Gamma^1) \to 0,$$

where $C_n(\Gamma^1)$ is in cohomological degree 0. We now consider positive and negative crossings. For every positive or negative crossing p, there are two different resolutions, either Γ^0 or Γ^1.



If p^- is a negative crossing we define $C_n(p^-)$ to be the chain complex

$$0 \to C_n(\Gamma^0)\{1-n\} \xrightarrow{\chi_0} C_n(\Gamma^1)\{-n\} \to 0,$$

and if p^+ is a positive crossing we define $C_n(p^+)$ to be the chain complex

$$0 \to C_n(\Gamma^1)\{n\} \xrightarrow{\chi_1} C_n(\Gamma^0)\{n-1\} \to 0,$$

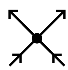
where $C_n(\Gamma^0)$ is always in cohomological degree 0. Now to a regular graph diagram D associate the complex of graded matrix factorizations

$$C_n(D) = \Big(\bigotimes_{L_j^i} L_j^i\Big) \otimes \Big(\bigotimes_p C_n(p)\Big) \otimes \Big(\bigotimes_q C_n(q)\Big)$$

where the first tensor product runs through all arcs in D that start and end at marks and that contain no crossings and no other marks, p runs through all the positive and negative crossings of D, and q runs through all singular crossings. The tensor products are over suitable polynomial $\mathbb{Q}$-algebras.

2. Reidemeister moves for graphs. We consider open regular graphs embedded in $B^3 \subset \mathbb{R}^3$ as graphs with rigid vertices. As explained in [2], a 4-valent graph with rigid vertices can be regarded as an embedding of a graph whose vertices have been replaced by rigid disks. Each disk has four strands attached to it, and the cyclic order of these strands is determined via the rigidity of the disk. An *RV-isotopy* or *rigid vertex isotopy* of the embedding of such a regular graph Γ in $\mathbb{R}^3$ consists in affine motions of the disks, coupled with topological ambient isotopies of the strands (corresponding to the edge of Γ). The notion of RV-isotopy is a mixture of mechanical (Euclidian) and topological concepts. It arises naturally in the building of models for graph embeddings, and it also arises naturally with regard to creating invariants of graph embeddings.

In [2], Kauffman derived a collection of moves, analogous to Reidemeister moves, that generates RV-isotopy for diagrams of 4-valent graph embeddings. As we are only interrested in 4-valent oriented graph embeddings whose oriented rigid vertex take the basic form



we will present the RV-moves in this case (see Figure 4).

3. Invariance under RV-moves. In [4], Khovanov and Rozansky have proved the invariance of $C_n(\Gamma)$ under type (I), (II) and (III) moves (see Figure 4). We prove invariance under type (IV) and (V). Invariance under type

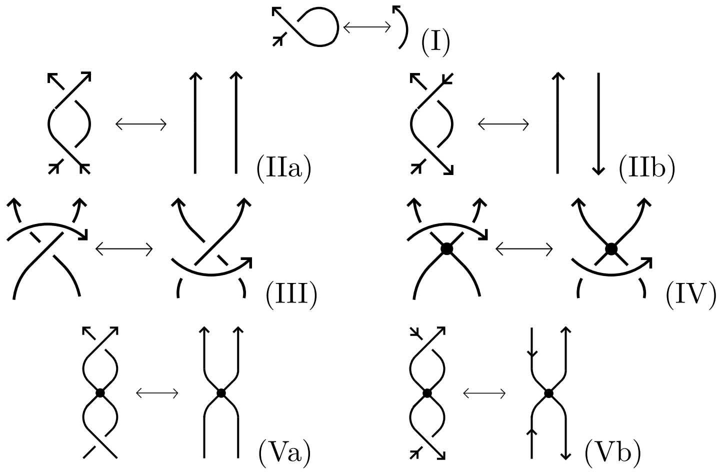


Fig. 4. Graph moves that generate rigid vertex isotopy

(IV) follows directly from the proof of invariance under (III). We will use at many levels the proofs of Khovanov and Rozansky (see [4]). All isomorphisms under graded matrix factorizations below are in homotopy categories hmf.

