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On the lengths of bad sequences of monomial ideals over polynomial rings

by

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Abstract. We give bad (with respect to the reverse inclusion ordering) sequences of monomial ideals in two variables with Ackermannian lengths and extend this to multiple recursive lengths for more variables.

1. Introduction. Quite recently Diane Maclagan [4] has proven the following interesting theorem:

THEOREM 1.1. *Every infinite sequence of monomial ideals in a polynomial ring contains an ideal that is a subset of an ideal that occurs earlier in that sequence.*

Monomial ideals play an important role in commutative algebra and algebraic combinatorics. Because the above theorem has several applications in computer algebra, it is of interest to study the logical and combinatorial issues surrounding it. Aschenbrenner and Pong [1] did this extensively from the viewpoint of the theory of well partial orders and they computed several related interesting ordinal invariants. We complement this, in particular Proposition 3.25 of their paper which concerns a finitary version of Maclagan's theorem (Theorem 2.1 in this note). We show that already in two variables there are bad sequences with linear complexity bounds which have non-primitive recursive lengths. We also extend this, for arbitrary n, to n-fold recursive lower bounds of such lengths with higher numbers of variables.

This is somewhat surprising because upper bounds for the lengths of increasing chains of ideals with linear complexity bounds that arise from the similarly shaped Hilbert basis theorem are primitive recursive for any fixed number of variables (Moreno Socías [5]).

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An interesting consequence of our result is that Theorem 2.1 is one of the rare examples of finitary theorems arising from practice that are not provable in $I\Sigma_2$, the theory of Peano Arithmetic with the induction axioms limited to Σ_2 formulas (this follows from a result from proof theory that provides upper bounds to provable existence; for an introduction to this area we recommend Wilfried Buchholz' lecture notes on his web-page [2]).

2. Preliminaries. We assume basic familiarity with ordinals below ε_0 and their Cantor normal forms, abbreviated CNF. For those ordinals in CNF we take fundamental sequences:

$$\begin{aligned} \alpha + 1[i] &= \alpha, \\ \alpha + \omega^{\alpha+1} \cdot (m+1)[i] &= \alpha + \omega^{\alpha+1} \cdot m + \omega^{\alpha} \cdot i, \\ \alpha + \omega^{\gamma} \cdot (m+1)[i] &= \alpha + \omega^{\gamma} \cdot m + \omega^{\gamma[i]}, \end{aligned}$$

for limit γ. *Primitive recursive functions* are functions $\mathbb{N}^d \to \mathbb{N}$ built from constant functions, projections and the successor function using composition and primitive recursion. *Multiple recursive functions* are also closed under multiple nested recursion as in [6]. We call a function *Ackermannian* if it eventually dominates every primitive recursive function. We use the fast growing hierarchy

$$F_0(i) = i + 1, \qquad F_{\alpha+1}(i) = F_{\alpha}^{(i)}(i), \qquad F_{\gamma}(i) = F_{\gamma[i]}(i).$$

In this definition the (i) in the exponent denotes i-fold composition. The function F_ω is Ackermannian and every multiple recursive function can be bounded by an F_α with $\alpha < \omega^{\omega^{\omega}}$.

For a field K, we consider ideals in the polynomial ring $K[X_0, \ldots, X_d, Y]$; *monomial ideals* are ideals generated by monomials. If we have a set G of monomials we denote by G the ideal generated by G. Observe that a monomial is an element of $\langle G \rangle$ if and only if there is a monomial from G that divides it. For a set G of generators set $|G| := \max\{n_0 + \cdots + n_d + m : X_0^{n_0} \ldots X_d^{n_d} Y^m \in G\}$. For an ideal I we take as its complexity $|I| := \min\{|G| : \langle G \rangle = I\}$.

