

Characterization of σ -porosity via an infinite game

by

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Abstract. Let X be an arbitrary metric space and P be a porosity-like relation on X . We describe an infinite game which gives a characterization of σ - P -porous sets in X . This characterization can be applied to ordinary porosity above all but also to many other variants of porosity.

1. Introduction. The theory of porous and σ -porous sets forms an important part of real analysis and Banach space theory for more than forty years. It originated in 1967 when E. P. Dolženko used for the first time the term ‘porous set’ and proved that some sets of his interest are σ -porous (see [2]). Since then the porosity has been widely used, especially in differentiation theory (see [4] for an example). A useful fact is that every σ -porous set in \mathbb{R}^n is of the first category and has Lebesgue measure zero. In many cases, it is more convenient to prove that a given set is σ -porous than to prove that it is small in the sense of both category and measure. On the other hand, not every set of the first category and measure zero is σ -porous. This was first proved by L. Zajíček in [6] (although E. P. Dolženko stated it without proof earlier).

The connection between σ -porosity and infinite games was first shown by M. Zelený in [8]. He defined an infinite game which is very similar to the well known Banach–Mazur game, and using that game, he characterized both sets which can be covered by countably many closed uniformly porous sets and σ -very porous sets. He also found a sufficient condition for σ -porosity in terms of games.

For this work, very inspirational was the infinite game $H(A)$ of Farah and Zapletal (see [3, Example 4.20]). Let us endow the Cantor space $\{0, 1\}^{\mathbb{N}}$ with the metric $d(x, y) = 1/k$ where k is the least such that $x(k) \neq y(k)$. For $n \in \mathbb{N}$ and $t \in \{0, 1\}^n$, let $U_t = \{y \in \{0, 1\}^{\mathbb{N}} : y \text{ is an extension of } t\}$.

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The Farah–Zapletal game associated with a set $A \subseteq \{0, 1\}^{\mathbb{N}}$ is defined as follows:

Lasker	(S_1^1)	(S_2^1, S_2^2)	(S_3^1, S_3^2, S_3^3)	...
Steinitz	x_1	x_2	x_3	

On the first move, Lasker plays a system S_1^1 (possibly empty) consisting of some sets of the form U_t where $t \in \{0, 1\}$. Then, Steinitz plays $x_1 \in \{0, 1\}$. On the second move, Lasker plays two systems S_2^1 and S_2^2 , each consisting of some sets of the form U_t where $t \in \{0, 1\}^4$. Again, Steinitz plays $x_2 \in \{0, 1\}$. On the n th move, Lasker plays systems $S_n^1, S_n^2, \dots, S_n^n$ consisting of some sets of the form U_t where $t \in \{0, 1\}^{n^2}$, and Steinitz plays $x_n \in \{0, 1\}$. After a run of this game, we get a point $x = (x_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ constructed by Steinitz and a σ -porous set

$$C = \bigcup_{k=1}^{\infty} \left(\left\{ y \in \{0, 1\}^{\mathbb{N}} : \{0, 1\}^{\mathbb{N}} \setminus \bigcup_{n=k}^{\infty} \bigcup S_n^k \text{ is porous at } y \right\} \setminus \bigcup_{n=k}^{\infty} \bigcup S_n^k \right)$$

constructed by Lasker. Steinitz wins if $x \in A \setminus C$, Lasker wins in the opposite case. By [3, Claim 4.21], Lasker has a winning strategy in the game $H(A)$ if and only if the set A is σ -porous.

Later, D. Rojas-Rebolledo generalized the ideas from [3] and managed to find a similar game which characterizes σ -porosity and also σ -strong porosity in any zero-dimensional compact metric space (see [5]).

