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Thorn orthogonality and domination in unstable theories

by

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Abstract. We study orthogonality, domination, weight, regular and minimal types in the contexts of rosy and super-rosy theories.

1. Introduction and preliminaries. There are several questions that motivated this research. First, it is natural to extend the concepts of domination, regularity and weight to rosy theories (as has already been done in the simple unstable context). One reason for doing this is "decomposition" theorems: one would like to analyze an arbitrary type in terms of types that can be studied and classified more easily: regular (admitting a pregeometry), minimal, etc. We prove several results of this kind. These provide a complementary picture to the recent work of Assaf Hasson and the first author [4] where minimal types in super-rosy theories are investigated. For example, the two articles combined throw some light on types in theories interpretable in o-minimal structures.

Another motivation came from our desire to understand and develop the concept of *strong dependence* [9]. It has recently become clear that this notion is strongly connected to weight. In [11] the second author shows that every strongly dependent type has rudimentarily finite generically stable weight. Hence a stable theory is strongly dependent precisely when every type has finite weight. The latter conclusion has also been observed by Adler [1], who studied the notion of "burden", which generalizes weight and makes sense in any theory. A related concept (within the context of dependent theories) is investigated by the authors in [7]. A natural question is: given a dependent theory with a good enough independence relation, does strong dependence always imply finite "weight"? More precisely, is thorn-weight finite in a strongly dependent rosy theory? We give a positive answer to this question.

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Several directions pursued in this paper require a delicate analysis of existence of mutually indiscernible (sometimes þ-Morley) sequences. Claims of this form are proved in Section 2.

The paper is organized as follows:

We start by defining notions related to forking and þ-forking, quoting some of the relevant results and proving others that will be needed throughout the paper.

Most of the paper is devoted to understanding þ-orthogonality and the role of þ-weight-1, þ-regular, and þ-minimal types in rosy, super-rosy, and finite $U^{þ}$-rank structures. We show many results analogous to those in stable (and simple) theories, and conclude with a strong decomposition theorem for types of finite rank in rosy theories. As already mentioned, this result suggests that analysis of minimal types (as is done e.g. in [4]) leads to understanding of all types in a rosy theory of finite rank (e.g., a theory interpretable in an o-minimal structure).

Section 2 defines þ-weight and relates it to existence of mutually indiscernible þ-Morley sequences, which helps us understand þ-weight better in strongly dependent and, more generally, strong (rosy) theories.

Section 3 gives proofs of certain basic results on thorn-weight, thorn-domination and regularity. Many of these proofs follow the lines of classical ones, but we still go through them carefully, and where the proofs diverge, we give alternative proofs for the þ-forking context or explain how to bridge the gaps. In this section we also show that every type in a strong rosy theory has finite thorn-weight.

Section 4 is devoted to decomposition results of an arbitrary type to "geometric" objects—þ-regular types in super-rosy theories and þ-minimal types in theories of finite $U^{þ}$-rank.

We have recently learnt that Hans Adler has also given (in an unpublished note) a proof of the fact that in a rosy theory, rudimentarily finite þ-weight implies finite þ-weight (Theorem 3.14). Both his and our proofs of this fact are mostly based on Wagner's argument [12] for simple theories, which is itself a generalization of Hyttinen's results [5] in the stable context.

In contrast, the analysis of finite rank theories in Section 4 is not close to the existing proofs for stable and simple theories. Several useful technical tools applicable in this and related contexts are developed, the main one being Proposition 4.6. We believe that these tools should have many applications. Sharper results related to coordinatization can be found in a subsequent work [2].

1.1. Notations and assumptions. Given a theory T, we will work inside its monster model denoted by $\mathfrak{C}$. By "monster" we mean that all cardinals we mention are "small" (i.e. smaller than the saturation of $\mathfrak{C}$), all sets are small subsets of $\mathfrak{C}$, all models are small elementary submodels of $\mathfrak{C}$, and truth values of all formulae and all types are calculated in $\mathfrak{C}$. We denote tuples (finite unless said otherwise) by lower case letters a, b, c etc., sets by A, B, C etc., models by M, N etc.

By $a \equiv_A b$ we mean $\text{tp}(a/A) = \text{tp}(b/A)$. Recall that this is equivalent to having $\sigma \in \text{Aut}(\mathfrak{C}/A)$ satisfying $\sigma(a) = b$.

Given an order type O, a sequence $I = \langle a_i : i \in O \rangle$ and $j \in O$, we often denote the set $\{a_i \colon i < j\}$ by $a_{<j}$. Similarly for $a_{\leq j}, a_{>j}$ etc. We also often identify the sequence I with the set $\bigcup I$; that is, when no confusion can arise we write $\text{tp}(a/I)$ etc.

We will write $a \mathop{\smile\!\!\!\!\!\!\;|}_A B$ for "$\text{tp}(a/AB)$ does not fork over A" even if T is not simple. Although non-forking is generally not an independence relation, we still find this notation convenient.

For simplicity we assume $T = T^{\text{eq}}$ for all theories T mentioned in this paper.

1.2. þ-forking. Since the paper deals with þ-forking and its properties, we will now define the basic concepts related to this notion. The following definitions and facts can be found in [6].

DEFINITION 1.1. Let $\varphi(x, y)$ be a formula, b be a tuple and C be any set.

- $\varphi(x, b)$ *strongly divides over* D if b is not algebraic over D and the set
$$\{\varphi(x, b')\}_{b' \models \text{tp}(b/D)}$$
is k-inconsistent for some $k \in \mathbb{N}$.
- $\varphi(\bar{x}, b)$ *þ-divides over* C if there is some $D \supset C$ such that $\varphi(x, b)$ strongly divides over D.
- $\varphi(x, b)$ *þ-forks over* C if there are finitely many formulae $\psi_1(x, b_1), \ldots,$ $\psi_n(x, b_n)$ such that $\varphi(x, b) \vdash \bigvee_i \psi_i(x, b_i)$ and $\psi_i(x, b_i)$ þ-divides over C for $1 \leq i \leq n$.

We define a theory to be *rosy* if it does not admit arbitrarily long þ-forking chains. By this we mean a series of types $\{p_\lambda(x)\}_{\lambda \in \Lambda}$ such that $p_\lambda(x)$ is a forking extension of $p_\sigma(x)$ for $\sigma < \lambda$ and Λ can have arbitrarily large cardinality (equivalently, one can ask for the existence of Λ with cardinality larger than $\beth_{|T|}$).

Naturally, we say that a (partial) type þ-divides/forks over a set A if it contains a formula which þ-divides/forks over A.

For strong dividing, it is not convenient to make the analogous definition and we will in general avoid speaking of strongly dividing types. If the set of parameters is finite, it is convenient make sure that the strongly dividing type uses "all" the parameters so that we are able to use algebraic closure much more efficiently.

Definition 1.2. Let $p(x,b)$ be a (partial) type over a finite tuple b. We say that $p(x,b)$ *strongly divides over a set* D if there is a formula $\varphi(x,b) \in p(x,b)$ which strongly divides over D.

Remark 1.1. Notice that the definition of strong dividing (for formulae) implies the following.

(i) If $\varphi(x,a)$ strongly divides over A, then for every $b \models \varphi(x,a)$ the type $\mathrm{tp}(a/Ab)$ is algebraic (whereas $\mathrm{tp}(a/A)$ is non-algebraic).

(ii) $\varphi(x,a)$ strongly divides over A if and only if

- $a \notin \mathrm{acl}(A)$,
- for every infinite non-constant indiscernible sequence $\langle a_i : i < \omega\rangle$ in $\mathrm{tp}(a/A)$, the set $\{\varphi(x,a_i) : i < \omega\}$ is inconsistent.

(iii) Let a, b, A be such that $a \notin \mathrm{acl}(A)$ and A is finite. Then $\mathrm{tp}(b/Aa)$ strongly divides over A if and only if $a \in \mathrm{acl}(Ab')$ for any $b' \models \mathrm{tp}(b/Aa)$.

(iv) If $\varphi(x,a)$ strongly divides over A and $B \supset A$ is such that $a \notin \mathrm{acl}(B)$ then $\varphi(x,a)$ strongly divides over B.

Proof. (i) Suppose $\varphi(x,a)$ strongly divides over A (so in particular $a \notin \mathrm{acl}(A)$), and let $b \models \varphi(x,a)$. By the definition, there are only finitely many $a_1, \ldots, a_{k-1}$ (say, $a_1 = a$) in $\mathrm{tp}(a/A)$ such that $\varphi(b,a_i)$. In particular, there are only finitely many realizations of $\mathrm{tp}(a/Ab)$, as required.

(ii) The "only if" direction is clear. For the "if" direction, suppose that $\varphi(x,a)$ does not strongly divide over A, but $a \notin \mathrm{acl}(A)$. Then for every $k < \omega$ there is a subset $\{a_1, \ldots, a_k\}$ of $\mathrm{tp}(a/A)$ such that $\exists x \bigwedge_{i=1}^{k} \varphi(x,a_i)$. By compactness, for any cardinal μ there is a sequence $\langle a_\alpha : \alpha < \mu\rangle$ of realizations of $\mathrm{tp}(a/A)$ such that $\exists x \bigwedge_{i=1}^{k} \varphi(x,a_{\alpha_i})$ for every $\alpha_1 < \cdots < \alpha_k < \mu$. By Fact 1.5 below there is such an infinite (non-constant) indiscernible sequence.

(iii) The "only if" direction follows from (i). On the other hand, assume that $a \notin \mathrm{acl}(A)$ and $\mathrm{tp}(b/Aa)$ does not strongly divide over A, but for any $b' \models \mathrm{tp}(b/Aa)$ we have $a \in \mathrm{acl}(Ab')$. By (ii), for every formula $\varphi(x,a) \in \mathrm{tp}(b/Aa)$ there is an indiscernible sequence $\langle a_i : i < \omega\rangle$ in $\mathrm{tp}(a/A)$ such that $\{\varphi(x,a_i) : i < \omega\}$ is consistent. Let $p(x,a) = \mathrm{tp}(b/Aa)$. By compactness, there is an indiscernible sequence $\langle a_i : i < \omega\rangle$ in $\mathrm{tp}(a/A)$ such that $q(x) = \bigcup_{i<\omega} p(x,a_i)$ is consistent (and moreover $a_0 = a$). Let $b' \models q(x)$. Clearly $a = a_0 \notin \mathrm{acl}(Ab')$, since $a_i \equiv_{Ab'} a_0$ for all i. This contradicts the assumptions.

(iv) follows easily from (ii). ∎

Recall that a formula $\varphi(x,y)$ is called *stable* if it does not have the order property (see [10]).

Fact 1.2. *If a stable formula $\varphi(x,y)$ witnesses that a type $p(x,a)$ forks over A, then there is a φ-formula witnessing that $p(x,a)$ þ-forks over A. In* *particular, in any stable theory the concepts of þ-forking and forking coincide.*

Proof. This is Lemma 5.1.1 in [6]. ■

As with stable theories, for many of our results we will need the existence of a global rank based on the independence notion, which in this case corresponds to þ-forking.

