FUNDAMENTA
MATHEMATICAE
216 (2012)

Fixed points for positive permutation braids

by

Michał Misiurewicz (Indianapolis, IN) and
Ana Rodrigues (Stockholm)

Abstract. Making use of the Nielsen fixed point theory, we study a conjugacy invariant of braids, which we call the level index function. We present a simple algorithm for computing it for positive permutation cyclic braids.

1. Introduction. Let f and g be orientation preserving disk homeomorphisms with periodic orbits P and Q respectively. For simplicity, we will consider only orbits contained in the interior of the disk; this can be also achieved simply by extending a homeomorphism to a larger disk. Two such pairs (f, P) and (g, Q) are *equivalent* if f is conjugate (in the dynamical systems meaning) to some $\widetilde{g}$ via a homeomorphism that maps P to Q and $\widetilde{g}$ is isotopic to g relative to Q (that is, via an isotopy that fixes the points of Q). Equivalence classes are called *braid types* ([2]) or *patterns* ([5], [6]). Here we will consider *oriented* braid types, where the conjugacy mentioned above is orientation preserving.

Thus, we are looking at the *mapping classes* of the homeomorphism relative to the periodic orbit. To realize the connection with braids, explaining the name "braid types", take the suspension flow of the homeomorphism with a periodic orbit P, and then the trajectory of any point $x \in P$ (cut at the level 0) can be identified with a braid. Since we pass from a 3-dimensional picture, where the points of the orbit are in the interior of the disk, to a basically 2-dimensional one, where they are ordered on an interval, the braid corresponding to an orbit is defined only up to an algebraic conjugacy. We will use for those braids also the name (*oriented*) *braid type of* P.

Because of our motivation, braids with cyclic permutations are of special interest to us. We will call them *cyclic braids*. We will study positive

2010 *Mathematics Subject Classification*: Primary 20F36; Secondary 37E15.
Key words and phrases: conjugacy invariants of braids, positive permutation braids, braid types of periodic orbits, Nielsen fixed point theory.

DOI: 10.4064/fm216-2-4

permutation braids, which are braid types for disk homeomorphisms obtained by "thickening" interval maps. If f is a continuous interval map with a finite union of periodic orbits (*fupo* in short) P' then a homeomorphism of a "thick interval", which is homeomorphic to a disk, can be associated to it. Let us denote by P the corresponding fupo of this homeomorphism. The interval defines the natural ordering on P', so when considering a braid associated to P, we have a natural choice of a braid. This braid is a *positive permutation braid* ([4], [11]), that is, a braid with all crossings positive and each pair of strands crossing at most once. Its permutation is the same as the permutation of the points of the orbit P' of the interval map.

Let us recall some basic notions from the Nielsen fixed point theory (for an exposition see [9]). Let X be a connected compact polyhedron, $f : X \to X$ be a self-map and let $\mathrm{Fix}(f) = \{x \in X : f(x) = x\}$. An equivalence relation on the set of fixed points of f can be defined. We say that two fixed points x and y of $f : X \to X$ belong to the same *fixed point class* if there is a path α joining them such that $f(\alpha)$ is homotopic to α keeping endpoints fixed during the homotopy. The equivalence classes with respect to this relation are *essential* if the sum of indices of fixed points in the class is non-zero. The *Nielsen number* $N(f)$ is defined as the number of essential fixed point classes. The Nielsen number is a homotopy invariant which gives a lower bound for the cardinality of $\mathrm{Fix}(f)$. The *fixed point index* provides an algebraic count of fixed points in an open set. In particular, it is known that if x is a saddle, then its index is -1 if both eigenvalues are positive and 1 if both eigenvalues are negative.

Invariants of braid conjugacies are useful in the study of braid types. In a recent paper [10], we dealt with such invariants, which we called *turning numbers*. Their simplest interpretation is as linking numbers of components of the nth power of the braid (where n is the number of strands of the braid). Although they distinguish between non-conjugate positive permutation cyclic braids with up to seven strands, we found an example of two braids with eight strands which have the same turning numbers but are not conjugate. In this paper we define the *level index function*, another invariant of braid conjugacy. It allows us to distinguish between the braids in the above example (they can be distinguished in several other ways, but this may be the quickest one).

Similarly to [10], we consider our results as a contribution to the study of periodic orbits of disk homeomorphisms, while they may be less important from the point of view of braid theory or knot theory.

To get the level index function, we are essentially abelianizing the Nielsen fixed point theory. While this technique is known (see, e.g., [3], [7], [8], [13]), we did not find in the literature such an invariant. Even if it can be treated as a special case of some more complicated one, here we describe it in elementary terms and provide easy means for its computation for a dynamically interesting class of braids.