THEOREM 2.1. *For every l, d there exists an M such that for every sequence $I_0, \ldots, I_M$ of monomial ideals in $K[X_0, \ldots, X_d, Y]$ with $|I_i| \le l + i$ for all $i \le M$, there exist $i < j < M$ with $I_i \supseteq I_j$.*

Proof. Maclagan's theorem with König's lemma. ∎

Denote such an M by $M_d(l)$. The remainder of this note will prove:

MAIN THEOREM 2.2. *We have:*

(1) *M_0 is Ackermannian.*
(2) *$(d, l) \mapsto M_d(l)$ is not multiply recursive.*

3. Two variables. We first examine the case $d = 0$ using ordinals below ω^{k+1}.

DEFINITION 3.1. For $\alpha = \omega^{\alpha_0} \cdot m_0 + \cdots + \omega^{\alpha_r} \cdot m_r$ and $\beta = \omega^{\beta_0} \cdot n_0 + \cdots + \omega^{\beta_s} \cdot n_s$ in CNF, with $m_0, \ldots m_r, n_1, \ldots, n_s > 0$, we write $\alpha \prec \beta$ if there is j such that $\alpha_i = \beta_i$, $m_i = n_i$ for $i < j$, $\alpha_j = \beta_j$ and $m_j < n_j$.

We will create $\prec$-descending sequences to prove Theorem 2.2. For two variables it is also possible to do this using $<$-descending sequences, but we will use some results from this section later, where $<$-descending sequences do not work. Notice that $\alpha \prec \beta$ implies $\alpha < \beta$ and that in this section the α_i and β_i are in $\mathbb{N}$. To relate $\prec$-descending sequences to bad sequences of monomial ideals take, for $\alpha < \omega^\omega$,

$$\Phi(\alpha) = \langle X^{\alpha_0} Y^{m_0}, \ldots, X^{\alpha_r} Y^{m_0 + \cdots + m_r} \rangle,$$

where each of the generators has the form $X^{\alpha_i} Y^{m_0 + \cdots + m_i}$.

LEMMA 3.2. *If* $\alpha \prec \beta < \omega^{k+1}$, *then* $\Phi(\alpha) \not\subseteq \Phi(\beta)$.

Proof. We use the notation from Definition 3.1. Take j such that $\alpha_i = \beta_i$, $m_i = n_i$ for $i < j$, $\alpha_j = \beta_j$ and $m_j < n_j$. Suppose, for a contradiction, that $p = X^{\alpha_j} Y^{m_0 + \cdots + m_j} \in \Phi(\beta)$. Then there is some $i \in \{0, \ldots, s\}$ such that $X^{\beta_i} Y^{n_0 + \cdots + n_i}$ divides p, i.e. $a_j \geq \beta_i$ and $m_0 + \cdots + m_j \geq n_0 + \cdots + n_i$. From the first inequality and $\alpha_j = \beta_j$ we obtain $j \leq i$. If $j = i$ then the second inequality and $m_k = n_k$ for $k < j$ yields $m_j \geq n_j$; if $j < i$ then we similarly obtain $n_{j+1} + \cdots + n_i \leq 0$. In both cases we have a contradiction. ∎

This implies that to show that M_0 is Ackermannian it is sufficient to construct certain $\prec$-descending sequences of Ackermannian lengths:

DEFINITION 3.3. Take $|\alpha| = m_0 + \cdots + m_r$. Define $L_\alpha(l)$ to be the maximum length of a sequence $\gamma_0 \succ \cdots \succ \gamma_L$ of ordinals, $<$-below α with $|\gamma_i| \leq l + i$.

COROLLARY 3.4. $M_0(l + k) \geq L_{\omega^{k+1}}(l)$.

LEMMA 3.5. $L_{\omega^{2k+2}}(l + 2) \geq F_k(l)$ *for* $l > 2$.

Proof. We construct sequences that show this by recursion on k. (This proof has been inspired by a similar lemma about sequences of fixed length in [3].)

For $k = 0$: $\omega \cdot (l + 1), \omega \cdot l + 2, \omega \cdot l + 1, \omega \cdot (l - 1) + 5, \ldots, \omega + 1$ is a $\prec$-descending sequence of length greater than $l + 1 = F_0(l)$.