DEFINITION 1.3. Let M be a model. We define the $\mathrm{U}^{\text{þ}}$-*rank* to be the foundation rank of the order given by the þ-forking relation on types consistent with M. A theory T will be called *super-rosy* whenever the $\mathrm{U}^{\text{þ}}$-rank of any type in any model of T is ordinal valued.

FACT 1.3. *Let T be a super-rosy theory and let a, b, A be subsets of a model M of T. Then*

$$\mathrm{U}^{\text{þ}}(\mathrm{tp}(b/aA)) + \mathrm{U}^{\text{þ}}(\mathrm{tp}(a/A)) \leq \mathrm{U}^{\text{þ}}(\mathrm{tp}(ab/A)) \leq \mathrm{U}^{\text{þ}}(\mathrm{tp}(b/aA)) \oplus \mathrm{U}^{\text{þ}}(\mathrm{tp}(a/A)).$$

Proof. Theorem 4.1.10 in [6]. ■

We will need the following easy but important observation. It will allow us to understand how far we need to extend the types to get þ-dividing from þ-forking and strong dividing from þ-dividing; it will be the key to the proof of the decomposition theorem for a type of finite þ-rank in Section 4. The proof is quite close to the proof of Lemmas 3.1, 3.2 and 3.4 in [3]. However, we prove (and need) a slightly different result, so we include a proof.

OBSERVATION 1.4. *Let M be a model of a rosy theory T, and let a, b, A be tuples (and sets) in M. Then the following hold:*

(i) *Let $p(x, a)$ be a type over Aa which þ-forks over A. Then there is a non-þ-forking extension $p(x, a, a')$ of $p(x, a)$ such that $p(x, a')$ þ-divides over A.*
(ii) *Let a, b be tuples and A be a set such that $\mathrm{tp}(b/Aa)$ þ-divides over A. Then there is some e and some finite $a_0 \in \mathrm{dcl}(Aa)$ such that $b \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{Aa} e$ and such that $\mathrm{tp}(b/Aea)$ contains a formula $\phi(x, a_0)$ which strongly divides over Ae. In particular, if $\mathrm{tp}(b/Aa)$ is a type of ordinal valued $\mathrm{U}^{\text{þ}}$-rank, then $\mathrm{U}^{\text{þ}}(\mathrm{tp}(b/Aae)) < \mathrm{U}^{\text{þ}}(\mathrm{tp}(b/Ae))$.*

Proof. (i) Let $p(x, a)$ be as in (i) and let $b \models p(x, a)$. By definition, there are finitely many formulae $\varphi_i(x, a_i)$ such that

$$p(x, a) \vdash \bigvee_{i=1}^{n} \varphi_i(x, a_i)$$

and $\varphi(x, a_i)$ þ-divides over A. By extension of þ-independence we know that there are $a'_1, \ldots, a'_n \models \mathrm{tp}(a_1 \ldots a_n/Aa)$ such that $b \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{Aa} a'_1 \ldots a'_n$.

So $\operatorname{tp}(b/Aaa'_1 \dots a'_n) \vdash \varphi_m(x, a'_m)$ for some m; defining $a' := a'_m$ and $p(x, a, a') := \operatorname{tp}(b/Aaa')$, we see by construction that $p(x, a, a')$ is as desired.

(ii) Let a, b and A be as in (ii). By the definition of þ-dividing there is some e' and some $\varphi(x, a_0) \in \operatorname{tp}(b/Aa)$ such that $\varphi(x, a_0)$ strongly divides over Ae'. Note that in particular $a_0 \notin \operatorname{acl}(Ae')$.

Let $e \models \operatorname{tp}(e'/Aa)$ be such that $b \mathop{\smile\!\!\!\!\!\vert}^{þ}_{Aa} e$. Since $e \models \operatorname{tp}(e'/Aa)$ and $a_0 \in \operatorname{dcl}(Aa)$, strong dividing is preserved. Moreover, $a \notin \operatorname{acl}(Ae)$. ∎

Finally, we will prove the well known Fact 1.7 which will simplify a lot of the proofs. Before we start with the proof, we will need several combinatorial statements.

The following classical result is originally due to Morley, although it is often referred to as "Erdős–Rado argument" since it is an easy consequence of the Erdős–Rado theorem and compactness:

FACT 1.5. *Let λ be a cardinal. Then there exists $\mu > \lambda$ such that for every set A of cardinality λ and a sequence $\langle a_i : i < \mu \rangle$ of tuples there exists an ω-type $q(x_0, x_1, \dots)$ of an A-indiscernible sequence such that for every $n < \omega$ there exist $i_1 < \dots < i_n < \mu$ such that the restriction of q to the first n variables equals* $\operatorname{tp}(a_{i_1} \dots a_{i_n}/A)$.

We will sometimes denote μ as above by $\mu(\lambda)$.

REMARK 1.6. Let A be a set, $A \subseteq B$, and I an A-indiscernible sequence. Then there exists I' with $I' \equiv_A I$ such that I' indiscernible over B.

Proof. First extend I to be long enough so that Fact 1.5 can be applied to it with $\lambda = |B| + |T|$. Then there exists I' indiscernible over B such that every n-type of I' over B "appears" in I. In particular I' has the same type over A as I (since I was A-indiscernible and $A \subseteq B$). ∎

FACT 1.7. *Let $a \mathop{\smile\!\!\!\!\!\vert}^{þ}_{A} B$. Then there is a þ-Morley sequence I over B based on A starting with a.*

Proof. First, construct a non-þ-forking sequence $I' = \langle a'_i : i < \mu \rangle$ in $\operatorname{tp}(a/B)$ based on A starting with A by the standard construction, that is, $a'_0 = a$, $a'_i \equiv_B a$, $a'_i \mathop{\smile\!\!\!\!\!\vert}^{þ}_{A} Ba'_{<i}$. Moreover, make μ large enough so that using Erdős–Rado (more precisely, Fact 1.5, see also Remark 1.6) one can find I which is an ω-sequence, B-indiscernible, and every n-type of I over B "appears" in I'. Clearly I is a þ-Morley sequence over B based on A. Moreover, since every element of I' satisfies $\operatorname{tp}(a/B)$, so does every element of I, so by applying an automorphism over B we may assume that I starts with a. ∎

1.3. Dependence, strong dependence and dp-minimality. Recall that a theory T is called *dependent* if there does not exist a formula which exemplifies the independence property. We are mostly going to use the following equivalent definition:

FACT 1.8. *T is dependent if and only if there do not exist an indiscernible sequence $I = \langle a_i : i < \lambda\rangle$, a formula $\varphi(x, y)$ and $\bar{b}$ such that both*

$$\{i\colon\ \models \varphi(a_i, b)\} \quad \textit{and} \quad \{i\colon\ \models \neg\varphi(a_i, b)\}$$

are unbounded in λ.

The following definitions were motivated by the notions of strong dependence of Shelah (see e.g. [9]) and appear in [11] and [7]. In the definitions below we denote tuples by $\bar{x}, \bar{a}$ (in order to stress the difference between singletons and finite tuples of arbitrary length).

DEFINITION 1.4.

(i) A *randomness pattern* of depth κ for a (partial) type p over a set A is an array $\langle \bar{b}^\alpha_i\colon \alpha < \kappa,\ i < \omega\rangle$ and formulae $\varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ for $\alpha < \kappa$ such that

 (a) the sequences $I^\alpha = \langle \bar{b}^\alpha_i\colon i < \omega\rangle$ are mutually indiscernible over A, that is, I^α is indiscernible over $AI^{\neq\alpha}$,
 (b) $\mathrm{len}(\bar{b}^\alpha_i) = \mathrm{len}(\bar{y}_\alpha)$,
 (c) for every $\eta \in {}^\kappa\omega$, the set

 $$\Gamma_\eta = \{\varphi_\alpha(\bar{x}, \bar{b}^\alpha_{\eta(\alpha)})\colon \alpha < \kappa\} \cup \{\neg\varphi_\alpha(\bar{x}, \bar{b}^\alpha_i)\colon \alpha < \kappa, i < \omega, i \neq \eta(\alpha)\}$$

 is consistent with p.

(ii) A (partial) type p over a set A is called *strongly dependent* if there does not exist a randomness pattern for p of depth ω.

(iii) The *dependence rank* (dp-rank) of a (partial) type p over a set A is the supremum of all κ such that there exists a randomness pattern for p of depth κ.

(iv) A (partial) type over a set A is called *dp-minimal* if the dp-rank of p is 1. In other words, p is dp-minimal if there does not exist a randomness pattern for p of depth 2.

(v) A theory is called *strongly dependent/dp-minimal* if the partial type $x = x$ is (here x is a singleton).

(vi) Let T be dependent. A type p is called *strongly generically stable* if it is strongly dependent and generically stable.

REMARK 1.9. Note that by mutual indiscernibility, in clause (c) of the definition of a randomness pattern it is enough to demand that the set

$$\{\varphi_\alpha(\bar{x}, \bar{b}^\alpha_0)\colon \alpha < \kappa\} \cup \{\neg\varphi_\alpha(\bar{x}, \bar{b}^\alpha_i)\colon \alpha < \kappa,\ 1 \leq i < \omega\}$$

is consistent with p.

Remark 1.10. Note that Shelah basically shows in [9, Observation 1.7] that if there exists a type $p(\bar{x})$ which is not strongly dependent, then there exists such a type $p'(x)$ with x being a *singleton*. Therefore if there exists a non-strongly dependent type, then T is not strongly dependent and the definitions above make sense.

Note that if in the definition of a randomness pattern all formulae are the same, we get the independence property:

Observation 1.11. *A theory T is dependent if and only if it does not admit a randomness pattern of some/any infinite depth with $\varphi_\alpha(\bar{x},\bar{y}) = \varphi(\bar{x},\bar{y})$ for all α if and only if T does not admit a randomness pattern of depth $|T|^+$.*

Proof. By compactness. ∎

A related notion, which will be convenient for us to consider, was investigated by Adler in [1]. We are going to use a slightly different terminology (some of it comes from [7]).

Definition 1.5.

(i) A *dividing pattern* of depth κ for a (partial) type p over a set A is an array $\langle \bar{b}^\alpha_i : \alpha < \kappa,\ i < \omega \rangle$ and formulae $\varphi_\alpha(\bar{x},\bar{y}_\alpha)$ for $\alpha < \kappa$ such that
  - (a) the sequences $I^\alpha = \langle \bar{b}^\alpha_i : i < \omega \rangle$ are mutually indiscernible over A, that is, I^α is indiscernible over $AI^{\neq\alpha}$,
  - (b) $\mathrm{len}(\bar{b}^\alpha_i) = \mathrm{len}(\bar{y}_\alpha)$,
  - (c) for every $\eta \in {}^\kappa\omega$, the set $\{\varphi_\alpha(\bar{x},\bar{b}^\alpha_{\eta(\alpha)}) : \alpha < \kappa\}$ is consistent with p,
  - (d) for every $\alpha < \kappa$, the set $\{\varphi_\alpha(x, b^\alpha_i) : i < \omega\}$ is inconsistent with p.