3.1. Invariance under (IV). As pointed out by Wu [6], Khovanov–Rozansky's proof of invariance under Reidemeister (III) can be simplified by using Bar-Natan's algebraic trick [1], i.e. by using the fact that the homotopy equivalence used for the proof of invariance under Reidemeister move (IIa) is a strong deformation retraction. If we think in the proof that way, then the proof of invariance under (IV) is contained in the one of (III).

We need to show that $C_n(\Gamma)$ and $C_n(\Gamma')$ are isomorphic for Γ, Γ' in Figure 5. The diagram Γ has four resolutions, denoted by Γ_{ij} for $i, j \in \{0,1\}$

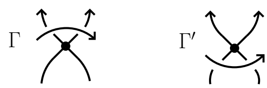


Fig. 5. Type (IV) move

(see Figure 6). The complex $C_n(\Gamma)\{-2n\}$ has the form

$$0 \to C_n(\Gamma_{00}) \xrightarrow{\partial^{-2}} \begin{pmatrix} C_n(\Gamma_{01})\{-1\} \\ C_n(\Gamma_{10})\{-1\} \end{pmatrix} \xrightarrow{\partial^{-1}} C_n(\Gamma_{11})\{-2\} \to 0$$

with $C_n(\Gamma_{11})\{-2\}$ in cohomological degree 0. This complex is shown in Figure 6.

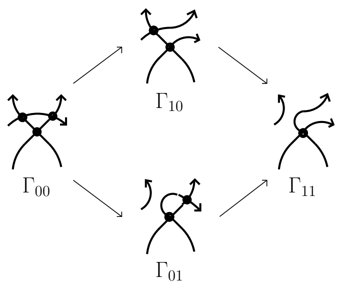


Fig. 6. Four resolutions of Γ in a type (IV) move

Khovanov and Rozansky [4] proved the following isomorphism:

$$C_n(\Gamma_{01}) \cong C_n(\Gamma_{11})\{+1\} \oplus C_n(\Gamma_{11})\{-1\}. \tag{3.1}$$

Furthermore, they proved that

$$C_n(\Gamma_{00}) \cong C_n(\Gamma_{11}) \oplus \Upsilon, \tag{3.2}$$

where Υ is defined in [4, Prop. 33].

The differential ∂^{-2} is injective on $C_n(\Gamma_{11}) \subset C_n(\Gamma_{00})$. In fact, the map to $C_n(\Gamma_{01})\{-1\}$ is injective (which follows from the inclusion $C_n(\Gamma_{11}) \subset C_n(\Gamma_{00})$ and the proof of invariance under (IIa), see [4]). The graded matrix factorization $\partial^{-2}(C_n(\Gamma_{00}))$ is a direct summand of $C_n^{-1}(\Gamma)\{-2n\}$. Thus $C_n(\Gamma)\{-2n\}$ contains a contractible summand

$$0 \to C_n(\Gamma_{11}) \xrightarrow{\partial^{-2}} C_n(\Gamma_{11}) \to 0. \tag{3.3}$$

The direct sum decomposition (3.1) can be chosen so that

$$C_n(\Gamma_{01})\{-1\} \cong p_{01}\partial^{-2} C_n(\Gamma_{11}) \oplus C_n(\Gamma_{11})\{-2\},$$

where p_{01} is the projection of $C_n^{-1}(\Gamma)\{-2n\}$ onto $C_n(\Gamma_{01})\{-1\}$. The differential ∂^{-1} is injective on $C_n(\Gamma_{11})\{-2\} \subset C_n(\Gamma_{01})\{-1\}$. Furthermore, the image of $C_n(\Gamma_{11})\{-2\} \subset C_n(\Gamma_{01})\{-1\}$ under ∂^{-1} is a direct summand of $C_n^0(\Gamma)\{-2n\}$. Hence the complex $C_n(\Gamma)\{-2n\}$ contains a contractible direct summand isomorphic to

$$0 \to C_n(\Gamma_{11})\{-2\} \xrightarrow{\partial^{-1}} C_n(\Gamma_{11})\{-2\} \to 0 \tag{3.4}$$

