Nonnormality points of $\beta X \setminus X$

by

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Abstract. Let $X$ be a crowded metric space of weight $\kappa$ that is either $\kappa^{\omega}$-like or locally compact. Let $y \in \beta X \setminus X$ and assume GCH. Then a space of nonuniform ultrafilters embeds as a closed subspace of $(\beta X \setminus X) \setminus \{y\}$ with $y$ as the unique limit point. If, in addition, $y$ is a regular $z$-ultrafilter, then the space of nonuniform ultrafilters is not normal, and hence $(\beta X \setminus X) \setminus \{y\}$ is not normal.

1. Introduction. An important theorem about the structure of $\beta X \setminus X$ when $X$ is discrete is due to Bešlagić and van Douwen [1].

Theorem 1.1 (Bešlagić and van Douwen [1]). Assume GCH. Let $\kappa$ be an infinite cardinal, and let $X$ be the discrete space of cardinality $\kappa$. Let $y$ be any point of $\beta X \setminus X$. Then the space of nonuniform ultrafilters on $\kappa^+$ embeds in $(\beta X \setminus X) \setminus \{y\}$ as a closed subset. Hence neither $(\beta X \setminus X) \setminus \{y\}$ nor $\beta X \setminus \{y\}$ is normal.

Recent research has extended nonnormality point results to nondiscrete spaces. For example:

Theorem 1.2 (Logunov [9] and Terasawa [12], independently). If $X$ is a crowded metrizable space space, then $\beta X \setminus \{y\}$ is not normal for all $y \in \beta X \setminus X$.

Theorem 1.3 (Logunov [10]). If $X$ is a crowded realcompact locally compact metrizable space space, and $y$ is not a P-point, then $(\beta X \setminus X) \setminus \{y\}$ is not normal for all $y \in \beta X \setminus X$.

Logunov and Terasawa prove their results without extra axioms of set theory. They prove that $\beta X \setminus \{y\}$ or $(\beta X \setminus X) \setminus \{y\}$ is not normal, but do not embed closed subspaces of nonuniform ultrafilters. Our results are closer to those of Bešlagić and van Douwen.

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Theorem 1.4. Let $X$ be a metric space of weight $\kappa$ without isolated points that is either $\kappa^\omega$-like or locally compact. Let $y \in \beta X \setminus X$. Assume GCH. Then a space of nonuniform ultrafilters embeds as a closed subspace of $(\beta X \setminus X) \setminus \{y\}$ with $y$ as the unique limit point. If $y$ is a regular $\omega$-ultrafilter, then neither $(\beta X \setminus X) \setminus \{y\}$ nor $\beta X \setminus \{y\}$ is normal.

2. Topological spaces. All spaces $X$ are Tihkonov, and hence have a Stone–Čech compactification $\beta X$. We consider a point of $\beta X$ to be a $\omega$-ultrafilter on $X$. We identify a point $x$ of $X$ with the $\omega$-ultrafilter $\tilde{x}$, the collection of all zero sets of $X$ of which $x$ is an element, so that $X$ is embedded as a subspace of $\beta X$. When $f$ is a bounded, continuous function from $X$ to $\mathbb{R}$, we denote the unique extension of $f$ by $\beta f$.

A space is called crowded if it has no isolated points. The topology, weight, and Lindelöf number of a space $X$ are denoted $\tau(X)$, $w(X)$, and $L(X)$. We use the letters $\kappa$, $\lambda$, $\theta$, etc. to denote infinite cardinals and the discrete spaces of that cardinality. We say that a space $X$ is $\kappa^\omega$-like if $X$ is metrizable, nowhere locally compact, and every nonempty open subset of $X$ has weight $\kappa$.

Lemma 2.1. Let $X$ be a $\kappa^\omega$-like metrizable space and let $Z$ be a subset of $X$ with $w(Z) = \lambda < \kappa$. There is a $\lambda^\omega$-like closed subset $Y$ of $X$ containing $Z$.

Proof. Set $Z_1 = Z$. Given $Z_n$ with $L(Z_n) = \lambda$, choose $\mathcal{V}_n \in [\tau(X)]^\lambda$ such that $Z_n \subseteq \bigcup \mathcal{V}_n$ and $\text{diam} \, V < 1/n$ for all $V \in \mathcal{V}$. Choose $Z_{n+1}$ such that $Z_n \subseteq Z_{n+1}$, $|Z_{n+1} \setminus Z_n| \leq \lambda$ (hence $L(Z_{n+1}) = \lambda$), and for all $V \in \mathcal{V}_n$ there is $E \in [V \cap Z_{n+1}]^\lambda$ which is closed discrete (hence $w(V \cap Z_{n+1}) = \lambda$). Set $Y_0 = \bigcup_{n \in \mathbb{N}} Z_n$; note that $w(Y) \leq \lambda$ because $\{V \cap Y_0 : (\exists n) \, V \in \mathcal{V}_n\}$ is a base for $Y_0$.

Let $y \in W$ be open in $Y_0$. There are $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ such that $y \in V \cap Z_{n+1} \subseteq W$. Then $w(W) \geq w(V \cap Z_{n+1}) = \lambda$. Finally, set $Y = \text{cl} \, Y_0$. ~

3. Regular $\omega$-ultrafilters. The next result tells us for which cardinals $\theta$ the space of nonuniform ultrafilters is not normal.

Lemma 3.1. Let $\text{NU}(\theta)$ denote the subspace of $\beta \theta$ of nonuniform ultrafilters. That is, $\text{NU}(\theta) = \{y \in \beta \theta : (\exists Z \in \theta) \, |Z| < \theta\}$.

1. (\text{III}) If $\theta$ is regular and not a strong limit cardinal (in particular, if $\theta = \kappa^+$), then $\text{NU}(\theta)$ is not normal.

2. (\text{III}) If $\theta$ is singular, then $\text{NU}(\theta)$ is not normal.

3. (\text{III}) The space $\text{NU}(\theta)$ is normal if and only if $\theta$ is weakly compact.

In the proof of Theorem 1.1, the reaping number $\tau(\kappa)$ of $\kappa$ is defined, and the space $\text{NU}(\tau(\kappa))$ is embedded in $(\beta \kappa \setminus \kappa) \setminus \{y\}$. The inequalities

\[\tau(\kappa) \leq \omega^\omega \leq \text{cof}(\omega^\omega) \leq \text{cf}(\omega^\omega) = \omega_1 \leq 2^{\text{cf}(\omega^\omega)} = 2^{\omega_1} \leq 2^{\text{cof}(\omega^\omega)} \leq 2^{\text{cf}(\omega^\omega)} = 2^{\omega_1} \leq 2^{2^{\omega_1}} = 2^{2^{\omega_1}}.\]
\( \kappa < \tau(\kappa) \leq 2^\kappa \) hold in ZFC, so GCH gives \( \tau(\kappa) = \kappa^+ \), and the embedded space is not normal.

In the proof of Theorem 1.4 we consider a point \( y \) of \( \beta X \setminus X \). The analog of \( \tau(\kappa) \) is \( \theta_y \), a cardinal which depends on the point \( y \) (not just the space \( X \)). The upper bound \( \theta_y \leq 2^\kappa \) is proved as in [1], but the lower bound \( \kappa < \theta_y \) requires assuming that \( y \) is a regular \( z \)-ultrafilter.

**Definition 3.2.** Let \( y \) be a \( z \)-ultrafilter on a space \( X \). We say that \( y \) is \( \kappa \)-regular if there is a subset \( Z \) of \( y \) such that \( Z \) is locally finite and \( |Z| = \kappa \). We say that \( y \) is regular if \( y \) is \( \omega(X) \)-regular.