(ii) A (partial) type p over a set A is called *strong* if there does not exist a dividing pattern for p of depth $\kappa = \omega$.

(iii) A theory is called *strong* if every finitary type is strong.

Remark 1.12. As in Remark 1.9, note that by mutual indiscernibility, in clause (c) of the definition of a dividing pattern it is enough to demand that the set $\{\varphi_\alpha(\bar{x},\bar{b}^\alpha_0) : \alpha < \kappa\}$ is consistent with p.

The reader is encouraged to have a look at [1] for the discussion of strong theories. A theory is strong and dependent if and only if it is strongly dependent (as suggested by the name), and this is the case we are mostly interested in; but there are also strong theories which are simple unstable, and even SOP_2.

A version of the following easy lemma was proven by the authors in [7] in order to establish the connection between randomness and dividing patterns.

It is also implicit in some proofs in [1]. For completeness, we include a proof suggested by the referee (which is somewhat different from the one in [7]).

Lemma 1.13.

(i) *Let $p(x)$ be a type over a set A, let $I = \langle b_i \rangle_{i \in O}$ be a sequence indiscernible over A, and let $\varphi(x, y)$ be a formula such that $p(x) \cup \varphi(x, b_i)$ is consistent for some (all) i and $\{\varphi(x, b_i)\}_{i \in O}$ is k-inconsistent with p for some $k \in \mathbb{N}$. Then*

$$p(x) \cup \{\varphi(x, b_l)\} \cup \{\neg\varphi(x, b_i)\}_{i \neq l}$$

is consistent for all l.

(ii) *Any dividing pattern is also a randomness pattern.*

(iii) *Clause* (ii) *also holds when the depth $n < \omega$ is replaced with any cardinal κ.*

Proof. (i) Without loss of generality, $O = \mathbb{Q}$ and $l = 0$. Let $a \models p \cup \phi(x, b_0)$. Since $\{\varphi(x, b_i) : i \in O\}$ is inconsistent with p, by compactness and indiscernibility there are only finitely many $q_j \in O$ such that $a \models \varphi(b, b_{q_j})$. Let O' be the set which results after removing from O those finitely many q_j. Then O' is order-isomorphic to O; by indiscernibility there is an Ab_0-automorphism σ moving $\langle b_q : q \in O' \rangle$ to $\langle b_q : q \in O \rangle$. Then $\sigma(a)$ satisfies $p \cup \varphi(x, b_0) \cup \neg\varphi(x, b_q) : q \neq 0$, as required.

(ii) follows from (i) by induction, and (iii) follows from (ii) by compactness. ■

2. þ-weight, crisscrossed þ-forking, and indiscernible sequences. The purpose of this section is to define the notion of þ-weight and to relate it to the existence of certain mutually indiscernible sequences, which will lead to the conclusion (Theorem 2.7) that in a strong rosy theory every type has rudimentarily finite þ-weight.

Throughout the section we will assume that T is rosy.

2.1. þ-weight. We define þ-pre-weight and þ-weight of a type p. We will denote them by $\mathrm{pwt}^{\text{þ}}(p)$ and $\mathrm{wt}^{\text{þ}}(p)$. Note that Fact 1.2 implies that in stable theories þ-weight coincides with the usual notion of weight.

Definition 2.1.

- Let $p(x)$ be any type over some set A. We will say that $a, \langle b_i \rangle_{i=1}^n$ witnesses $\mathrm{pwt}^{\text{þ}}(p(x)) \geq n$ (*þ-pre-weight of* p is at least n) if $a \models p(x)$, $\langle b_i \rangle_{i=1}^n$ is A-þ-independent and $a \not\mathop{\smile\hspace{-0.9em}\mid}^{\text{þ}}_A b_i$ for all i, j. If n is maximal such that such a witness exists, we will say that $a, \langle b_i \rangle_{i=1}^n$ witnesses $\mathrm{pwt}^{\text{þ}}(p(x)) = n$ and that p *has þ-pre-weight* n.

- We say that a type p has *finite* þ-pre-weight if $\mathrm{pwt}^{þ}(p) < \omega$. We say that a type p has *rudimentarily finite* þ-pre-weight if one cannot find an infinite witness $\{b_i : i < \omega\}$ as above.
- Let $p(x)$ be any type over some set A. We will say that $a, B, \langle b_i \rangle_{i=1}^{n}$ witnesses $\mathrm{wt}^{þ}(p(x)) \geq n$ (*þ-weight of p* is at least n) if $a \models p(x)$, $a \mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} B$, $\langle b_i \rangle_{i=1}^{n}$ is B-þ-independent and $a \not\mathop{\smile\!\!\!\!\!\!|}^{þ} b_i$ for all i, j. If n is maximal such that such a witness exists, we will say that $a, B, \langle b_i \rangle_{i=1}^{n}$ witnesses $\mathrm{wt}^{þ}(p(x)) = n$ and that p has *þ-weight n*.
- We say that a type p has *finite* þ-weight if $\mathrm{wt}^{þ}(p) < \omega$. We say that a type p has *rudimentarily finite* þ-weight if every non-þ-forking extension of p has rudimentarily finite pre-weight.

It follows from the definition that $\mathrm{wt}^{þ}(p) \geq n$ if and only if there exists a non-þ-forking extension of p with þ-pre-weight at least n.

Notice also that one could define infinite þ-pre-weight and weight as usual, but we will be concerned only with finite þ-weights in this paper.

2.2. Crisscrossed forking. In order to establish the main result of this section, Theorem 2.7, we find it convenient to relate weight to the following concept of "crisscrossed" (þ-)forking. This notion might be familiar to some readers from an earlier preprint of the authors, not intended for publication (an older version of [7]).

DEFINITION 2.2.

(i) We say that a tuple $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ and a set A witness *n-crisscrossed strong-dividing* (n-cc-strong-dividing) if $\models \exists x \bigwedge_i \varphi_i(x, \bar{a}^i)$, $\varphi_i(x, \bar{a}^i)$ strongly divides over A and $\bar{a}^i \mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} \langle \bar{a}^j \rangle_{j \neq i}$ for all i.

(ii) We say that a tuple $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ and a set A witness *n-crisscrossed þ-dividing* (n-cc-þ-dividing) if $\models \exists x \bigwedge_i \varphi_i(x, \bar{a}^i)$, $\varphi_i(x, \bar{a}^i)$ þ-divides over A and $\bar{a}^i \mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} \langle \bar{a}^j \rangle_{j \neq i}$ for all i.

(iii) We say that a tuple $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ and a set A witness *n-crisscrossed þ-forking* (n-cc-þ-forking) if $\models \exists x \bigwedge_i \varphi_i(x, \bar{a}^i)$, $\varphi_i(x, \bar{a}^i)$ þ-forks over A and $\bar{a}^i \mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} \langle \bar{a}^j \rangle_{j \neq i}$ for all i.

(iv) We say that T admits n-cc-þ-forking (or þ-dividing or strong dividing) if there exists a tuple $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ witnessing n-cc-þ-forking (or þ-dividing or strong forking) over A, with x being a *singleton.*

(v) Let p be a type over a set A. We say that a tuple $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ witnesses *n-cc-þ-forking* (or þ-dividing or strong dividing) *in p* if $\langle \varphi_i(x, \bar{a}^i) \rangle_{i<n}$ and A witness n-cc-þ-forking (or þ-dividing or strong dividing) and the formula $\bigwedge_i \varphi_i(x, \bar{a}^i)$ is consistent with p.

(vi) We say that a type $p \in \mathrm{S}(A)$ admits n-cc-þ-forking (or þ-dividing or strong dividing) if there exists a tuple $\langle \varphi_i(x, \bar{a}^i)\rangle_{i<n}$ witnessing n-cc-þ-forking (or þ-dividing or strong dividing) in p.

REMARK 2.1. The following are direct consequences of the definitions.

(i) T admits n-cc-þ-forking if and only if there exists a set A and a type $p \in \mathrm{S}_1(A)$ which admits n-cc-þ-forking.
(ii) Let T be rosy. Then a type $p \in \mathrm{S}(A)$ does not admit n-cc-þ-forking if and only if it has pre-þ-weight less than n.
(iii) So a rosy T does not admit n-cc-þ-forking if and only if every 1-type has pre-þ-weight less than n if and only if every 1-type has þ-weight less than n.

The following technical result is quite useful.

THEOREM 2.2. *The following are equivalent for any $p \in \mathrm{S}(A)$:*

(i) *p admits n-cc-þ-forking.*
(ii) *p admits n-cc-þ-dividing.*
(iii) *There is an extension $p(x, B)$ of $p(x)$ such that $p(x, B)$ admits n-cc-strong dividing.*

Proof. It is clear that if an extension of a type has a witness for n-cc-strong dividing then the same tuple is a witness of n-cc-þ-dividing, and any witness for n-cc-þ-dividing is a witness of n-cc-þ-forking. We will prove that (i) implies (iii) for $n = 2$. The general case will follow by a straightforward induction on n using the properties of þ-forking in rosy theories.

(i)⇒(iii). Let

$$\{\varphi(x, \bar{a}), \psi(x, \bar{b})\}, A$$

be a 2-cc-þ-forking witness for p. By definition there are finitely many formulae $\varphi_i(x, \bar{a}_i)$, $\psi_j(x, \bar{b}_j)$ and tuples $\bar{c}, \bar{d}$ such that

$$\varphi(x, \bar{a}) \models \bigvee_{i=1}^{k_a} \varphi_i(x, \bar{a}_i), \qquad \psi(x, \bar{b}) \models \bigvee_{i=1}^{k_a} \psi_i(x, \bar{b}_i), \tag{1}$$

and

$$\begin{aligned} &\varphi_i(x, \bar{a}_i) \text{ strongly divides over } A\bar{c} \text{ for all } i, \\ &\psi_j(x, \bar{b}_j) \text{ strongly divides over } A\bar{d} \text{ for all } j. \end{aligned} \tag{2}$$

By hypothesis $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \bar{b}$, so by extension of þ-independence (on both sides) we may assume that

$$\bar{a}\bar{a}_i\bar{c} \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \bar{b}\bar{b}_j\bar{d}$$

(we only need to preserve $\mathrm{tp}(\bar{a}\bar{a}_i\bar{c}/A)$ and $\mathrm{tp}(\bar{b}\bar{b}_j\bar{d}/A)$ to preserve the implications and inconsistency required to witness strong dividing).