This paper is organized as follows. In Section 2 we say how to deal with the fixed point theory for a punctured disk (which is not compact). In Section 3 we introduce *winding sums* and *lapses*. In Section 4 we define the level index function and explain the main idea of the paper. In Section 5 we present a simple way to pass from an interval map with a fupo to a positive permutation braid with the same permutation. In Section 6 we introduce an algorithm to compute the level index function for positive permutation cyclic braids. Finally in Section 7 we show that in the example mentioned earlier the braids can be distinguished by the level index function.

2. Fixed point theory for a punctured disk. We need to use the Nielsen fixed point theory for a disk punctured at a finite number of points. However, normally the theory is presented for compact spaces. A solution is to use the results of the paper [12]. It works in our setup, provided all maps considered, including the level maps of homotopies involved, belong to the class $\mathcal{F}$ of maps for which the set of fixed points in the punctured disks is compact. Since we puncture the disk in the points of a finite union of periodic orbits P, a map belongs to $\mathcal{F}$ if and only if each point of P which is a fixed point is not a point of accumulation of fixed points. This will be satisfied for instance if each fixed point which belongs to P is attracting. To achieve this, we prove the following lemma.

LEMMA 2.1. *Let B be a ball in $\mathbb{R}^2$, centered at the origin O, with radius R. Let $f : B \to \mathbb{R}^2$ be a homeomorphism of B onto its image, such that $f(O) = O$. Then there exists a continuous map $\Phi : [0, R_0] \times B \to \mathbb{R}^2$, where*

$$R_0 = \min(\inf\{\|f(x)\| : \|x\| = R\}, R)/2,$$

such that $\Phi_0 = f$; for $t > 0$ the origin is an attracting fixed point of Φ_t (where $\Phi_t(x) = \Phi(t, x)$); for all t the map Φ_t is a homeomorphism of B onto $f(B)$ and $\Phi_t = f$ on the boundary of B.

Proof. Set $\psi(r) = \sup\{\|f(x)\| : \|x\| \le r\} + r$. This function is well defined for $r \in [0, R]$ and $\psi(0) = 0$. Moreover, ψ is strictly increasing. For $t \in [0, R_0]$ we set

$$\varphi_t(s) = \begin{cases} \psi^{-1}(s) & \text{if } 0 \le s \le t, \\ \psi^{-1}(t) + (s - t)(2t - \psi^{-1}(t))/t & \text{if } t < s \le 2t, \\ s & \text{if } s > 2t. \end{cases}$$

Clearly, $\varphi_t(s)$ depends continuously on (t, s). Moreover, if $r > 0$ then, by the definition of ψ, we have $\psi(r) > r$, so $\psi^{-1}(t) < t$ for $t > 0$. Therefore, by the definition of φ_t, we see that φ_t is strictly increasing.

Let $x \in B \setminus \{O\}$. By the definition of ψ, we have $\|f(x)\| \le \psi(r) - r$ for all $r \ge \|x\|$. In particular, this holds for $r = \|x\|$, so $\|f(x)\| \le \psi(\|x\|) - \|x\| < \psi(\|x\|)$. Let $t \ge \psi(\|x\|)$. Then $\varphi_t(\|f(x)\|) < \varphi_t(\psi(\|x\|)) = \|x\|$.

Set

$$\Phi_t(x) = \varphi_t(\|f(x)\|) \cdot \frac{f(x)}{\|f(x)\|}$$

if $x \neq O$ and $\Phi_t(O) = O$. Since $\varphi_t(s)$ depends continuously on (t, s), also $\Phi_t(x)$ depends continuously on (t, x) if $x \neq O$. However, $\|\Phi_t(x)\| = \varphi_t(\|f(x)\|) < \|x\|$ if $\|x\| \le \psi^{-1}(t)$, so this continuous dependence includes also $x = O$. Moreover, from this inequality it follows that if $t > 0$ then O is an attracting fixed point of Φ_t.

For each t the map Φ_t is a composition of the map $x \mapsto \varphi_t(\|x\|) \cdot x/\|x\|$, which is a homeomorphism onto its image, with f. Therefore Φ_t is a homeomorphism onto its image. If x belongs to the boundary of B, then $\|x\| = R$, so $\|f(x)\| \ge 2R_0 \ge 2t$. Therefore $\varphi_t(\|f(x)\|) = \|f(x)\|$, so $\Phi_t(x) = f(x)$. From this and the fact that Φ_t is a homeomorphism onto its image, it follows that $\Phi_t(B) = f(B)$. ∎

Thus, locally we can change our homeomorphism by a homotopy in such a way that every point of P that is a fixed point is attracting. In particular, we get the following corollary.