If \( X \) is a discrete space of cardinality \( \kappa \), then any ultrafilter on \( X \) is a \( z \)-ultrafilter (because every subset of \( X \) is an open set, a closed set, and a \( z \)-set). In this case, a \( \kappa \)-regular ultrafilter is exactly an \( (\omega, \kappa) \)-regular ultrafilter as defined in [2]. The notion of regular ultrafilter appears implicitly in papers from the mid-1950’s, for example [5].

**Theorem 3.3 ([7, Section 12.7]).** Let \( \kappa \) be an infinite cardinal. There is a maximal ideal \( M \) in \( C(\kappa) \) such that \( |C(\kappa)/M| > \kappa \). In fact, no set of power at most \( \kappa \) is cofinal in the ordered field \( C(\kappa)/M \). If \( 2^\kappa = \kappa^+ \), then \( \text{cf}(C(\kappa)/M) = |C(\kappa)/M| = 2^\kappa \).

**Proof.** Because \( \kappa \) is infinite, there is a bijection \( \alpha \mapsto s_\alpha \) from \( \kappa \) to \([\kappa]^{<\omega}\). For each \( \alpha \in \kappa \), set \( Z_\alpha = \{ \gamma \in \kappa : \alpha \in s_\gamma \} \). By construction, \( \{Z_\alpha : \alpha \in \kappa \} \) has the finite intersection property: if \( s = s_\gamma \in [\kappa]^{<\omega} \), then \( \gamma \in \bigcap\{Z_\alpha : \alpha \in s\} \). Extend \( \{Z_\alpha : \alpha \in \kappa \} \) to a \( z \)-ultrafilter \( y \), and set \( M = \{ f \in C(\kappa) : f^{-1}\{0\} \subseteq y \} \).

Given \( B = \{g_\alpha : \alpha < \kappa \} \subseteq C(\kappa) \), define

\[
 f(\gamma) = 1 + \max\{g_\alpha(\gamma) : \alpha \in s_\gamma \}.
\]

The maximum exists because \( s_\gamma \) is finite, and \( f \) is continuous because \( \kappa \) is discrete. Let \( g_\alpha \in B \) be arbitrary. For every \( \gamma \in Z_\alpha \),

\[
 g_\alpha(\gamma) \leq \max\{g_{\alpha'} : \alpha' \in s_\gamma \} < f(\gamma). \tag*{■}
\]

We can generalize the previous theorem to show that if \( X \) is a paracompact space, and \( y \) is a regular \( z \)-ultrafilter on \( X \), then \( C(X)/M_y \) has cofinality greater than \( \kappa \), where \( M_y \) is the maximal ideal of functions \( f \) such that \( \{x \in X : f(x) = 0\} \subseteq y \). We have also generalized the notion of “\( \kappa^+ \)-good” to \( z \)-ultrafilters and proved the analogous theorem. If \( y \) is a \( \kappa^+ \)-good \( z \)-ultrafilter on a paracompact space \( X \), then \( C(X)/M_y \) is an \( \eta_\alpha \)-set, where \( \kappa^+ = \aleph_\alpha \).

**Definition 3.4.** Let \( \text{UR}(\kappa) \) be the assertion that every uniform ultrafilter on a set of cardinality \( \kappa \) is \( \kappa \)-regular. Let \( \text{UR} \) assert that \( \text{UR}(\kappa) \) holds for every infinite \( \kappa \). Informally, we read \( \text{UR} \) as “every uniform ultrafilter is regular”.
The most familiar example of a nonregular ultrafilter is a countably complete free ultrafilter on a measurable cardinal. Hence UR implies that there are no measurable cardinals. Like the assumption that there are no measurable cardinals, UR is safe. The assumption of Theorem 1.4, GCH + UR, is a consequence of V = L. Hence UR does not imply that ZFC is consistent. On the other hand, it has been shown that \( \neg \text{UR} \) does imply that ZFC is consistent. In fact, it is plausible to conjecture that \( \neg \text{UR} \) is equiconsistent with “there exists a measurable cardinal”. See [3].

**Lemma 3.5.** Assume UR(\( \kappa \)). That is, every uniform ultrafilter \( p \) on a set of cardinality \( \kappa \) is \( \kappa \)-regular. Let \( X \) be a metrizable space of weight \( \kappa \) which is locally compact. Then every uniform z-ultrafilter \( y \) on \( X \) is \( \kappa \)-regular.

**Proof.** Let \( C \) be the collection of open subsets of \( X \) that have compact closure. Because \( X \) is locally compact, \( C \) covers \( X \). Let \( R \) be a locally finite open refinement of \( C \).

We claim that \( |R| = \kappa \). Since \( y \) is free, \( X \) is not compact and therefore \( R \) cannot be finite. Hence if \( \kappa = \omega \) then \( |R| = \kappa = \omega \). Suppose that \( \kappa > \omega \). Let \( B \) be a base for \( X \) of cardinality \( \kappa \). Because \( R \) is locally finite, \( |R| \leq |B| = \kappa \).

In the other direction, if \( R \in \mathcal{R} \), then \( L(R) = \omega \). Hence \( \kappa = L(\bigcup \mathcal{R}) \leq |\mathcal{R}| \cdot \omega \).

For each \( Z \in y \), set \( U(Z) = \{ U \in R : U \cap Z \neq \emptyset \} \). Observe that \( p^0 = \{ U(Z) : Z \in y \} \cup \{ X \setminus S : S \in [\mathcal{R}]^{<\kappa} \} \) has the finite intersection property, and extend it to a uniform ultrafilter \( p \) on \( \mathcal{R} \).

Because \( p \) is \( \kappa \)-regular, there is a point finite collection \( \{ U_\alpha : \alpha \in \kappa \} \subset p \). For each \( \alpha \), set \( Z_\alpha = \text{cl}\bigcup U_\alpha \). We now show that \( Z_\alpha \in y \). Let \( Z \in y \) be arbitrary. The collections \( U_\alpha \) and \( U(Z) \) are both members of \( p \), so \( U_\alpha \cap U(Z) \neq \emptyset \). Let \( U \in U_\alpha \cap U(Z) \). Then \( U \cap Z \neq \emptyset \) and therefore \( \bigcup U_\alpha \cap Z \neq \emptyset \). Hence \( Z_\alpha \cap Z \neq \emptyset \), so \( Z_\alpha \in y \).

We have shown that \( \{ Z_\alpha : \alpha \in \kappa \} \) is a subset of \( y \); we must show that it is locally finite. Because \( \mathcal{R} \) is locally finite, for each \( x \in X \) there is an open set \( V \) such that \( x \in V \) and \( \{ U \in \mathcal{R} : V \cap U \neq \emptyset \} \) is finite. Then \( \{ \alpha \in \kappa : (\exists U \in U_\alpha) \ V \cap U \neq \emptyset \} \) is finite, and we are done.

In the result above, the hypothesis “\( X \) is locally compact” can be replaced with the cumbersome “Let \( X \) have a cover \( C \) of open sets of weight less than \( \lambda \), for some regular cardinal \( \lambda \) less than or equal to \( \kappa \)”.

**4. Pi-bases.** In our constructions we will use locally finite pairwise disjoint collections \( \xi \) of open sets. The collections will come from an appropriate \( \pi \)-base. Following Terasawa we use \( \xi^* \) to denote \( \bigcup \xi \). Observe that such a collection \( \xi \) is locally finite and maximal disjoint if and only if \( \xi^* \) is dense in \( X \).
Proposition 4.1 (Terasawa). Let $X$ be a crowded metrizable space. Then $X$ has a $\pi$-base

$$\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$$

such that

1. $\mathcal{B}_n$ is a locally finite, maximal disjoint family of nonempty open sets;
2. $\mathcal{B}_n$ refines $\mathcal{B}_{n-1}$;
3. for each $B \in \mathcal{B}_{n-1}$, there are three sets $B^{(i)} \in \mathcal{B}_n$, $i = 0, 1, 2$, such that $\text{cl} B^{(i)} \subset B$ and $\text{cl} B^{(i)} \cap \text{cl} B^{(j)} = \emptyset$ for $i \neq j$;
4. every open cover of $X$ is refined by a locally finite, maximal disjoint subfamily of $\mathcal{B}$.