Let (X, d) be a nonempty metric space and let P be a porosity-like relation on X (the definition can be found in Section 2). In this paper, we associate an infinite game $G(A)$ (inspired by the game from [3]) with any subset A of X . This is a game between Boulder and Sisyfos (we follow the terminology of J. Zapletal) where Boulder has a similar role to Steinitz in the game above, and Sisyfos corresponds to Lasker. The game is defined as follows:

Boulder	B_1	B_2	B_3	...
Sisyfos	(S_1^1)	(S_2^1, S_2^2)	(S_3^1, S_3^2, S_3^3)	

On the first move, Boulder plays a nonempty open set $B_1 \subseteq X$ such that $\text{diam } B_1 < \infty$, and Sisyfos plays an open set $S_1^1 \subseteq B_1$. On the second move, Boulder plays a nonempty open set B_2 such that $B_2 \subseteq B_1$ and $\text{diam } B_2 \leq \frac{1}{2} \text{diam } B_1$, and Sisyfos plays open sets $S_2^1 \subseteq B_2$ and $S_2^2 \subseteq B_2$. On the n th move, $n > 1$, Boulder plays a nonempty open set B_n such that $B_n \subseteq B_{n-1}$ and $\text{diam } B_n \leq \frac{1}{2} \text{diam } B_{n-1}$, and Sisyfos plays open sets $S_n^1 \subseteq B_n, \dots, S_n^n$

$\subseteq B_n$. Sisyfos wins the run if at least one of the following two conditions is satisfied:

- (i) $\bigcap_{n=1}^{\infty} B_n \cap A = \emptyset$,
- (ii) $\bigcap_{n=1}^{\infty} B_n = \{x\}$ and there exists $m \in \mathbb{N}$ such that $x \in X \setminus \bigcup_{n=m}^{\infty} S_n^m$ and $P(x, X \setminus \bigcup_{n=m}^{\infty} S_n^m)$.

Boulder wins in the opposite case.

In Section 3, we characterize σ - P -porous sets in X via this game by proving the following theorem.

THEOREM 1.1. *Sisyfos has a winning strategy in the game $G(A)$ if and only if A is a σ - P -porous set.*

Since virtually all types of porosities can be considered as porosity-like relations (namely ordinary porosity, symmetric porosity in \mathbb{R} , strong porosity, right and left porosity), this is a more general result than in [3] and [5] as regards the assumptions both on the metric space X and on the porosity-like relation P .

An application of this characterization can be found in [1]. The game $G(A)$ (now for a subset A of a compact metric space) is modified there to a more complicated form. This modified game still characterizes σ -porosity and can be used to prove that in any given locally compact metric space with a porosity-like relation P satisfying some additional conditions, every analytic subset which is not σ - P -porous has a compact subset which is not σ - P -porous. Here, for P , we can substitute e.g. ordinary porosity and symmetric porosity (see also [9] and [10]) but also strong porosity.

2. Preliminaries. Let M be a nonempty set and $n \in \mathbb{N}$. We denote by M^n the set of all sequences $s = (s_1, \dots, s_n)$ of length n from M . We also set $M^0 = \{\emptyset\}$ where \emptyset is the *empty sequence* and

$$M^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N} \cup \{0\}} M^n.$$

If $s \in M^n$ and $t \in M^m$ then the *concatenation* of s and t is the sequence $s \wedge t = (s_1, \dots, s_n, t_1, \dots, t_m) \in M^{n+m}$. In the obvious way, we also understand the *infinite concatenation* $s_1 \wedge s_2 \wedge s_3 \wedge \dots$ of a sequence $(s_n)_{n=1}^{\infty}$ of elements of $M^{<\mathbb{N}}$.

For a nonempty subset B of a metric space (X, d) , we set $\text{diam } B = \sup\{d(a, b) : a \in B, b \in B\}$.

DEFINITION 2.1. Let X be a metric space and let $P \subseteq X \times 2^X$ be a relation between points of X and subsets of X . Then P is called a *point-set relation on X* . The symbol $P(x, A)$ where $x \in X$ and $A \subseteq X$ means that $(x, A) \in P$.

The point-set relation P on X is called a *porosity-like relation* if the following conditions hold for every $A \subseteq X$ and $x \in X$:

- (P1) if $B \subseteq A$ and $P(x, A)$ then $P(x, B)$,
- (P2) $P(x, A)$ if and only if there exists $r > 0$ such that $P(x, A \cap B(x, r))$,
- (P3) $P(x, A)$ if and only if $P(x, \bar{A})$.

If P is a porosity-like relation on X , $A \subseteq X$ and $x \in X$, we say that

- A is *P -porous at x* if $P(x, A)$,
- A is *P -porous* if it is P -porous at every point $x \in A$,
- A is *σ - P -porous* if it is a countable union of P -porous sets.

An example of a porosity-like relation is the ordinary porosity (or just porosity).

DEFINITION 2.2. Let X be a metric space, $A \subseteq X$, $x \in X$ and $R > 0$. Let us define

$$\gamma(x, R, A) = \sup\{r > 0 : \text{there exists } z \in B(x, R) \text{ with } B(z, r) \cap A = \emptyset\},$$

$$p(x, A) = \limsup_{R \rightarrow 0^+} \frac{\gamma(x, R, A)}{R}.$$

We say that

- A is *porous at x* if $p(x, A) > 0$,
- A is *porous* if it is porous at every $x \in A$,
- A is *σ -porous* if it is a countable union of porous sets.

REMARK 2.3. To be more exact, if we speak about ordinary porosity as a particular case of a porosity-like relation, we mean the following: if we define $(x, A) \in P$ to mean that A is porous at x , then P is a porosity-like relation on X , as can be easily verified.

We will need the following theorem.

THEOREM 2.4 ([7, Lemma 3]). *Let X be a metric space, P be a porosity-like relation on X and $A \subseteq X$. Then A is σ - P -porous if and only if for every $x \in A$ there exists $r > 0$ such that $B(x, r) \cap A$ is σ - P -porous.*

DEFINITION 2.5. Let X be a topological space. A system \mathcal{V} of subsets of X is said to be

- *discrete* if for every $x \in X$ there exists a neighborhood of x which intersects at most one set from the system \mathcal{V} ,
- *σ -discrete* if it is a countable union of discrete systems.

We will use the existence of a σ -discrete basis of open sets in a metric space. This is guaranteed by the following well known theorem.

THEOREM 2.6. *Let X be a metrizable topological space. Then X has an open basis which is σ -discrete.*

3. Proof of the main theorem. In this section, we prove Theorem 1.1. Let us fix a nonempty metric space (X, d) , a porosity-like relation P on X and $A \subseteq X$ throughout this section.

We say that a finite (possibly empty) sequence (B_1, \dots, B_i) of nonempty open sets in X is *good* if $B_{n+1} \subseteq B_n$, $\text{diam } B_1 < \infty$ and $\text{diam } B_{n+1} \leq \frac{1}{2} \text{diam } B_n$, $n = 1, \dots, i-1$. So a finite sequence of nonempty open sets in X is good if and only if Boulder can play the set B_n on his n th move, $n = 1, \dots, i$ (this is clearly independent of Sisyfos' moves). If $T = (B_1, \dots, B_i)$ is a good sequence, we say that a run of the game $G(A)$ is *T-compatible* if Boulder played B_1, \dots, B_i in sequence on his first i moves.

If Boulder played the sets B_n , $n \in \mathbb{N}$, in a run of $G(A)$ and $\bigcap_{n=1}^{\infty} B_n = \{x\}$ then x is called an *outcome* of the game. If Sisyfos wins the game by satisfying (ii) for some $m \in \mathbb{N}$, then every such m is called a *witness of Sisyfos' victory*.

Let ρ be a strategy for Sisyfos in the game $G(A)$. For $m \in \mathbb{N} \cup \{0\}$ and a good sequence $T = (B_1, \dots, B_i)$, we denote by M_m^T the set of all

$$x \in \begin{cases} A & \text{if } i = 0, \\ A \cap B_i & \text{if } i > 0, \end{cases}$$

such that in every run V of $G(A)$ such that

- the outcome of V is x ,
- V is T -compatible,
- Sisyfos followed the strategy ρ ,

all the witnesses of Sisyfos' victory (if any) are greater than m . The set M_m^T also depends on the strategy ρ . This will not cause any difficulties since if we speak about this set later, the strategy ρ will be fixed.