COROLLARY 2.2. *Every orientation preserving homeomorphism of $\mathbb{D}$ with a fupo P is homotopic rel. P to a homeomorphism from $\mathcal{F}$.*

REMARK 2.3. In Lemma 2.1 instead of one homeomorphism F we can take a one-parameter family of homeomorphisms $f^{(u)}$, depending continuously on $u \in [0, 1]$. Then we get a one-parameter family of maps $\Phi^{(u)}$, depending continuously on u. The only changes that we have to make in the proof are that when defining R_0 and ψ, we take the infimum and supremum respectively, also over $u \in [0, 1]$.

In view of the above remark, we get the next corollary.

COROLLARY 2.4. *Two orientation preserving homeomorphisms of $\mathbb{D}$ with a fupo P that are homotopic rel. P are also homotopic rel. P via maps from $\mathcal{F}$.*

Proof. Locally in neighborhoods of fixed points which are in P we first apply $\Phi_t^{(0)}$ with t growing from 0 to R_0 (where each such point plays the role of O in Lemma 2.1), and $f^{(0)}$ outside the neighborhoods. Then we apply $\Phi_{R_0}^{(u)}$ with u growing from 0 to 1 inside, and the original homotopy $f^{(u)}$ outside. Finally, we apply $\Phi_t^{(1)}$ with t decreasing from R_0 to 0 inside and $f^{(1)}$ outside. ∎

3. Winding sums and lapses. Let $\mathbb{D}$ be the closed unit disk in the plane and $f : \mathbb{D} \to \mathbb{D}$ its orientation preserving homeomorphism. Let P be a periodic orbit of f contained in the interior of $\mathbb{D}$. For any closed curve γ in $\mathbb{D} \setminus P$ and a point $x \in P$ there is an integer $\mathrm{w}(\gamma, x)$ called the *winding number of γ around x*. It measures how many times γ goes around x (the counterclockwise direction is positive, the clockwise is negative). It can be defined using the tools of complex analysis (it is called sometimes the *index of x with respect to γ* there). It can also be defined in topological terms. If S^1 is the unit circle, then we can consider γ as the map $\Gamma : S^1 \to \mathbb{D} \setminus P$. Then

$$s \mapsto \frac{\Gamma(s) - x}{\|\Gamma(s) - x\|}$$

is a continuous map of S^1 to itself, and $\mathrm{w}(\gamma, x)$ is the degree of this map.

Now we define the *winding sum of γ around P* by

$$\mathrm{ws}(\gamma, P) = \sum_{x \in P} \mathrm{w}(\gamma, x).$$

We need more notation. While a moment ago we were treating a closed curve γ as a map defined on the circle, usually we will treat it as a continuous map of a closed interval to $\mathbb{D} \setminus P$. That is, we choose some point on the curve that serves as its beginning and end. Of course the winding numbers and winding sum do not depend on this choice. Now, for curves (including closed curves) γ and δ for which the end of γ is the same as the beginning of δ, we denote by $\gamma\delta$ their concatenation. By γ' we will denote the inverse of γ, that is, the same curve as γ but with the reverse orientation. By $f(\gamma)$ we will denote the image of γ under f, that is, the curve $f \circ \gamma$. Note that

(a) $f(\gamma\delta) = f(\gamma)f(\delta)$,
(b) $f(\gamma') = (f(\gamma))'$,
(c) $(\gamma\delta)' = \delta'\gamma'$.

The following simple properties of the winding sum follow immediately from the analogous properties of winding numbers:

(a) if closed curves γ and δ in $\mathbb{D} \setminus P$ are homotopic then $\mathrm{ws}(\gamma) = \mathrm{ws}(\delta)$,
(b) $\mathrm{ws}(\gamma\delta) = \mathrm{ws}(\gamma) + \mathrm{ws}(\delta)$,
(c) $\mathrm{ws}(\gamma') = -\mathrm{ws}(\gamma)$.

The next property requires a proof.

LEMMA 3.1. *Let γ be a loop in $\mathbb{D} \setminus P$. Then* $\mathrm{ws}(f(\gamma)) = \mathrm{ws}(\gamma)$.

Proof. Let $P = \{p_0, \dots, p_{n-1}\}$ with $f(p_i) = p_{i+1}$ for $i = 0, \dots, n-1$ (the addition in the indices is modulo n). Let δ_i be a very small circle around p_i, with the counterclockwise orientation. Then the winding number of δ_i around p_i is 1, while around all other p_j is 0. Therefore $\mathrm{ws}(\delta_i) = 1$. Moreover, since

f is an orientation preserving homeomorphism, $f(\delta_i)$ is homotopic to δ_{i+1}, so $\mathrm{ws}(f(\delta_i)) = 1$.