Since $\varphi(x,\bar a)\wedge\psi(x,\bar b)$ is consistent with p, it is clear from (1) that the conjunction $\varphi_i(x,\bar a_i)\wedge\psi_j(x,\bar b_j)$ is consistent with p for some i,j. By monotonicity of þ-forking independence we know that $\bar a_i \mathop{\smile\!\!\!\!\!\!|}\,_A \bar b_j$, so (2) implies that $(\varphi_i(x,\bar a_i),\psi_j(x,\bar b_j)),A$ is a witness for cc-þ-dividing. This is enough to show (ii).

To complete the proof of (i)⇒(iii), notice that monotonicity of þ-forking implies in particular that $\bar a_i \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{Ac} d$, $\bar b_j \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{Ad} c$, and $\bar a_i \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{Acd} \bar b_j$.

Since by definition $\bar a_i\notin \operatorname{acl}(Ac)$ and $\bar b_j\notin\operatorname{acl}(Ad)$ we see that $\bar a_i,\bar b_j\notin \operatorname{acl}(Acd)$: e.g., $\bar a_i\notin\operatorname{acl}(Ac)$, but $\bar a_i \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{Ac} d$, so $\bar a_i\notin\operatorname{acl}(Acd)$.

So

$$\begin{array}{ll} & \varphi(x,\bar a_i) \text{ strongly divides over } Acd,\\ (3) & \psi(x,\bar b_j) \text{ strongly divides over } Acd,\\ & \bar a_i \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{Acd} \bar b_j. \end{array}$$

Let $B:=Acd$, let $p(x,B,\bar a_i,\bar b_j)$ be a non-þ-forking extension of $p(x)\cup\{\varphi(x,\bar a_i)\cup\psi(x,\bar b_j)\}$ and let $p(x,B)$ be the restriction of $p(x,B,\bar a_i,\bar b_j)$ to B. All the conditions in the definition of 2-cc-strong dividing are satisfied, which completes the proof of the theorem. ∎

2.3. Finite þ-weight and strong dependence. The main goal of this subsection is to characterize, in rosy theories, strong dependence in terms of the þ-pre-weight. In order to do this, we will need to prove the existence of mutually þ-Morley sequences. The procedures will also bring some light on what is needed to characterize strong dependence within dependent theories (or Adler's "strongness" within arbitrary theories) in terms of weight with respect to some independence notion.

OBSERVATION 2.3. *Let $\{I^i : i<n\}$ be sequences such that I^i is a þ-Morley sequence over $AI^{<i}$ based on A. Then I^i is a non-þ-forking sequence over $AI^{\neq i}$ based on A.*

Proof. We need to prove that $\bar a^i_j \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \bar a^i_{<j} I^{\neq i}$ where we define $I^i=\langle \bar a^i_j : j<\mu_i\rangle$.

By the assumptions, $\bar a^i_j \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \bar a^i_{<j} I^{<i}$ for all i,j. Hence by transitivity and finite character of þ-forking, we have $I^{>i} \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A I^{\leq i}$ for all i, in particular $I^{>i} \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \bar a^i_{\leq j} I^{<i}$ for all i,j. By transitivity again, combining $\bar a^i_j \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \bar a^i_{<j} I^{<i}$ and $I^{>i} \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \bar a^i_{\leq j} I^{<i}$, we have $\bar a^i_{\leq j} \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A I^{\neq i}$.

Therefore, since $\bar a^i_j \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \bar a^i_{<j}$, we get $\bar a^i_j \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \bar a^i_{<j} I^{\neq i}$, as required. ∎

LEMMA 2.4. *Let $\{\bar a^i : i<n\}$ be a set of tuples and let $\{I^i : i<n\}$ be sequences such that*

- *for each $i < n$ the sequence I^i is $AI^{<i}a^{>i}$-indiscernible,*
- *I^i starts with $\bar{a}^i$.*

Then there exist sequences $\{J^i : i < n\}$ such that

- *for each $i < n$ the sequence J^i is $AJ^{\neq i}$-indiscernible,*
- *$I^i \equiv_{Aa^i} J^i$. So in particular, J^i starts with $\bar{a}^i$.*

Moreover, if I^i are þ-Morley sequences over $AI^{<i}a^{>i}$ based on A, then we can make J^i þ-Morley over $AI^{\neq i}$ based on A.

Proof. Exactly the same construction is used to prove both parts of the lemma. To avoid being repetitive, we will prove the "moreover" part. The proof of the first part is the same, except that without the extra assumptions we cannot get the stronger conclusion. So assume the sequence of I^i's is a þ-Morley sequence over $AI^{<i}a^{>i}$ based on A.

We need to make sure that I^i can be made indiscernible over $AI^{\neq i}$ and not only over $AI^{<i}a^{>i}$. So assume that $\mathrm{len}(I^i) = \mu_i = \mu(\sum \mu_{<i} + |A| + |T|)$ as in Fact 1.5. We will make our way "backwards", that is, by downward induction on i, starting with $i = n$.

Assume that for $\ell > i$, I^ℓ are þ-Morley ω-sequences over $AI^{\neq \ell}$ based on A, whereas for $\ell \leq i$ we still have I^ℓ of length μ_ℓ which are þ-Morley sequences over $AI^{<\ell}\bar{a}^{>\ell}$ based on A, non-þ-forking over $AI^{\neq \ell}$ (we have the last assumption by Observation 2.3).

By Fact 1.5 we can find J^i which is an indiscernible ω-sequence over $AI^{\neq i}$ such that every n-type of J^i over $AI^{\neq i}$ "appears" in I^i. So in particular J^i has the same type over $AI^{<i}\bar{a}^{>i}$ as I^i. Moreover, since þ-forking has finite character, J^i is non-þ-forking over $AI^{\neq i}$.

Notice that given a finite tuple $\bar{b}$ in J^i the question of whether for some $\bar{\alpha} = \alpha_1 < \cdots < \alpha_k < \omega$ and $\bar{\beta} = \beta_1 < \cdots < \beta_k < \omega$ we have $\bar{a}^\ell_{\bar{\alpha}} \equiv_{\bar{b}I^{\neq \ell, i}} \bar{a}^\ell_{\bar{\beta}}$ amounts to the same question over *some* $\bar{b}'$ in I^i. Since these were indiscernible, we find that for any $\ell > i$ the sequence I^ℓ is still indiscernible over $AI^{\neq \ell, i}J^i$. Using a similar argument one can also make sure that for $\ell \neq i$, I^ℓ is still a non-þ-forking sequence over $AI^{\neq \ell, i}J^i$.

So J^i satisfies all the requirements, except that we need the first element of it to be $\bar{a}^i$. Note, though, that the first element of J^i has the same type over $I^{<i}\bar{a}^{>i}$ as $\bar{a}^i$. So applying an automorphism over $I^{<i}\bar{a}^{>i}$, we obtain a new J^i that starts with $\bar{a}^i$ and a new I^ℓ for $\ell > i$ which have all the required properties, completing the proof of the inductive step. ∎

LEMMA 2.5. *Let $\{\bar{a}^i : i < n\}$ be a set of tuples which is þ-independent over a set A. Then there exist sequences $\{I^i : i < n\}$ such that*

- *for each $i < n$ the sequence I^i is a þ-Morley sequence over $AI^{\neq i}$ based on A. So I^i is $AI^{\neq i}$-indiscernible and $I^i \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A I^{\neq i}$,*
- *I^i starts with $\bar{a}^i$.*

Proof. We construct sequences I^i such that I^i is a þ-Morley sequence over $AI^{<i}\bar{a}^{>i}$ based on A. By Lemma 2.4, this is enough to obtain the desired conclusion. The construction is by induction on $i < n$.

The case $i = 0$ follows from Fact 1.7.

So let $i > 0$, and assume that $I^{<i}$ already exist. Note that $I^0 \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \bar{a}^{>0}$ and $I^1 \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A I^0\bar{a}^{>1}$, hence $I^0 I^1 \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A a^{>1}$. Continuing, we see that $I^{<i} \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \bar{a}^{\geq i}$. By symmetry and transitivity, $\bar{a}^i \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A I^{<i}\bar{a}^{>i}$, and we can apply Fact 1.7 again. ∎

We are now able to prove that strong dependence implies boundedness (by ω) of cc-strongly dividing patterns and of þ-weight.

Proposition 2.6. *If a type p admits an n-cc-strong-dividing witness, then* dp-rank$(p) \geq n$.

Proof. Let $\langle \psi_i(x, \bar{a}^i)\rangle_{i<n}$ and a set A witness n-cc-strong dividing, that is, $\bigwedge_i \psi_i(x, \bar{a}^i)$ is consistent with p, $\psi_i(x, \bar{a}^i)$ strongly divides over A and $\bar{a}^i \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A \langle \bar{a}^j\rangle_{j\neq i}$ for all i.

By the definition of strong dividing, $\bar{a}^i \notin \operatorname{acl}(A)$. Since $\{\bar{a}^i : i < n\}$ is þ-independent, we can build as in Lemma 2.5 sequences $I^i = \langle \bar{a}^i_j : j < \omega\rangle$ such that

- I^i is a þ-Morley sequence over $AI^{\neq i}$ based on A,
- $\bar{a}^i_0 = \bar{a}^i$.

For each $i < n$ and $k < \omega$ denote $\psi^k_i(x) = \psi^k_i(x, \bar{a}^i_{<k}) = \bigwedge_{j<k} \psi_i(x, \bar{a}^i_j)$. Note that since $\psi_i(x, \bar{a}^i_0)$ strongly divides over A, for some $k < \omega$ the formula $\psi^k_i(x)$ is inconsistent.

So we clearly have a þ-dividing pattern (see Definition 1.5) for p of depth n; applying Lemma 1.13(ii), we are done. ∎

Theorem 2.7. *If T is strongly dependent* (*and rosy*) *then every* (*finitary*) *type has rudimentarily finite þ-weight. If T is dp-minimal then every* 1*-type has þ-weight* 1. *Moreover, the conclusion is true if we just assume that T is strong and rosy.*

Proof. This now follows easily from Remark 2.1, Theorem 2.2 and Proposition 2.6.

For the "moreover" part note that Lemma 2.5 only assumes rosiness, and in Proposition 2.6 we show, in fact, existence of a dividing pattern. ∎

The next section will be devoted to showing the equivalence between "rudimentarily finite þ-weight" and "finite þ-weight".

3. þ-orthogonality and þ-regularity in rosy structures. The first part of this section is devoted to developing the analogous notions of domination, orthogonality, weight and regularity in the þ-forking context and the properties such notions have under different hypotheses. We will start with the relevant definitions.

Throughout the section we will assume that T is rosy.

DEFINITION 3.1.