Suppose $y \in \beta X \setminus X$. Terasawa remarks that the $\pi$-base in Proposition 4.1 can be easily modified so that

$$y \notin \text{cl}_{\beta X} B \quad \text{for all } B \in \mathcal{B}.$$  \(\#\)

This property of $\mathcal{B}$ was not, however, necessary in his proof that $\beta X \setminus \{y\}$ is not normal; the butterfly sets did not need to be subsets of $\beta X \setminus X$. To show that $(\beta X \setminus X) \setminus \{y\}$ is not normal, our construction will require closed subsets of $\beta X \setminus X$. The following propositions define a $\pi$-base $\mathcal{B}$ for two types of metric spaces. For $X$ locally compact, $(\#)$ is true for $\mathcal{B}$ for any $y \in \beta X \setminus X$. For $X$ $\kappa\omega$-like, given $y \in \beta X \setminus X$, we construct $\mathcal{B}$ so that $(\#)$ is satisfied.

We say that a $\pi$-base $\mathcal{B}$ for a crowded metric space is nice if it satisfies (1), (2) and (4) in Proposition 4.1. In Section 5 we use the properties of a nice $\pi$-base to construct locally finite collections. In the sections after 5 we use a nice $\pi$-base with the additional properties (3) and $(\#)$.

The proofs of the next two results are omitted because they follow easily from Proposition 4.1.

Proposition 4.2. Let $X$ be a locally compact crowded metrizable space. Then $X$ has a $\pi$-base

$$\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$$

such that

1. $\mathcal{B}_n$ is a locally finite, maximal disjoint family such that $\text{cl}_X B$ is compact for each $B \in \mathcal{B}$;
2. $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$ and $|\{B' \in \mathcal{B}_{n+1} : B' \subset B\}| = 4$ for all $B \in \mathcal{B}_n$;
3. for $B \in \mathcal{B}_n$ there are $B^0, B^1 \in \mathcal{B}_{n+1}$ such that $\text{cl} B^0 \cap \text{cl} B^1 = \emptyset$ and $\text{cl} B^0, \text{cl} B^1 \subset B$;
4. every open cover of $X$ is refined by a locally finite, maximal disjoint subfamily of $\mathcal{B}$.  \(\Box\)
Proposition 4.3. Let $\kappa$ be an infinite cardinal and let $X$ be a $\kappa^\omega$-like metric space. Let $y$ be a free $z$-ultrafilter on $X$. Then $X$ has a $\pi$-base

$$B = \bigcup_{n \in \omega} B_n$$

such that

1. $B_n$ is a locally finite, maximal disjoint family of nonempty open sets;
2. $B_n$ refines $B_{n-1}$;
3. $|B_0| = \kappa$ and for each $B \in B_{n-1}$, there are sets $B^{(\eta)} \in B_n$, $\eta \in \kappa$, such that $\text{cl } B^{(\eta)} \subseteq B$ and $\text{cl } B^{(\eta)} \cap \text{cl } B^{(\eta')} = \emptyset$ for $\eta \neq \eta'$;
4. every open cover of $X$ is refined by a locally finite, maximal disjoint subfamily of $B$;
5. $\text{cl } B \notin y$ for all $B \in B$.

5. Locally finite collections and cofinalities. Let $X$ be a crowded metrizable space with a nice $\pi$-base $\mathcal{B}$. Let $\Xi$ be the collection of maximal pairwise disjoint, locally finite collections $\xi \subset B$.

Remark 1. For each $B, B' \in B$, if $B \cap B' \neq \emptyset$ then either $B = B'$, $B \subset B'$ or $B' \subset B$.

Remark 2. If $\xi, \eta \in \Xi$ and $B \in \xi$, then since both $\xi^*$ and $\eta^*$ are dense in $X$, because of Remark 1, there is $B' \in \eta$ such that either $B = B'$, $B \subset B'$ or $B' \subset B$.

Fix a free $z$-ultrafilter $y$ on $X$ and let $\tau_y$ be the collection of open neighborhoods of $y$ in $\beta X$. Let $\mathcal{N}_y = \{X \cap O : y \in O, O \in \tau(\beta X)\}$. The collection $\mathcal{N}_y$ is a free open filter on $X$. We write $\mathcal{N}_y^*$ for the collection of open subsets $U$ of $X$ that are dense in some $N \in \mathcal{N}_y$, that is, $N \subset \text{cl } U$. Using $\mathcal{N}_y^*$, we define a strict partial order $<_y$ on $\Xi$. For $\xi, \eta \in \Xi$ let $L(\xi, \eta) = \{B \in \xi : B' \subset B \text{ for some } B' \in \eta\}$. Define $\xi <_y \eta$ if $L(\xi, \eta)^* \subset \mathcal{N}_y^*$.

The lemma below is analogous to Theorem 3.3.

Lemma 5.1. Let $\kappa = w(X)$. Suppose $y \in \beta X \setminus X$ is a regular $z$-ultrafilter. Any subset $\{\xi_\gamma : \gamma \in \lambda\}$ of $\Xi$ where $\lambda \leq \kappa$ is bounded.

Proof. Let $\{\xi_\gamma : \gamma \in \kappa\} \subset \Xi$. We construct $\xi \in \Xi$ such that $\xi_\gamma <_y \xi$ for all $\gamma \in \kappa$. Let $\{Z_\gamma : \gamma \in \kappa\} \subset y$ be a locally finite subcollection of $y$. Since $X$ is paracompact, there is a locally finite collection $\mathcal{W} = \{W_\gamma : \gamma \in \kappa\}$ of open subsets of $X$ such that $Z_\gamma \subset W_\gamma$ for all $\alpha \in \kappa$ (see [4] Remark 5.1.19]). Note that $W_\gamma \in \mathcal{N}_y$. For each $x \in X$ let $F_x = \{\gamma : x \in \text{cl } W_\gamma\}$ and set $U_0^x = X \setminus \bigcup\{\text{cl } W_\gamma : \gamma \notin F_x\} = X \setminus \text{cl}(\bigcup\{W_\gamma : \gamma \notin F_x\})$. For $\gamma \in F_x$ let $C(x, \gamma) = \{B \in \xi_\gamma : x \in \text{cl } B\}$ and set $C_x = \bigcup\{C(x, \gamma) : \gamma \in F_x\}$. Define $U_x = U_0^x \setminus \bigcup\{\text{cl } B : B \in \xi_\gamma \setminus C_x, \gamma \in F_x\}$. Since $\xi_\gamma$ is locally finite, $U_x$ is an open neighborhood of $x$. Choose a finite set $E_x \subset X \setminus \{x\}$ such
that \(|E_x \cap B| \geq 1\) for each \(B \in \mathcal{C}_x\). Let \(V_x = U_x \setminus E_x\). For \(B, B' \in \mathcal{B}\), observe that if \(B \subseteq V_x\), \(\gamma \in F_x\), \(B' \in \xi\), and \(B \cap B' = \emptyset\) then \(B \subseteq B'\).

The collection \(\mathcal{V} = \{V_x : x \in X\}\) is an open cover of \(X\). Let \(\xi \in \Xi\) be a maximal locally finite collection refining \(\mathcal{V}\). Suppose \(\gamma \in \kappa\). We will show that \(L(\xi, \xi)^*\) contains \(W_\gamma \cap \xi^* \cap \xi_{\gamma^*}\), and is therefore dense in \(W_\gamma\), and hence \(\xi \triangleleft y \xi\).

Let \(x' \in W_\gamma \cap \xi^* \cap \xi_{\gamma^*}\). So, there are \(x \in X\), \(B \in \xi\), and \(B' \in \xi\) such that \(B \subseteq V_x\) and \(x' \in B \cap B'\). Since \(V_x \cap W_\gamma = \emptyset\) it must be that \(\gamma \in F_x\). Following a previous observation, \(B \subseteq B'\). Hence \(x' \in L(\xi, \xi)^*\).

If we assume that \(2^\kappa = \kappa^+\), we may write \(\Xi\) as \(\{\xi_\gamma : \gamma \in \kappa^+\}\). We define \(\{\xi_\gamma : \gamma \in \kappa^+\}\) by induction, using Lemma 5.1 to define \(\xi_\gamma\) greater than \(\{\xi_\alpha : \alpha < \gamma\} \cup \{\xi_\gamma\}\). The result is a \(<_y\)-increasing sequence \(\{\xi_\gamma : \gamma \in \kappa^+\}\) cofinal in \(\Xi\).

If \(y\) is a remote point, then the partial order \((\Xi, <_y)\) is a total order. We can show, without using axioms beyond ZFC, that if \(y\) is a remote point, then the cofinality of \((\Xi, <_y)\) is equal to the cofinality of \(C(X)/M_y\).

6. \(H\)'s and \(L\)'s. Suppose \(y\) is a \(z\)-ultrafilter on a crowded metric space \(X\) with weight \(\kappa\). Following Logunov [9] and Terasawa [12], in this section we use a cofinal sequence from \(\Xi\) to define a sequence of closed sets intersecting to \(y\).

Suppose \(\{\xi_\gamma : \gamma \in \theta_y\}\) is a cofinal \(<_y\)-increasing sequence in \(\Xi\). We note now that \(\theta_y \leq 2^\kappa\) and make extra assumptions on \(\theta_y\) later. Without loss of generality we may assume that \(\xi_0 \cap B_0 = \emptyset\). If \(\xi_0 \cap B_0 = \emptyset\), replace \(\xi_0\) with \((\xi_0 \setminus B_0) \cup \{B \in B_1 : (\exists B' \in \xi_0 \cap B_0) B \subseteq B'\}\). Let \(\mathcal{N}_\gamma = \{U \subseteq \xi_\gamma : U^* \in \mathcal{N}_y\}\) and let

\[
H_\gamma = \bigcap \{\text{cl}_X U^* : U \in \mathcal{N}_\gamma\}.
\]

**Claim.** For each \(\gamma \in \theta_y\), \(y \in H_\gamma\).

**Proof.** If \(U^*\) and \(V^*\) are dense in \(N\) and \(N'\) from \(\mathcal{N}_y\), then \(U^* \cap V^*\) is dense in \(N \cap N'\), which is also in \(\mathcal{N}_y\). Hence, \(\mathcal{N}_\gamma\) is a filter on \(\xi_\gamma\). Every \(U \in \mathcal{N}_y\) is dense in some \(N \in \mathcal{N}_y\), the trace of a neighborhood of \(y\) on \(X\). Therefore, \(y \in \text{cl}_X U\) for all \(U \in \mathcal{N}_y\).

**Claim.** For each \(\gamma \in \theta_y\), \(H_\gamma \subseteq \beta X \setminus X\).

**Proof.** By Proposition 4.3(5), for any \(B \in \xi_\gamma\), since \(y \notin \text{cl}_X B\) it must be that \(\xi_\gamma \setminus \{B\} \in \mathcal{N}_\gamma\). Fix \(x \in X\). Since \(\xi_\gamma\) is locally finite, \(\mathcal{U} = \{B \in \xi_\gamma : x \in \text{cl}_X B\}\) is finite and hence \(\xi_\gamma \setminus \mathcal{U} \in \mathcal{N}_\gamma\). Also, \(x \notin \text{cl}_X (\xi_\gamma \setminus \mathcal{U})^*\) and therefore \(x \notin H_\gamma\).

**Claim.** If \(\gamma' < \gamma\) then \(H_\gamma \subseteq H_{\gamma'}\).
Proof. Let $\gamma' < \gamma$ and let $U \in \mathcal{N}_{\gamma'}$. We will show that $H_\gamma \subset \text{cl}_B U^*$. Since $\gamma' < \gamma$, $\xi_{\gamma'} <_y \xi_\gamma$ and therefore $L(\xi_{\gamma'}, \xi_\gamma) \in \mathcal{N}_{\gamma'}$. Hence $U \cap L(\xi_{\gamma'}, \xi_\gamma) \in \mathcal{N}_{\gamma'}$. Since $U$, $L(\xi_{\gamma'}, \xi_\gamma) \subset \xi_{\gamma'}$ we find that $U^* \cap L(\xi_{\gamma'}, \xi_\gamma)^* = (U \cap L(\xi_{\gamma'}, \xi_\gamma))^*$. Let $W = U \cap L(\xi_{\gamma'}, \xi_\gamma)$ and $V = \{V \in \xi_\gamma : V \cap U \neq \emptyset \text{ for some } U \in W\}$. Since $\xi_{\gamma'}$ is dense in $X$, $\text{cl}_X V^* \supset W^*$. Furthermore, $V \in \xi_\gamma$, $U \in L(\xi_{\gamma'}, \xi_\gamma)$ and $V \cap U \neq \emptyset$ imply that $V \subset U$. Therefore $V^* \subset W^*$ and hence $\text{cl}_X V^* = \text{cl}_X W^* \subset \text{cl}_B U^*$. 

Claim. $\bigcap\{H_\gamma : \gamma \in \theta_y\} = \{y\}$.

Proof. We have seen that $y \in \bigcap\{H_\gamma : \gamma \in \theta_y\}$. Let $O' \in \tau_y$. We will find $\gamma \in \theta_y$ such that $H_\gamma \subset O'$. Let $W', U' \in \tau_y$ be such that

$$\text{cl}_B W' \subset W' \subset \text{cl}_B U' \subset O'.\
Let O = O' \cap X, U = U' \cap X and W = W' \cap X. So, \text{cl}_X W \subset U \subset \text{cl}_X U \subset O.\
Let V = X \setminus \text{cl}_X W. Then \{U, V\} is an open cover of X. By Proposition 4.2 there is $\xi \in \mathcal{Z}$ that refines $\{U, V\}$. Let $\gamma \in \theta_y$ be such that $\xi <_y \xi_{\gamma'}$. Note that $W \in \mathcal{N}_y$. Since $\xi <_y \xi_{\gamma'}$ we have $L(\xi, \xi_{\gamma'})^* \in \mathcal{N}_y$ and $W \cap L(\xi, \xi_{\gamma'})^* \in \mathcal{N}_y$. Let $\hat{W} = W \cap L(\xi, \xi_{\gamma'})^*$ and let $V = \{B \in \xi_\gamma : B \cap \hat{W} \neq \emptyset\}$. Since $\xi_{\gamma'}^*$ is dense in $X$ and $\hat{W}$ is open, $\text{cl}_X V^* \supset \hat{W}$. Hence $V^* \in \mathcal{N}_y$. On the other hand, if $B \in V$ then $B \cap L(\xi, \xi_{\gamma'})^* \neq \emptyset$ and therefore $B \subset B'$ for some $B' \in \xi$. Since $\xi$ refines $\{U, V\}$, either $B \subset B' \subset U$ or $B \subset B' \subset V$. Since $B \cap W \neq \emptyset$, it cannot be the case that $B \subset V$. Therefore $B \subset U$ and hence $V^* \subset U$ and $\text{cl}_X V^* \subset \text{cl}_X U \subset O$. Then, since $X$ is normal, $\text{cl}_X V^* \subset O'$. Since $V \in \xi_\gamma$ and $V^* \in \mathcal{N}_y$ we have $H_\gamma \subset \text{cl}_X V^* \subset O'$ as desired. 

Next, we will use the cofinal sequence to inductively define a pair of locally finite collections, $\mathcal{L}_0$ and $\mathcal{L}_1$, from $\mathcal{B}$ such that $\text{cl}(\mathcal{L}_0)^* \cap \text{cl}(\mathcal{L}_1)^* = \emptyset$. In this induction, we must do $\theta_y$ many tasks, and each step of the induction can have at most $\kappa$ predecessors. Therefore, we assume $2^\kappa = \kappa^+$ to get $\theta_y \leq \kappa^+$. The constructions of the $\mathcal{L}$’s for the two types of spaces are not the same. However, in either case, the pairs will be used for the same purpose: to “split” the $H_\gamma$’s.

6.1. $X$ is locally compact. We are able to arrange the cofinal sequence of collections $\{\xi_\gamma : \gamma \in \theta_y\}$ as “step functions”, which makes the definition of the $\mathcal{L}$’s easier than in the $\kappa^\omega$-like case. List $\mathcal{B}_0 = \{B_{\alpha, \emptyset} : \alpha \in \kappa\}$ and $\mathcal{B}_i = \{B_{\alpha, \sigma} : \alpha \in \kappa, \sigma \in 4^i\}$ in such a way that $B_{\alpha, \sigma} \subset B_{\alpha, \sigma'}$ if $\sigma$ extends $\sigma'$. We may assume that for $\alpha \in \kappa$ and $\sigma \in 4^i$, $\text{cl}_X B_{\alpha, \sigma \cap 0} \cap \text{cl}_X B_{\alpha, \sigma \cap 1} = \emptyset$ and $\text{cl}_X B_{\alpha, \sigma \cap 0} \cap \text{cl}_X B_{\alpha, \sigma \cap 1} \subset B_{\alpha, \sigma}$. Notice that the collections $\xi$ from $\mathcal{Z}$ that have the property that $B_{\alpha, \sigma}, B_{\alpha, \sigma'} \in \xi$ implies $|\sigma| = |\sigma'|$ form an unbounded set in $\mathcal{Z}$. To see this, let $\xi' \in \mathcal{Z}$ and let $n(\alpha) = \max\{|\sigma| : B_{\alpha, \sigma} \in \xi'\} + 1$. 

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Then the collection $\xi = \{B_{\alpha,\sigma} : \alpha \in \kappa, \sigma \in n(\alpha)4\}$ has the property that $\xi >_y \xi'$ since $L(\xi', \xi) = \xi'$.

Therefore, we may assume that $\{\xi_\gamma : \gamma \in \theta_y\}$ is a sequence of collections that have the property that for each $\gamma \in \theta_y$ and $\alpha \in \kappa$ if $B_{\alpha,\sigma}, B_{\alpha,\sigma'} \in \xi_\gamma$ then $|\sigma| = |\sigma'|$. For each $\gamma \in \theta_y$ define the function $L(\gamma, \cdot) : \kappa \to \omega$ such that $\xi_\gamma = \{B_{\alpha,\sigma} : \alpha \in \kappa, \sigma \in n(\gamma,\alpha)4\}$. Notice that for any $\gamma' < \gamma < \theta_y$ the set $L(\xi_\gamma, \xi_{\gamma'})^*$ is dense in $\{B_{\alpha} : \alpha \in S\}^*$ for any nonempty set $S \subset \kappa$.

Defining the $L^{\omega}_\gamma$'s. For $\gamma \in \theta_y$ and $i = 0, 1$ define $L^{\omega}_\gamma = \{B_{\alpha,\sigma^i} : \alpha \in \kappa, \sigma \in n(\gamma,\alpha)4\}$.

**Claim.** For all $\gamma \in \theta_y$, $cl_{\beta X}(\bigcup L^{0}_\gamma) \cap cl_{\beta X}(\bigcup L^{1}_\gamma) = \emptyset$.

**Proof.** For each $\alpha \in \kappa$ and $\sigma \in i4$, $cl_X(B_{\alpha,\sigma^0} \cap cl_X B_{\alpha,\sigma^1} = \emptyset$. Also, $B_{\alpha,\sigma} \cap B_{\alpha,\beta} = \emptyset$ for $\sigma \neq \beta \in n(\gamma,\alpha)4$, and for $i = 0, 1$ we have $cl_X B_{\alpha,\sigma^i} \subset B_{\alpha,\sigma}$ and $cl_X B_{\alpha,\beta^i} \subset B_{\alpha,\beta}$. Therefore $cl_X B_{\alpha,\sigma^i} \cap cl_X B_{\alpha,\beta^j} = \emptyset$ for $i, j = 0, 1$. So,

$$\bigcup \{cl_X B_{\alpha,\sigma^0} : \sigma \in n(\gamma,\alpha)4\} \cap \bigcup \{cl_X B_{\alpha,\sigma^0} : \sigma \in n(\gamma,\alpha)4\} = \emptyset.$$

Now, since $\{B_{\alpha,\emptyset} : \alpha \in \kappa\}$ is a locally finite family and since $cl_X B_{\alpha,\sigma^i} \subset B_{\alpha,\emptyset}$ for each $\sigma \in \bigcup_n \omega \cdot 4$ and $i = 0, 1$, we have

$$cl_X \left(\bigcup L^{0}_\gamma\right) \cap cl_X \left(\bigcup L^{1}_\gamma\right) = \bigcup \{cl_X B_{\alpha,\sigma^0} : \sigma \in n(\gamma,\alpha)4, \alpha \in \kappa\}$$

$$\cap \bigcup \{cl_X B_{\alpha,\sigma^1} : \sigma \in n(\gamma,\alpha)4, \alpha \in \kappa\} = \emptyset.$$

Finally, since $cl_X(\bigcup L^{0}_\gamma) \cap cl_X(\bigcup L^{1}_\gamma) = \emptyset$ we conclude that $cl_{\beta X}(\bigcup L^{0}_\gamma) \cap cl_{\beta X}(\bigcup L^{1}_\gamma) = \emptyset$.

Since $cl_{\beta X}(\bigcup L^{0}_\gamma) \cap cl_{\beta X}(\bigcup L^{1}_\gamma) = \emptyset, y$ can be in at most one of $cl_{\beta X}(\bigcup L^{0}_\gamma)$ or $cl_{\beta X}(\bigcup L^{1}_\gamma)$. Without loss of generality, assume $y \notin cl_{\beta X}(\bigcup L^{0}_\gamma)$ for each $\gamma \in \theta_y$.

Consider a finite collection $\{\xi_{\gamma_i} : i \in m\} \subset \{\xi_\gamma : \gamma \in \theta_y\}$ such that $\gamma_i < \gamma_j$ for $i < j \leq m$ and let $U(i, j) = L(\xi_{\gamma_i}, \xi_{\gamma_j})$. It is the case that $U(i, j)^* \in \mathcal{N}_y$ for each $i < j$ and hence $U = \bigcap \{U(i, j)^* : i < j \leq m\} \in \mathcal{N}_y$. For any $B \in \xi_{\gamma_0}$ such that $B \cap U \neq \emptyset$ we observe that $\{B' \in \gamma_i : B' \subset B\}$ refines $\{B' \in \gamma_j : B' \subset B\}$ whenever $0 < j < i \leq m$.

A special case of the following claim, in particular when $\Phi$ is constant, is proven in [12, Lemma 3] and in [9, Proposition 6].

**Claim 6.1.** For any $\rho < \theta_y$ and $\Phi : D \subset [\rho, \theta_y) \to 2$, the collection $\{H_\rho\} \cup \{cl_{\beta X}(\bigcup L^{\Phi(\gamma)}_\gamma) : \gamma \in D\}$ has nonempty intersection.
Proof. Let $\rho < \theta_y$ and $\Phi : D \to 2$ for some $D \subseteq [\rho, \theta_y)$. We will show that $\{cl_{iX} U^* : U \in \mathcal{N}_\rho\} \cup \{cl_{iX}(\bigcup L_{\gamma}^{\Phi(\gamma)} : \gamma \geq \rho\}$ has the finite intersection property. Let $U_1, \ldots, U_n \in \mathcal{N}_\rho$ and let $\gamma_1, \ldots, \gamma_m \in \gamma$ be such that $\gamma_m \geq \cdots \geq \gamma_1 \geq \rho$. Since $\mathcal{N}_\rho$ is a filter, $\mathcal{U} = \bigcap \{U_i : 1 \leq i \leq n\} \in \mathcal{N}_\rho$ and therefore $V = U^* \in \mathcal{N}_\rho$. For $i < j \leq m$, let $U(i, j)^* = L(\xi_{\gamma_i}, \xi_{\gamma_j})$ and notice that $U = \bigcap \{U(i, j)^* : i < j \leq m\} \subseteq \mathcal{N}_\rho$. Let $B_{\alpha, \sigma} \in \mathcal{V}$ be such that $B_{\alpha, \sigma} \subset V$ and $B_{\alpha, \sigma} \cap U \neq \emptyset$. As noted before, $\{B \in \gamma_i : B \subset B_{\alpha, \sigma}\}$ refines $\{B \in \gamma_j : B \subset B_{\alpha, \sigma}\}$ whenever $0 < j < i \leq m$. Define $\sigma' \in n(\gamma_m, \alpha)+1$ as follows: $\sigma'|_{n(\rho, \alpha)} = \sigma$, $\sigma'(n(\gamma_i, \alpha) + 1) = \Phi(\gamma_i)$ for each $1 \leq i \leq m$ and $\sigma'(k) = 0$ otherwise. Then $B_{\alpha, \sigma'} \subset B_{\alpha, \sigma} \cap U \neq \emptyset$. Furthermore, $B_{\alpha, \sigma'} \subset \bigcup L_{\gamma_i}^{\Phi(\gamma_i)}$ since $\sigma'$ extends $\sigma'$ and hence $B_{\alpha, \sigma'} \subset \mathcal{U}$. If $B_{\alpha, \sigma'} \subset \bigcup L_{\gamma_i}^{\Phi(\gamma_i)}$ then $\sigma'$ extends $\sigma'$ and hence $B_{\alpha, \sigma'} \subset \mathcal{U}$. If $B_{\alpha, \sigma'} \subset \bigcup L_{\gamma_i}^{\Phi(\gamma_i)}$ then $\sigma'$ extends $\sigma'$ and hence $B_{\alpha, \sigma'} \subset \mathcal{U}$.

6.2. $X$ is $\kappa^\omega$-like. Consider a finite collection $\{\xi_{\gamma_i} : i \in \mathbb{N}\} \subseteq \{\xi_{\gamma} : \gamma \in \theta_y\}$ such that $\gamma_i < \gamma_j$ for $i < j \leq n$ and let $U(i, j) = L(\xi_{\gamma_i}, \xi_{\gamma_j})$. It is the case that $U(i, j)^* \in \mathcal{N}_\rho$ for each $i < j$ and hence $U = \bigcap \{U(i, j)^* : i < j \leq n\} \subseteq \mathcal{N}_\rho$. It is tempting to assume that, as in the locally compact case, $\{B \in \xi_{\gamma_0} : B \subset cl U\} \neq \emptyset$. However, there may not exist $B \in \xi_{\gamma_0}$ such that $\{B' \in \gamma_i : B' \subset B\}$ refines $\{B' \in \gamma_i : B' \subset B\}$ whenever $0 < j < i \leq n$.

Defining the $L_i$'s. We define $\{L_i : \gamma \in \theta_y\}$ by induction on $\gamma \in \theta_y$.

Let $P = \{p : dom(p) \in [\theta_y]^{<\omega}, ran(p) \subseteq 2\}$. Let $\gamma_p = \max(dom(p))$ and $n(p) = |p|$. Define $p|_i$ to be the function $p$ restricted to the first $i$ elements of $dom(p)$. We say $B \in \mathcal{B}$ and $p \in \mathcal{P}$ are aligned if for each $\gamma \in dom(p)$ and $B' \in \xi_{\gamma}$ such that $B' \cap B \neq \emptyset$, we have $B' \subseteq B$. We will define $L(B, p)$ for each $B$ and $p$ and set

$$L_\gamma = \bigcup \{L(B, p) : \gamma = \gamma \text{ and } p(\gamma) = i\}.$$ 

If $B$ and $p$ are not aligned, set $L(B, p) = \emptyset$.

Stage $\gamma = 0$. There are two $p \in P$ with $dom(p) = \{0\}$, namely $p^0 = \{(0, 0)\}$ and $p^1 = \{(0, 1)\}$. Notice that $B \in \mathcal{B}$ is aligned with $p^0$ or $p^1$ if there exists $B' \in \xi_0$ such that $B' \subseteq B$, and that there are $\kappa$ such $B$. List as $\{(B_\nu, p_\nu) : \nu \in \kappa\}$ all pairs $(B, p)$ such that $p = p^0$ or $p = p^1$ and $B$ is aligned with $p$, so that each $(B, p)$ appears in the list $\kappa$ times. We will define a sequence $\{L(\nu) : \nu \in \kappa\}$ and for each $p$ and $B$ aligned with $p$, we will set $L(B, p) = \{L(\nu) : (B, p) = (B_\nu, p_\nu)\}$.

Suppose we have defined $L(\mu) \in \mathcal{B}$ for each $\mu < \nu$ such that $L(\mu) \cap V_\mu \subseteq B_\mu$ where $V_\mu$ is some element of $\xi_0$. Also assume that if $L(\mu), L(\mu') \subseteq V \in \xi_0$, then $\mu = \mu'$. We now define $L(\nu)$. For each $\nu \in \xi_0$ such that $V \cap B_\nu \neq \emptyset$ there is $\eta \in \kappa$ such that $V \subseteq B_\eta$. Furthermore, since $\xi_0^*$ is dense in $X$, for each $\eta \in \kappa$ there is $V \in \xi_0$ such that $V \subseteq B_\eta$. For each $\mu < \nu$,
$L(\mu)$ is contained in an element $V$ of $\xi_0$ and $|\nu| < \kappa$. Therefore, there are $\kappa$ many $\eta \in \kappa$ such that for all $\mu < \nu$, $B_\mu^\nu \cap L(\mu) = \emptyset$. So, let $\eta_0$ be one such $\eta$ and choose $L(\nu) \in B$ so that $L(\nu) \subset V_\nu \subset B_\nu^\nu \subset B_\nu$ for some $V_\nu \in \xi_0$.

For $p = p^0$ or $p^1$ and each $B$ aligned with $p$, set

$$\mathcal{L}(B, p) = \{L(\nu) : (B, p) = (B_\nu, p_\nu)\}.$$  

Let

$$\mathcal{L}_i^0 = \bigcup \{\mathcal{L}(B, p) : p = p^i \text{ and } B \text{ is aligned with } p\}.$$  

Notice that if $L(\nu), L(\mu) \subset B^i \in \xi_0$ then $\nu = \mu$. So, since $\xi_0$ is locally finite, $\text{cl}(\bigcup \mathcal{L}_0^i)$ is disjoint from $\text{cl}(\bigcup \mathcal{L}_1^i)$. Since each $(B, p)$ is listed $\kappa$ times, $|\{\nu : L(B_\nu, p_\nu) \subset B\}| = \kappa$. Consequently, $|\{\eta \in \kappa : \text{there is } L \in \mathcal{L}(B, p), \text{ } L \subset B_\eta^\eta\}| = \kappa$.

**Induction Hypothesis.** Let $B$ and $p$ be aligned such that $\gamma_p \leq \gamma$ and $n(p) > 1$. Then, for $\kappa$ many $\eta \in \kappa$, there is a sequence $\{L_i : 0 \leq i < n(p), L_i \in \mathcal{L}(B, p|i)\}$ such that

$$L_{n(p) - 1} \subset L_{n(p) - 2} \subset \cdots \subset L_0 \subset B^n \subset B.$$  

Also, for each $\gamma' < \gamma$, $\text{cl}(\bigcup \mathcal{L}_0^\gamma')$ is disjoint from $\text{cl}(\bigcup \mathcal{L}_\gamma^1)$.

**Stage $\gamma$.** Consider all $(B, p)$ such that $\gamma_p = \gamma$ and $B$ is aligned with $p$. We have assumed $2^\kappa = \kappa^+$. So, $\gamma < \kappa^+$ and hence there are $\leq \kappa$ many $p$ with $\gamma_p = \gamma$. Therefore, we can list the collection of such $(B, p)$ as $\{(B_\nu, p_\nu) : \nu \in \kappa\}$ in such a way that each $(B, p)$ appears $\kappa$ times. Assume we have defined $L(\mu) \in B$ for each $\mu < \nu$ so that $L(\mu) \subset V_\mu \subset B_\mu$ where $V_\mu$ is some element of $\xi_\gamma$. Also assume that if $L(\mu), L(\mu') \subset V \in \xi_\gamma$, then $\mu = \mu'$. Let $\eta \in \kappa$ be such that there is $\{L_i : 0 \leq i < n(\nu), L_i \in \mathcal{L}(B_\nu, p_\nu|i)\}$ with $L_{n(\nu) - 1} \subset L_{n(\nu) - 2} \subset \cdots \subset L_0 \subset B^n_\nu \subset B_\nu$. Since we have defined $L(\mu)$ for $|\nu| < \kappa$ many $\mu$, by the inductive hypothesis we may also assume that $\eta$ satisfies $B_\eta \cap L(\mu) = \emptyset$ for all $\mu < \nu$.

Let $V \in \xi_\gamma$ be such that $L_{n(\nu) - 1} \cap V = \emptyset$. Let $L(\nu)$ be an element of $B$ such that

$$L(\nu) \subset (V \cap L_{n(\nu) - 1}) \subset L_{n(\nu) - 2} \subset \cdots \subset L_0 \subset B^n_\nu \subset B_\nu.$$  

Set $\mathcal{L}(B, p) = \{L(\nu) : (B_\nu, p_\nu) = (B, p)\}$ and observe that

$$\left(\bigcup \mathcal{L}(B, p)\right) \cap \bigcap \{\bigcup \mathcal{L}(B, p|i) : i < n(p)\} \neq \emptyset.$$  

Now, set $\mathcal{L}_\gamma^i = \bigcup \mathcal{L}(B, p) : \gamma_p = \gamma$ and $p(\gamma) = i\}$. This concludes stage $\gamma$. 

For each $p$ and $B$ aligned with $p$, we have

$$\left(\bigcup \mathcal{L}(B, p)\right) \cap \bigcap \{\bigcup \mathcal{L}(B, p|i) : i < n(p)\} \neq \emptyset.$$  

Therefore, if $\text{dom}(p) \setminus \{\gamma_p\} = \{\gamma_i : 1 \leq i < n(p)\}$, we deduce that $\bigcap \{\mathcal{L}_{\gamma_i}^{p(\gamma_i)} : i < n(p)\} \cap B \neq \emptyset$.  

**Nonnormality points of $\beta X \setminus X**
Claim 6.2. For any \( \rho < \theta_y \) and \( \Phi : D \subset [\rho, \theta_y) \rightarrow 2 \), the collection 
\( \{H_\rho\} \cup \{cl_{\beta X}(\bigcup L^\phi(\gamma)) : \gamma \in D\} \) has nonempty intersection.

Proof. Let \( \rho < \theta_y \) and \( \Phi : D \rightarrow 2 \) for some \( D \subset [\rho, \theta_y) \). We will show 
that \( \{cl_{\beta X} U^* : U \in \mathcal{N}_\rho\} \cup \{cl_{\beta X}(\bigcup L^\phi(\gamma)) : \gamma \geq \rho\} \) has the finite intersection property. Let \( U_1, \ldots, U_n \in \mathcal{N}_\rho \) and let \( \gamma_1, \ldots, \gamma_m \in D \) be such that \( \gamma_m > \cdots > \gamma_1 > \rho \). For each \( i \leq m \), \( L(\xi_\rho, \xi_{\gamma_i}) \in \mathcal{N}_\rho \) since \( \xi_{\gamma_i} > \xi_\rho \). Hence, \( U = \bigcap\{U_i : 1 \leq i \leq n\} \cap \bigcap\{L(\xi_\rho, \xi_{\gamma_i}) : 1 \leq i \leq m\} \in \mathcal{N}_\rho \). Let \( p \) be the function \( \Phi \) restricted to \( \{\gamma_i : 1 \leq i \leq m\} \). Note that if \( B \in \mathcal{U} \) then \( B \) is aligned with \( p \). From the previous construction we conclude that \( \bigcap\{\bigcup L^\phi_{\gamma_i} : i \leq m\} \cap B \neq \emptyset \). \( \blacksquare \)

7. Theorems

Theorem 7.1. Let \( X \) be a crowded metrizable space of weight \( \kappa \) that is 
either \( \kappa^\omega \)-like 
or locally compact. Let \( y \in \beta X \setminus X \). Suppose that \( 2^\kappa = \kappa^+ \) 
and \( \theta_y^{\theta_y} = \theta_y \). Then there is a closed copy of \( NU(\theta_y) \) in \( (\beta X \setminus X) \setminus \{y\} \).

Proof. We follow the argument found in [1] to embed \( NU(\theta_y) \) into 
(\( \beta X \setminus X \) \setminus \{y\}), using the \( \mathcal{L}_\gamma \)'s to play the role of the reaping sets.

The induction. Denote by \( \theta_y \) the discrete space of size \( \theta_y \). We define a 1-1 function \( g \) from \( \theta_y \) into a compact subset of \( \beta X \setminus X \) such that 
\((1) \ y \in cl_{\beta X} g[A] \) if and only if \( |A| = \theta_y \).
\((2) \) If \( A, B \in \theta_y, A \cap B = \emptyset \) then \( cl_{\beta X} g[A] \cap cl_{\beta X} g[B] = \emptyset \).

By assumption, we have \( \theta_y^{\theta_y} = \theta_y \). List \( \theta_y \cup \{(A, B) : A, B \in \theta_y, A \cap B = \emptyset \} \) as \( \{T_\eta : \eta \in \theta_y\} \) in such a way that if \( T_\eta = (A, B) \), then 
\( \eta \geq \sup(A \cup B) \), and if \( T_\eta \in \theta_y \), then \( \eta \geq T_\eta \). For \( \rho \in \theta_y \) let \( D_\rho = \{\eta : T_\eta = (A, B) \) and \( \rho \in A \cup B \} \cup \{\eta : \rho \in T_\eta\} \). Note that \( D_\rho \subset [\rho, \theta_y) \).

For each \( \rho \in \theta_y \) we define \( \Phi_\rho : D_\rho \rightarrow 2 \) and choose \( g(\rho) \) to be any element 
of \( K_\rho := \bigcap\{\{H_\rho\} \cup \{cl_{\beta X}(\bigcup L^\phi_{\gamma_i}) : \gamma \in D_\rho\}\} \). We define \( \Phi_\rho \) by induction.

Let \( \eta \in \theta_y \) and assume we have defined \( \Phi_{\eta \cap D_\rho} \). If \( T_\eta \in \theta_y \), let \( \Phi_{\beta}(\eta) = 0 \) 
for all \( \beta < T_\eta \). If \( T_\eta = (A, B) \), let \( \Phi_{\beta}(\eta) = 0 \) for all \( \beta \in A \) and let \( \Phi_{\beta}(\eta) = 1 \) 
for all \( \beta \in B \). By Claims 6.1 and 6.2, \( K_\rho \neq \emptyset \) for each \( \rho \in \theta_y \), so we may 
choose \( g(\rho) \in K_\rho \).

To show (1), let \( A \subset \theta_y \) be such that \( |A| < \theta_y \). There is \( \gamma \in \theta_y \) with \( A \subset [0, \gamma) \). Let \( \eta \) satisfy \( T_\eta = \gamma \). Note that \( \eta \geq \gamma \). For any \( \rho < \gamma = T_\eta \), \( \Phi_\rho(\eta) = 0 \). 
So, for \( \rho \in A, K_\rho \subset \mathcal{L}_\eta^0 \). But \( y \notin cl_{\beta X}(\bigcup \mathcal{L}_\eta^0) \). Hence, \( y \notin cl_{\beta X} g[A] \). For the other direction, let \( A \subset \theta_y \) be such that \( |A| = \theta_y \). Since \( \theta_y \) is regular, \( A \) is 
unbounded in \( \theta_y \). Let \( U \in \mathcal{N} \). There is \( \gamma \in \theta_y \) such that \( H_\gamma \subset U \). For \( \rho \geq \gamma \), 
g(\rho) \in H_\rho \subset H_\gamma \subset U \). Hence \( y \in cl_{\beta X} g[A] \).

To show (2), let \( A, B \in \theta_y \subset \theta_y \) be such that \( A \cap B = \emptyset \). Let \( \eta \) be such that 
\( T_\eta = (A, B) \). Then, for each \( \rho \in A, \Phi_\rho(\eta) = 0 \), and for each \( \rho \in B, \Phi_\rho(\eta) = 1 \).
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Hence $g(\rho) \in K_\rho \subset \text{cl}_{\beta X} \left( \bigcup L^0_\eta \right)$ for $\rho \in A$ and $g(\rho) \in K_\rho \subset \text{cl}_{\beta X} \left( \bigcup L^1_\eta \right)$ for $\rho \in B$. But $\text{cl}_{\beta X} \left( \bigcup L^0_\eta \right) \cap \text{cl}_{\beta X} \left( \bigcup L^1_\eta \right) = \emptyset$. Hence $\text{cl}_{\beta X} g[A] \cap \text{cl}_{\beta X} g[B] = \emptyset$. Note (2) implies $g$ is one-to-one.

Since $\theta_y$ is discrete, $g$ is continuous. Extend $g$ to $\beta g : \beta \theta_y \to \beta X \setminus X$. It follows from Bešlagić and van Douwen’s [1, Lemma 2.2] that the image of $\beta g$ is a closed subset of $(\beta X \setminus X) \setminus \{y\}$ which is homeomorphic to $\text{NU}(\theta_y)$. ■

**Theorem 7.2.** $(2^\kappa = \kappa^+)$ Let $X$ be a metric space of weight $\kappa$ that is either crowded locally compact or $\kappa^\omega$-like. Any regular $z$-ultrafilter is a nonnormality point of $\beta X \setminus X$.

**Proof.** Since $y$ is regular, by Lemma 5.1 $\theta_y > \kappa$. By the hypothesis, $\theta_y = \kappa^+ = 2^\kappa$ and hence $\theta_y$ is regular and not a strong limit. By Lemma 3.1 $\text{NU}(\theta_y)$ is not normal. Hence, by Theorem 7.1, $y$ is a nonnormality point of $\beta X \setminus X$. ■

**Corollary 7.3.** Suppose GCH+UR. Let $X$ be a crowded locally compact metric space. Then each $y \in \beta X \setminus X$ is a nonnormality point of $\beta X \setminus X$.

**Proof.** We have seen that if $y \in \beta X \setminus X$ is uniform then it is a nonnormality point of $\beta X \setminus X$. Suppose that $y \in \beta X \setminus X$ is not uniform. That is, there exists $Z \in y$ for which $w(Z) < w(X)$. Let $Z \in y$ be such that $\lambda = w(Z)$ is minimum. Then $y$ is a uniform $z$-ultrafilter on the set $Z$, and by UR, it is regular. However, it may be the case that $Z$ has isolated points. We aim to find a crowded locally compact closed subset $Y$ of $X$ with weight $\lambda$ such that $Z \subset Y$. There is a cover of $Z$ consisting of sets $\text{cl} B$ from a subcollection $Z$ of $B_0$ of size $\lambda$. Let $Y = \bigcup \{ \text{cl} B : B \in Z \}$. Since $B_0$ is locally finite, $Y$ is closed. Each $B \in Z$ is crowded and has compact closure, so $Y$ is crowded locally compact.

So, $y \in \text{cl}_{\beta X} Y$. Since $X$ is normal and $Y$ is closed, $Y$ is $C^*$-embedded in $X$. Therefore, $\beta Y = \text{cl}_{\beta X} Y$ and $y|_Y$ is uniform on $Y$. So, by the theorem, $y$ is a nonnormality point of the set $\left( \text{cl}_{\beta X} Y \right) \setminus Y$ and hence a nonnormality point of $\beta X \setminus X$. ■

**8. Questions.** Gillman’s question [6], which started research in this area, is still not completely answered.

**Problem 8.1.** Let $X$ be $\mathbb{N}$. Let $y$ be any point of $\beta X \setminus X$. Without extra axioms of set theory, is $(\beta X \setminus X) \setminus \{y\}$ not normal? If yes, what if $X$ is any discrete space? If yes, what if $X$ is any metrizable space?

There are many ways that our work can be extended. For example

**Problem 8.2.** Assume GCH. For every crowded metrizable space $X$ and every $y \in \beta X \setminus X$, is $(\beta X \setminus X) \setminus \{y\}$ not normal?
Katětov (see [4, 5.5.10]) showed that if there is a nonrealcompact metrizable (more generally, paracompact) space, then there is a measurable cardinal. In other words, if there is a countably complete free $z$-ultrafilter on a metrizable (more generally, paracompact) space, then there is a countably complete free ultrafilter on a set. Is there an analogue for nonregular ultrafilters?

**Problem 8.3.** If there is a nonregular ultrafilter on a metrizable (more generally, paracompact) space, is there a nonregular ultrafilter on a set?

**Problem 8.4.** What can be proved about $\theta_y$ and the normality of $(\beta X \setminus X) \setminus \{y\}$ when $y$ is a nonregular $z$-ultrafilter?

We do not know whether it is possible that $\theta_y$ is an uncountable weakly compact cardinal. It is possible that $\theta_y = \omega$. For example, let $q$ be a $\kappa$-complete ultrafilter on a measurable cardinal $\kappa$. Let $X$ be $\kappa \times \mathbb{R}$. Then $X$ is crowded, locally compact, metrizable. (If a nowhere locally compact example is wanted, we can use $\mathbb{Q}$ in place of $\mathbb{R}$.) For $r \in \mathbb{R}$ let $e_r : \kappa \to X$ be defined by $e_r(\alpha) = (\alpha, r)$, and let $\beta e_r : \beta \kappa \to \beta X$ be the extension. Let $y$ be $\beta e_0(q)$. Then $\theta_y = \omega$. In fact, $\{\beta e_{1/n}(q) : n \in \mathbb{N}\}$ is a sequence converging to $y$. We can show that $(\beta X \setminus X) \setminus \{y\}$ is not normal. Observe that neither Theorem 1.3 ($X$ is not realcompact) nor Theorem 1.4 ($y$ is nonregular) applies here.

References

Nonnormality points of $\beta X \setminus X$


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