Choose a point $x \in \mathbb{D} \setminus P$ and for each i a curve ζ_i from x to a point y_i on δ_i (which we will consider the beginning and end of δ_i). Then $\xi_i = \zeta_i \delta_i \zeta_i'$ is a loop homotopic to δ_i and the loops ξ_i are generators of the fundamental group of $\mathbb{D} \setminus P$ with the base point x. This means that γ is homotopic to some loop of the form $\eta_1 \dots \eta_k$, where $\eta_j \in \{\xi_0, \dots, \xi_{n-1}, \xi_0', \dots, \xi_{n-1}'\}$. In particular,

$$\mathrm{ws}(\gamma) = \sum_{j=1}^{k} \mathrm{ws}(\eta_j).$$

For each i, the loop ξ_i is homotopic to δ_i, so $f(\xi_i)$ is homotopic to $f(\delta_i)$. Therefore $\mathrm{ws}(f(\xi_i)) = 1 = \mathrm{ws}(\xi_i)$ and $\mathrm{ws}(f(\xi_i')) = -1 = \mathrm{ws}(\xi_i')$. Thus, $\mathrm{ws}(f(\eta_j)) = \mathrm{ws}(\eta_j)$ for all j. Hence,

$$\mathrm{ws}(f(\gamma)) = \sum_{j=1}^{k} \mathrm{ws}(f(\eta_j)) = \sum_{j=1}^{k} \mathrm{ws}(\eta_j) = \mathrm{ws}(\gamma). \ \blacksquare$$

Let now $x, y \in \mathbb{D} \setminus P$ be fixed points of f. If γ is a curve beginning at x and ending at y then $\gamma f(\gamma')$ is a loop, so we can consider $\mathrm{ws}(\gamma f(\gamma'))$. We will call it the *lapse* of the pair (x, y) and denote it by $\ell(x, y)$. We shall show that it is independent of the choice of the curve γ, so it is well defined. Note that if $x \neq y$ then γ is not a loop, so we cannot use Lemma 3.1, and thus $\ell(x, y)$ may be non-zero.

LEMMA 3.2. *Let $x, y \in \mathbb{D} \setminus P$ be fixed points of f and let γ, δ be curves beginning at x and ending at y. Then* $\mathrm{ws}(\gamma f(\gamma')) = \mathrm{ws}(\delta f(\delta'))$.

Proof. We have

$$-\mathrm{ws}(\delta f(\delta')) = \mathrm{ws}((\delta f(\delta'))') = \mathrm{ws}(f(\delta)\delta').$$

Therefore it is enough to show that $\mathrm{ws}(\gamma f(\gamma') f(\delta)\delta') = 0$. However, if we change the beginning and end of the loop $\gamma f(\gamma') f(\delta)\delta'$ from x to y, we get the loop $\delta'\gamma f(\gamma') f(\delta)$, which has the same winding sum. Since $\delta'\gamma$ and $f(\gamma')f(\delta)$ are loops, we find by Lemma 3.1 that

$$\begin{aligned}\mathrm{ws}(\delta'\gamma f(\gamma') f(\delta)) &= \mathrm{ws}(\delta'\gamma) + \mathrm{ws}(f(\gamma')f(\delta)) = \mathrm{ws}(\delta'\gamma) + \mathrm{ws}(f((\delta'\gamma)')) \\ &= \mathrm{ws}(\delta'\gamma) + \mathrm{ws}((\delta'\gamma)') = 0. \ \blacksquare\end{aligned}$$

It turns out that lapses are additive.

LEMMA 3.3. *If $x, y, z \in \mathbb{D} \setminus P$ are fixed points of f then $\ell(x, z) = \ell(x, y) + \ell(y, z)$.*

Proof. Let γ be a curve beginning at x and ending at y, and δ be a curve beginning at y and ending at z. Then $\gamma\delta$ begins at x and ends at z. If we change the beginning and end of the loop $\gamma\delta f(\delta') f(\gamma')$ from x to y, we get the loop $f(\gamma')\gamma\delta f(\delta')$, which has the same winding sum. In the same way we can replace the loop $f(\gamma')\gamma$ by the loop $\gamma f(\gamma')$ with the same winding sum. Therefore

$$\begin{aligned}\ell(x,z) &= \mathrm{ws}(\gamma\delta f((\gamma\delta)')) = \mathrm{ws}(\gamma\delta f(\delta')f(\gamma')) = \mathrm{ws}(f(\gamma')\gamma\delta f(\delta'))\\ &= \mathrm{ws}(f(\gamma')\gamma) + \mathrm{ws}(\delta f(\delta')) = \mathrm{ws}(\gamma f(\gamma')) + \mathrm{ws}(\delta f(\delta'))\\ &= \ell(x,y) + \ell(y,z). \ \blacksquare\end{aligned}$$

COROLLARY 3.4. *If $x, y \in \mathbb{D} \setminus P$ are fixed points of f then $\ell(x,x) = 0$ and $\ell(y,x) = -\ell(x,y)$.*

4. Global point of view. Let us now look at all fixed points of f and corresponding lapses. Assume that the set $\mathrm{Fix}(f)$ of fixed points of f is finite. Because of additivity, we can treat the lapses as differences of a *potential* function on $\mathrm{Fix}(f)$, defined up to an additive constant. We will denote it by Φ.