- Two types $p(x)$ and $q(x)$ are *almost þ-orthogonal* if they are over a common domain B and for any tuples $a \models p'$ and $b \models q'$ we have $a \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_B b$. This is denoted by $p \perp^{\text{þ}}_a q$.
- Two types p and q are *þ-orthogonal* if any non-þ-forking extensions p' and q' of p and q respectively to a common domain are almost þ-orthogonal. This is denoted by $p \perp^{\text{þ}} q$.
- Let A be a set, and a, b tuples. We say that *a þ-dominates b over A* if for every c the relation $b \not\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A c$ implies $a \not\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A c$. In this case we write $b \lhd^{\text{þ}}_A a$.
- We say that a, b are *th-domination equivalent over A* if they dominate each other over A. Clearly, this is an equivalence relation. In this case we write $a \bowtie^{\text{þ}}_A b$.
- Let $p(x)$ and $q(x)$ be types over A and B respectively. We will say that $p(x)$ *þ-dominates* $q(x)$ if there are realizations a, b of p and q respectively such that $a \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A B$, $b \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_B A$ and $b \lhd^{\text{þ}}_{A\cup B} a$. If $A = B$, we say that p þ-dominates q *over A*.
- We say that types p and q are *þ-equidominant* if there are non-forking extensions p', q' of p, q respectively to a common domain C and realizations $a' \models p', b' \models q'$ which are domination equivalent over C. In this case we write $p \bowtie^{\text{þ}} q$.

REMARK 3.1. Note that equidominance is not (in general) an equivalence relation on types. Note also that if two types dominate each other, they are not necessarily equidominant (even if the domination is over the same set A of parameters), not even in stable theories. The problem is that whereas dominance on elements (over a set A) is transitive, dominance on types is generally not. See Section 5.2 of [12] for a further discussion of this matter and examples.

3.1. Basic properties. Here we list the basic properties of þ-weight, þ-domination and þ-orthogonality. Some of the results and proofs in this subsection are very similar, and sometimes completely analogous to the results in simple theories (see Section 5.2 of [12]).

LEMMA 3.2.

(i) *If* $a \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A} b$ *then* $\mathrm{wt}^{\text{þ}}(a/A) = \mathrm{wt}^{\text{þ}}(a/Ab)$.

(ii) $\mathrm{wt}^{\text{þ}}(ab/A) \leq \mathrm{wt}^{\text{þ}}(a/A) + \mathrm{wt}^{\text{þ}}(b/A)$. *Equality holds whenever* $a \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A} b$.

Proof. The proofs are the same as the proofs of Lemmas 5.2.3 and 5.2.4 in [12], replacing instances of forking by þ-forking. ∎

The following is very easy:

OBSERVATION 3.3. *Suppose that a is þ-dominated by b over a set A, and $A' \supseteq A$ is such that $a \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A} A'$ and $b \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A} A'$. Then a is þ-dominated by b over A'.*

OBSERVATION 3.4. *Suppose* $a \lhd^{\text{þ}}_{A} b$. *Then* $\mathrm{wt}^{\text{þ}}(a/A) \leq \mathrm{wt}^{\text{þ}}(b/A)$.

Proof. Assume that $\mathrm{wt}^{\text{þ}}(a/A) \geq n$. Then there are $A', \{c_i : i < n\}$ witnessing this; that is, $a \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A} A'$, $\{c_i : i < n\}$ is an A'-þ-independent set, and $a \not\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A} c_i$ for all i. Let $b' \equiv_{Aa} b$ be such that $b' \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{Aa} A'$. So $ab' \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A} A'$, hence by the previous observation, a is dominated by b' over A'. So $b' \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A'} c_i$ for all i. In particular, $\mathrm{wt}^{\text{þ}}(b/A) \geq \mathrm{wt}^{\text{þ}}(b'/A') \geq n$, as required. ∎

OBSERVATION 3.5.

- *If $p, q \in \mathrm{S}(A)$ are not almost þ-orthogonal and* $\mathrm{pwt}^{\text{þ}}(q) = 1$, *then p dominates q over A.*
- *The relation $p \not\perp_A q$ is an equivalence relation on types over A of þ-pre-weight* 1.

Proof. Easy (see 5.2.11 and 5.2.12 in [12]). ∎

The following two lemmas are easy but very useful.

LEMMA 3.6. *Assume $b \lhd^{\text{þ}}_{A} a$. Then there exists B containing A such that $a \mathop{\smile\!\!\!\!\!\!|}_{A} B$ (hence $b \mathop{\smile\!\!\!\!\!\!|}_{A} B$) such that $ab \lhd^{\text{þ}}_{B} a$.*

Proof. We try to choose by induction on $\alpha < |T|^+$ an increasing and continuous sequence of sets A_α such that $A_0 = A$ and for all α we have:

- $ab \not\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A_\alpha} A_{\alpha+1}$,
- $a \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A_\alpha} A_{\alpha+1}$ (hence $b \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A_\alpha} A_{\alpha+1}$).

By local character of þ-independence, there is $\alpha < |T|^+$ such that it is impossible to choose $A_{\alpha+1}$. Denote $B = A_\alpha$. It is easy to see that all the requirements are satisfied. ∎

LEMMA 3.7. *Assume that $ab \lhd^{\text{þ}}_{A} a$, $a \not\mathop{\smile\!\!\!\!\!\!|}^{þ}_{A} c$, $b \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A} c$ and* $\mathrm{wt}^{\text{þ}}(\mathrm{tp}(c/A)) = 1$. *Then $bc \lhd^{\text{þ}}_{A} a$.*

Proof. Assume $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} d$. Since $ab \lhd^{þ}_{A} a$, we have $ab \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} d$, hence $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{Ab} d$. Let $A' = Ab$. Then $c \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} A'$ and $c \not\mathop{\smile\!\!\!\!\!\mid}^{þ}_{A'} a$ (otherwise, by transitivity $c \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} ab$). Since $\mathrm{wt}^{þ}(c/A) = 1$, clearly $c \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A'} d$ (otherwise, remembering that $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A'} d$, we would conclude that a, d witness $\mathrm{pwt}^{þ}(c/A') \geq 2$). Hence $bc \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} d$, as required. ■

Observation 3.8. *Let $p \in \mathrm{S}(B)$ and $a, B, b_1, \ldots, b_n$ witness $\mathrm{pwt}^{þ}(p) = n$. Then $a \lhd^{þ}_{B} b_1 \ldots b_n$.*

Proof. Assume $c \not\mathop{\smile\!\!\!\!\!\mid}^{þ}_{B} a$ and $c \mathop{\smile\!\!\!\!\!\mid}^{þ}_{B} b_1 \ldots b_n$. Then the set $\{c, b_1, \ldots, b_n\}$ is B-þ-independent, and it witnesses $\mathrm{pwt}^{þ}(a/B) \geq n + 1$, a contradiction. ■

3.2. From rudimentarily finite to finite. We will now prove that if a type has rudimentarily finite þ-weight, it has finite þ-weight. As with stable theories, in order to show this we found it necessary to prove the very interesting fact that a type of (rudimentarily) finite þ-weight is þ-equidominant with a finite free product of þ-weight-1 types.

A good start would be showing that every type of rudimentarily finite weight is "related" (in terms of non-þ-orthogonality) to þ-weight-1 types. The following two lemmas generalize Hyttinen's results from [5] on types in a stable theory, and we adapt his technique to the rosy context.

Lemma 3.9. *Let $p \in \mathrm{S}(A)$, and assume that*

(i) *$a, A', \{b_1, \ldots, b_n\}$ witness $\mathrm{wt}^{þ}(p) \geq n$. That is, $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} A'$, $\{b_1, \ldots, b_n\}$ are þ-independent over A' and $a \not\mathop{\smile\!\!\!\!\!\mid}^{þ}_{A'} b_i$ for all i.*

(ii) *There is no C extending A' such that the following three conditions hold:*

  (a) *$a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A'} C$,*

  (b) *$b_1 \ldots b_{n-1} \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A'} C b_n$,*

  (c) *$b_n \not\mathop{\smile\!\!\!\!\!\mid}^{þ}_{A'} C$.*

Then

(1) *Whenever $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A'} c$ and $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A' b_n} c$, we have $b_n \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A'} c$.*

(2) *If, furthermore, $\mathrm{wt}^{þ}(\mathrm{tp}(b_n/A')) > 1$, then there are B and b'_n, b'_{n+1} such that $a, B, \{b_1, \ldots, b_{n-1}, b'_n, b'_{n+1}\}$ witness $\mathrm{wt}^{þ}(p) \geq n + 1$.*

Proof. (1) Assume $b_n \not\mathop{\smile\!\!\!\!\!\mid}^{þ}_{A'} c$ but $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A'} c$ and $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A' b_n} c$. Without loss of generality $c \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A' b_n a} b_1 \ldots b_{n-1}$, hence $c \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A' b_n} b_1 \ldots b_{n-1}$. Let $C = A'c$. It is easy to see that (a)–(c) above hold for C (e.g. (b) holds by symmetry and transitivity), contradicting assumption (ii) of the lemma.

(2) Assume $\mathrm{wt}^{þ}(\mathrm{tp}(b_n/A')) > 1$. This means that there are $B \supseteq A'$ and c, d such that

- $b_n \mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{A'} B$,
- $c \mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{B} d$,
- $b_n \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{B} c$ and $b_n \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{B} d$.

Again we may assume without loss of generality $ab_1 \dots b_{n-1} \mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{A'b_n} Bcd$. It is easy to see that the assumptions of the lemma still hold after replacing A' with B. So part (1) holds as well. In particular, since $b_n \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{B} c$ and $b_n \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{B} d$, whereas $a \mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{Bb_n} c$ and $a \mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{Bb_n} d$, we have $a \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{B} c$ and $a \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{B} d$. Choosing $b'_n = c$, $b'_{n+1} = d$, we are done. ■

Lemma 3.10. *Let $p \in \mathrm{S}(A)$ be a type of rudimentarily finite þ-weight. Then p is non-þ-orthogonal to a type of þ-weight* 1.

Moreover, suppose that $a \models p, B = \{b_i : i < m\}, d$ are such that $a, A, \{b_i : i < m\} \cup \{d\}$ witness $\mathrm{wt}^{þ}(p) \geq m+1$. *Then there exist $D \supseteq A$ and d' such that*

- $\mathrm{wt}^{þ}(d'/A') = 1$,
- $a, D, \{b_i : i < m\} \cup \{d'\}$ *witness* $\mathrm{wt}^{þ}(p) \geq m+1$.

Proof. By considering a non-þ-forking extension it is clear that the lemma follows from the "moreover" part.

We will prove that if the conclusion fails we can witness that p has rudimentarily infinite þ-weight, thus contradicting the hypothesis of the lemma.

Assume towards a contradiction that the conclusion fails and construct by induction on $n \geq m$ sets A_n, B_n and tuples d_n such that

- $B_n = \{b_i : i < n\}$, so $|B_n| = n$,
- $A_m = A$, $B_m = B$, $d_m = d$,
- the sequences $\langle A_n : n < \omega \rangle$ and $\langle B_n : n < \omega \rangle$ are increasing,
- $a, A_n, B_n = \{b_i : i < n\} \cup \{d_n\}$ witness $\mathrm{wt}^{þ}(p) \geq n+1$.

The case $n = m$ is given, so suppose we have A_n, B_n and d_n as above.

By local character of þ-independence, we can replace A by A' satisfying the assumptions of Lemma 3.9 with b_n there replaced by our d: if given some A' there exists a C as in (ii) of Lemma 3.9 above, it satisfies all the requirements of A' in (i), so we can replace A' with C and continue; local character of þ-forking and the fact that $d \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{A'} C$ guarantee that the process will eventually stop. So by Lemma 3.9 (and the assumption towards contradiction), we can "split" d into two elements b_n and d_{n+1}, that is, find A_{n+1}, b_n, d_{n+1} such that $a, A_{n+1}, \{b_i : i < n\} \cup \{d_{n+1}\}$ witness $\mathrm{wt}^{þ}(p) \geq n+1$, as required.

Let $A_\omega = \bigcup_{n<\omega} A_n$ and $B_\omega = \bigcup_{n<\omega} B_n$. Clearly, B_ω is an *infinite witness* for $\mathrm{wt}^{þ}(p) \geq \aleph_0$, contradicting p having rudimentarily finite weight.

Since this construction contradicts our hypothesis, we know that for some n we have $\mathrm{wt}^{\text{þ}}(d_n/A_n) = 1$. But then $D = A_n, d' = d_n$ satisfy the conditions required in the conclusion of the lemma. ∎

We are finally ready to prove that a type of rudimentarily finite þ-weight has finite þ-weight. The proof will be based on Observation 3.11, but first we make the following (temporary) definition.

DEFINITION 3.2. Let $p = \mathrm{tp}(a/A)$ be any type.

We will say that a witness $a, A, \{b_i : i < m\}$ is a *nice witness of* $\mathrm{wt}^{\text{þ}}(p) \geq m$ if $ab_0 \ldots b_{m-1} \lhd_A a$ and $\mathrm{wt}^{\text{þ}}(b_i/A) = 1$ for all i.

We will say that a witness $a, A, \{b_i : i < m\}$ of $\mathrm{wt}^{\text{þ}}(p) \geq m$ *is contained* in a witness $a, A', \{b_i : i < n\}$ of $\mathrm{wt}^{\text{þ}}(p) \geq n$ if $A \subset A'$, $(b_i)_{i<n} \mathop{\smile\!\!\!\!\!\!|}_A A'$, and $m \leq n$. We say that the first witness is *properly contained* in the second one if $m < n$.

We will say that a (nice) witness is *maximal* if it is not properly contained in any other (nice) witness.

OBSERVATION 3.11. *Let $p = \mathrm{tp}(a/A)$ be a type of rudimentarily finite weight. Then every witness $a, A, \{b_i : i < m\}$ of $\mathrm{wt}^{\text{þ}}(p) \geq m$ is contained in a* maximal *witness $a, A', \{b_i : i < n\}$. Even more, every nice witness $a, A, \{b_i : i < m\}$ of $\mathrm{wt}^{\text{þ}}(p) \geq m$ is contained in a witness $a, A', \{b_i : i < n\}$ to $\mathrm{wt}^{\text{þ}}(p) \geq n$ which is maximal among all nice witnesses.*

Proof. The proof is precisely the same as the proof of Lemma 3.10 above: If there is no maximal witness, then we can construct by induction on $n < \omega$ increasing witnesses $A_n, B_n = \{B_i : i < n\}$; taking the unions of these sets, we get a contradiction. ∎

Notice that, a priori, this does not mean that every such maximal witness has the same size, or that there are no different such witnesses of finite unbounded cardinalities so that the þ-weight of p could still be infinite.

The proof of the following lemma shows that the size of any nice maximal witness (in particular with $\mathrm{wt}^{\text{þ}}(b_i) = 1$) is the same finite number n, which must *a posteriori* be equal to $\mathrm{wt}^{\text{þ}}(p)$; that every type of rudimentarily finite weight has finite weight follows as an easy corollary.

LEMMA 3.12. *Let p be a type of rudimentarily finite þ-weight. Then any maximal nice witness $a, A', \{b_i : i < m\}$ of $\mathrm{wt}^{\text{þ}}(p) \geq m$ satisfies $a \bowtie^{\text{þ}}_{A'} b_0 \ldots b_{n-1}$.*

Proof. Let a, A' and $b_0 \ldots b_{n-1}$ be as in the statement of the lemma. It is clearly enough to make sure that $a \lhd^{\text{þ}}_{A'} b_0 \ldots b_{n-1}$.

So suppose $a \not\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A'} c$ but $b_0 \ldots b_{n-1} \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{A'} c$. Then by definition $a, A', B \cup \{c\}$ witness $\mathrm{wt}^{\text{þ}}(p) \geq n+1$. By Lemma 3.10 there are D, c' such that $a, D, B \cup \{c'\}$ witness $\mathrm{wt}^{\text{þ}}(p) \geq n+1$ and $\mathrm{wt}^{\text{þ}}(c'/D) = 1$. By Lemma 3.7,

$b_0 \dots b_{n-1}c' \triangleleft^{\text{þ}}_{A'} a$. By Lemma 3.6 we may assume $ab_0 \dots b_{n-1}c' \triangleleft^{\text{þ}}_{A'} a$. So $a, A', b_0 \dots b_{n-1}c'$ is a nice witness of $\mathrm{wt}^{\text{þ}}(p) \geq n+1$, contradicting the maximality of $a, A', b_0 \dots b_{n-1}$. ■

The following easy observation shows that nice witnesses exist.

OBSERVATION 3.13. *Let $p = \mathrm{tp}(a/A)$ be a non-algebraic type of rudimentarily finite weight. Then there exists a nice witness of $\mathrm{wt}^{\text{þ}}(p) \geq 1$.*

Proof. By Lemma 3.10 we can find b with $\mathrm{wt}^{\text{þ}}(b/A') = 1$ where A' is the domain over which $a \not\downarrow^{\text{þ}}_{A'} b$. Since $\mathrm{wt}^{\text{þ}}(b/A') = 1$ and $a \not\downarrow^{\text{þ}}_{A'} b$, Observation 3.5 implies that $b \triangleleft^{\text{þ}}_{A} a$. Finally, we can assume $ab \triangleleft^{\text{þ}}_{A'} a$ by Lemma 3.6, which finishes the proof. ■

We have finally reached our goal.

THEOREM 3.14. *Let $p \in \mathrm{S}(A)$ be a non-algebraic type of rudimentarily finite þ-weight. Then $\mathrm{wt}^{\text{þ}}(p) < \aleph_0$ and p is þ-equidominant with a finite free product of þ-weight-1 types. More precisely, there exist $a, A', \{b_i : i < n\}$ such that*

- *$a, A', \{b_i : i < n\}$ witness that $\mathrm{wt}^{\text{þ}}(p) \geq n$,*
- *$\mathrm{wt}^{\text{þ}}(b_i/A') = 1$ for all i,*
- *$a \bowtie^{\text{þ}}_{A'} b_0 \dots b_{n-1}$.*

Proof. Let a, A', $B = \{b_i : i < n\}$ be such that

(i) $a, A', \{b_i : i < n\}$ witness that $\mathrm{wt}^{\text{þ}}(p) \geq n$,
(ii) $\mathrm{wt}^{\text{þ}}(b_i/A') = 1$ for all i,
(iii) $aB \triangleleft^{\text{þ}}_{A'} a$,
(iv) $\{b_i : i < n\}$ is maximal satisfying (i)–(iii). In other words, if there are $A'' \supseteq A'$, $B'' \supseteq B$ satisfying (i)–(iii), then $B'' = B$.

In other words, a, A', B is a maximal nice witness for $\mathrm{wt}^{\text{þ}}(p) \geq n$. It is easy to see that such A', B exist: Observation 3.13 gives us a non-empty B_0 satisfying (i)–(iii). Since p has rudimentarily finite weight, by Observation 3.11 we know that B_0 is contained in a maximal B, as required in (i)–(iv) above.

By Lemma 3.12, $a \bowtie^{\text{þ}}_{A'} b_0 \dots b_{n-1}$. By Lemma 3.2 and Observation 3.4 it follows that p has finite weight n. ■

Reading carefully the above proof, we obtain the following more precise statement.

COROLLARY 3.15. *Let p be a type of rudimentarily finite þ-weight. Then $\mathrm{wt}^{\text{þ}}(p) = n$ for some $n < \omega$, and any maximal nice witness $a, A', \{b_i : i < m\}$ of $\mathrm{wt}^{\text{þ}}(p) \geq m$ satisfies $m = n$ and $a \bowtie^{\text{þ}}_{A'} b_0 \dots b_{n-1}$.*

COROLLARY 3.16. *In a strongly dependent (and even strong) rosy theory, every type has finite þ-weight.*

Proof. By Theorems 2.7 and 3.14. ∎

3.3. þ-regular types. We will finish this section by investigating basic properties of þ-regular types. Their definition is the analogue of the definition of regular types in the stable and simple context.

DEFINITION 3.3. A type $r(x)$ over A is *þ-regular* if given any $B \supset A$, a þ-forking extension $q(x) \in \mathrm{S}(B)$ of $r(x)$, and a non-þ-forking extension $p(x) \in \mathrm{S}(B)$ of $r(x)$, the extension $q(x)$ is almost þ-orthogonal to $p(x)$.

We originally proved the following desired property of þ-regular types, assuming finite þ-weights, as an easy corollary of the definition of þ-regularity and the results we have so far in this section. The referee then pointed out the following proof which makes no use of finite þ-weight.

PROPOSITION 3.17. *A þ-regular type has þ-weight* 1.

Proof. Let $p(x) := \mathrm{tp}(a/A)$ be a þ-regular type, and assume towards a contradiction that we have a witness $a, \{b_1, b_2\}$ for $\mathrm{pwt}^{\text{þ}}(p) \geq 2$.

Let $\langle a_i \rangle_{i \leq \omega}$ be a þ-Morley sequence in $\mathrm{tp}(a/Ab_1)$ with $a_\omega = a$ and $\langle a_i \rangle_{i<\omega} \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{Ab_1 a} b_2$. Notice that, since $\langle a_i \rangle_{i<\omega} \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{Ab_1} a$, we have $\langle a_i \rangle_{i<\omega} \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{Ab_1} ab_2$ and in particular

$$b_2 \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{A} \langle a_i \rangle_{i<\omega}. \tag{4}$$

But $\mathrm{tp}(a/A\langle a_i \rangle_{i<\omega} b_1)$ is finitely satisfiable in $A\langle a_i \rangle_{i<\omega}$, so $a \mathop{\smile\!\!\!\!\!\mid}_{A\langle a_i \rangle_{i<\omega}} b_1$ and $a \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{A\langle a_i \rangle_{i<\omega}} b_1$. Hence $a \not\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{A} \langle a_i \rangle_{i<\omega}$.

Let n be minimal with $a \not\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{A} \langle a_i \rangle_{i<n}$. By minimality (and the fact that $\mathrm{tp}(a/\langle a_i \rangle_{i<k}) = \mathrm{tp}(a_k/\langle a_i \rangle_{i<k})$ for all k), the sequence $\langle a_i \rangle_{i<n}$ is independent over A. But it is also independent over Ab_2 by (4) so by transitivity $a \not\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{Ab_2} \langle a_i \rangle_{i<n}$. But $\mathrm{tp}(a/Ab_2)$ is a forking extension of p and $\mathrm{tp}(a_k/\langle a_i \rangle_{i<k} b_2)$ is a non-forking extension of $p(x)$ for all $0 \leq k \leq n-1$, which implies that at some point we will contradict þ-regularity of p. ∎

We conclude by pointing out the following unsurprising but important property of a regular type:

OBSERVATION 3.18. *Let $p \in \mathrm{S}(A)$ be a þ-regular type. Define (as usual), for a tuple $\bar{c}$ of realizations of p,*

$$\mathrm{cl}_p(\bar{c}) = \{a \models p \colon a \not\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{A} \bar{c}\}.$$

Then $(p^{\mathfrak{C}}, \mathrm{cl}_p)$ is a pregeometry.

Proof. The proof is quite easy and it is the same as the standard proof of the analogous result for (forking) regular types. ∎

REMARK 3.19. We should mention that the converse of Observation 3.18 is true assuming stability of $p(x)$ (see [8]). In the general—rosy—context, however, we have been unable to either prove it or show a counterexample.

4. Super-rosy theories and types of finite $\mathrm{U}^{þ}$-rank. The goal of this section is proving that under reasonable assumptions, any type can be "decomposed" into a finite product of "geometric" types.

4.1. Exchange and decomposition in types of finite weight. Recall that in Theorem 3.14 we in particular proved the following.

THEOREM 4.1. *Let $p \in \mathrm{S}(A)$ be such that $\mathrm{wt}^{þ}(p) = n$. Then there exists a set B with $A \subseteq B$ and $b_1, \ldots, b_n$ þ-independent over B such that $p \bowtie \mathrm{tp}(b_1 \ldots b_n/B)$ and $\mathrm{wt}^{þ}(b_i/B) = 1$.*

We will improve this statement by replacing þ-weight-1 types in the conclusion by regular types (in the super-rosy context) and þ-minimal types (in the finite rank context).

LEMMA 4.2 (Exchange Lemma). *Let $a, b_1, \ldots, b_n$ be a þ-weight-1 witness of $\mathrm{wt}^{þ}(\mathrm{tp}(a/A)) = n$. Let q be a type with $\mathrm{dom}(q) \supset A$ such that q is not þ-orthogonal to $\mathrm{tp}(b_n/A)$ and $\mathrm{wt}^{þ}(q) = 1$. Then there is some $b \models q$ and some B such that $a, B, \langle b_1, \ldots, b_{n-1}, b\rangle$ witness $\mathrm{tp}(a/A)$ has þ-weight n. Moreover, if $a \bowtie^{þ}_{A} b_1 \ldots b_n$, then we can find B such that both $a \bowtie^{þ}_{B} b_1 \ldots b_{n-1}b_n$ and $a \bowtie^{þ}_{B} b_1 \ldots b_{n-1}b$.*

Proof. Let $b' \models q$, B' be such that $b_n \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} B'$, $b' \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} B'$ and $b' \not\mathop{\smile\!\!\!\!\!\mid}^{þ}_{B'} b_n$ (such b' and B' exist as $\mathrm{tp}(b_n/A)$ and q are not þ-orthogonal).

Without loss of generality $Bb \mathop{\smile\!\!\!\!\!\mid}^{þ}_{Ab_n} ab_1 \ldots b_{n-1}$. In particular, $ab_1 \ldots b_n \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} B$ and $b_1 \ldots b_{n-1} \mathop{\smile\!\!\!\!\!\mid}^{þ}_{B} b_n b$, and so the set $\{b, b_1, \ldots, b_{n-1}\}$ is independent over B.

Now if $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{B} b$, then a, b witness $\mathrm{wt}^{þ}(\mathrm{tp}(b_n/B)) \geq 2$, which contradicts our assumptions (via Lemma 3.2). So $a \not\mathop{\smile\!\!\!\!\!\mid}^{þ}_{B} b$ and by the definition $a, B, \langle b_1, \ldots, b_{n-1}, b\rangle$ witnesses $\mathrm{tp}(a/A)$ has þ-weight n.

For the "moreover" part, assume that $a \bowtie^{þ}_{A} b_1 \ldots b_n$. Recall that by Observations 3.6 and 3.3 we may assume $ab_1 \ldots b_n \lhd^{þ}_{A} a$ (that is, first replace A with some A' such that $a \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} A'$, $b_1 \ldots b_n \mathop{\smile\!\!\!\!\!\mid}^{þ}_{A} A'$, and $ab_1 \ldots b_n \lhd^{þ}_{A'} a$, and then find B), hence (by 3.3 again) $ab_1 \ldots b_n \lhd^{þ}_{B} a$. By Lemma 3.7, $b_1 \ldots b_{n-1} b \lhd^{þ}_{B} a$. Finally by Observation 3.8 we have $b_1 \ldots b_{n-1} b_n \bowtie^{þ}_{B} a$ and $b_1 \ldots b_{n-1} b \bowtie^{þ}_{B} a$, as required. ∎

4.2. þ-regularity and decomposition in the super-rosy case. As in the super-stable case, we first prove the existence of "many" þ-regular types in a super-rosy theory, which makes the theory of þ-regular types relevant. We will also point out that all super-rosy types in a rosy theory have finite þ-weight (hence the results of the previous section apply in the super-rosy context).

PROPOSITION 4.3. *Let T be super-rosy. Then every type p with domain A is non-þ-orthogonal to a þ-regular type q with domain $B \supset A$.*

Proof. The proof is a variation of the proof of Proposition 5.1.11 in [12].

Let $\mathcal{P}$ be the set of types r such that $\mathrm{dom}(r) = B \supset A$ and r is not almost þ-orthogonal to p, and let q be a type in $\mathcal{P}$ of minimal $\mathrm{U}^{\text{þ}}$-rank. Let a', b be realizations of p, q respectively such that $a' \perp\!\!\!\perp^{\text{þ}}_A B$ and $a' \not\perp\!\!\!\perp^{\text{þ}}_B b$.

Suppose q is not þ-regular so there is some c', b', C' such that $c', b' \models q$, $b' \perp\!\!\!\perp^{\text{þ}}_B C'$, $c' \not\perp\!\!\!\perp^{\text{þ}}_B C'$ and $b' \not\perp\!\!\!\perp^{\text{þ}}_{C'} c'$.

Since $\mathrm{tp}(b'/B) = \mathrm{tp}(b/B) = q$ there is an automorphism fixing B and sending c', b', C' to elements c, b, C and let $a \models p$ realize a non-þ-forking extension of $\mathrm{tp}(a'/bB)$ to bBC. So $a \perp\!\!\!\perp^{\text{þ}}_{bB} C$, and since $b \perp\!\!\!\perp^{\text{þ}}_B C$, by transitivity we have $ab \perp\!\!\!\perp^{\text{þ}}_B C$, which implies that $a \perp\!\!\!\perp^{\text{þ}}_B C$; it follows that $a \perp\!\!\!\perp^{\text{þ}}_A C$ (recall that $a' \perp\!\!\!\perp^{\text{þ}}_A B$ and $\mathrm{tp}(a/B) = \mathrm{tp}(a'/B)$).

Notice also that $a \not\perp\!\!\!\perp^{\text{þ}}_C b$ (as $a \not\perp\!\!\!\perp^{\text{þ}}_B b$ and $a \perp\!\!\!\perp^{\text{þ}}_B C$).

So we have $a \perp\!\!\!\perp^{\text{þ}}_A C$, $a \not\perp\!\!\!\perp^{\text{þ}}_C b$, $b \perp\!\!\!\perp^{\text{þ}}_B C$, $c \not\perp\!\!\!\perp^{\text{þ}}_B C$ and $b \not\perp\!\!\!\perp^{\text{þ}}_C c$. In particular

$$\mathrm{U}^{\text{þ}}(\mathrm{tp}(c/C)) < \mathrm{U}^{\text{þ}}(\mathrm{tp}(c/B)) = \mathrm{U}^{\text{þ}}(\mathrm{tp}(b/B))$$

and

$$\mathrm{U}^{\text{þ}}(\mathrm{tp}(b/Cc)) < \mathrm{U}^{\text{þ}}(\mathrm{tp}(b/C)) = \mathrm{U}^{\text{þ}}(\mathrm{tp}(b/B));$$

by minimality of $\mathrm{U}^{\text{þ}}(\mathrm{tp}(b/B))$ (among all types in $\mathcal{P}$) we deduce that $\mathrm{tp}(c/C)$ and $\mathrm{tp}(b/Cc)$ are not in $\mathcal{P}$; so in particular $a \perp\!\!\!\perp^{\text{þ}}_C c$ and $a \perp\!\!\!\perp^{\text{þ}}_{Cc} b$. By transitivity $a \perp\!\!\!\perp^{\text{þ}}_C bc$ and $a \perp\!\!\!\perp^{\text{þ}}_C b$, a contradiction. ∎

PROPOSITION 4.4. *Let $p(x)$ be a type such that*

$$\mathrm{U}^{\text{þ}}(p) = \sum_{i=1}^{k} \omega^{\alpha_i} n_i.$$

Then p has þ-weight at most $\sum_{i=1}^{k} n_i$.

Proof. This is word for word the same proof as for Theorem 5.2.5 in [12] using the þ-forking version of Lascar's inequalities (Fact 1.3). ∎

As an easy corollary we obtain the following theorem which strengthens Theorem 4.1 in the super-rosy context.

Theorem 4.5.

- *Any super-rosy type has finite þ-weight.*
- *Let T be super-rosy and $p \in \mathrm{S}(A)$. Then $\mathrm{wt}^{\text{þ}}(p) = n$ for some $n < \omega$ and there exists a set B with $A \subseteq B$ and $b_1, \ldots, b_n$ þ-independent over B such that $p \bowtie \mathrm{tp}(b_1 \ldots b_n/B)$ and $\mathrm{tp}(b_i/B)$ are þ-regular.*

Proof. The first item follows immediately from Proposition 4.4.

To prove the second item, notice first that $\mathrm{wt}^{\text{þ}}(p)$ is finite by Proposition 4.4. Now apply Theorem 4.1 combined with existence of þ-regular types (Proposition 4.3) and the Exchange Lemma (Lemma 4.2), recalling that by Proposition 3.17, þ-regular types have þ-weight 1. ∎

4.3. Types of finite $\mathrm{U}^{\text{þ}}$-rank. The following proposition is an interesting result with many consequences in theories of finite $\mathrm{U}^{\text{þ}}$-rank.

Proposition 4.6.

- *Let $p(x) = \mathrm{tp}(b/A)$ be any type such that $\mathrm{U}^{\text{þ}}(p) = \alpha + 1$. Then there is a finite tuple a and a tuple e such that $\mathrm{U}^{\text{þ}}(\mathrm{tp}(b/Aa)) = \alpha$, $b \mathop{\smile\!\!\!\!\!\!\,|}^{\text{þ}}_{A} e$, $b \mathop{\smile\!\!\!\!\!\!\,|}^{\text{þ}}_{Aa} e$, $\mathrm{tp}(b/Aa)$ contains a formula which strongly divides over Ae and $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a/Ae)) = 1$.*
- *If $p(x) = \mathrm{tp}(b/A)$ is any type of þ-rank $\alpha + 1$ then there is a non-þ-forking extension $\mathrm{tp}(b/Ae)$ of p and a finite tuple $a \in \mathrm{acl}(Abe)$ such that $\mathrm{tp}(a/Ae)$ is minimal.*

Proof. The second item follows immediately from the first one. To prove the first item, notice that we can choose a' finite so that $p(x, a')$ over Aa' is a þ-dividing extension of $p(x)$ and $\mathrm{U}^{\text{þ}}(\mathrm{tp}(b/Aa')) = \alpha$. By Observation 1.42 there is some e and some finite $a \in \mathrm{dcl}(Aa')$ such that $b \mathop{\smile\!\!\!\!\!\!\,|}^{\text{þ}}_{Aa'} e$ and $\mathrm{tp}(b/Aa'e)$ contains a formula $\phi(x, a)$ which strongly divides over Ae, so in particular $a \in \mathrm{acl}(Abe)$. Note that

$$\alpha = \mathrm{U}^{\text{þ}}(\mathrm{tp}(b/Aa)) = \mathrm{U}^{\text{þ}}(\mathrm{tp}(b/Aae)) < \mathrm{U}^{\text{þ}}(\mathrm{tp}(b/Ae)) \leq \mathrm{U}^{\text{þ}}(\mathrm{tp}(b/A)) = \alpha + 1,$$

hence $\mathrm{U}^{\text{þ}}(\mathrm{tp}(b/Ae)) = \mathrm{U}^{\text{þ}}(\mathrm{tp}(b/A)) = \alpha + 1$; in particular, $b \mathop{\smile\!\!\!\!\!\!\,|}^{\text{þ}}_{A} e$. By Lascar's inequalities we know that

$$\mathrm{U}^{\text{þ}}(\mathrm{tp}(ba/Ae)) = \mathrm{U}^{\text{þ}}(\mathrm{tp}(b/Ae)) + \mathrm{U}^{\text{þ}}(\mathrm{tp}(a/Abe)) = \alpha + 1 + 0 = \alpha + 1$$

and

$$\mathrm{U}^{\text{þ}}(b/Aae) + \mathrm{U}^{\text{þ}}(\mathrm{tp}(a/Ae)) \leq \mathrm{U}^{\text{þ}}(\mathrm{tp}(ba/Ae)) \leq \mathrm{U}^{\text{þ}}(b/Aae) \oplus \mathrm{U}^{\text{þ}}(\mathrm{tp}(a/Ae)).$$

So

$$\alpha + \mathrm{U}^{\text{þ}}(\mathrm{tp}(a/Ae)) \leq \alpha + 1 \leq \alpha \oplus \mathrm{U}^{\text{þ}}(\mathrm{tp}(a/Ae)),$$

and the result follows. ∎

Notice that Proposition 4.6 provides the inductive step, in theories of finite $U^{þ}$-rank, for any property which is closed under non-þ-forking restrictions and coordinatized types (in the sense that if a type p is coordinatized by types having the property, then p must have the property). This has nice consequences (it was strongly used, for example, in [4]). Some of the more direct consequences include the following.

COROLLARY 4.7. *Let $p(x)$ be any type of finite $U^{þ}$-rank. Then $p(x)$ is non-þ-orthogonal to a þ-minimal type.*

Proof. By Proposition 4.6 given $p(x) = \mathrm{tp}(b/A)$ of finite $U^{þ}$-rank, there is a non-þ-forking extension $\mathrm{tp}(b/Ae)$ and an element $a \in \mathrm{acl}(Abe)$ such that $\mathrm{tp}(a/Ae)$ is þ-minimal. Clearly $\mathrm{tp}(b/Ae)$ and $\mathrm{tp}(a/Ae)$ are non-almost-þ-orthogonal. ∎

COROLLARY 4.8. *Let $p \in S(A)$ be a type of finite $U^{þ}$-rank. Then $\mathrm{wt}^{þ}(p) = n$ for some finite n and there is a set B with $A \subseteq B$, and $b_1, \ldots, b_n$ independent over B such that $p \bowtie \mathrm{tp}(b_1 \ldots b_n/B)$ and $U^{þ}(\mathrm{tp}(b_i/B)) = 1$.*

Proof. The fact that a type of finite $U^{þ}$-rank has finite þ-weight follows from Proposition 4.4. The rest of the assertions follow from Theorem 4.1 using the Exchange Lemma and Corollary 4.7. ∎

We will conclude this section by making some remarks about Proposition 4.6. Let us first recall the following definition.

DEFINITION 4.1. We will say that $\mathrm{tp}(a/A)$ is *coordinatizable by þ-minimal types* if there are $a_0, a_1, \ldots, a_N$ such that $a_n \in \mathrm{acl}(Aa)$ for all n, $a_N = a$, and $\mathrm{tp}(a_{n+1}/Aa_0 \ldots a_n)$ has $U^{þ}$-rank one.

We will say that $\mathrm{tp}(a/A)$ is *coordinatizable by minimal types* if there are $a_0, a_1, \ldots, a_N$ such that $a_n \in \mathrm{acl}(Aa)$ for all n, $a_N = a$, and $\mathrm{tp}(a_{n+1}/Aa_0 \ldots a_n)$ has U-rank one.

Whenever $a_0, a_1, \ldots, a_N$ witness that $\mathrm{tp}(a/A)$ is coordinatizable by (þ-) minimal types, we will say that $a_0, a_1, \ldots, a_N$ is a (þ-)*coordinatizing sequence* of $\mathrm{tp}(a/A)$.

At first glance, it would appear that one could coordinatize a non-þ-forking extension of any type of finite $U^{þ}$-rank by repeatedly applying Proposition 4.6. However, this would prove a coordinatization theorem in the stable case, which is known not to be true, as the following example shows.

EXAMPLE 4.9. Let $\mathcal{L} := \{L, E\}$ be such that L is a ternary relation and E a binary relation and let T be the theory that states that E is an equivalence relation with infinitely many infinite classes and such that L defines an affine space on each E-class (so $L(x, y, z) \Rightarrow E(x, y) \wedge E(x, z) \wedge E(y, z)$). A natural model M of this theory is a sheaf of affine planes indexed by a line, where $E(x,y)$ if and only if x and y are in the same plane and $L(x,y,z)$ happens whenever x,y,z are collinear points in the same E-class.

Let g be a $\emptyset$-generic E-class in M and a a g-generic point in g. The conclusion of Proposition 4.6 applied to the type $\mathrm{tp}(a/g)$ can be seen in the following way: Let b be any point in g such that $a \mathop{\smile\!\!\!\!\!\!\,|}^{\text{þ}}_{g} b$ and let l be the line through a and b (so that $l := \{x\colon L(x,a,b)\}$). Then $\mathrm{tp}(a/gb)$ is a non-forking extension of $\mathrm{tp}(a/g)$, $l \in \mathrm{acl}(ab)$ and $\mathrm{tp}(l/gb)$ is a þ-minimal type.

Going back to coordinatization, if we try to coordinatize $\mathrm{tp}(a/\emptyset)$ the first step is $\mathrm{tp}(a/g)$ and $\mathrm{tp}(g/\emptyset)$. The next step, however, would be to coordinatize the non-þ-forking extension $\mathrm{tp}(a/gb)$ of $\mathrm{tp}(a/g)$. But $b \not\mathop{\smile\!\!\!\!\!\!\,|}^{\text{þ}}_{\emptyset} a$ (and the reader can check that this is true for every possible b we can choose) so this does not help at all in trying to coordinatize $\mathrm{tp}(a/\emptyset)$, or any non-þ-forking extension of it. In fact, it is not hard to check that $\mathrm{tp}(a/\emptyset)$ cannot be coordinatized in terms of þ-minimal types.

This gives a superstable (even ω-stable) example where no coordinatization is possible, which illustrates the limitations of Proposition 4.6 to get a full coordinatization result for super-rosy theories. This means that even under very strong assumptions on the theory, one cannot hope to analyze a finite rank type from the minimal components in terms of Definition 4.1.

This was overcome first by Zilber for uncountably categorical theories and later by Hrushovski for superstable theories. They proved coordinatization results, but instead of requiring each of the "intermediate" types to be minimal, they required first that each of the types be either minimal or strongly related to a strongly minimal set ("almost strongly minimal"), and second that one could control the automorphism groups of the types via a definable group (the "binding group"). This allowed one to have a very deep control of the types in terms of minimal types and to have strong combinatorial manipulations that eventually lead to group and field existence theorems which were some of the most important results in geometric stability theory.

That said, the question of whether or not a finite rank type can be coordinatized as defined in Definition 4.1 is still quite interesting, and when possible, would allow a stronger analysis than the one provided by Zilber and Hrushovski. This makes the understanding of the limitations of Proposition 4.6 quite important. The main issue (with Proposition 4.6) is that we have no control over the parameter e we need to pass from þ-dividing to strong dividing. In the affine space, for example, this e cannot be overlooked, nor can we have any control over where it comes from.

This has two main consequences. On the one hand, once we try to use Proposition 4.6 inductively and coordinatize $\operatorname{tp}(b/Aae)$, the f we need can be taken to be such that $b \mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{Aa} f$ but there is no hope to find one with $b \mathop{\smile\!\!\!\!\!\!|}\nolimits^{þ}_{A} f$. Another consequence is that we can only coordinatize a non-þ-forking extension of types of rank $\alpha + i$ by types of rank α and rank i when $i = 1$, but we cannot do the same for $i > 1$ without further assumptions.

It seems that this lack of control over the choice of e could be somewhat overcome if we had extra assumptions (definable choice seems be the right notion), but even this assumption seems not enough to get any coordinatization-like result beyond possibly the finite $\mathrm{U}^{þ}$-rank case. However, coordinatization is such a useful tool, and the connections with definable choice are so unclear, that even results assuming finite $\mathrm{U}^{þ}$-rank would be quite interesting. Some progress on this question has recently been made in [2].

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