Let Boulder and Sisyfos play a run of the game $G(A)$. Let

$$V = (B_1, \mathcal{S}_1, B_2, \mathcal{S}_2, \dots), \quad \mathcal{S}_n = (S_n^1, S_n^2, \dots, S_n^n), \quad n \in \mathbb{N},$$

where Boulder played B_n and Sisyfos played $S_n^1, S_n^2, \dots, S_n^n$ on the n th move of the run. Then we will refer to the run itself by V and if we speak about B_n or S_n^m , $m \in \{1, \dots, n\}$, $n \in \mathbb{N}$, we use the symbols $B_n(V)$ and $S_n^m(V)$, respectively.

First of all, we prove the following two lemmata. Lemma 3.1 is well known at least for ordinary porosity.

LEMMA 3.1. *Let \mathcal{V} be a σ -discrete system of σ -P-porous sets in X . Then $\bigcup \mathcal{V}$ is also σ -P-porous.*

Proof. Let $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ where \mathcal{V}_n is a discrete system for every $n \in \mathbb{N}$. Let us fix $n \in \mathbb{N}$ and $x \in X$. There exists $r > 0$ such that $B(x, r)$ intersects at most one set from \mathcal{V}_n . Therefore $B(x, r) \cap \bigcup \mathcal{V}_n$ is σ -P-porous. By Theorem

2.4, the set $\bigcup \mathcal{V}_n$ is σ - P -porous. Finally, so is

$$\bigcup \mathcal{V} = \bigcup_{n=1}^{\infty} \bigcup \mathcal{V}_n. \blacksquare$$

LEMMA 3.2. *Let ρ be a strategy for Sisyfos in the game $G(A)$. Let*

$$T_0 = (B_1, \dots, B_i)$$

be a good sequence of nonempty open sets and let $m \in \mathbb{N} \cup \{0\}$. Then there exist a P -porous set $N_m^{T_0}$ and a σ -discrete system \mathcal{E} of subsets of X such that

- (a) $M_m^{T_0} = N_m^{T_0} \cup \bigcup \mathcal{E}$,
- (b) *for every $E \in \mathcal{E}$, there exists a finite sequence T of nonempty open sets in X such that $T_0 \wedge T$ is good and $E \subseteq M_{m+1}^{T_0 \wedge T}$.*

Proof. Whenever we speak about a run of the game $G(A)$ in this proof, we suppose that Sisyfos followed the strategy ρ . Let us denote

$$Z = \bigcup \{S_n^{m+1}(V) : n \geq m+1, V \text{ is a } T_0\text{-compatible run of } G(A)\}.$$

For every $x \in Z$, let us fix $n(x) \geq m+1$ and a T_0 -compatible run $V(x)$ of $G(A)$ such that x lies in the open set $S_{n(x)}^{m+1}(V(x))$. For $x \in Z$, let us denote

$$T(x) = (B_{i+1}(V(x)), B_{i+2}(V(x)), \dots, B_{n(x)}(V(x))).$$

Now, whenever $y \in S_{n(x)}^{m+1}(V(x))$ for some $x \in Z$ and V' is a $T_0 \wedge T(x)$ -compatible run with outcome y then V' coincides with $V(x)$ in its first $n(x)$ moves, in particular $S_{n(x)}^{m+1}(V') = S_{n(x)}^{m+1}(V(x))$, and so $y \notin X \setminus \bigcup_{n=m+1}^{\infty} S_n^{m+1}(V')$ and $m+1$ is not a witness of Sisyfos' victory in V' . Thus, if $y \in S_{n(x)}^{m+1}(V(x)) \cap M_m^{T_0}$ then also $y \in M_{m+1}^{T_0 \wedge T(x)}$, so we have

$$(1) \quad S_{n(x)}^{m+1}(V(x)) \cap M_m^{T_0} \subseteq M_{m+1}^{T_0 \wedge T(x)}.$$

Now, if \mathcal{B} is a σ -discrete basis of open sets in X (whose existence is guaranteed by Theorem 2.6) then the system

$$\mathcal{E}' = \{G \in \mathcal{B} : G \subseteq S_{n(x)}^{m+1}(V(x)) \text{ for some } x \in Z\}$$

is a σ -discrete covering of Z . We define

$$\mathcal{E} = \{M_{m+1}^{T_0}\} \cup \{G \cap M_m^{T_0} : G \in \mathcal{E}'\}, \quad N_m^{T_0} = M_m^{T_0} \setminus (Z \cup M_{m+1}^{T_0}).$$

The system \mathcal{E} is obviously σ -discrete and $M_m^{T_0} = N_m^{T_0} \cup \bigcup \mathcal{E}$. Moreover, if $E \in \mathcal{E}$ then either $E = M_{m+1}^{T_0} = M_{m+1}^{T_0 \wedge \emptyset}$ or $E = G \cap M_m^{T_0}$ for some $G \in \mathcal{E}'$. In the latter case, there exists $x \in Z$ such that

$$G \subseteq S_{n(x)}^{m+1}(V(x))$$

and so

$$E \subseteq S_{n(x)}^{m+1}(V(x)) \cap M_m^{T_0} \subseteq M_{m+1}^{T_0 \wedge T(x)},$$

where the last inclusion is due to (1).

It only remains to show that $N_m^{T_0}$ is P -porous. Let us choose $x \in N_m^{T_0}$ arbitrarily. Then $x \in M_m^{T_0} \setminus M_{m+1}^{T_0}$ and so there exists a T_0 -compatible run V of $G(A)$ with outcome x such that $m + 1$ is a witness of Sisyfos' victory in V , in particular

$$P\left(x, X \setminus \bigcup_{n=m+1}^{\infty} S_n^{m+1}(V)\right).$$

But

$$N_m^{T_0} \subseteq X \setminus Z \subseteq X \setminus \bigcup_{n=m+1}^{\infty} S_n^{m+1}(V),$$

and so it follows from (P1) (see Definition 2.1) that $P(x, N_m^{T_0})$. ■

Proof of Theorem 1.1. First, let us assume that $A = \bigcup_{n=1}^{\infty} A_n$ where A_n is a P -porous set for every $n \in \mathbb{N}$. On his n th move, let Sisyfos play $S_n^j = \emptyset$ for $j < n$ and $S_n^n = B_n \setminus \overline{A_n}$. Let Boulder and Sisyfos play a run of $G(A)$ with Sisyfos following the above strategy. We may assume that $\bigcap_{n=1}^{\infty} B_n = \{x\}$ and $x \in A$ because otherwise Sisyfos wins by (i). Then there exists $m \in \mathbb{N}$ such that $x \in A_m$. We have

$$(2) \quad X \setminus \bigcup_{n=m}^{\infty} S_n^m = \overline{A_m} \cup (X \setminus B_m)$$

and therefore

$$x \in A_m \subseteq X \setminus \bigcup_{n=m}^{\infty} S_n^m.$$

Furthermore, P -porosity of A_m implies $P(x, A_m)$. But this is equivalent to $P(x, \overline{A_m})$ by (P3) (see Definition 2.1), and by (P2), this is equivalent to $P(x, \overline{A_m} \cup (X \setminus B_m))$ since $x \in B_m$. So we have $P(x, X \setminus \bigcup_{n=m}^{\infty} S_n^m)$ by (2). Therefore, Sisyfos wins by (ii) with m as a witness, and the strategy described is winning.

Now, let us assume that Sisyfos has a winning strategy ρ in $G(A)$. Let us denote $E_0 = A$. By Lemma 3.2, we have

$$(3) \quad A = E_0 = M_0^{\emptyset} = N_0^{\emptyset} \cup \bigcup \mathcal{E}$$

where N_0^{\emptyset} is P -porous and \mathcal{E} is a σ -discrete system of subsets of X such that for every $E_1 \in \mathcal{E}$, there exists a good sequence $T(E_1)$ such that $E_1 \subseteq M_1^{T(E_1)}$. Now, for every $E_1 \in \mathcal{E}$ we have

$$(4) \quad E_1 \subseteq M_1^{T(E_1)} = N_1^{T(E_1)} \cup \bigcup \mathcal{E}^{E_1}$$

where $N_1^{T(E_1)}$ is P -porous and \mathcal{E}^{E_1} is a σ -discrete system of subsets of X such that for every $E_2 \in \mathcal{E}^{E_1}$, there exists a finite sequence $T(E_1, E_2)$ of nonempty open sets such that $T(E_1) \wedge T(E_1, E_2)$ is good and $E_2 \subseteq M_2^{T(E_1) \wedge T(E_1, E_2)}$. Suppose that for some $k \in \mathbb{N}$, we already have $E_1 \in \mathcal{E}$, $E_2 \in \mathcal{E}^{E_1}, \dots, E_k \in \mathcal{E}^{E_1, E_2, \dots, E_{k-1}}$ and finite sequences $T(E_1), T(E_1, E_2), \dots, T(E_1, \dots, E_k)$ such that

$$H := T(E_1) \wedge T(E_1, E_2) \wedge \dots \wedge T(E_1, \dots, E_k)$$

is good and $E_k \subseteq M_k^H$. Then

$$E_k \subseteq M_k^H = N_k^H \cup \bigcup \mathcal{E}^{E_1, \dots, E_k}$$

where N_k^H is P -porous and $\mathcal{E}^{E_1, \dots, E_k}$ is a σ -discrete system of subsets of X such that for every $E_{k+1} \in \mathcal{E}^{E_1, \dots, E_k}$, there exists a finite sequence $T(E_1, \dots, E_{k+1})$ of nonempty open sets such that the sequence

$$H \wedge T(E_1, \dots, E_{k+1}) = T(E_1) \wedge T(E_1, E_2) \wedge \dots \wedge T(E_1, \dots, E_{k+1})$$

is good and

$$E_{k+1} \subseteq M_{k+1}^{T(E_1) \wedge T(E_1, E_2) \wedge \dots \wedge T(E_1, \dots, E_{k+1})}.$$

By iterating this process, we get a system of P -porous sets

$$\mathcal{U} = \left\{ N_k^{T(E_1) \wedge T(E_1, E_2) \wedge \dots \wedge T(E_1, \dots, E_k)} \cap \bigcap_{i=0}^k E_i : \right. \\ \left. k \in \mathbb{N} \cup \{0\}, E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots, E_k \in \mathcal{E}^{E_1, \dots, E_{k-1}} \right\}.$$

We show that $A \subseteq \bigcup \mathcal{U}$. Suppose that this is not true and so there exists $x \in A \setminus \bigcup \mathcal{U}$. By (3), there exists $E_1 \in \mathcal{E}$ such that $x \in E_1 \subseteq M_1^{T(E_1)}$. By (4), there exists $E_2 \in \mathcal{E}^{E_1}$ such that $x \in E_2 \subseteq M_2^{T(E_1) \wedge T(E_1, E_2)}$. Next, there exists $E_3 \in \mathcal{E}^{E_1, E_2}$ such that $x \in E_3 \subseteq M_3^{T(E_1) \wedge T(E_1, E_2) \wedge T(E_1, E_2, E_3)}$. Continuing, we find a sequence $(E_k)_{k=1}^\infty$ where $E_1 \in \mathcal{E}$ and $E_k \in \mathcal{E}^{E_1, \dots, E_{k-1}}$ for $k > 1$ such that

$$x \in E_k \subseteq M_k^{T(E_1) \wedge T(E_1, E_2) \wedge \dots \wedge T(E_1, \dots, E_k)}$$

for every $k \in \mathbb{N}$. Therefore Boulder can play a run of $G(A)$ in the following way. He plays all the sets from $T(E_1)$ in sequence on his first moves, then all the sets from $T(E_1, E_2)$ and so on. (If there exists $k_0 \in \mathbb{N} \cup \{0\}$ such that all the sequences $T(E_1, \dots, E_k)$, $k > k_0$, are empty then the sequence

$$T(E_1) \wedge T(E_1, E_2) \wedge \dots = T(E_1) \wedge T(E_1, E_2) \wedge \dots \wedge T(E_1, \dots, E_{k_0})$$

is finite. Then Boulder can finish the run arbitrarily so that the outcome is x .) After such a run, the outcome is x and no $m \in \mathbb{N}$ can be a witness of Sisyfos' victory since $x \in M_m^{T(E_1) \wedge T(E_1, E_2) \wedge \dots \wedge T(E_1, \dots, E_m)}$ for every $m \in \mathbb{N}$. This contradicts the assumption that the strategy ρ is winning for Sisyfos.

By (P1), it suffices to show that $\bigcup \mathcal{U}$ is a σ - P -porous set. For $k \in \mathbb{N} \cup \{0\}$ and $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots, E_k \in \mathcal{E}^{E_1, \dots, E_{k-1}}$, let us denote

$$Q(E_1, \dots, E_k) = N_k^{T(E_1) \wedge T(E_1, E_2) \wedge \dots \wedge T(E_1, \dots, E_k)} \cap \bigcap_{i=0}^k E_i.$$

Then $\bigcup \mathcal{U} = \bigcup_{k=0}^\infty \bigcup \mathcal{U}_k$ where

$$\mathcal{U}_k = \{Q(E_1, \dots, E_k) : E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots, E_k \in \mathcal{E}^{E_1, \dots, E_{k-1}}\}.$$

It is obviously sufficient to prove that $\bigcup \mathcal{U}_k$ is σ - P -porous for every $k \in \mathbb{N} \cup \{0\}$. For $k = 0$ we know that $\bigcup \mathcal{U}_0 = N_0^0$, which is a P -porous set. Now suppose that $k > 0$. To finish the proof, it suffices to prove the following claim and use it for $j = 1$.

CLAIM 3.3. *For every $j \in \{1, \dots, k\}$ and for every $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots, E_{j-1} \in \mathcal{E}^{E_1, \dots, E_{j-2}}$, the set*

$$\bigcup_{E_j \in \mathcal{E}^{E_1, \dots, E_{j-1}}} \bigcup_{E_{j+1} \in \mathcal{E}^{E_1, \dots, E_j}} \dots \bigcup_{E_k \in \mathcal{E}^{E_1, \dots, E_{k-1}}} Q(E_1, \dots, E_k)$$

is σ - P -porous.

Proof. For $j = k$ and every $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots, E_{k-1} \in \mathcal{E}^{E_1, \dots, E_{k-2}}$, the set

$$\bigcup_{E_k \in \mathcal{E}^{E_1, \dots, E_{k-1}}} Q(E_1, \dots, E_k)$$

is the union of a σ -discrete system (since $\mathcal{E}^{E_1, \dots, E_{k-1}}$ is σ -discrete) of P -porous sets (since $N_k^{T(E_1) \wedge T(E_1, E_2) \wedge \dots \wedge T(E_1, \dots, E_k)}$ is P -porous). By Lemma 3.1, this set is σ - P -porous.

Let us assume that the assertion holds for $j + 1$ where $j \in \{1, \dots, k - 1\}$ and let $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots, E_{j-1} \in \mathcal{E}^{E_1, \dots, E_{j-2}}$ be given. Then

$$\bigcup_{E_j \in \mathcal{E}^{E_1, \dots, E_{j-1}}} \left(\bigcup_{E_{j+1} \in \mathcal{E}^{E_1, \dots, E_j}} \dots \bigcup_{E_k \in \mathcal{E}^{E_1, \dots, E_{k-1}}} Q(E_1, \dots, E_k) \right)$$

is the union of a σ -discrete system (since $\mathcal{E}^{E_1, \dots, E_{j-1}}$ is σ -discrete) of σ - P -porous sets (the assumption for $j + 1$). By Lemma 3.1, it is also σ - P -porous. ■

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