Thus, the situation is now as follows. There is a potential function $\Phi : \mathrm{Fix}(f) \to \mathbb{Z}$ and if x, y are fixed points of f then

$$\ell(x,y) = \Phi(y) - \Phi(x).$$

For each fixed point x of f we consider its *index* $\mathrm{Ind}(f,x)$. Recall that in particular if x is a saddle, then its index is -1 if both eigenvalues are positive, and 1 if both eigenvalues are negative.

For each level of the potential we can take the sum of indices of fixed points with this potential. In such a way we get the *level index* function $\mathrm{LI}_f : \mathbb{Z} \to \mathbb{Z}$ given by

$$\mathrm{LI}_f(n) = \sum_{x \in \Phi^{-1}(n)} \mathrm{Ind}(f,x).$$

As always with the index, all this can be done also in the case when the set $\mathrm{Fix}(f)$ is not finite. Remember that it is really defined up to a shift in the domain.

In the general Nielsen fixed point theory, a homotopy $H : X \times [0,1] \to X$ joining $f : X \to X$ with $g : X \to X$ establishes a one-to-one correspondence between fixed point classes of f and g. We will use the notation h_t for slices of H, that is, $H(x,t) = h_t(x)$. We will also consider a *fat homotopy* $\mathbf{H} : X \times [0,1] \to X \times [0,1]$, given by $\mathbf{H}(x,t) = (H(x,t),t)$. The correspondence mentioned above can be detected by looking at a fixed point x of f and a fixed point $\widehat{x}$ of g; they are in corresponding fixed point classes if $(x,0)$ and $(\widehat{x},1)$ are in the same fixed point class of $\mathbf{H}$ (this is, in fact, the content of Theorem 2.7 of [9] and Lemma 2 of [12]).

Now we can prove that the lapses are preserved by homotopies.

Theorem 4.1. *Let P be a common fupo of two orientation preserving homeomorphisms $f, g : \mathbb{D} \to \mathbb{D}$, with $f|_P = g|_P$. Assume that they are homotopic rel. P via a homotopy H. Let $x, y \in \mathbb{D} \setminus P$ be fixed points of f and $\widehat{x}, \widehat{y}$ be fixed points of g, in fixed point classes corresponding via H to the fixed point classes of x, y, respectively. Then $\ell(\widehat{x}, \widehat{y}) = \ell(x, y)$.*

Proof. Since $(x, 0)$ and $(\widehat{x}, 1)$ are in the same fixed point class of $\mathbf{H}$, there exists a path γ in $(\mathbb{D} \setminus P) \times [0, 1]$ from $(x, 0)$ to $(\widehat{x}, 1)$ such that $H(\gamma)$ is homotopic to γ with endpoints fixed. Similarly, there exists a path δ in $(\mathbb{D} \setminus P) \times [0, 1]$ from $(y, 0)$ to $(\widehat{y}, 1)$ such that $H(\delta)$ is homotopic to δ with endpoints fixed. Finally, take a path Γ from $(x, 0)$ to $(y, 0)$ in $(\mathbb{D} \setminus P) \times \{0\}$ and its image $H(\Gamma)$. Then $\ell(x, y)$ is equal to the winding sum of the projection of $\Gamma H(\Gamma')$ to $\mathbb{D} \setminus P$.

Now consider the path $\beta = \gamma' \Gamma \delta$ and its projection α to $(\mathbb{D} \setminus P) \times \{1\}$. Clearly, α and β are homotopic with endpoints fixed. Thus, $H(\alpha)$ and $H(\beta)$ are also homotopic with endpoints fixed. Therefore, the loop $\alpha H(\alpha')$ is homotopic to the loop $\beta H(\beta')$. We have $\beta H(\beta') = \gamma' \Gamma \delta H(\delta') H(\Gamma') H(\gamma)$. However, the loops $\delta H(\delta')$ and $H(\gamma)\gamma'$ are nullhomotopic, so $\beta H(\beta')$ is homotopic to the loop $\Gamma H(\Gamma')$. Thus, the loops $\alpha H(\alpha')$ and $\Gamma H(\Gamma')$ are homotopic. Hence their projections to $\mathbb{D} \setminus P$ are homotopic. However, $\ell(x, y)$ is equal to the winding sum of the projection of $\Gamma H(\Gamma')$ to $\mathbb{D} \setminus P$ and $\ell(\widehat{x}, \widehat{y})$ is equal to the winding sum of the projection of $\alpha H(\alpha')$ to $\mathbb{D} \setminus P$. This proves that $\ell(\widehat{x}, \widehat{y}) = \ell(x, y)$. ∎