For $k + 1$: Take sequences $\omega^{2k+2} > \gamma_0^i \succ \cdots \succ \gamma_{L_i}^i$ of length $F_k^{(i+1)}(l)$ with $|\gamma_j^i| \leq F_k^{(i)}(l) + 2 + j$ (from IH, with i ranging from 0 to $l - 1$). The idea of the construction of the new sequence is to glue these sequences together $l - 1$ times, using the length of γ^i to guarantee that the condition on the complexity bounds is met when appending γ^{i+1}:

$$\begin{gathered}\omega^{2k+3}\cdot(l+2),\ldots,\omega^{2k+3}\cdot 3,\\ \omega^{2k+3}+\omega^{2k+2}\cdot(l+1)+l,\ldots,\omega^{2k+3}+\omega^{2k+2}\cdot(l+1)+1,\\ \omega^{2k+3}+\omega^{2k+2}\cdot l+\gamma^0_0,\ldots,\omega^{2k+3}+\omega^{2k+2}\cdot l+\gamma^0_{L_0},\\ \vdots\\ \omega^{2k+3}+\omega^{2k+2}+\gamma^{l-1}_0,\ldots,\omega^{2k+3}+\omega^{2k+2}+\gamma^{l-1}_{L_{l-1}}.\end{gathered}$$

The first two lines of this construction are there to ensure that the third line can start with complexity $2l+3$. The next $l-1$ lines then ensure that the last line has length $F^{(l)}_k(l)=F_{k+1}(l)$. More precisely we take the sequence:

(1) If $0\le i<l$:
$$\beta_i=\omega^{2k+3}\cdot(l+2-i),$$
(2) If $l\le i<2l$:
$$\beta_i=\omega^{2k+3}+\omega^{2k+2}\cdot(l+1)+(2l-i),$$
(3) For $0\le a<l$, if
$$l+F^{(0)}_k(l)+\cdots+F^{(a)}_k(l)\le i<l+F^{(0)}_k(l)+\cdots+F^{(a+1)}_k(l),$$
where $b=i-(l+F^{(0)}_k(l)+\cdots+F^{(a)}_k(l))$, then
$$\beta_i=\omega^{2k+3}+\omega^{2k+2}\cdot l+\gamma^a_b.$$

This $\prec$-descending sequence has length $l+F^{(0)}_k(l)+\cdots+F^{(l)}_k(l)>F^{(l)}_k(l)$. It remains to check the complexities (using the bounds on i in the three cases):

(1) $|\beta_i|=l+2-i\le l+2+i$,
(2) $|\beta_i|=1+l+1+2l-i=3l+2-i=l+2+(2l-i)\le l+2+l$
$\le l+2+i$,
(3) $|\beta_i|=1+l+|\gamma^a_b|=1+l+F^{(a)}_k(l)+2+b$
$=1+l+F^{(a)}_k(l)+2+i-(2l+F^{(1)}_k(l)+\cdots+F^{(a)}_k(l))$
$=l+2+1+i-(l+F^{(0)}_k(l)+\cdots+F^{(a-1)}_k(l))$
$\le l+2+1+F^{(a)}_k(l)\le l+2+i$. ∎

COROLLARY 3.6. *M_0 is Ackermannian.*

4. More variables. We generalise this technique to prove the second part of Theorem 2.2. For $\alpha<\omega^{\omega^{d+1}}$ in CNF $\alpha=\omega^{\alpha_0}\cdot m_0+\cdots+\omega^{\alpha_k}\cdot m_k$, where $\alpha_i=\omega^d\cdot n^d_i+\cdots+\omega^0\cdot n^0_i$, we take

$$\Psi(\alpha)=\langle X_d^{n_0^d}\cdots X_0^{n_0^0}Y^{m_0},\ldots,X_d^{n_k^d}\cdots X_0^{n_k^0}Y^{m_0+\cdots+m_k}\rangle,$$

where the generators have the form $X_d^{n_i^d} \cdots X_0^{n_i^0} Y^{m_0+\cdots+m_i}$. First we explain why we do not use $<$-descending sequences.

LEMMA 4.1. *There exist $\bar{\alpha} < \alpha < \omega^{\omega^2}$ such that $\Psi(\bar{\alpha}) \subseteq \Psi(\alpha)$.*

Proof. Take, for example,

$$\alpha = \omega^{\omega\cdot 2+1} \cdot 4 + \omega^{\omega\cdot 1+1} \cdot 5, \qquad \bar{\alpha} = \omega^{\omega\cdot 2+1} \cdot 4$$

or

$$\alpha = \omega^{\omega\cdot 3+1} \cdot 7 + \omega^{\omega\cdot 1+1} \cdot 5, \qquad \bar{\alpha} = \omega^{\omega\cdot 2+1} \cdot 15 + \omega^{\omega\cdot 1+1} \cdot 100.$$

In the first case the generators of $\Psi(\bar{\alpha})$ are also generators of $\Psi(\alpha)$. In the latter case we see $X_1X_0Y^{115} = Y^{103} \cdot X_1X_0Y^{12}$ and $X_1^2X_0Y^{15} = X_1Y^3 \cdot X_1X_0Y^{12}$, so the generators of $\Psi(\bar{\alpha})$ are in $\Psi(\alpha)$. ∎

Therefore we cannot use this ordering on the ordinals. If we examine the examples shown in the lemma we see two problems:

(1) $\Psi(\omega^{\alpha_0} \cdot m_0 + \cdots + \omega^{\alpha_i} \cdot m_i) \subseteq \Psi(\omega^{\alpha_0} \cdot m_0 + \cdots + \omega^{\alpha_k} \cdot m_k)$, but $\omega^{\alpha_0} \cdot m_0 + \cdots + \omega^{\alpha_i} \cdot m_i < \omega^{\alpha_0} \cdot m_0 + \cdots + \omega^{\alpha_k} \cdot m_k$ (if $i < k$).
(2) If $\alpha = \omega^{\alpha_0} \cdot m_0 + \cdots + \omega^{\alpha_k} \cdot m_k)$ and $\bar{\alpha} = \omega^{\bar{\alpha}_0} \cdot \bar{m}_0 + \cdots + \omega^{\bar{\alpha}_{\bar{k}}} \cdot \bar{m}_{\bar{k}}$ then $\bar{\alpha}_i = \alpha_i$, $\bar{m}_i = m_i$ (for $i < j$), $\bar{\alpha}_j < \alpha_j$ does not guarantee $\Psi(\bar{\alpha}) \not\subseteq \Psi(\alpha)$, even though $\bar{\alpha} < \alpha$ is true.

The ordering $\prec$ does not pose these problems.

DEFINITION 4.2. For $\bar{\alpha} = \omega^{\bar{\alpha}_0} \cdot \bar{m}_0 + \cdots + \omega^{\bar{\alpha}_k} \cdot \bar{m}_k$ and $\alpha = \omega^{\alpha_0} \cdot m_0 + \cdots + \omega^{\alpha_l} \cdot m_l$ in CNF, with $\bar{m}_1, \ldots \bar{m}_k, m_0, \ldots m_l > 0$, we write $\bar{\alpha} \prec \alpha$ if there is j such that $\bar{\alpha}_i = \alpha_i$, $\bar{m}_i = m_i$ for $i < j$, $\bar{\alpha}_j = \alpha_j$ and $\bar{m}_j < m_j$.

This is precisely Definition 3.1, with some notational differences to prevent confusion. Notice that here also $\bar{\alpha} \prec \alpha$ implies $\bar{\alpha} < \alpha$ but that $\bar{\alpha}_i, \alpha_i < \omega^\omega$ need not be natural numbers. The following lemma tells us that this is indeed an ordering we can use:

LEMMA 4.3. *If $\bar{\alpha} \prec \alpha < \omega^{\omega^\omega}$ then $\Psi(\bar{\alpha}) \not\subseteq \Psi(\alpha)$.*

Proof. Take $\bar{\alpha} \prec \alpha$ (as in Definition 4.2). Then $p := X_d^{\bar{n}_j^d} \cdots X_0^{\bar{n}_j^0} Y^{\bar{m}_0+\cdots+\bar{m}_j} \notin \Psi(\alpha)$ for the j from Definition 4.2. To see this, note that, for $i < j$, because $\alpha_i > \bar{\alpha}_j$, the generator $X_d^{n_i^d} \cdots X_0^{n_i^0} Y^{m_0+\cdots+m_i}$ has X_r-degree (for some r) too high to be a factor of p. If $i \geq j$, because $m_j > \bar{m}_j$, then $X_d^{n_i^d} \cdots X_0^{n_i^0} Y^{m_0+\cdots+m_i}$ has Y-degree too high to be a factor of p. ∎

Take a new complexity for ordinals $\alpha < \omega^{\omega^\omega}$ in CNF:

$$|\alpha| = \max_{0\leq i\leq k} (n_i^d + \cdots + n_i^0) + m_0 + \cdots + m_k.$$

DEFINITION 4.4. $L_\alpha^f(l)$ is the maximum length of a sequence $\gamma_0 \succ \ldots \succ \gamma_L$, $<$-below α, with $|\gamma_i| \leq l + f(i)$ for all $i \leq L$. We say that such a sequence shows that $L^f_\alpha(l) \geq L-1$ (here the length need not be the maximum possible).

COROLLARY 4.5. *$M_d(l) \geq L^{\mathrm{id}}_{\omega^{\omega^{d+1}}}(l)$.*

So we can concentrate again on obtaining long $\prec$-descending sequences. Recall Lemma 3.5, which has an important consequence for $L^{\mathrm{id}}_{\omega^\omega}$ (taking the new definition for complexity into account):

LEMMA 4.6. *$L^{\mathrm{id}}_{\omega^{2k+2}}(l+2k+3) \geq F_k(l)$, hence $L^{\mathrm{id}}_{\omega^\omega}$ is Ackermannian.*

We now loosen the complexity bounds on the sequences to get nice lower bounds on the lengths which range over the multiply recursive functions. This is helpful because it allows construction of long sequences with little worry about these complexities. Later we use the above lemma to get back the stricter bounds on complexities whilst retaining similar lower bounds on the lengths.

DEFINITION 4.7. For $\alpha = \omega^d \cdot k_d + \cdots + \omega^0 \cdot k_0 < \omega^\omega$,

$$f_\alpha(i) := (i+3)^{d+1} \cdot k_d + \cdots + (i+3) \cdot k_0.$$

LEMMA 4.8. *$f_\alpha(i) \geq f_{\alpha[l]}(i)$ for all $i \geq l$ and limit α.*

Proof. Use the fact that $(i+3)^{j+1} \geq (i+3)^j \cdot l$:

$$\begin{aligned} f_\alpha(i) &= (i+3)^{d+1} \cdot k_d + \cdots + (i+3)^{j+1} \cdot (k_j+1) \\ &\geq (i+3)^{d+1} \cdot k_d + \cdots + (i+3)^{j+1} \cdot k_j + (i+3)^j \cdot l = f_{\alpha[l]}(i). \ \blacksquare \end{aligned}$$

The function f_α is specifically chosen to permit constructing long sequences with little effort and to have reasonably low growth rate (in the sense that the number of elements in $\{i : f_\alpha^{-1}(i) = l\}$ is primitive recursive in l).

THEOREM 4.9. *$L^{f_\alpha}_{\omega^\alpha}(f_\alpha(l)) \geq F_\alpha(l)$ for all $l > 2$ and $\alpha \geq \omega$.*

Proof. By recursion on α we construct sequences $\mathrm{Seq}(\alpha, l)$ which show this.

For $\alpha = \omega$: We take a sequence that has been constructed for Lemma 4.6 which shows that $L^{\mathrm{id}}_{\omega^{2l+2}}(l+2l+3) \geq F_l(l)$.

For $\alpha+1$: We define $\mathrm{Seq}(\alpha+1, l)$ to be

$$\begin{gathered} \omega^\alpha \cdot (l+1) + \beta^1_0, \ldots, \omega^\alpha \cdot (l+1) + \beta^1_{R_1}, \\ \omega^\alpha \cdot (l) + \beta^2_0, \ldots, \omega^\alpha \cdot (l) + \beta^2_{R_2}, \\ \vdots \\ \omega^\alpha \cdot (1) + \beta^l_0, \ldots, \omega^\alpha \cdot (1) + \beta^l_{R_l} \end{gathered}$$

where $\beta^i_0, \ldots, \beta^i_{R_i} = \mathrm{Seq}(\alpha, F^{(i-1)}_\alpha(l))$.

For α limit: $\mathrm{Seq}(\alpha, l) = \mathrm{Seq}(\alpha[l], l)$.

Note that for $\mathrm{Seq}(\omega, l)$ we used the sequence constructed for Lemma 4.6; leafing back to the proof of Lemma 3.5 (as $l > 2$) we see that this sequence starts with l elements of decreasing complexity. Using induction on α we conclude that all $\mathrm{Seq}(\alpha, l)$ start with l elements of decreasing complexity.

Furthermore the largest ω^γ in the CNF of the lead element in $\mathrm{Seq}(\alpha, l)$ has complexity greater than or equal to $|\omega^\alpha| - 1$ (this can also be shown by induction on α).

We now demonstrate that these are appropriate sequences by showing, by induction on α, that $\mathrm{Seq}(\alpha, l)$ has the right length and complexity bounds.

For $\alpha = \omega$: Because $f_\omega(i) = (i+3)^2 > \mathrm{id}(i)$ and $f_\omega(l) > (l+3)\cdot l > l + 2l + 3$ this sequence shows $L^{f_\omega}_{\omega^\omega}(f_\omega(l)) \geq L^{f_\omega}_{\omega^\omega}((l+3)\cdot l) \geq F_l(l) = F_\omega(l)$.

For $\alpha + 1$: $\mathrm{Seq}(\alpha+1, l)$ has length $\geq F^{(l)}_\alpha(l)$ (by IH). Furthermore if $\omega^\alpha \cdot (l + 2 - j) + \beta^j_i$ has position c in the sequence, then

$$c \geq F^{(0)}_\alpha(l) + \cdots + F^{(j-1)}_\alpha(l) + i.$$

We examine the complexity of this element:

$$\begin{aligned} |\omega^\alpha \cdot (l+2-j) + \beta^j_i| &\leq l + 2 + f_\alpha(F^{(j-1)}_\alpha(l)) + f_\alpha(i) \\ &\leq l + 2 + f_\alpha(l) + f_\alpha(c) \leq f_{\alpha+1}(l) + f_{\alpha+1}(c). \end{aligned}$$

Here the first inequality uses the second part of the note above.

For α is limit: $\mathrm{Seq}(\alpha, l)$ has length $\geq F_{\alpha[l]}(l) = F_\alpha(l)$. The complexity bounds are guaranteed by Lemma 4.8 and the first part of the note above. ∎

The following lemmas ensure that we can use this result:

Lemma 4.10. $L^f_\alpha(l + k) \geq L^{f+k}_\alpha(l)$.

Lemma 4.11. $L^f_{\omega^{\omega^{d+1}}}(l + k + 2) \geq L^{k\cdot f}_{\omega^{\omega^{d+1}}}(l)$.

Proof. Take a sequence $\beta_0 \succ \cdots \succ \beta_R$ of length $L^{k\cdot f}_{\omega^{\omega^{d+1}}}(l)$. Then the sequence defined by

$$\alpha_i = \omega^{\omega^d} \cdot \beta_{\lfloor i/k \rfloor} + k + 1 - i\%k$$

shows the lemma. ∎

Lemma 4.12. *If $f : i \mapsto g(i)^k$ ($g \geq id$), then $L^{2\cdot g}_{\omega^{\omega^{d+1}}}(2\cdot l + 1) \geq L^f_{\omega^{\omega^{d+1}}}(l)$ for sufficiently large l.*

Proof. Lemma 4.6 shows that $L^{\mathrm{id}}_{\omega^\omega}$ is Ackermannian, hence, for large enough l we know $L^{\mathrm{id}}_{\omega^\omega}(l) \geq \sharp\{i : l = \lfloor \sqrt[k]{i} \rfloor\}$. Take sequences $\gamma^i_0 \succ \cdots \succ \gamma^i_T$ of length $L^{\mathrm{id}}_{\omega^\omega}(l + i)$ and $\beta_0 \succ \cdots \succ \beta_R$ of length $L^f_{\omega^{\omega^{d+1}}}(l)$ and define

$$\alpha_i = \omega^{\omega^d} \cdot \beta_{\lfloor \sqrt[k]{i} \rfloor} + \gamma^{\lfloor \sqrt[k]{i} \rfloor}_{i - \lfloor \sqrt[k]{i} \rfloor^k}.$$

Then the α_i's show the lemma. ∎

These combine with Theorem 4.9 to prove part (2) of Theorem 2.2.

COROLLARY 4.13. *For each d there exists a primitive recursive h_d such that*

$$M_d(h_d(l)) \geq F_{\omega^{d+1}}(l)$$

for sufficiently large l, hence $(d,l) \mapsto M_d(l)$ is not multiply recursive.

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