After splitting off contractible direct summands (3.3) and (3.4), the complex $C_n(\Gamma)\{-2n\}$ reduces to the complex C defined by

$$0 \to \Upsilon \xrightarrow{\partial^{-2}} C_n(\Gamma_{10})\{-1\} \to 0.$$

Since both $C_n(\Gamma)\{-2n\}$ and $C_n(\Gamma')\{-2n\}$ contain $C_n(\Gamma_{10})\{-1\}$, [4, Prop. 33] ensures that we can perform exactly the same reduction to $C_n(\Gamma')\{-2n\}$. Finally, we conclude that $C_n(\Gamma) \cong C_n(\Gamma')$.

3.2. Invariance under (Va). This invariance can be obtained as a consequence of Lemma 4.10 from Rasmussen [5]. We detail the proof.

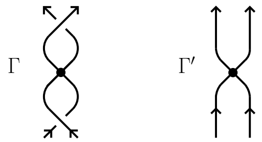


Fig. 7. Type (Va) move

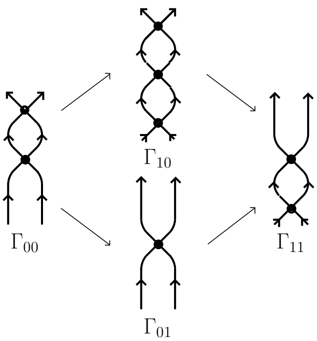


Fig. 8. Four resolutions of Γ in a type (Va) move

We need to show that $C_n(\Gamma)$ and $C_n(\Gamma')$ are isomorphic for the graphs Γ, Γ' shown in Figure 7. The diagram Γ has four resolutions, denoted by Γ_{ij} for $i, j \in \{0, 1\}$ and shown in Figure 8. The complex $C_n(\Gamma)$ has the form

$$0 \to C_n(\Gamma_{00})\{+1\} \xrightarrow{\partial^{-1}} \begin{pmatrix} C_n(\Gamma_{01}) \\ C_n(\Gamma_{10}) \end{pmatrix} \xrightarrow{\partial^{0}} C_n(\Gamma_{11})\{-1\} \to 0$$

where $C_n(\Gamma_{01})$ and $C_n(\Gamma_{10})$ are in cohomological degree 0. We have depicted this complex in Figure 8. Since Γ_{00} and Γ_{11} are isotopic, $C_n(\Gamma_{00})$ and $C_n(\Gamma_{11})$ are isomorphic. Khovanov and Rozansky [4] proved that

$$C_n(\Gamma_{10}) \cong C_n(\Gamma_{00})\{+1\} \oplus C_n(\Gamma_{00})\{-1\}. \tag{3.5}$$

Khovanov–Rozansky's proof of invariance under (IIa) ensures that the differential ∂^{-1} is injective on $C_n^{-1}(\Gamma_{00})$. The direct sum decomposition (3.5) can be chosen so that

$$C_n(\Gamma_{10}) \cong p_{10}\partial^{-1}C_n(\Gamma_{00})\{+1\} \oplus C_n(\Gamma_{00})\{-1\}.$$

Thus, $C_n(\Gamma)$ contains a contractible summand

$$0 \to C_n(\Gamma_{00})\{+1\} \xrightarrow{\partial^{-1}} C_n(\Gamma_{00})\{+1\} \to 0. \tag{3.6}$$

Furthermore, we have

$$C_n(\Gamma_{11})\{-1\} \cong C_n(\Gamma_{01}) \oplus C_n(\Gamma_{01})\{-2\}. \tag{3.7}$$

The differential ∂^0 is surjective onto $C_n(\Gamma_{01}) \subset C_n(\Gamma_{11})\{-1\}$. Thus $C_n(\Gamma)$ contains a contractible summand

$$0 \to C_n(\Gamma_{01}) \xrightarrow{\partial^0} C_n(\Gamma_{01}) \to 0. \tag{3.8}$$

After splitting off the contractible direct summands (3.6) and (3.8), the complex $C_n(\Gamma)$ reduces to a complex C of the form

$$0 \to C_n(\Gamma_{11})\{-1\} \xrightarrow{\partial^0} C_n(\Gamma_{01})\{-2\} \to 0$$

Since Γ' and Γ_{01} are isotopic, the decomposition (3.7) ensures that C is homotopy equivalent to

$$0 \to C_n(\Gamma') \to 0.$$

3.3. Invariance under (Vb). We need to show that $C_n(\Gamma)$ and $C_n(\Gamma')$ are isomorphic for the graphs Γ, Γ' shown in Figure 9. The diagram Γ has

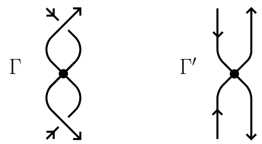


Fig. 9. Type (Vb) move

four resolutions, denoted by Γ_{ij} for $i, j \in \{0, 1\}$ and shown in Figure 10. The complex $C_n(\Gamma)$ has the form

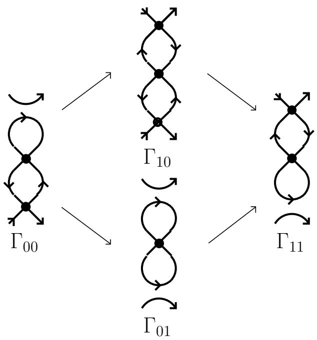


Fig. 10. Four resolution of Γ in a type (Vb) move

$$0 \to C_n(\Gamma_{00})\{+1\} \xrightarrow{{}^t(\partial^{-1,0},\partial^{-1,1})} \begin{pmatrix} C_n(\Gamma_{10}) \\ C_n(\Gamma_{01}) \end{pmatrix} \xrightarrow{(\partial^{0,0},\partial^{0,1})} C_n(\Gamma_{11})\{-1\} \to 0$$

where $C_n(\Gamma_{01})$ and $C_n(\Gamma_{10})$ are in cohomological degree 0. We have depicted this complex in Figure 10.

Applying Khovanov–Rozansky's results, we have the isomorphisms

$$C_n(\Gamma_{00}) \cong \bigoplus_{i=0}^{n-2} C_n(G_{00})\{2-n+2i\}\langle 1\rangle, \tag{3.9}$$

$$C_n(\Gamma_{01}) \cong \bigoplus_{i=0}^{n-2} C_n(G_{01})\{2-n+2i\}\langle 1\rangle, \tag{3.10}$$

$$C_n(\Gamma_{11}) \cong \bigoplus_{i=0}^{n-2} C_n(G_{11})\{2-n+2i\}\langle 1\rangle, \tag{3.11}$$

$$C_n(\Gamma_{10}) \cong \Big(\bigoplus_{i=0}^{n-3} C_n(G_{11})\{3-n+2i\}\langle 1\rangle\Big) \oplus C_n(\Gamma'), \tag{3.12}$$

where G_{00}, G_{01} and G_{11} are depicted in Figure 11.

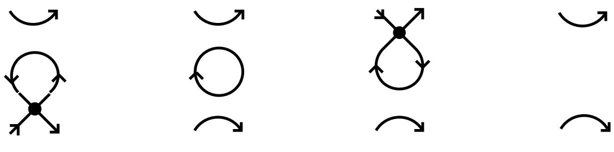


Fig. 11. The graphs G_{00}, G_{01}, G_{11}, G

We can twist the direct sum decompositions (3.9), (3.10) and (3.11) so that $\partial^{-1,1}$ and $\partial^{0,1}$ have diagonal form with respect to the decompositions $\partial^{-1,1} = \sum_{i=0}^{n-2} \partial_i^{-1,1}$ and $\partial^{0,1} = \sum_{i=0}^{n-2} \partial_i^{0,1}$. The proof of invariance under (I) by Khovanov and Rozansky implies that $\partial_i^{-1,1}$ is split injective and $\partial_i^{0,1}$ is split surjective for all $i \in [\![0, n-2]\!]$. Hence $\partial^{-1,1}$ is split injective and $\partial^{0,1}$ is split surjective. Denote by δ_i^{-1} the restriction of δ^{-1} to $C_n(G_{00})\{3-n+2i\}\langle 1\rangle$ and by δ_i^0 the composition of δ^0 with the projection onto $C_n(G_{11})\{1-n+2i\}\langle 1\rangle$. Since the category hmf_w has splitting idempotents (see [4, p. 46]), we can decompose $C_n^0(\Gamma)$ as the direct sum

$$C_n^0(\Gamma) \cong \Big(\bigoplus_{i=0}^{n-2} Im(\partial_i^{-1})\Big) \oplus \Big(\bigoplus_{i=0}^{n-2} Y_1^i\Big) \oplus Y_2$$

in such a way that ∂_i^0 restricts to an isomorphism from Y_1^i to $C_n(G_{11})\{1-$

$n+2i\}$ for all $i = 0, \dots, n-2$ and $\partial_i^0(Y_2) = 0$. Therefore, $C_n(\Gamma)$ is isomorphic to the direct sum of complexes

$$\begin{array}{ccccc} 0 & \to & Y_2 & \to & 0, \\ 0 \to C_n(G_{00})\{3-n+2i\} & \xrightarrow{\cong} & \mathrm{Im}(\partial_{\mathrm{i}}^{-1}) & \to & 0, \\ 0 & \to & Y_1^i & \xrightarrow{\cong} & C_n(G_{11})\{1-n+2i\} \to 0. \end{array}$$

We can decompose further the sum decompositions (3.9)–(3.11) to obtain

$$C_n(\Gamma_{00}) \cong \bigoplus_{i=0}^{n-2} \bigoplus_{j=0}^{n-2} C_n(G)\{4-2n+2(i+j)\}, \tag{3.13}$$

$$C_n(\Gamma_{01}) \cong \bigoplus_{i=0}^{n-2} \bigoplus_{j=0}^{n-1} C_n(G)\{3-2n+2(i+j)\}, \tag{3.14}$$

$$C_n(\Gamma_{11}) \cong \bigoplus_{i=0}^{n-2} \bigoplus_{j=0}^{n-2} C_n(G)\{4-2n+2(i+j)\}, \tag{3.15}$$

$$C_n(\Gamma_{10}) \cong \Big(\bigoplus_{i=0}^{n-3} \bigoplus_{j=0}^{n-2} C_n(G)\{5-2n+2(i+j)\} \Big) \oplus C_n(\Gamma'), \tag{3.16}$$

where G is the rightmost graph in Figure 11. From formulas (3.13) to (3.16) we obtain

$$C_n^0(\Gamma) \cong C_n(\Gamma_{01}) \oplus C_n(\Gamma_{10}) \cong C_n(\Gamma_{00})\{+1\} \oplus C_n(\Gamma_{11})\{-1\} \oplus C_n(\Gamma').$$

The category hmf_w is Krull–Schmidt; this implies that $Y_2 \cong C_n(\Gamma')$. Therefore, the complexes $C_n(\Gamma)$ and $0 \to C_n(\Gamma') \to 0$ are isomorphic. This concludes our proof of the invariance under type (Vb) moves. Theorem 1 follows.

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References

[1] D. Bar-Natan, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. 9 (2005), 1443–1499.

[2] L. Kauffman, *Invariants of graphs in three-space*, Trans. Amer. Math. Soc. 311 (1989), 697–710.

[3] L. Kauffman and P. Vogel, *Link polynomials and a graphical calculus*, J. Knot Theory Ramif. 1 (1992), 59–104.

[4] M. Khovanov and L. Rozansky, *Matrix factorizations and link homology*, Fund. Math. 199 (2008), 1–91.
[5] J. Rasmussen, *Khovanov–Rozansky homology of two-bridge knots and links*, Duke Math. J. 136 (2007), 551–583.
[6] H. Wu, *Braids, transversal knots and the Khovanov–Rozansky theory*, Trans. Amer. Math. Soc. 360 (2008), 3365–3389.

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