Now we can explain the main idea of our paper. Suppose we have an orientation preserving disk homeomorphism $f : \mathbb{D} \to \mathbb{D}$ with a fupo P. By Corollary 2.2, modifying f by a homotopy rel. P if necessary, we may assume that $f \in \mathcal{F}$. By Corollary 2.4, we may modify all homotopies rel. P so that their slices are in $\mathcal{F}$. Thus, we can use the Nielsen fixed point theory in its full strength.

When we refer to fixed point classes, we mean an orientation preserving disk homeomorphism with a fupo restricted to the disk minus that fupo. According to the Nielsen fixed point theory, we have finitely many essential fixed point classes. If two fixed points are in the same fixed point class then there is a path γ joining them such that its image is homotopic to γ with endpoints fixed, so the lapse between those points is 0. Thus, we can speak of the lapses between fixed point classes. By the additivity of lapses, they are given by a potential function, again defined not for individual fixed points, but for fixed point classes. Remember that the potential is always defined only up to a shift by an integer. In such a way, we get a lot of information: for each essential fixed point class we know its potential and index; we will call this information *fpinfo* for short.

Now replace our homeomorphism by another one, homotopic to it rel. P. By the results of [12] and by Theorem 4.1, fpinfo does not change. If we apply to our homeomorphism an orientation preserving conjugacy, then of course the fpinfo also does not change. Note that if the conjugacy reverses orientation, then lapses change sign, so the potential also changes sign. Thus, we will stick in the definition of the braid type, which we will now call *oriented braid type*, to orientation preserving conjugacies. This shows that fpinfo depends only on the oriented braid type of (f, P). In particular, since every braid is an oriented braid type of some fupo of an orientation preserving disk homeomorphism, fpinfo is an object assigned to a braid. Moreover, if two braids are conjugate, then they are oriented braid types of the same fupo of the same orientation preserving disk homeomorphism, and therefore fpinfo is an invariant of conjugacy for braids.

This looks like a nice and powerful theory, but it has a drawback. Namely, determining the fpinfo of a braid may be complicated. In the definition of Nielsen equivalence *there exists* a path with a certain property, and this is essential. On the other hand, in order to determine the lapse between two fixed points, one can use an *arbitrary* path. This makes it a lot simpler and easier to use in concrete cases. Thus, if we forget about some information contained in fpinfo and stick only to the level index function, then we will get a conjugacy invariant for braids, which is less potent, but much easier to compute. Going from fpinfo to level index amounts to taking sums of indices for fixed point classes with the same potential, so fpinfo determines level index (and, in particular, level index is an invariant of braid conjugacy).

5. Connection between interval maps and positive permutation braids. We will present a simple way to pass from an interval map with a fupo whose points are permuted in a given way to the positive permutation braid with the same permutation. We will pass through a certain disk homeomorphism, whose properties we will use later. In particular, we want this homeomorphism to have only "the same" fixed points as the interval map.

Let $g : I \to I$ be a continuous map of a closed interval I into itself and let $P' \subset I$ be a fupo with permutation σ. That is, $P' = \{x_1, \dots, x_n\}$ with $x_1 < \dots < x_n$ and $g(x_i) = x_{\sigma(i)}$ for $i = 1, \dots, n$. Since we do not care much what g does outside of P' (remember that this is an interval map, not a disk homeomorphism), we may assume that I, P' and g are of some special form. Namely, $I = [0, 1]$, $x_i = (i-1)/(n-1)$ for $i = 1, \dots, n$ and g is the "connect the dots" map; that is, between two adjacent points of P' the map g is affine. The problem with this map is that when two adjacent points of P' are fixed points, then the whole interval joining them consists of fixed points. We want to have only finitely many fixed points, so on such an interval we adjust g by moving its graph above the diagonal on the open interval.

Now we start constructing a disk homeomorphism. We start with the square I^2 and squeeze it strongly towards the diagonal $\varDelta = \{(x,x) : x \in I\}$. To be more precise, we fix a very small $\varepsilon > 0$ and apply the map $(u,v) \mapsto (u, \varepsilon v)$ in coordinates $u = x+y$, $v = x-y$. We call this map F_1. Thus, F_1 maps homeomorphically I^2 to the rhombus with vertices

$$(0,0), \left(\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right), (1,1), \left(\frac{1+\varepsilon}{2}, \frac{1-\varepsilon}{2}\right).$$

Note that F_1 is the identity on the diagonal.

Now we make the next step and define the map F_2 that moves the points horizontally. We set $F_2(x,y) = (g(y)+x-y, y)$. It is a homeomorphism of $F_1(I^2)$ onto its image. On the diagonal we have $F_2(y,y) = (g(y), y)$, so the image of the diagonal is the graph of g, reflected through the diagonal. The image of $F_1(I^2)$ is contained in a small neighborhood of the image of the diagonal, although it is not necessarily contained in I^2.

The next map, F_3, moves the points vertically. Ideally, we would like just to project everything vertically onto the diagonal. Then on the diagonal the composition $F_3 \circ F_2$ would be $(y,y) \mapsto (g(y), g(y))$, so practically it would be just g. Unfortunately, we cannot do this, because our map has to be a homeomorphism. Therefore we will just approximate the vertical projection onto the diagonal by a homeomorphism onto its image. Writing a formula is possible, but not practical. The important thing is to describe where $F_2(\varDelta)$ is mapped. The points of $F_2(\varDelta) \cap \varDelta$ (that is, the points of the form (x,x) with $g(x) = x$) will not be moved. The points $(g(y), y)$, where $y \in P'$, will be moved to $(g(y), g(y))$. The rest of $F_2(\varDelta)$ will be mapped close to the diagonal, with the order in each vertical fiber preserved. This map can be extended to a homeomorphism F_3 of $F_2(I^2)$ onto its image, that does not change the first coordinates of points, and maps $F_2(I^2)$ to a small neighborhood of the diagonal.

The construction of F_2 and $F_3 \circ F_2$ is illustrated in Figure 1. We mark the diagonal and the points of $P = \{(x,x) : x \in P'\}$ for the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 2 & 4 & 5 & 8 & 1 & 6 \end{pmatrix}.$$

The left part shows the image of the diagonal under F_2 and the right part the image of the diagonal under $F_3 \circ F_2$. If ε from the definition of F_1 is very small, the images of I^2 under $F_2 \circ F_1$ and under $F_3 \circ F_2 \circ F_1$, respectively, are practically indistinguishable from the images of the diagonal.

Set $f = F_3 \circ F_2 \circ F_1$. At this moment f is defined only on I^2 and maps it homeomorphically onto a small neighborhood of $\varDelta$. Add to the domain of f small closed balls centered at $(0,0)$ and $(1,1)$, so that the new domain contains $f(I^2)$ in its interior. Now we can extend f to a homeomorphism of

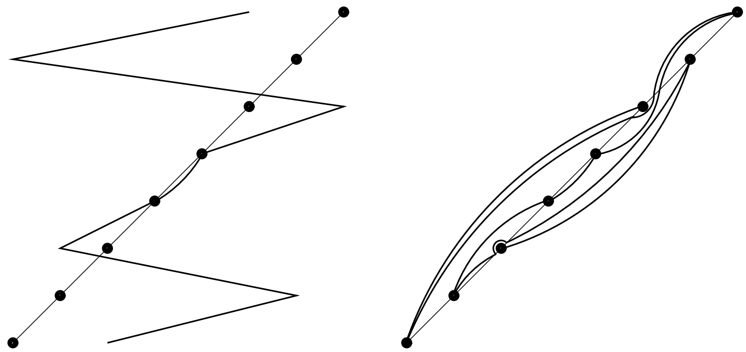

Fig. 1. Images of the diagonal under F_2 and $F_3 \circ F_2$

this new domain A onto its image, so that $f(A)$ is contained in the interior of A. We do not want to add in this way any fixed points. If $f(0,0) \neq (0,0)$ and $f(1,1) \neq (1,1)$, then this clearly can be done. Suppose that $f(0,0) = (0,0)$. Then, according to our construction, a small neighborhood of $(0,0)$ intersected with I^2 is mapped by f strictly inside I^2, except the point $(0,0)$. Thus, again we can make our extension without adding any fixed points. The same applies to the point $(1,1)$.

Now we have the set A, homeomorphic to a closed disk, and its homeomorphic image $f(A)$, contained in its interior (in fact, both sets have pretty regular shapes). Thus there exists a homeomorphism $\varphi : A \to D_2$ (by D_r we denote the closed disk in $\mathbb{R}^2$, centered at the origin, of radius r), such that $\varphi(f(A)) = D_1$. The map $G = \varphi \circ f \circ \varphi^{-1}$ is a homeomorphism from D_2 onto D_1. It is clear that we can extend it to a homeomorphism $G : D_3 \to D_3$ that has no fixed points in $D_3 \setminus D_2$. Similarly, we can extend φ to a homeomorphism of some disk $\mathbb{D}$ containing A onto D_3. Then $\varphi^{-1} \circ G \circ \varphi$ is an extension of f to a homeomorphism $f : \mathbb{D} \to \mathbb{D}$, and this new f has no fixed points outside I^2.

Now we investigate the fixed points of f in I^2.

Lemma 5.1. *The map f has no fixed points outside Δ. A point $(x,x) \in \Delta$ is a fixed point of f if and only if x is a fixed point of g.*

Proof. If x is a fixed point of g, then (x,x) is a fixed point of f by the construction of f. Again by the construction, if $(x,y) \in I^2$, then the point $f(x,y)$ is close to $(g((x+y)/2), g((x+y)/2))$. Thus, if (x,y) is a fixed point of f, then it has to be close to some point (t,t), where t is a fixed point of g. Thus, in order to complete the proof, it is enough to show that there are no fixed points of f close to the ones that we already identified.

If a fixed point x of g does not belong to P', then g is expanding in its neighborhood (or it reverses orientation at x). Thus, f is expanding in a neighborhood of (x, x) in the approximate direction of Δ (or it reverses orientation on Δ, which gives the same result) and contracting very strongly in the perpendicular direction. Therefore there are no other fixed points of f in a small neighborhood of (x, x). If $x \in P'$, then on each side of (x, x) in the direction of Δ where the adjacent point of P is not fixed, the situation is the same as above. If the adjacent point of P is also fixed, first note that in a small neighborhood of (x, x) the map $F_3 \circ F_2$ moves points to the right and up. Above the diagonal, F_1 moves points to the right and down, so f moves points to the right; below the diagonal, F_1 moves point to the left and up, so f moves points up. Thus, the only possible fixed points of f can be on the diagonal, but there we know exactly what is going on: only (x, x) is fixed (in the small neighborhood we are talking about). ∎

Remark 5.2. While in the above proof the sizes of neighborhoods are not defined precisely, we consider it a sufficiently rigorous proof. Making everything "machine checkable" would require adding several pages of epsilons, which nobody would read anyway.

Let us look a little closer into the nature of the fixed points of f that do not belong to P. We can easily compute their indices (in the sense of fixed point theory).

Lemma 5.3. *Let x be a fixed point of g that does not belong to P'. Then* $\mathrm{Ind}(f, (x, x))$ *is* -1 *if g preserves orientation at x and* 1 *if g reverses orientation at x.*

Proof. In the first case there is expansion in the direction of the diagonal and contraction in a transverse direction. In both directions the orientation is preserved, so $\mathrm{Ind}(f, (x, x)) = -1$. In the second case there is orientation reversal in both directions (remember that the two-dimensional orientation is preserved), so $\mathrm{Ind}(f, (x, x)) = 1$. ∎

Finally, we check that the braid type of P is correct.

Lemma 5.4. *Let the permutation of P' be σ. Then the braid corresponding to (f, P) is the positive permutation braid with permutation σ.*

Proof. When we construct the suspension of f, we look at the three-dimensional picture from the front, which corresponds to looking at the two-dimensional pictures like Figure 1 from below. Thus, the order of the points is the same for P' and P, so the permutation of the braid we get is σ. As we follow the suspension flow, the points of P move as described by the definitions of F_2 and F_3. That is, first they move horizontally (and up, because it is a suspension!), and if we compare two strands, the one that starts to the left is closer to us. Thus in this part of the movement (corresponding to F_2), all crossings are positive (we are free to choose the interpretation of the braid, so let the strand "on top" mean closer to us and "positive crossing" mean the strand starting on the left going on top). Moreover, all movements are monotone, so two strands cross at most once. Taking into account that in the second part of the movement (corresponding to F_3) there is no left-right movement, so no more crossings occur, this proves that the braid we get is a positive permutation one. ■

6. Positive permutation cyclic braids. Now we present a simple way to compute the level index function for positive permutation cyclic (*ppc* for short) braids. This can also be done for positive permutation braids that are not cyclic, but for cyclic ones the algorithm is simpler, and after all, our main motivation is to study just periodic orbits, not fupos.

Let us look at the orientation preserving disk homeomorphisms with a periodic orbit whose braid type can be represented as a positive permutation braid. According to the construction from the preceding section, when we pay attention mainly to the points of the periodic orbit, we can visualize it as in the diagram of Figure 2. We draw a diagonal with the points of the periodic orbit P on it (see (a)). