FUNDAMENTA
MATHEMATICAE
214 (2011)

On countable dense and strong n-homogeneity

by

Jan van Mill (Amsterdam)

Abstract. We prove that if a space X is countable dense homogeneous and no set of size $n-1$ separates it, then X is strongly n-homogeneous. Our main result is the construction of an example of a Polish space X that is strongly n-homogeneous for every n, but not countable dense homogeneous.

1. Introduction. *Unless otherwise stated, all spaces under discussion are separable and metrizable.*

Recall that a space X is *countable dense homogeneous* (CDH) if, given any two countable dense subsets D and E of X, there is a homeomorphism $f\colon X \to X$ such that $f(D) = E$. The first result in this area is due to Cantor, who showed that the reals are CDH. Fréchet [23] and Brouwer [5], independently, proved that the same is true for the n-dimensional Euclidean space $\mathbb{R}^n$. In 1962, Fort [22] proved that the Hilbert cube is also CDH, and in 1969 Bessaga and Pełczyński [3] showed that strongly locally homogeneous (SLH) Polish spaces are CDH (see also [4, p. 139, Theorem 7.1]).

Countable dense sets are pushed to one another by the standard 'back-and-forth' method. The problem that one faces is how to ensure that a certain inductively constructed sequence of homeomorphisms converges to a homeomorphism of the space under consideration.

The topological sum of the 1-sphere $\mathbb{S}^1$ and the 2-sphere $\mathbb{S}^2$ is an example of a CDH-space which is not homogeneous. Bennett [2] proved that for connected spaces, countable dense homogeneity implies homogeneity (see also [29, 1.6.8]). So for connected spaces, countable dense homogeneity can be thought of as a strong form of homogeneity. In [36], Ungar made this precise for locally compact spaces by proving the following elegant result.

2010 *Mathematics Subject Classification*: Primary 57S05; Secondary 54H15, 54F45.
Key words and phrases: countable dense homogeneous, (strongly) n-homogeneous, continuous action, Polish space, counterexample.

DOI: 10.4064/fm214-3-2

Theorem 1.1 (Ungar). *Let X be a locally compact space such that no finite set separates it. Then the following statements are equivalent:*

(a) *X is* CDH.
(b) *X is n-homogeneous for every n.*
(c) *X is strongly n-homogeneous for every n.*

Let us comment on Ungar's proof. First of all, the equivalence (b)⇔(c) follows from Corollary 3.10 in his earlier paper [35]. The assumption of local connectivity there is superfluous since all one needs for the proof is the existence of a Polish group which makes the space under consideration n-homogeneous for all n; here a space is called *Polish* if its topology is generated by a complete metric. That no finite set separates X is essential for (b)⇔(c) as $\mathbb{S}^1$ demonstrates. Ungar's proofs of the implications (a)⇒(c) and (c)⇒(a) were both based (among other things) on the celebrated Effros Theorem from [17] (see also [1] and [30]) on transitive actions of Polish groups on Polish spaces. In the proof of (c)⇒(a) the Effros Theorem controls the inductive process, and in the proof of (a)⇒(c) it allows the use of the Baire Category Theorem. Moreover, (c)⇒(a) is true for all locally compact spaces, additional connectivity assumptions are not needed for the proof, and $\mathbb{S}^1$ again demonstrates that (a)⇒(c) is false without them.

The aim of this paper is to investigate whether Theorem 1.1 can be improved. The first question that comes to mind is whether the assumption on local compactness can be relaxed to that of completeness. In recent years it has become clear that there are delicate topological differences in the homogeneity properties of locally compact and non-locally compact Polish spaces. It is for example a trivial result that for each homogeneous locally compact space there exists a Polish group acting transitively on it. For Polish spaces this need not be true, as was shown in van Mill [32]. It turns out that a transitive action by a Polish group on a Polish space is a very strong homogeneity property of that space because of the Effros Theorem. Locally compact homogeneous spaces have this property and the proof of Ungar's Theorem 1.1 heavily depends on it.

Let the group G act on the space X. We say that a subset H of G *makes X* CDH provided that for all countable dense subsets D and E of X there is an element $g \in H$ such that $gD = E$. So, informally speaking, H witnesses the fact that X is CDH. Similarly, we say that H *makes X n-homogeneous* provided that for all subsets F and G of X of size n there exists $g \in H$ such that $gF = G$. We finally say that H *makes X strongly n-homogeneous* if given any two n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of distinct points of X, there exists $g \in H$ such that $gx_i = y_i$ for every $i \le n$.

Our first result is that the implication (a)⇒(c) in Ungar's Theorem holds for all spaces, in essence even without connectivity assumptions.

Theorem 1.2. *If the group G makes the infinite space X* CDH *and no set of size $n-1$ separates X, then G makes X strongly n-homogeneous.*

Observe that this improves Bennett's result quoted above that a connected CDH-space is homogeneous.

The main result in this paper is that the assumption of local compactness in the implication (c)⇒(a) in Ungar's Theorem is essential, even when dealing with Polish spaces.

Example 1.3. There is a Polish space $\mathscr{X}$ which is strongly n-homogeneous for every n, but not CDH.

In $\mathscr{X}$ it is possible to perform all back-and-forth steps in the standard process to push a given countable dense set onto another. But it is not possible to ensure that the inductively constructed sequence of homeomorphisms converges to a homeomorphism, despite the fact that $\mathscr{X}$ is completely metrizable.

The space $\mathscr{X}$ is a variation of the example in van Mill [32]. It is situated in the product of two Cantor sets where one of the factors is endowed with a stronger topology in which convergence is equivalent to ordinary convergence and convergence in 'norm'. In contrast to the space in [32], the pathology in $\mathscr{X}$ is not based upon connectivity but upon the pathology present in Erdős' space from [20]. We will give more information about this in the Appendix to this paper.

For some recent results on countable dense homogeneity, see [25], [21], [31].

2. Preliminaries

(A) Notation. We use 'countable' for 'at most countable'. For a set X and $n \in \mathbb{N}$, $[X]^{<n}$ and $[X]^n$ denote $\{A \subseteq X : |A| < n\}$ and $\{A \subseteq X : |A| = n\}$, respectively. In addition, $[X]^{<\omega}$ signifies the collection of all finite subsets of X. If X is a set then $\mathscr{P}(X)$ denotes its power set.

An *Egyptian fraction* is a finite sum of positive distinct unit fractions. Any positive rational number has representations as an Egyptian fraction with arbitrarily many terms and with arbitrarily large denominators. This can be proved by repeatedly using the identity

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}.$$

For more information, see e.g. Campbell [7] and Hoffman [24, p. 153].

(B) Topology. A subset of a space X is called *clopen* if it is both closed and open.

We let $\mathscr{H}(X)$ denote the group of homeomorphisms of a space X. We say that X is *strongly locally homogeneous* (abbreviated SLH) if it has a base

$\mathscr{B}$ such that for all $B \in \mathscr{B}$ and $x, y \in B$ there is an element $f \in \mathscr{H}(X)$ that is supported on B (that is, f is the identity outside B) and moves x to y. As we mentioned in §1, Bessaga and Pełczyński [3] showed that strongly locally homogeneous Polish spaces are CDH (see also [4, p. 139, Theorem 7.1]). This is done by the standard 'back-and-forth' method. An inspection of the proof shows that the 'back-and-forth' method allows one to simultaneously push countably many pairwise disjoint countable dense sets in place.

THEOREM 2.1 (Bessaga and Pełczyński). *Let X be* SLH *and Polish, and let $\{D_i : i \in \mathbb{N}\}$ and $\{E_i : i \in \mathbb{N}\}$ be sequences of pairwise disjoint countable dense subsets of X. Then there is a homeomorphism $f\colon X \to X$ such that $f(D_i) = E_i$ for every i.*

Observe that the standard Cantor set $\{0,1\}^{\mathbb{N}}$ has the property that all of its nonempty clopen subsets are homeomorphic. Hence $\{0,1\}^{\mathbb{N}}$ is SLH and so the conclusion of Theorem 2.1 holds for $X = \{0,1\}^{\mathbb{N}}$.

Consider the power set $\mathscr{P}(\mathbb{N})$ with the symmetric difference $\triangle$ as group operation. We equip $\mathscr{P}(\mathbb{N})$ with the standard Cantor set topology that comes with identification with $\{0,1\}^{\mathbb{N}}$. That is, a basic clopen subset of $\mathscr{P}(\mathbb{N})$ has the form

$$[\sigma, \tau] = \{A \in \mathscr{P}(\mathbb{N}) : \sigma \subseteq A,\ \tau \cap A = \emptyset\},$$

where σ and τ are arbitrary disjoint finite subsets of $\mathbb{N}$. On $\mathscr{P}(\mathbb{N})$ we use the standard product metric

$$d(A, B) = \sum_{n \in A \triangle B} 2^{-n}.$$

Let $\Upsilon = \mathscr{P}(\mathbb{N}) \times \mathscr{P}(\mathbb{N})$. We let $\pi_i\colon \Upsilon \to \mathscr{P}(\mathbb{N})$ denote the projection onto the ith coordinate ($i = 1, 2$).

A function $f\colon X \to [-\infty, \infty]$ is called *lower semicontinuous* (abbreviated LSC) if $f^{-1}((t, \infty])$ is open in X for every $t \in \mathbb{R}$.

Let $A \subseteq X$. We say that A *separates* X provided that $X \setminus A$ is disconnected. And A is called a *C-set in X* if A can be written as an intersection of clopen subsets of X. If A is a C-set, then there clearly is a *countable* family $\mathscr{C}$ consisting of clopen subsets of X such that $A = \bigcap \mathscr{C}$. A C-set is closed being an intersection of closed sets.

A space X is called *totally disconnected* if for every pair of distinct points $x_1, x_2 \in X$ there exist clopen sets C_1, C_2 such that $x_1 \in C_1$, $x_2 \in C_2$ and $C_1 \cap C_2 = \emptyset$.

If d is a metric on X, then by the symbol $x_n \to_d x$ we mean that the sequence $(x_n)_n$ converges to x in the topology on X induced by d.

We shall frequently use the well-known fact that a G_δ-subset of a Polish space is Polish itself. For details, see [18, 4.3.23].

(C) Actions of groups. Let $a\colon G\times X\to X$ be an action of a group G on the space X. For every $g\in G$, the function $x\mapsto a(g,x)$ is a homeomorphism of X. We use gx as an abbreviation for $a(g,x)$. This notation is sometimes slightly confusing, especially if G is a group of homeomorphisms of some space. The action is called *transitive* if for all $x,y\in X$ there exists $g\in G$ such that $gx=y$. For every $x\in X$ we let Gx denote the *orbit* of x, i.e., $Gx=\{gx : g\in G\}$. If $A\subseteq X$, then

$$G_A=\{f\in G : (\forall\, x\in A)(f(x)=x)\}.$$

That is, G_A is the *stabilizer subgroup* of A.

LEMMA 2.2. *Let the group G act on the infinite set X. Then the following statements are equivalent for every $n\geq 1$:*

(a) *G makes X strongly n-homogeneous.*
(b) *For every $F\in[X]^{n-1}$, the group G_F acts transitively on $X\setminus F$.*

Proof. It is clear that (a)⇒(b). We prove $(\mathrm{b})_n\Rightarrow(\mathrm{a})_n$ by induction on n. For $n=1$ there is nothing to prove. So assume that our statement holds for $n-1$, where $n\geq 2$, and that X satisfies $(\mathrm{b})_n$. Since X is infinite it satisfies $(\mathrm{b})_{n-1}$ and hence $(\mathrm{a})_{n-1}$ by our inductive hypothesis. Let $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ be arbitrary n-tuples of distinct points of X. By what we have just observed, there is an element $g_0\in G$ such that $g_0x_i=y_i$ for every $i\leq n-1$. Put $F=\{y_1,\ldots,y_{n-1}\}$. By $(\mathrm{b})_n$ there is an element $g_1\in G_F$ such that $g_1g_0x_n=y_n$. So we conclude that $g_1g_0x_i=y_i$ for every $i\leq n$. ∎

3. Proof of Theorem 1.2

PROPOSITION 3.1. *Let X be a space. Suppose that G is a subset of $\mathscr{H}(X)$ that makes X CDH. If $F\subseteq X$ is finite and $D,E\subseteq X\setminus F$ are countable and dense in X, then there are elements $\alpha,\beta\in G$ such that $\alpha\restriction F=\beta\restriction F$ and $(\alpha^{-1}\circ\beta)(D)\subseteq E$.*

Proof. Let h_0 be an arbitrary element in G. Suppose $\{h_\beta : \beta<\alpha\}\subseteq G$ have been constructed for some $\alpha<\omega_1$. Now by CDH, pick $h_\alpha\in G$ such that

$$h_\alpha(F\cup E)=\bigcup_{\beta<\alpha}h_\beta(D). \tag{†}$$

For $1\leq\alpha<\omega_1$, let T_α be a nonempty finite subset of $[1,\alpha)$ such that $h_\alpha(F)\subseteq\bigcup_{\beta\in T_\alpha}h_\beta(D)$. We claim that, for $T\colon[1,\omega_1)\to[\omega_1]^{<\omega}$ defined by $T(\alpha)=T_\alpha$, the fiber $T^{-1}(A)$ is uncountable for some $A\in[\omega_1]^{<\omega}$. If the latter were not true, there would exist a strictly increasing sequence $\{\alpha_n\}_n$ of countable ordinal numbers such that $T^{-1}(A)\subseteq\alpha_{n+1}$ for each $A\in T([1,\alpha_n])$. Then, letting $\alpha=\sup_n\alpha_n$ and $A=T(\alpha)$, one can find n such that $A\subseteq\alpha_n$;

hence $A \in T([1, \alpha_n])$, which contradicts $\alpha \in T^{-1}(A) \subseteq \alpha_{n+1}$. (This is of course nothing but the standard argument in the proof of the Pressing Down Lemma.) So pick an $A \in [\omega_1]^{<\omega}$ for which $B = T^{-1}(A)$ is uncountable. Then $h_\alpha(F) \subseteq \bigcup_{\beta \in A} h_\beta(D)$ for every $\alpha \in B$. Since $\bigcup_{\beta \in A} h_\beta(D)$ is countable, and B is uncountable, we may assume without loss of generality that $h_\alpha \restriction F = h_\beta \restriction F$ for all $\alpha, \beta \in B$. Hence if $\alpha, \beta \in B$ are such that $\beta < \alpha$, then $h_\alpha \restriction F = h_\beta \restriction F$, and by (†), $(h_\alpha^{-1} \circ h_\beta)(D) \subseteq E$. ∎

Corollary 3.2. *Let X be a space without isolated points. Assume that the group G makes X* CDH. *Then for every finite subset $F \subseteq X$, every G_F-invariant subset of $X \setminus F$ is open.*

Proof. For $x \in X \setminus F$, let $Y = G_F x$. To show that Y has nonempty interior, assume that this is not the case. Then we may pick a countable dense set D in X which is contained in $X \setminus (F \cup Y)$. By Proposition 3.1, there exists $h \in G_F$ such that $h(D \cup \{x\}) \subseteq D$, a contradiction because $h(x) \in G_F x = Y$ and $h(x) \in D \subseteq X \setminus Y$. So Y is open, being an orbit. ∎

Proof of Theorem 1.2. Assume that G makes the space X CDH and no set of size $n-1$ separates X. By Lemma 2.2 all we need to show is that for every $F \in [X]^{n-1}$ the group G_F acts transitively on $X \setminus F$. By Corollary 3.2 every orbit $G_F x$ for $x \in X \setminus F$ is open. Since orbits are disjoint, they are clopen. So we are done by connectivity. ∎

4. The harmonic Erdős space. For $A \in \mathscr{P}(\mathbb{N})$, we put $\varphi(A) = \sum_{n \in A} 1/n$ if $A \neq \emptyset$, and $\varphi(\emptyset) = 0$. Then $\varphi\colon \mathscr{P}(\mathbb{N}) \to [0, \infty]$ is an LSC measure. Observe that φ is badly discontinuous. Define

$$\mathscr{E} = \{A \in \mathscr{P}(\mathbb{N}) : \varphi(A) < \infty\}.$$

By $\mathscr{E}_{\mathrm{w}}$ we denote $\mathscr{E}$ with the topology it inherits from $\mathscr{P}(\mathbb{N})$. Observe that $\mathscr{E}_{\mathrm{w}}$ is a dense F_σ-subgroup of $\mathscr{P}(\mathbb{N})$ since φ is LSC.

Let $\mathscr{E}_{\mathrm{s}}$ denote the subspace of all vectors $x = (x_n)_n$ in the real Hilbert space ℓ^1 such that $x_n \in \{0, 1/n\}$ for all n. As sets, $\mathscr{E}_{\mathrm{w}}$ and $\mathscr{E}_{\mathrm{s}}$ can be identified in the obvious way, but as topological spaces they are different. Observe that $\mathscr{E}_{\mathrm{s}}$ is closed in ℓ^1 and hence is a Polish space, but $\mathscr{E}_{\mathrm{w}}$ is not Polish. If $A, B \in \mathscr{E}$ and x_A and x_B denote their corresponding vectors in ℓ^1, then

$$\|x_A - x_B\| = \sum_{n \in A \triangle B} 1/n = \varphi(A \triangle B).$$

Hence $\mathscr{E}_{\mathrm{s}}$ can and will be thought of as $\mathscr{E}$ with the topology induced by the metric $\varrho(A, B) = \varphi(A \triangle B)$. Observe that for every $A, B \in \mathscr{E}$ we have $d(A, B) \leq \varrho(A, B)$. Hence the topology on $\mathscr{E}$ induced by ϱ is finer than the topology on $\mathscr{E}$ induced by d. This is trivial anyway since convergence in ℓ^1 is equivalent to pointwise convergence and convergence in norm.

The space $\mathscr{E}_{\mathrm{s}}$ is called the *harmonic Erdős space* and was introduced by Dijkstra [10]. For more details, see §8(B) below.

For $t \in (0, \infty]$ and $A \in \mathscr{E}_{\mathrm{s}}$, put

$$U(A,t) = \{B \in \mathscr{E}_{\mathrm{s}} : \varphi(A \triangle B) < t\},$$
$$B(A,t) = \{B \in \mathscr{E}_{\mathrm{s}} : \varphi(A \triangle B) \le t\},$$
$$F(A,t) = \{B \in U(A,t) : B \text{ is finite}\}.$$

We emphasize that these sets are subspaces of $\mathscr{E}_s$. Hence $F(A,t)$ is dense in $U(A,t)$. Moreover, $U(A,t)$ is open in $B(A,t)$ but not necessarily dense (see §8(B)), and $U(A,\infty) = \mathscr{E}_{\mathrm{s}}$.

The following trivialities will be used a few times in what follows.

LEMMA 4.1.

(a) *Let $(A_n)_n$ be an increasing sequence of subsets of $\mathbb{N}$ such that $A = \bigcup_{n=1}^{\infty} A_n \in \mathscr{E}$. Then $A_n \to_{\varrho} A$.*

(b) *Let σ and τ be finite subsets of $\mathbb{N}$, and let $r \in (0,\infty)$. Then*

$$U(\emptyset, r) \cap [\sigma,\tau] = U(\sigma, r-\delta) \cap [\sigma,\tau],$$

where $\delta = \varphi(\sigma)$.

(c) *Let $[\sigma_1,\tau_1],[\sigma_2,\tau_2],\dots$ be a decreasing sequence of basic clopen sets in $\mathscr{P}(\mathbb{N})$ such that $\{A\} = \bigcap_{n\in\mathbb{N}}[\sigma_n,\tau_n]$. Then $\lim_{n\to\infty}\varphi(\sigma_n) = \varphi(A)$.*

Proof. For (a), let $\varepsilon > 0$. Since $\varphi(A) < \infty$, there exists N such that $\varphi(\{a \in A : a > N\}) < \varepsilon$. Let n_0 be so large that $\{a \in A : a \le N\} \subseteq A_{n_0}$. Then for $n \ge n_0$ we have

$$\varrho(A, A_n) = \varphi(A \triangle A_n) \le \varphi(\{a \in A : a > N\}) < \varepsilon,$$

as required.

For (b), assume that $A \in U(\emptyset, r) \cap [\sigma,\tau]$. Then $\sigma \subseteq A$, hence $\varphi(A \setminus \sigma) < r-\delta$, i.e., $\varrho(A,\sigma) = \varphi(A \triangle \sigma) = \varphi(A \setminus \sigma) < r - \delta$. Conversely, if $A \in U(\sigma, r-\delta) \cap [\sigma,\tau]$, then $\sigma \subseteq A$, hence $r-\delta > \varrho(A,\sigma) = \varphi(A \triangle \sigma) = \varphi(A \setminus \sigma)$, i.e., $\varphi(A) < r$.

For (c), simply observe that $\sigma_1 \subseteq \sigma_2 \subseteq \dots \subseteq A$ and $A = \bigcup_n \sigma_n$. ∎

That $\dim \mathscr{E}_{\mathrm{s}} = 1$ can quite easily be proved by the method of Erdős [20]. We will prove a stronger result that will be exactly what we will need in §7.

LEMMA 4.2. *Let $t \in (0,\infty)$, and let C be a relatively clopen subset of $U(\emptyset, t)$. Then for every $F \in C$ and $N \in \mathbb{N}$ there is a sequence $m_1 < m_2 < \cdots$ in $\mathbb{N} \setminus F$ such that*

(1) *$m_1 > N$,*
(2) *for all n, $F \cup \{m_1,\dots,m_n\} \in C$,*
(3) *$\varphi(F \cup \{m_1, m_2, \dots\}) = t$.*

Proof. First observe that since $\varphi(F) < t$ we have $\varphi(\mathbb{N} \setminus F) = \infty$. As a consequence, $E = \{m \in \mathbb{N} \setminus F : m > N\}$ is infinite. Since $F \in C$ and C is open in $\mathscr{E}_{\mathrm{s}}$ and the sequence $\{F \cup \{m\})_{m \in E}$ converges to F in $\mathscr{E}_{\mathrm{s}}$, there exists $m \in E$ such that $F \cup \{m\} \in C$. Let m_1 be the first such m. Similarly, having constructed m_1 through m_n, let m_{n+1} be the first $m > m_n$ in E such that $F \cup \{m_1, \ldots, m_n\} \cup \{m\} \in C$. Put $X = \{m_n : n \in \mathbb{N}\}$ and $Y = F \cup X$. Since $\varphi(F \cup \{m_1, \ldots, m_n\}) < t$ for all n, clearly, $\varphi(Y) \leq t$.

All we need to prove is that $\varphi(Y) = t$. Striving for a contradiction, assume that $\varphi(Y) < t$, i.e., $Y \in U(\emptyset, t)$. Since $\varphi(X) < \infty$ and $\varphi(E) = \infty$ there exists an infinite subset G of $\mathbb{N}$ and for every $n \in G$ an element $z_n \in E$ such that $m_n < z_n < m_{n+1}$. Observe that $F \cup \{m_1, \ldots, m_n\} \to_{\varrho} F \cup X = Y \in U(\emptyset, t)$ (Lemma 4.1(a)). Hence $Y \in C$ since C is closed in $U(\emptyset, t)$. Moreover, for every $n \in G$ we have $F \cup \{m_1, \ldots, m_n\} \cup \{z_n\} \notin C$ by construction. Since $z_n \to \infty$ and hence $\varphi(\{z_n\}) \to 0$,

$$F \cup \{m_1, \ldots, m_n\} \cup \{z_n\} \to_{\varrho} F \cup X = Y.$$

Since C is open in $\mathscr{E}_{\mathrm{s}}$ and hence a neighborhood of Y, $F \cup \{m_1, \ldots, m_n\} \cup \{z_n\} \in C$ for almost all n. But this contradicts the fact that $F \cup \{m_1, \ldots, m_n\} \cup \{z_n\} \notin C$ for every $n \in G$. ∎

Observe that this implies that every nonempty clopen subset C of $\mathscr{E}_{\mathrm{s}}$ is 'unbounded', i.e., $\sup\{\varphi(A) : A \in C\} = \infty$. Hence $\mathscr{E}_{\mathrm{s}}$ is nowhere zero-dimensional.

PROPOSITION 4.3. *Let $t \in (0, \infty)$. Then for every countable family $\mathscr{D}$ consisting of relative C-sets in $U(\emptyset, t)$ such that $F(\emptyset, t) \cap \bigcup \mathscr{D} = \emptyset$, the set $U(\emptyset, t) \setminus \bigcup \mathscr{D}$ is nowhere zero-dimensional.*

Proof. Pick an arbitrary $A \in Y = U(\emptyset, t) \setminus \bigcup \mathscr{D}$. There clearly exists $t' \in (0, \infty)$ such that $t' < t$ and $A \in U(\emptyset, t')$. Hence since $U(\emptyset, t')$ is an open neighborhood of A, to prove that Y is not zero-dimensional at A it suffices to prove that for every relatively clopen subset U of Y that contains A there exists $B \in U$ such that $\varphi(B) > t'$. So let U be an arbitrary relatively clopen subset of Y that contains A. We will prove something stronger than strictly needed, namely that if $s = \sup\{\varphi(B) : B \in U\}$, then $s = t$. Striving for a contradiction, assume that $s < t$. Enumerate $\mathscr{D}$ as $\{D_n : n \in \mathbb{N}\}$. Since the dense subset $F(\emptyset, t)$ of $U(\emptyset, t)$ is contained in Y, we may pick an element $G \in U \cap F(\emptyset, t)$. By recursion on n we will construct a finite subset F_n of $\mathbb{N} \setminus G$, an element $k_n \in \mathbb{N} \setminus G$ and a clopen neighborhood C_n of D_n in $U(\emptyset, t)$ such that

(1) $F_{n-1} \subseteq F_n$,
(2) $G \cup F_n \in U \setminus \bigcup_{i=1}^{n} C_i$, $G \cup F_n \cup \{k_n\} \in Y \setminus U$,
(3) $k_n > 2^n$.

Assume that F_i, C_i and k_i have been chosen up to n. Since $G \cup F_n \in U$, D_{n+1} is a C-set in $U(\emptyset, t)$ and $U \cap D_{n+1} = \emptyset$, there is a relatively clopen subset C_{n+1} of $U(\emptyset, t)$ such that $D_{n+1} \subseteq C_{n+1}$ but $G \cup F_n \notin C_{n+1}$. Hence $G \cup F_n \in U(\emptyset, t) \setminus \bigcup_{i=1}^{n+1} C_i$. By Lemma 4.2, there is a sequence $m_1 < m_2 < \cdots$ in $\mathbb{N} \setminus (G \cup F_n)$ such that

(4) $m_1 > 2^{n+1}$,
(5) for all n, $G \cup F_n \cup \{m_1, \ldots, m_n\} \in U(\emptyset, t) \setminus \bigcup_{i=1}^{n+1} C_i$,
(6) $\varphi(G \cup F_n \cup \{m_1, m_2, \ldots\}) = t$.

Since $\varphi(G \cup F_n) \leq s < t$, there exists $N \in \mathbb{N}$ such that

$$\varphi(G \cup F_n \cup \{m_1, \ldots, m_N\}) > s = \sup\{\varphi(B) : B \in U\}.$$

Let $i \leq N$ be the first element such that $G \cup F_n \cup \{m_1, \ldots, m_i\} \notin U$. If $i = 1$ put $F_{n+1} = F_n$ and $k_{n+1} = m_1$, and if $i > 1$ put $F_{n+1} = F_n \cup \{m_1, \ldots, m_{i-1}\}$ and $k_{n+1} = m_i$. Then $G \cup F_{n+1} \in U \setminus \bigcup_{i=1}^{n+1} C_i$, and since $G \cup F_{n+1} \cup \{k_{n+1}\}$ is finite and belongs to $U(\emptyset, t)$, it is an element of $Y \setminus U$. Hence our choices are as required.

Put $F = \bigcup_{n=1}^{\infty} F_n$. Observe that $G \cup F_n \to_{\varrho} G \cup F$ (Lemma 4.1(a)). Since $G \cup F_n \in U$ for every n, it follows that $\varphi(G \cup F) \leq s < t$. Hence $G \cup F \in U(\emptyset, t)$. Moreover, (2) clearly implies that $G \cup F \notin \bigcup_{i=1}^{\infty} C_i$, i.e., $G \cup F \in Y$. Hence $G \cup F \in U$ since U is closed in Y. Now $k_n \to \infty$ and hence $\varphi(\{k_n\}) \to 0$, so $G \cup F_n \cup \{k_n\} \to_{\varrho} G \cup F$. This means that $G \cup F_n \cup \{k_n\} \in U$ for almost all n, which contradicts (2). ∎

The following result can be proved by the method used in Dijkstra, van Mill and Steprāns [14, Theorem 3.1].

Proposition 4.4. *Let $t \in [0, \infty)$, and let $\mathbb{P}$ be any zero-dimensional Polish space. Then every nonempty closed subset X of $\mathbb{P} \times \mathscr{E}_{\mathrm{s}}$ that is contained in $\mathbb{P} \times B(\emptyset, t)$ is somewhere zero-dimensional.*

5. Homeomorphisms with control. We show that certain clopen subsets of $\mathscr{P}(\mathbb{N})$ are homeomorphic by 'norm preserving' homeomorphisms. That all nonempty clopen subsets of $\mathscr{P}(\mathbb{N})$ are homeomorphic is trivial since $\mathscr{P}(\mathbb{N})$ is a Cantor set. The work done here is to ensure that the constructed homeomorphisms are continuous in the stronger (Erdős space) topologies considered later.

Lemma 5.1. *Let $U \subseteq \mathscr{P}(\mathbb{N})$ be a nonempty clopen set. Then there is a finite subset α of $\mathbb{N}$ such that $\alpha \in U$ and $\min\{\varphi(A) : A \in U\} = \varphi(\alpha)$.*

Proof. There is a partition $\mathscr{V}$ of U consisting of nonempty basic clopen sets. There are for every $V \in \mathscr{V}$ disjoint finite subsets σ_V and τ_V of $\mathbb{N}$ such that $V = [\sigma_V, \tau_V]$. Observe that for every $V \in \mathscr{V}$, $\varphi(\sigma_V) =$

$\min\{\varphi(A) : A \in [\sigma_V, \tau_V]\}$. Hence, since $\mathscr{V}$ is finite,

$$\min\{\varphi(A) : A \in U\} = \min\{\varphi(\sigma_V) : V \in \mathscr{V}\}$$

exists and is attained at a point of the form σ_V for some $V \in \mathscr{V}$. ∎

For a nonempty clopen subset U of $\mathscr{P}(\mathbb{N})$, it will be convenient to denote $\min\{\varphi(A) : A \in U\}$ by $M(U)$. We also put $M(\emptyset) = \infty$. Hence by Lemma 5.1 it follows that $M(U)$ is rational for every nonempty clopen subset U of $\mathscr{P}(\mathbb{N})$. Moreover, if $[\sigma, \tau]$ is a basic clopen set, then σ is the unique element of $[\sigma, \tau]$ at which $M([\sigma, \tau])$ is attained.

Observe that the point at which the 'minimum' of a clopen set is attained need not be unique. Also observe that if U and V are clopen subsets of $\mathscr{P}(\mathbb{N})$ such that $U \subseteq V$, then $M(V) \le M(U)$.

Let $\mathscr{C}$ denote the collection of all nonempty clopen subsets C of $\mathscr{P}(\mathbb{N})$ such that $M(C)$ is attained at a single point. Hence $\mathscr{C}$ contains all nonempty basic clopen subsets of $\mathscr{P}(\mathbb{N})$.

We will prove that all elements of $\mathscr{C}$ are homeomorphic via homeomorphisms with control. We use among other things a method essentially due to Charatonik [8] and Bula and Oversteegen [6] for proving that the Lelek fan is unique, with refinements that can be found in Dijkstra and van Mill [13, §6].

Lemma 5.2. *Let $E, F \in \mathscr{C}$, and let $\varepsilon > 0$. Then there are clopen partitions $\mathscr{U}$ and $\mathscr{V}$ of E and F, respectively, and a bijection $\xi : \mathscr{U} \to \mathscr{V}$ such that*

(1) *both $\mathscr{U}$ and $\mathscr{V}$ consist entirely of elements of $\mathscr{C}$,*
(2) $\operatorname{mesh} \mathscr{U} < \varepsilon$, $\operatorname{mesh} \mathscr{V} < \varepsilon$,
(3) *for every $U \in \mathscr{U}$, $M(U) - M(E) = M(\xi(U)) - M(F)$.*

Proof. By Lemma 5.1, there are finite subsets σ_0 and σ_1 of $\mathbb{N}$ such that $\sigma_0 \in E$, $M(E) = \varphi(\sigma_0)$, $\sigma_1 \in F$, and $M(F) = \varphi(\sigma_1)$. Select partitions $\mathscr{A} = \{A_0, \ldots, A_m\}$ and $\mathscr{B} = \{B_0, \ldots, B_n\}$ of E respectively F into nonempty basic clopen sets such that $\sigma_0 \in A_0$, $\sigma_1 \in B_0$, $\operatorname{mesh} \mathscr{A} < \varepsilon$ and $\operatorname{mesh} \mathscr{B} < \varepsilon$. We may assume without loss of generality that $m, n \ge 1$. Let $B_0 = [\alpha, \beta]$. Now $\sigma_1 \in [\alpha, \beta] \subseteq F$, hence $\alpha \subseteq \sigma_1$, and so $\alpha = \sigma_1$ since $\varphi(\alpha) \ge \varphi(\sigma_1) = M(F)$.

Since $E \in \mathscr{C}$, $M(A_i) > M(E) = \varphi(\sigma_0)$ for $1 \le i \le m$. Hence by the remarks in §2(A), we may pick nonempty pairwise disjoint finite subsets $\gamma_1, \ldots, \gamma_m$ of $\mathbb{N} \setminus (\alpha \cup \beta)$ such that for every $1 \le i \le m$,

$$M(A_i) - \varphi(\sigma_0) = \varphi(\gamma_i).$$

For $1 \le i \le m$, put $V_i = [\alpha \cup \gamma_i, \beta \cup \bigcup_{j=1}^{i-1} \gamma_j]$. The collection of basic clopen sets $\{V_1, \ldots, V_m\}$ is pairwise disjoint since $\gamma_i \neq \emptyset$ for every i, and clearly $M(V_i) = \varphi(\alpha) + \varphi(\gamma_i) = \varphi(\sigma_1) + \varphi(\gamma_i)$. Hence we conclude that for

$1 \le i \le m$,

$$M(A_i) - \varphi(\sigma_0) = M(V_i) - \varphi(\sigma_1).$$

Observe that $\sigma_1 = \alpha \notin \bigcup_{i=1}^m V_i$, and $V_i \subseteq [\alpha, \beta] = B_0$ for every $1 \le i \le m$.

Conversely, we can find nonempty pairwise disjoint basic clopen sets $U_1, \ldots, U_n$ contained in A_0 such that $\sigma_0 \notin \bigcup_{j=1}^n U_j$, and for $j \in \{1, \ldots, n\}$,

$$M(B_j) - \varphi(\sigma_1) = M(U_j) - \varphi(\sigma_0).$$

Define

$$\begin{aligned} \mathscr{U} &= \{A_1, \ldots, A_m, U_1, \ldots, U_n, A_0 \setminus \textstyle\bigcup_{i=1}^n U_i\}, \\ \mathscr{V} &= \{V_1, \ldots, V_m, B_1, \ldots, B_n, B_0 \setminus \textstyle\bigcup_{j=1}^m V_j\}, \end{aligned}$$

and $\xi\colon \mathscr{U} \to \mathscr{V}$ by

$$\begin{cases} \xi(A_i) = V_i & (1 \le i \le m), \\ \xi(U_j) = B_j & (1 \le j \le n), \\ \xi(A_0 \setminus \bigcup_{i=1}^n U_i) = B_0 \setminus \bigcup_{j=1}^m V_j. \end{cases}$$

Observe that all elements of $\mathscr{U}$ but at most one are nonempty basic clopen sets and hence belong to $\mathscr{C}$. Moreover, $\sigma_0 \in A_0 \setminus \bigcup_{i=1}^n U_i$, which clearly implies that $A_0 \setminus \bigcup_{i=1}^n U_i$ belongs to $\mathscr{C}$ as well. So $\mathscr{U} \subseteq \mathscr{C}$, and similarly $\mathscr{V} \subseteq \mathscr{C}$. It follows that $\mathscr{U}$, $\mathscr{V}$, and ξ are as required. ∎

THEOREM 5.3. *Let $E, F \in \mathscr{C}$. Then there is a homeomorphism $f\colon E \to F$ such that for every $A \in E$ we have $\varphi(A) - M(E) = \varphi(f(A)) - M(F)$.*

Proof. Let $\mathscr{U}_0 = \{E\}$, $\mathscr{V}_0 = \{F\}$, and $\xi_0\colon \mathscr{U}_0 \to \mathscr{V}_0$ be the obvious function. We construct by recursion on $n \ge 1$ clopen partitions $\mathscr{U}_n$ of E and $\mathscr{V}_n$ of F and a bijection $\xi_n\colon \mathscr{U}_n \to \mathscr{V}_n$ such that

(1) $\bigcup_{n\ge 0} \mathscr{U}_n \cup \bigcup_{n\ge 0} \mathscr{V}_n \subseteq \mathscr{C}$,
(2) $\mathscr{U}_n$ refines $\mathscr{U}_{n-1}$ and $\mathscr{V}_n$ refines $\mathscr{V}_{n-1}$,
(3) $\operatorname{mesh} \mathscr{U}_n \le 2^{-n}$,
(4) $\operatorname{mesh} \mathscr{V}_n \le 2^{-n}$,
(5) if $U \in \mathscr{U}_n$ and $U' \in \mathscr{U}_{n-1}$ are such that $U \subseteq U'$, then $\xi_n(U) \subseteq \xi_{n-1}(U')$,
(6) if $U \in \mathscr{U}_n$ and $U' \in \mathscr{U}_{n-1}$ are such that $U \subseteq U'$, then $M(U) - M(U') = M(\xi_n(U)) - M(\xi_{n-1}(U'))$.

Assume that $\mathscr{U}_n$, $\mathscr{V}_n$ and $\xi_n\colon \mathscr{U}_n \to \mathscr{V}_n$ have been constructed for some $n \ge 0$. It is clear that by applying Lemma 5.2 on each pair $(U, \xi_n(U))$ for $U \in \mathscr{U}_n$ separately, we get what we want for $n+1$. This completes the construction.

The sequences $(\mathscr{U}_n)_n$ and $(\mathscr{V}_n)_n$ define a homeomorphism $f\colon E \to F$ in the obvious way:

$$(\forall\, n \ge 0)(\forall U \in \mathscr{U}_n)[A \in U \Leftrightarrow f(A) \in \xi_n(U)].$$

We claim that f is as required. To see this, take an arbitrary element $A \in E$. For each $n \geq 0$, let U_n be the unique element in $\mathscr{U}_n$ containing A. Since φ is LSC, by (3) and Lemma 4.1(c) we clearly have

$$M(E) = M(U_0) \leq M(U_1) \leq \cdots \nearrow \varphi(A),$$

and similarly

$$M(F) = M(\xi_0(U_0)) \leq M(\xi_1(U_1)) \leq \cdots \nearrow \varphi(f(A)).$$

Since by (6) we have

$$M(U_n) - M(U_{n-1}) = M(\xi_n(U_n)) - M(\xi_{n-1}(U_{n-1}))$$

for $n \geq 1$, we are done. ∎

Corollary 5.4. *Let $E \in \mathscr{C}$. Then there is a homeomorphism $f\colon \mathscr{P}(\mathbb{N}) \to E$ such that for every $A \in \mathscr{P}(\mathbb{N})$ we have $\varphi(f(A)) - M(E) = \varphi(A)$.*

So all elements of $\mathscr{C}$ are homeomorphic to the model $\mathscr{P}(\mathbb{N})$ by 'norm preserving' homeomorphisms. See Dijkstra and van Mill [11] for similar results in infinite-dimensional topology.

Remark 5.5. Let f be the homeomorphism in Corollary 5.4, and let $r \in (0, \infty)$. Clearly, $f(U(\emptyset, r)) = U(\emptyset, r + M(E)) \cap E$.

6. The example: part 1. Let Σ denote the collection of all finite subsets of $\mathbb{N}$. Clearly, Σ is dense in $\mathscr{P}(\mathbb{N})$. Since $\mathscr{P}(\mathbb{N})$ has no isolated points, it is clear that we can split Σ into countably many sets that are dense in $\mathscr{P}(\mathbb{N})$, say $\Sigma_0, \Sigma_1, \Sigma_2, \ldots$. Let $\xi\colon \mathbb{N} \to \mathbb{Q}_+$ be an arbitrary bijection; here $\mathbb{Q}_+$ denotes the set of all strictly positive rational numbers. For every n, put

$$S_n = \{A \in \mathscr{P}(\mathbb{N}) : \xi(n) \leq \varphi(A)\}, \qquad T_n = \{A \in \mathscr{P}(\mathbb{N}) : \varphi(A) < \xi(n)\}.$$

Observe that $T_n = U(\emptyset, \xi(n))$ is open in $\mathscr{E}_{\mathrm{s}}$ and an F_σ-subset in $\mathscr{P}(\mathbb{N})$.

Consider the subspace

$$X = ((\mathscr{P}(\mathbb{N}) \setminus \Sigma) \times \mathscr{E}_{\mathrm{w}}) \cup \bigcup_{n \in \mathbb{N}} (\Sigma_n \times T_n)$$

of $\Upsilon = \mathscr{P}(\mathbb{N}) \times \mathscr{P}(\mathbb{N})$. Observe that $\pi_1(X) = \mathscr{P}(\mathbb{N}) \setminus \Sigma_0$ and $\pi_2(X) = \mathscr{E}_{\mathrm{w}}$. Hence $X \subseteq (\mathscr{P}(\mathbb{N}) \setminus \Sigma_0) \times \mathscr{E}_{\mathrm{w}}$.

By a basic clopen subset of X we mean any set of the form $([\sigma_0, \tau_0] \times [\sigma_1, \tau_1]) \cap X$, i.e, a basic clopen subset of Υ restricted to X.

(A) All nonempty basic clopen subsets of X are homeomorphic. Throughout, let σ_0, τ_0 and σ_1, τ_1 be pairs of disjoint finite subsets of $\mathbb{N}$. Our aim is to prove that the basic clopen subset $Y = ([\sigma_0, \tau_0] \times [\sigma_1, \tau_1]) \cap X$ of X is homeomorphic to X by a homeomorphism with 'control'. It will be convenient to introduce the following notation:

$$\delta = \varphi(\sigma_1), \qquad E = \{n \in \mathbb{N} : \xi(n) - \delta > 0\}.$$

Clearly,

$$\{\xi(n) - \delta : n \in E\} = \mathbb{Q}_+,$$

and $(\Sigma_n \times T_n) \cap ([\sigma_0, \tau_0] \times [\sigma_1, \tau_1]) \neq \emptyset$ if and only if $n \in E$.

PROPOSITION 6.1. *There is a product homeomorphism $F = (f, g)\colon \Upsilon \to [\sigma_0, \tau_0] \times [\sigma_1, \tau_1]$ such that*

(1) $F(X) = X \cap ([\sigma_0, \tau_0] \times [\sigma_1, \tau_1])$,
(2) *for every $B \in \mathscr{P}(\mathbb{N})$ we have $\varphi(g(B)) - \varphi(\sigma_1) = \varphi(B)$.*

Proof. Define a bijection $\eta\colon \mathbb{N} \to E$ by $\{\eta(n)\} = \xi^{-1}(\{\xi(n) + \delta\})$. By Theorem 2.1 there is a homeomorphism $f\colon \mathscr{P}(\mathbb{N}) \to [\sigma_0, \tau_0]$ such that

$$f(\Sigma_n) = \begin{cases} \Big(\Sigma_0 \cup \bigcup_{n \in \mathbb{N} \setminus E} \Sigma_n\Big) \cap [\sigma_0, \tau_0] & (n = 0), \\ \Sigma_{\eta(n)} \cap [\sigma_0, \tau_0] & (n \geq 1). \end{cases}$$

In addition, by Corollary 5.4 there is a homeomorphism $g\colon \mathscr{P}(\mathbb{N}) \to [\sigma_1, \tau_1]$ such that for every $B \in \mathscr{P}(\mathbb{N})$ we have $\varphi(g(B)) - \varphi(\sigma_1) = \varphi(B)$.

CLAIM 1. $f(\Sigma) = \Sigma \cap [\sigma_0, \tau_0]$.

Proof. This is trivial since

$$\begin{aligned} f(\Sigma) = f\Big(\Sigma_0 \cup \bigcup_{n \in \mathbb{N}} \Sigma_n\Big) &= \Big(\Sigma_0 \cup \bigcup_{n \in \mathbb{N} \setminus E} \Sigma_n \cup \bigcup_{n \in \mathbb{N}} \Sigma_{\eta(n)}\Big) \cap [\sigma_0, \tau_0] \\ &= \Big(\Sigma_0 \cup \bigcup_{n \in \mathbb{N} \setminus E} \Sigma_n \cup \bigcup_{n \in E} \Sigma_n\Big) \cap [\sigma_0, \tau_0] = \Sigma \cap [\sigma_0, \tau_0]. \ \blacklozenge \end{aligned}$$

CLAIM 2. $g(\mathscr{E}_{\mathrm{w}}) = [\sigma_1, \tau_1] \cap \mathscr{E}_{\mathrm{w}}$.

Proof. This is also trivial since $\varphi(\sigma_1) < \infty$, hence from $\varphi(g(B)) - \varphi(\sigma_1) = \varphi(B)$ we get $\varphi(B) < \infty \Leftrightarrow \varphi(g(B)) < \infty$. ♦

Now define $F\colon \Upsilon \to [\sigma_0, \tau_0] \times [\sigma_1, \tau_1]$ in the obvious way by $F(A, B) = (f(A), g(B))$.

CLAIM 3. $F(X) = X \cap ([\sigma_0, \tau_0] \times [\sigma_1, \tau_1])$.

Proof. Pick an arbitrary element $(A, B) \in X$. Then by Claim 1,

$$A \in \mathscr{P}(\mathbb{N}) \setminus \Sigma \ \Leftrightarrow\ f(A) \in [\sigma_0, \tau_0] \setminus f(\Sigma) = [\sigma_0, \tau_0] \setminus \Sigma,$$

hence by Claim 2,

$$(A, B) \in (\mathscr{P}(\mathbb{N}) \setminus \Sigma) \times \mathscr{E}_{\mathrm{w}} \ \Leftrightarrow\ (f(A), g(B)) \in ([\sigma_0, \tau_0] \setminus \Sigma) \times ([\sigma_1, \tau_1] \cap \mathscr{E}_{\mathrm{w}}).$$

Moreover, for every $n \in \mathbb{N}$ we have, by Remark 5.5 and Lemma 4.1(b),

$$\begin{aligned}
&(A,B) \in \Sigma_n \times T_n\\
&\quad\Leftrightarrow\ (f(A), g(B)) \in f(\Sigma_n) \times g(U(\emptyset, \xi(n)))\\
&\quad\Leftrightarrow\ (f(A) \in \Sigma_{\eta(n)} \cap [\sigma_0, \tau_0]) \,\&\, (g(B) \in U(\emptyset, \xi(n)+\delta) \cap [\sigma_1, \delta_1])\\
&\quad\Leftrightarrow\ (f(A) \in \Sigma_{\eta(n)} \cap [\sigma_0, \tau_0]) \,\&\, (g(B) \in U(\emptyset, \xi(\eta(n))) \cap [\sigma_1, \delta_1])\\
&\quad\Leftrightarrow\ (f(A), f(B)) \in (\Sigma_{\eta(n)} \times T_{\eta(n)}) \cap ([\sigma_0, \tau_0] \times [\sigma_1, \tau_1]).
\end{aligned}$$

Hence we are done since we observed above that $(\Sigma_n \times T_n) \cap ([\sigma_0, \tau_0] \times [\sigma_1, \tau_1]) \neq \emptyset$ if and only if $n \in E$. ♦

Since g satisfies (2) by construction, this completes the proof. ∎

This leads us to the following.

THEOREM 6.2. *Let $[\sigma_0^0, \tau_0^0] \times [\sigma_1^0, \tau_1^0]$ and $[\sigma_0^1, \tau_0^1] \times [\sigma_1^1, \tau_1^1]$ be basic clopen subsets of Υ. Then there is a product homeomorphism $F = (f, g)\colon [\sigma_0^0, \tau_0^0] \times [\sigma_1^0, \tau_1^0] \to [\sigma_0^1, \tau_0^1] \times [\sigma_1^1, \tau_1^1]$ such that*

(1) $F(X \cap ([\sigma_0^0, \tau_0^0] \times [\sigma_1^0, \tau_1^0])) = X \cap ([\sigma_0^1, \tau_0^1] \times [\sigma_1^1, \tau_1^1])$,
(2) *for every $B \in [\sigma_1^0, \tau_1^0]$ we have $\varphi(g(B)) - \varphi(\sigma_1^1) = \varphi(B) - \varphi(\sigma_1^0)$.*

Proof. By Proposition 6.1 there are product homeomorphisms $F_i = (f_i, g_i)\colon \Upsilon \to [\sigma_0^i, \tau_0^i] \times [\sigma_1^i, \tau_1^i]$ for $i = 0, 1$ such that

(3) $F_i(X) = X \cap ([\sigma_0^i, \tau_0^i] \times [\sigma_1^i, \tau_1^i])$,
(4) for every $B \in \mathscr{P}(\mathbb{N})$ we have $\varphi(g_i(B)) - \varphi(\sigma_1^i) = \varphi(B)$.

Define $F = F_1 \circ F_0^{-1}$, and observe that $F = (f_1 \circ f_0^{-1}, g_1 \circ g_0^{-1})\colon [\sigma_0^0, \tau_0^0] \times [\sigma_1^0, \tau_1^0] \to [\sigma_0^1, \tau_0^1] \times [\sigma_1^1, \tau_1^1]$ is a product homeomorphism such that

$$F(X \cap ([\sigma_0^0, \tau_0^0] \times [\sigma_1^0, \tau_1^0])) = F_1(X) = X \cap ([\sigma_0^1, \tau_0^1] \times [\sigma_1^1, \tau_1^1]),$$

and for every $B \in [\sigma_1^0, \tau_1^0]$,

$$\varphi(B) - \varphi(\sigma_1^0) = \varphi(g_1 \circ g_0^{-1}(B)) - \varphi(\sigma_1^1),$$

as required. ∎

(B) Homogeneity properties of X. It will be convenient to adopt the following notation. If $V = [\sigma_0, \tau_0] \times [\sigma_1, \tau_1]$ is a basic clopen subset of Υ, then $\mu(V)$ and $\nu(V)$ denote $\varphi(\sigma_0)$ and $\varphi(\sigma_1)$, respectively.

PROPOSITION 6.3. *For $A \in [\sigma, \tau]$ there are basic clopen sets $[\sigma_0, \tau_0]$, $[\sigma_1, \tau_1], \ldots$ such that*

(1) $[\sigma_0, \tau_0] = [\sigma, \tau]$, *and $[\sigma_n, \tau_n]$ is a proper subset of $[\sigma_{n-1}, \tau_{n-1}]$ for all $n \in \mathbb{N}$,*
(2) $\bigcap_{n \geq 0} [\sigma_n, \tau_n] = \{A\}$,

(3) *for every* $n \geq 0$ *we can write* $[\sigma_n, \tau_n] \setminus [\sigma_{n+1}, \tau_{n+1}]$ *as the disjoint union of basic clopen sets* F_n *and* G_n *such that* $\lim_{n\to\infty} M(F_n) = \varphi(A)$ *and* $\lim_{n\to\infty} M(G_n) = \varphi(A)$.

Proof. Put $\sigma_0 = \sigma$ and $\tau_0 = \tau$. We distinguish three subcases.

CASE 1: $A = \sigma$ or $\mathbb{N} \setminus A = \tau$. Assume first that $A = \sigma$. Enumerate $\mathbb{N} \setminus (A \cup \tau)$ as $\{b_n : n \in \mathbb{N}\}$ such that $b_n < b_m$ if $n < m$. For every $n \in \mathbb{N}$, put

$$\tau_n = \tau \cup \{b_1, \ldots, b_n\}.$$

Observe that $\bigcap_{n\geq 0}[A, \tau_n] = \{A\}$. For $n \geq 0$, $E_n = [A, \tau_n] \setminus [A, \tau_{n+1}] = [A \cup \{b_{n+1}\}, \tau_n]$ is the disjoint union of the basic clopen sets F_n and G_n, where

$$F_n = [A \cup \{b_{n+1}, b_{n+2}\}, \tau_n], \qquad G_n = [A \cup \{b_{n+1}\}, \tau_n \cup \{b_{n+2}\}].$$

Observe that $\lim_{n\to\infty} M(F_n) = \varphi(A)$ and $\lim_{n\to\infty} M(G_n) = \varphi(A)$.

Assume next that $A = \mathbb{N} \setminus \tau$. Then put $B = \tau$, apply what we just proved for B and $[\tau, \sigma]$, and take complements.

CASE 2: *A is finite but $A \neq \sigma$, or $\mathbb{N} \setminus A$ is finite but $\mathbb{N} \setminus A \neq \tau$.* Assume first that A is finite but $A \neq \sigma$. Write $A \setminus \sigma$ as $\{c_1, \ldots, c_N\}$ for some N such that $c_i < c_j$ if $i < j$. Then the collection of basic clopen sets $\{[\sigma, \tau], [\sigma \cup \{c_1\}, \tau], [\sigma \cup \{c_1, c_2\}, \tau], \ldots, [\sigma \cup \{c_1, \ldots, c_N\}, \tau]\}$ is strictly decreasing and $[\sigma \cup \{c_1, \ldots, c_N\}, \tau] = [A, \tau]$. Observe that every ring

$$[\sigma, \tau] \setminus [\sigma \cup \{c_1\}, \tau], \ldots, [\sigma \cup \{c_1, \ldots, c_{N-1}\}, \tau] \setminus [\sigma \cup \{c_1, \ldots, c_N\}, \tau]$$

is a nonempty basic clopen set and hence can be split into two disjoint nonempty basic clopen sets. So we can proceed from $[A, \tau]$ as we did in Case 1. By taking complements, the case that $A \setminus \tau$ is finite but $\mathbb{N} \setminus A \neq \tau$ follows from what we have just proved.

CASE 3: *A and $\mathbb{N} \setminus A$ are infinite.* Enumerate $A \setminus \sigma$ as $\{d_n : n \in \mathbb{N}\}$, where $d_n < d_m$ if $n < m$. In addition, write $\mathbb{N} \setminus (A \cup \tau)$ as $\{e_n : n \in \mathbb{N}\}$, where $e_n < e_m$ if $n < m$. Put

$$\sigma_n = \sigma \cup \{d_1, \ldots, d_n\}, \qquad \tau_n = \tau \cup \{e_1, \ldots, e_n\}.$$

Observe that $\bigcap_{n\geq 0}[\sigma_n, \tau_n] = \{A\}$. Also,

$$[\sigma_n, \tau_n] \setminus [\sigma_{n+1}, \tau_{n+1}] = F_n \cup G_n,$$

where

$$F_n = [\sigma_n, \tau_n \cup \{d_{n+1}\}], \qquad G_n = [\sigma_{n+1} \cup \{e_{n+1}\}, \tau_n].$$

Observe that $F_n \cap G_n = \emptyset$. Moreover, clearly, $\lim_{n\to\infty} M(F_n) = \varphi(A)$ and $\lim_{n\to\infty} M(G_n) = \varphi(A)$. ■

Corollary 6.4. *Let E be a clopen subset of Υ containing (A_0, A_1). Then $E \setminus \{(A_0, A_1)\}$ can be partitioned into basic clopen sets $V_1, V_2, \ldots$ such that $\lim_{n\to\infty} \mu(V_n) = \varphi(A_0)$ and $\lim_{n\to\infty} \nu(V_n) = \varphi(A_1)$.*

Proof. Since E can be partitioned into finitely may basic clopen subsets of Υ, we may assume without loss of generality that E is a basic clopen set itself, say

$$E = [\sigma^0, \tau^0] \times [\sigma^1, \tau^1].$$

By Proposition 6.3 we may pick for $i = 0, 1$ sequences of basic clopen sets $([\sigma^i_n, \tau^i_n])_{n\geq 0}$ such that

(1) $[\sigma^i_0, \tau^i_0] = [\sigma^i, \tau^i]$, and $[\sigma^i_n, \tau^i_n]$ is a proper subset of $[\sigma^i_{n-1}, \tau^i_{n-1}]$ for all $n \in \mathbb{N}$,
(2) $\bigcap_{n\geq 0}[\sigma^i_n, \tau^i_n] = \{A_i\}$,
(3) for every $n \geq 0$ we can write $[\sigma^i_n, \tau^i_n]\setminus[\sigma^i_{n+1}, \tau^i_{n+1}]$ as the disjoint union of basic clopen sets F^i_n and G^i_n such that $\lim_{n\to\infty} M(F^i_n) = \varphi(A_i)$ and $\lim_{n\to\infty} M(G^i_n) = \varphi(A_i)$.

Consider the following sequence of basic clopen sets:

$$V_1 = F^0_0 \times [\sigma^1_0, \tau^1_0], V_2 = G^0_0 \times [\sigma^1_0, \tau^1_0], V_3 = [\sigma^0_1, \tau^0_1] \times F^1_0, V_4 = [\sigma^0_1, \tau^0_1] \times G^1_0,$$

and generally

$$V_{4n-3} = F^0_n \times [\sigma^1_n, \tau^1_n], \quad V_{4n-2} = G^0_n \times [\sigma^1_n, \tau^1_n],$$
$$V_{4n-1} = [\sigma^0_n, \tau^0_n] \times F^1_n, \quad V_{4n} = [\sigma^0_n, \tau^0_n] \times G^1_n,$$

Observe that $[\sigma^0_1, \tau^0_1] \times [\sigma^1_1, \tau^1_1]$ is the complement of $V_1 \cup V_2 \cup V_3 \cup V_4$ in $[\sigma^0, \tau^0] \times [\sigma^1, \tau^1] = [\sigma^0_0, \tau^0_0] \times [\sigma^1_0, \tau^1_0]$, etc. Hence the V's partition $([\sigma^0, \tau^0] \times [\sigma^1, \tau^1]) \setminus \{(A_0, A_1)\}$ and are by (2), (3) and Lemma 4.1(c) as required. ∎

Theorem 6.5. *Let $(A_0, B_0), (A_1, B_1) \in X$. In addition, let E_0 and E_1 be clopen subsets of Υ containing (A_0, B_0) and (A_1, B_1), respectively. Then there are a continuous function $\alpha\colon E_0 \to \mathbb{R}$ and a homeomorphism $h\colon E_0 \to E_1$ such that*

(1) $h(A_0, B_0) = (A_1, B_1)$,
(2) $h(X \cap E_0) = X \cap E_1$,
(3) for $(C, D) \in E_0$ we have $\varphi(D) + \alpha(C, D) = \varphi(D')$, where $h(C, D) = (C', D')$.

Proof. By Corollary 6.4, $E_0 \setminus \{(A_0, B_0)\}$ can be partitioned into basic clopen sets $V_1, V_2, \ldots$ such that $\lim_{n\to\infty} \nu(V_n) = \varphi(B_0)$. Similarly, $E_1 \setminus \{(A_1, B_1)\}$ can be partitioned into basic clopen sets $W_1, W_2, \ldots$ such that $\lim_{n\to\infty} \nu(W_n) = \varphi(B_1)$. By Theorem 6.2 we may pick for every n a product homeomorphism $F_n = (f_n, g_n)\colon V_n \to W_n$ such that

(4) $F_n(X \cap V_n) = X \cap W_n$,
(5) for $(C, D) \in V_n$ we have $\varphi(g_n(D)) - \nu(W_n) = \varphi(D) - \nu(V_n)$.

Define $h\colon E_0 \to E_1$ in the obvious way by $h(A_0, B_0) = (A_1, B_1)$ and $h\restriction V_n = F_n\restriction V_n$ for every n. Then h is a homeomorphism and satisfies (1) and (2). For (3), define $\alpha\colon E_0 \to \mathbb{R}$ by

$$\alpha(C, D) = \begin{cases} \varphi(B_1) - \varphi(B_0) & ((C, D) = (A_0, B_0)), \\ \nu(W_n) - \nu(V_n) & ((C, D) \in V_n). \end{cases}$$

To prove that α is continuous, assume that $(C_i, D_i)_i$ is a sequence in E_0 converging to (A_0, B_0). We may assume without loss of generality that $(C_i, D_i) \in V_{n_i}$ and that $n_i \neq n_j$ for $i \neq j$. Hence, by construction,

$$\alpha(C_i, D_i) = \nu(W_{n_i}) - \nu(V_{n_i}) \to \varphi(B_1) - \varphi(B_0) = \alpha(A_0, B_0).$$

This proves continuity, and since (5) implies (3), the proof is complete. ∎

7. The example: part 2. Consider the space X in §6. This is not the example we are looking for since it is zero-dimensional being a subspace of $\mathscr{P}(\mathbb{N}) \times \mathscr{E}_{\mathrm{w}}$, and not Polish (cf. §4). So the plan is to improve its topology.

(A) The example. We consider X to be a subspace of $\mathscr{P}(\mathbb{N}) \times \mathscr{E}_{\mathrm{s}}$, and denote it by $\mathscr{X}$. We claim that $\mathscr{X}$ is the example we are looking for.

Lemma 7.1. *$\mathscr{X}$ is totally disconnected and Polish.*

Proof. That $\mathscr{X}$ is totally disconnected is clear since both $\mathscr{P}(\mathbb{N})$ and $\mathscr{E}_{\mathrm{s}}$ are. Observe that $X = ((\mathscr{P}(\mathbb{N}) \setminus \Sigma_0) \times \mathscr{E}_{\mathrm{w}}) \setminus \bigcup_{n\in\mathbb{N}}(\Sigma_n \times S_n)$, that Σ is countable, and that for every n, $S_n^* = S_n \cap \mathscr{E}$ is a closed subspace of $\mathscr{E}_{\mathrm{s}}$. Hence $\mathscr{X} = ((\mathscr{P}(\mathbb{N}) \setminus \Sigma_0) \times \mathscr{E}_{\mathrm{s}}) \setminus \bigcup_{n\in\mathbb{N}}(\Sigma_n \times S_n^*)$ is a G_δ-subset of the Polish space $\mathscr{P}(\mathbb{N}) \times \mathscr{E}_{\mathrm{s}}$ and is therefore Polish itself. ∎

(B) $\mathscr{X}$ is not CDH

Theorem 7.2. *There are a first category set A in $\mathscr{X}$ and a countable dense set B in $\mathscr{X}$ such that $f(B) \cap A \neq \emptyset$ for each homeomorphism $f\colon \mathscr{X} \to \mathscr{X}$.*

Proof. Indeed, let $A = \bigcup_{n\in\mathbb{N}}(\Sigma_n \times T_n)$. Observe that for every $\sigma \in \Sigma$ the set $\{\sigma\} \times \mathscr{E}_{\mathrm{s}}$ is a C-set in $\mathscr{P}(\mathbb{N}) \times \mathscr{E}_{\mathrm{s}}$ since $\mathscr{P}(\mathbb{N})$ is zero-dimensional, hence A is a countable union of (relative) C-sets in $\mathscr{X}$. Moreover, A is a first category subset of $\mathscr{X}$.

There clearly is a sequence $(\sigma_i)_i$ of finite subsets of $\mathbb{N}$ and a sequence of natural numbers $n_1 < n_2 < \cdots$ such that for every i,

(1) $\min \sigma_i > i$, $\sigma_i \in \Sigma_{n_i}$,
(2) $\lim_{i\to\infty} \xi(n_i) = 0$.

By (1) it follows that the sequence $(\sigma_i)_i$ converges to $\emptyset$ in $\mathscr{P}(\mathbb{N})$. Consequently, the sequence of 'vertical segments'

$$\{\sigma_i\} \times T_{n_i} = \{\sigma_i\} \times U(\emptyset, \xi(n_i))$$

converges to $(\emptyset, \emptyset)$ in $\mathscr{P}(\mathbb{N}) \times \mathscr{E}_{\mathrm{s}}$, that is, every neighborhood of $(\emptyset, \emptyset)$ in $\mathscr{P}(\mathbb{N}) \times \mathscr{E}_{\mathrm{s}}$ contains all but finitely many of its terms. Now put

$$B = \bigcup_{i=1}^{\infty} \{\sigma_i\} \times F(\emptyset, \xi(n_i)).$$

We claim that A and B are as required. Clearly, B is dense.

Striving for a contradiction, assume that $f\colon \mathscr{X} \to \mathscr{X}$ is a homeomorphism that 'frees' B from A, i.e., $f(B) \cap A = \emptyset$. Pick $t \in (0, \infty)$ so large that

$$\varphi(\pi_2(f(\emptyset, \emptyset))) < t.$$

Put $K_i = f(\{\sigma_i\} \times U(\emptyset, \xi(n_i))$ for every i. Since the set $\mathscr{P}(\mathbb{N}) \times U(\emptyset, t)$ is open in $\mathscr{P}(\mathbb{N}) \times \mathscr{E}_{\mathrm{s}}$, by the above remarks there exists i such that $K_i \subseteq \mathscr{P}(\mathbb{N}) \times U(\emptyset, t)$. Observe that since A is the union of a countable collection of C-sets in $\mathscr{X}$, $K_i \cap A$ is the union of a countable collection of C-sets in K_i which misses $f(\{\sigma_i\} \times F(\emptyset, \xi(n_i)))$. Hence $K_i \setminus A$ is nowhere zero-dimensional by Proposition 4.3. But this contradicts Proposition 4.4 since $K_i \setminus A$ is a 'bounded' closed subset of $(\mathscr{P}(\mathbb{N}) \setminus \Sigma) \times \mathscr{E}_{\mathrm{s}}$. ■

This clearly implies that $\mathscr{X}$ is not CDH. Since $\mathscr{X}$ is Polish, there is a countable dense subset D of $\mathscr{X}$ such that $D \cap A = \emptyset$. Hence by Theorem 7.2 there does not exist a homeomorphism $f\colon \mathscr{X} \to \mathscr{X}$ such that $f(B) = D$.

(C) $\mathscr{X}$ is strongly n-homogeneous for every n. It will be convenient to introduce the following notation. If $g \in \mathscr{H}(\Upsilon)$ and $(A, B) \in \Upsilon$, then A_g and B_g denote $\pi_1(g(A, B))$ and $\pi_2(g(A, B))$, respectively. Hence $g(A, B) = (A_g, B_g)$.

Let G denote the collection of all elements $g \in \mathscr{H}(\Upsilon)$ such that

(1) $g(X) = X$,
(2) there is a continuous function $\alpha(g)\colon \Upsilon \to \mathbb{R}$ such that for every $(A, B) \in \Upsilon$ we have

$$\varphi(B) + \alpha(g)(A, B) = \varphi(B_g). \qquad (\dagger)$$

Lemma 7.3. *For $g \in G$, the function $\alpha(g)$ in ($\dagger$) is unique.*

Proof. Suppose that there are continuous functions $\alpha, \beta\colon \mathscr{P}(\mathbb{N}) \to \mathbb{R}$ such that for every $(A, B) \in \mathscr{P}(\mathbb{N})$ we have $\varphi(B) + \alpha(A, B) = \varphi(B_g)$ and $\varphi(B) + \beta(A, B) = \varphi(B_g)$. Pick arbitrary finite sets $\sigma, \tau \subseteq \mathbb{N}$. As $\varphi(\tau) < \infty$ and α is bounded since $\mathscr{P}(\mathbb{N})$ is compact, $\varphi(\tau_g) = \varphi(\tau) + \alpha(\sigma, \tau) < \infty$. Similarly, $\varphi(\tau_g) = \varphi(\tau) + \beta(\sigma, \tau) < \infty$. Hence $\alpha(\sigma, \tau) = \beta(\sigma, \tau)$. Since $\{(\sigma, \tau) : \sigma, \tau \in [\mathbb{N}]^{<\omega}\}$ is dense in Υ, we get $\alpha = \beta$. ■

This result has some interesting consequences.

Corollary 7.4. *If $g, h \in G$, then $g \circ h \in G$ and $\alpha(g \circ h) = \alpha(h) + \alpha(g) \circ h$. Moreover, if $g \in G$, then $g^{-1} \in G$ and $\alpha(g^{-1}) = -\alpha(g) \circ g^{-1}$.*

Proof. Simply observe that for all $(A, B) \in \Upsilon$ we have $\varphi(B) + \alpha(h)(A, B) = \varphi(B_h)$ and $\varphi(B_h) + \alpha(g)(A_h, B_h) = \varphi((B_h)_g) = \varphi(B_{g \circ h})$. This gives

$$\varphi(B) + \alpha(h)(A, B) + \alpha(g)(A_h, B_h) = \varphi(B_{g \circ h}).$$

Moreover, for every $(A, B) \in \Upsilon$ we have $\varphi(B_{g^{-1}}) + \alpha(g)(A_{g^{-1}}, B_{g^{-1}}) = \varphi((B_{g^{-1}})_g) = \varphi(B)$. Hence we are done by Lemma 7.3. ∎

From this we conclude that G is a subgroup of $\mathscr{H}(\Upsilon)$.

We will now show that $\mathscr{X}$ is 'very' homogeneous.

Proposition 7.5. *G makes $\mathscr{X}$ strongly n-homogeneous for every n.*

Proof. Let $F \subseteq \mathscr{X}$ be finite, and pick arbitrary distinct (A_0, B_0) and (A_1, B_1) in $\mathscr{X} \setminus F$. There are disjoint clopen neighborhoods E_0 and E_1 of (A_0, B_0) respectively (A_1, B_1) in Υ such that $F \cap (E_0 \cup E_1) = \emptyset$. By Theorem 6.5, there are a continuous function $\alpha\colon E_0 \to \mathbb{R}$ and a homeomorphism $h\colon E_0 \to E_1$ such that

(1) $h(A_0, B_0) = (A_1, B_1)$,
(2) $h(X \cap E_0) = X \cap E_1$,
(3) for $(C, D) \in E_0$ we have $\varphi(D) + \alpha(C, D) = \varphi(D_h)$.

Now define $f\colon \Upsilon \to \Upsilon$ by

$$f(C, D) = \begin{cases} (C, D) & ((C, D) \notin E_0 \cup E_1), \\ h(C, D) & ((C, D) \in E_0), \\ h^{-1}(C, D) & ((C, D) \in E_1). \end{cases}$$

Then f is a homeomorphism, belongs to G, restricts to the identity on F, and $f(A_0, B_0) = (A_1, B_1)$. Hence we are done by Lemma 2.2. ∎

Hence $\mathscr{X}$ is indeed as promised in Example 1.3.

In §8(C) below we will show that G can be given the structure of a (separable metrizable) topological group that acts on $\mathscr{X}$ and hence by Proposition 7.5 makes $\mathscr{X}$ strongly n-homogeneous for every n.

8. Appendix

(A) The space $\mathscr{X}$. In the construction of $\mathscr{X}$ we applied, among other things, ideas and results that can be found in [32] and in the papers [13, 10] on Erdős spaces. The example in [32] is 1-dimensional, Polish, homogeneous but not 2-homogeneous, and has the curious property that it does not admit a transitive action by a (separable metrizable) topological group. The reason for this is that its components are wildly distributed. Since we aimed at a space here with stronger homogeneity properties, namely a space which is strongly n-homogeneous for every n, we had to give up nontrivial components. As a result, we win some and lose some: our example is not as 'bad' as the space in [32] since it admits, as we will show below, a transitive action by a (separable metrizable) topological group, but it does not admit such an action by a Polish group.

The space $\mathscr{X}$ is totally disconnected and 1-dimensional. From dimension-theoretic perspective this is best possible. Let us indicate that a similar space of dimension zero does not exist. It cannot be locally compact by the implication (c)⇒(a) in Theorem 1.1 (which is true without connectivity assumptions). Hence it is nowhere locally compact by homogeneity. But it is Polish, so if it is zero-dimensional then it is homeomorphic to the space of irrational numbers, which is known to be CDH. For details, see [29, Chapter 1].

(B) Erdős spaces. In 1940 Erdős [20] proved that the 'rational Hilbert space' $\mathfrak{E}$, which consists of all vectors in the real Hilbert space ℓ^2 that have only rational coordinates, has dimension one and is homeomorphic to its own square. This answered a question of Hurewicz who proved in [26] that for every compact space X and every 1-dimensional space Y we have $\dim(X \times Y) = \dim X + 1$ (see also [19, 1.9.E(b)]).

It is not difficult to prove that $\mathfrak{E}$ has dimension at most 1. Erdős [20] proved the surprising fact that every nonempty clopen subset of $\mathfrak{E}$ is unbounded, and hence that for no $x \in \mathfrak{E}$ does the open ball $\{y \in \mathfrak{E} : \|x - y\| < 1\}$ contain a nonempty clopen subset of $\mathfrak{E}$. This implies among other things that $\mathfrak{E}$ is nowhere zero-dimensional.

Erdős [20] also proved that the closed subspace $\mathfrak{E}_c$ of ℓ^2 consisting of all vectors such that every coordinate is in the convergent sequence $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ has the same property. The space $\mathfrak{E}_c$ is called the *complete Erdős space* and was shown in Dijkstra [10, Theorem 3] to be homeomorphic to the 'irrational' Hilbert space, which consists of all vectors in the real Hilbert space ℓ^2 that have only irrational coordinates. All nonempty clopen subsets of $\mathfrak{E}_c$ are unbounded just as the nonempty clopen subsets of $\mathfrak{E}$ are. In Proposition 4.3 we used Erdős' method to prove a stronger result that was crucial later.

Both $\mathfrak{E}$ and $\mathfrak{E}_c$ are totally disconnected. Metric and topological characterizations of $\mathfrak{E}_c$ can be found in Kawamura, Tymchatyn and Oversteegen [27] and Dijkstra and van Mill [12]. The space $\mathfrak{E}_c$ surfaces at many places. For example, as the set of endpoints of certain dendroids (among them, the Lelek fan), the set of endpoints of the Julia set of the exponential map, the set of endpoints of the separable universal $\mathbb{R}$-tree, line-free groups in Banach spaces and Polishable ideals on $\mathbb{N}$ (for details, see e.g. [27, 15, 16, 10, 12]). The space

$\mathfrak{E}$ surfaces e.g. as the homeomorphism group of a topological n-manifold with $n \geq 2$ consisting of those homeomorphisms that keep a given countable dense set fixed (Dijkstra and van Mill [13]).

The so-called *harmonic Erdős space* $\mathscr{E}_{\mathrm{s}}$, which we discussed in §4, was first identified by Dijkstra [10]. It is homeomorphic to the complete Erdős space $\mathfrak{E}_{\mathrm{c}}$ as well as to the irrational Erdős space E (Dijkstra [10]). The harmonic Erdős space turned out to be a particularly elegant model of $\mathfrak{E}_{\mathrm{c}}$ that is easy to work with.

Example 1.3 is inspired by the example in [32] with the role of the interval $(0, 1)$ taken by $\mathscr{E}_{\mathrm{s}}$.

(C) Some properties of the complete Erdős space. It was stated in [27] that $\mathfrak{E}_{\mathrm{c}}$ is CDH. It is however not SLH, as observed by Dijkstra. In fact, it is one of the very few known Polish CDH-spaces that are not SLH. The argument is simple. Indeed, let f be a homeomorphism of $\mathfrak{E}_{\mathrm{c}}$ that is supported on a nonempty bounded open set, say U. We claim that f is the identity. Indeed, pick distinct $x, y \in U$ such that $f(x) = y$. There are disjoint clopen subsets C_1 and C_2 of $\mathfrak{E}_{\mathrm{c}}$ such that $x \in C_1$ and $y \in C_2$. Then $f(C_1) \cap C_2$ is clopen, nonempty since it contains y, and contained in U. But this contradicts Erdős' result from [20]. See [9, Proposition 6.9] for a more general result. For a connected Polish CDH-space that is not SLH, see [31].

It is an open problem whether every compact CDH-space is SLH. Kennedy Phelps [28] proved that if a continuum is 2-homogeneous, and has a nontrivial homeomorphism that restricts to the identity on some nonempty open set, then it is SLH. Hence a CDH-continuum X with such a homeomorphism is SLH. Simply observe that X, being homogeneous by Bennett's result quoted above, has no cutpoints and consequently is 2-homogeneous by Theorem 1.2.

If we compare the harmonic Erdős space with the irrational Erdős space, then these are homeomorphic spaces but there are notable metric differences. For example, it is not difficult to show that for a fixed $t \in (0, \infty)$ the sphere

$$\{x \in E : \|x\| = t\}$$

has no isolated points. But if we consider for a fixed rational number $t \in (0, \infty)$ the corresponding 'sphere'

$$S = \{A \in \mathscr{E}_{\mathrm{s}} : \varphi(A) = t\},$$

then it has a dense set of isolated points. Simply consider the collection of all finite sets σ in $\mathbb{N}$ such that $\varphi(\sigma) = t$. We claim that this collection is dense in S and consists of isolated points of S. To prove this, take an arbitrary element $A \in S$, and let $\varepsilon > 0$. There is a finite subset $B \subseteq A$ such that $t - \delta < \varepsilon/2$, where $\delta = \varphi(B)$. Pick a finite set $C \subseteq \mathbb{N} \setminus B$ such that $\varphi(C) = t - \delta$ (see the remarks about Egyptian fractions in §2). Then $B \cup C$

is finite, $\varphi(B \cup C) = t$, and

$$\varrho(A, B \cup C) \leq \varphi(A \setminus B) + \varphi(C) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Moreover, $[B \cup C, \emptyset]$ is a clopen neighborhood of $B \cup C$ in $\mathscr{P}(\mathbb{N})$ and has the property that $[B \cup C, \emptyset] \cap S = \{B \cup C\}$.

Every point in $\mathscr{E}_{\mathrm{s}}$ has a neighborhood base consisting of sets that are compact in $\mathscr{P}(\mathbb{N})$. Indeed, we claim that for $A \in \mathscr{E}_{\mathrm{s}}$ and $t \in [0, \infty)$ the closed ball $B(A, t)$ in $\mathscr{E}_{\mathrm{s}}$ is closed in $\mathscr{P}(\mathbb{N})$. Take an arbitrary $B \in \mathscr{P}(\mathbb{N})$ such that $\varphi(A \triangle B) > t$. There are finite subsets σ of $B \setminus A$ and τ of $A \setminus B$ such that $\varphi(\sigma) + \varphi(\tau) > t$. Then $[\sigma, \tau]$ is a clopen neighborhood of B that misses $B(A, t)$, i.e., $\mathscr{P}(\mathbb{N}) \setminus B(A, t)$ is open.

This has nice consequences. Let $A \subseteq \mathscr{E}_{\mathrm{s}}$ be closed. Then $\mathscr{E}_{\mathrm{s}} \setminus A$ can be covered by a countable family consisting of closed balls that are compact in $\mathscr{P}(\mathbb{N})$. This means that A is a G_δ-subset of $\mathscr{E}_{\mathrm{w}}$. So if A is 'bounded' in the sense that $A \subseteq B(\emptyset, r)$ for some $r \in (0, \infty)$, then A_{w} is a G_δ-subset of a compact subset of $\mathscr{P}(\mathbb{N})$ and hence is Polish. For this observation, and many similar ones, see [13].

This bitopological aspect of Erdős spaces was captured by Oversteegen and Tymchatyn [34] by the introduction of the class of almost zero-dimensional spaces of which $\mathfrak{E}_{\mathrm{c}}$ is a universal element. It was crucial in obtaining the topological characterizations of Erdős spaces in [13] and [12].

(D) Actions on $\mathscr{X}$. Let X be a compact space. It is well-known, and easy to prove, that $\mathscr{H}(X)$ endowed with the compact-open topology is a Polish group and that the natural action $\mathscr{H}(X) \times X \to X$ is continuous. Hence if X is a homogeneous compact space then it admits a Polish group acting transitively on it. By using one-point compactifications, the same holds for locally compact homogeneous spaces. For details, see [29, §1.3].

For a compact space X we let $C(X)$ denote the Banach space of all real-valued continuous functions on X with the standard supremum norm. Let $a\colon G \times X \to X$ be a continuous action. Define $\bar{a}\colon G \times C(X) \to C(X)$ by $\bar{a}(g, f) = f \circ g$. In the proof of Proposition 8.1 below we will need the well-known and easy to prove fact that $\bar{a}$ defines a continuous action of G on $C(X)$. We will present the (standard) proof for the sake of completeness.

Assume that $(g_n)_n$ is a sequence in G converging to an element $g \in G$, and let $(f_n)_n$ be a sequence in $C(X)$ converging to an element $f \in C(X)$. We have to prove that $(f_n \circ g_n)_n$ converges to $f \circ g$. Pick $\varepsilon > 0$. There is an element $N_1 \in \mathbb{N}$ such that $\|f - f_n\| < \varepsilon/6$ for every $n \geq N_1$. By compactness of X, there is also a neighborhood U of e in G such that for every $x \in X$ and $h \in U$ we have $|f(x) - f(hx)| < \varepsilon/2$. Pick $N_2 \in \mathbb{N}$ such that $g_n g^{-1} \in U$ for every $n \geq N_2$. Now if $x \in X$ is arbitrary, $n \geq N_1$ and $h \in U$, then $|f_n(x) - f_n(hx)| \leq |f_n(x) - f(x)| + |f(x) - f(hx)| + |f(hx) - f_n(hx)| <$

$\varepsilon/6+\varepsilon/2+\varepsilon/6 = 5\varepsilon/6$. Take an arbitrary $x \in X$, let $n \geq N = \max(N_1, N_2)$, and put $y = gx$. We then have $|f(gx) - f_n(g_n x)| = |f(y) - f_n(g_n g^{-1} y)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(g_n g^{-1} y)| < \varepsilon/6 + 5\varepsilon/6 = \varepsilon$. So for $n \geq N$ we have $\|f \circ g - f_n \circ g_n\| = \sup_{x\in X} |(f \circ g)(x) - (f_n \circ g_n)(x)| = \sup_{x\in X} |f(gx) - f_n(g_n x)| \leq \varepsilon$, as required.

Now consider the subgroup G of $\mathscr{H}(\Upsilon)$ that was defined in the previous section. The function $\alpha\colon G \to C(\Upsilon)$ induces a function $e\colon G \to \mathscr{H}(\Upsilon)\times C(\Upsilon)$ in the obvious way by $e(g) = (g, \alpha(g))$. We topologize G by the subspace topology τ that $e(G)$ inherits from $\mathscr{H}(\Upsilon) \times C(\Upsilon)$. That is, a sequence $(g_n)_n$ in G converges to an element $g \in G$ in the topology τ if and only if $(g_n)_n$ converges to g in $\mathscr{H}(\Upsilon)$ and $(\alpha(g_n))_n$ converges to $\alpha(g)$ in the Banach space $C(\Upsilon)$. Observe that (G, τ) is separable and metrizable.

PROPOSITION 8.1. *(G, τ) is a topological group.*

Proof. Let $(g_n)_n$ and $(h_n)_n$ be sequences in (G, τ) converging to g respectively h. Then $g_n \to g$ and $h_n \to h$ in $\mathscr{H}(\Upsilon)$ and hence $g_n \circ h_n \to g \circ h$ in $\mathscr{H}(\Upsilon)$. By the above remarks, the function $\mathscr{H}(\Upsilon)\times C(\Upsilon) \to C(\Upsilon)$ defined by $(g, f) \mapsto f \circ g$ is a continuous action of $\mathscr{H}(\Upsilon)$ on $C(\Upsilon)$. Since $\alpha(h_n) \to \alpha(h)$ and $\alpha(g_n) \to \alpha(g)$ in $C(\Upsilon)$, we deduce by Corollary 7.4 that

$$\alpha(g_n \circ h_n) = \alpha(h_n) + \alpha(g_n) \circ h_n \to \alpha(h) + \alpha(g) \circ h = \alpha(g \circ h)$$

in $C(\Upsilon)$. We therefore conclude that $g_n \circ h_n \to g \circ h$ in (G, τ). That $g_n^{-1} \to g^{-1}$ in (G, τ) follows by a similar reasoning. ∎

It is clear that G acts on X by the natural action a defined by $(x, g) \mapsto g(x)$. We claim that a remains continuous if we strengthen the topologies on both X and G.

PROPOSITION 8.2. *$a\colon (G, \tau) \times \mathscr{X} \to \mathscr{X}$ is continuous.*

Proof. Assume that $(g_n)_n$ is a sequence in (G, τ) converging to g, and let $(A_n, B_n)_n$ be a sequence in $\mathscr{X}$ converging to (A, B). Then $(g_n)_n$ converges to g in G and $(A_n, B_n)_n$ converges to (A, B) in X. Hence $(g_n(A_n, B_n))_n$ converges to $g(A, B)$ in X. Moreover, $(\alpha(g_n))_n$ converges to $\alpha(g)$ in $C(\Upsilon)$. Hence $(\alpha(g_n)(A_n, B_n))_n$ converges to $\alpha(g)(A, B)$. In addition, $\varphi(B_n) \to \varphi(B)$. Since $\varphi(B_n) + \alpha(g_n)(A_n, B_n) = \varphi((B_n)_{g_n})$ for every n, we see that $\varphi((B_n)_{g_n}) \to \varphi(B) + \alpha(g)(A, B) = \varphi(B_g)$. Therefore $\varphi(g_n(A_n, B_n)) \to \varphi(g(A, B))$, which means that $g_n(A_n, B_n) \to g(A, B)$ in $\mathscr{X}$. ∎

Hence by Proposition 7.5 it follows that the topological group (G, τ) makes $\mathscr{X}$ strongly n-homogeneous for every n. By Theorem 7.2 and [33, Proposition 3.4] it follows that there does not exist a Baire group with an action on $\mathscr{X}$ which is both continuous and transitive. In particular, there is no such Polish group.

By the previous remark, $\mathscr{X}$ is not a topological group. It was shown in [15] that $\mathfrak{E}_c$ *is* a topological group. We saw above that $\mathfrak{E}_c$ is CDH in contrast with $\mathscr{X}$ (Theorem 7.2). Hence there are at least two simple topological properties that distinguish between $\mathfrak{E}_c$ and $\mathscr{X}$.

References

[1] F. D. Ancel, *An alternative proof and applications of a theorem of E. G. Effros*, Michigan Math. J. 34 (1987), 39–55.
[2] R. Bennett, *Countable dense homogeneous spaces*, Fund. Math. 74 (1972), 189–194.
[3] C. Bessaga and A. Pełczyński, *The estimated extension theorem, homogeneous collections and skeletons, and their applications to the topological classification of linear metric spaces and convex sets*, Fund. Math. 69 (1970), 153–190.
[4] —, —, *Selected Topics in Infinite-Dimensional Topology*, Monograf. Mat. 58, PWN–Polish Sci. Publ., Warszawa, 1975.
[5] L. E. J. Brouwer, *Some remarks on the coherence type* η, Proc. Akad. Amsterdam 15 (1913), 1256–1263.
[6] W. D. Bula and L. G. Oversteegen, *A characterization of smooth Cantor bouquets*, Proc. Amer. Math. Soc. 108 (1990), 529–534.
[7] P. Campbell, *A practical approach to Egyptian fractions*, J. Recreational Math. 10 (1977–78), 81–86.
[8] W. J. Charatonik, *The Lelek fan is unique*, Houston J. Math. 15 (1989), 27–34.
[9] J. J. Dijkstra, *Characterizing stable complete Erdős space*, Israel J. Math., to appear.
[10] —, *On homeomorphism groups of Menger continua*, Trans. Amer. Math. Soc. 357 (2005), 2665–2679.
[11] J. J. Dijkstra and J. van Mill, *Topological equivalence of discontinuous norms*, Israel J. Math. 128 (2002), 177–196.
[12] —, —, *Characterizing complete Erdős space*, Canad. J. Math. 61 (2009), 124–140.
[13] —, —, *Erdős space and homeomorphism groups of manifolds*, Mem. Amer. Math. Soc. 208 (2010), no. 979.
[14] J. J. Dijkstra, J. van Mill, and J. Steprāns, *Complete Erdős space is unstable*, Math. Proc. Cambridge Philos. Soc. 137 (2004), 465–473.
[15] T. Dobrowolski and J. Grabowski, *Subgroups of Hilbert spaces*, Math. Z. 211 (1992), 657–669.
[16] T. Dobrowolski, J. Grabowski, and K. Kawamura, *Topological type of weakly closed subgroups in Banach spaces*, Studia Math. 118 (1996), 49–62.
[17] E. G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. 81 (1965), 38–55.
[18] R. Engelking, *General Topology*, 2nd ed., Heldermann, Berlin, 1989.
[19] —, *Theory of Dimensions Finite and Infinite*, Heldermann, Lemgo, 1995.
[20] P. Erdős, *The dimension of the rational points in Hilbert space*, Ann. of Math. 41 (1940), 734–736.
[21] I. Farah, M. Hrušák, and C. X. Martínez Ranero, *A countable dense homogeneous set of reals of size* $\aleph_1$, Fund. Math. 186 (2005), 71–77.
[22] M. K. Fort Jr., , *Homogeneity of infinite products of manifolds with boundary*, Pacific J. Math. 12 (1962), 879–884.
[23] M. Fréchet, *Les dimensions d'un ensemble abstrait*, Math. Ann. 68 (1910), 145–168.

[24] P. Hoffman, *The Man Who Loved Only Numbers, The Story of Paul Erdős and the Search for Mathematical Truth*, Hyperion Books, New York, 1998.

[25] M. Hrušák and B. Zamora Avilés, *Countable dense homogeneity of definable spaces*, Proc. Amer. Math. Soc. 133 (2005), 3429–3435.

[26] W. Hurewicz, *Sur la dimension des produits Cartésiens*, Ann. of Math. 36 (1935), 194–197.

[27] K. Kawamura, L. G. Oversteegen, and E. D. Tymchatyn, *On homogeneous totally disconnected 1-dimensional spaces*, Fund. Math. 150 (1996), 97–112.

[28] J. Kennedy Phelps, *A condition under which 2-homogeneity and representability are the same in continua*, ibid. 121 (1984), 89–98.

[29] J. van Mill, *The Infinite-Dimensional Topology of Function Spaces*, North-Holland, Amsterdam, 2001.

[30] —, *A note on the Effros Theorem*, Amer. Math. Monthly 111 (2004), 801–806.

[31] —, *A countable dense homogeneous space with a dense rigid open subspace*, Fund. Math. 201 (2008), 91–98.

[32] —, *Homogeneous spaces and transitive actions by Polish groups*, Israel J. Math. 165 (2008), 133–159.

[33] —, *Analytic groups and pushing small sets apart*, Trans. Amer. Math. Soc. 361 (2009), 5417–5434.

[34] L. G. Oversteegen and E. D. Tymchatyn, *On the dimension of certain totally disconnected spaces*, Proc. Amer. Math. Soc. 122 (1994), 885–891.

[35] G. S. Ungar, *On all kinds of homogeneous spaces*, Trans. Amer. Math. Soc. 212 (1975), 393–400.

[36] —, *Countable dense homogeneity and n-homogeneity*, Fund. Math. 99 (1978), 155–160.

Jan van Mill
Department of Mathematics
Faculty of Sciences
VU University Amsterdam
De Boelelaan 1081[a]
1081 HV Amsterdam, The Netherlands
E-mail: j.van.mill@vu.nl

Received 10 April 2010;
in revised form 26 August 2011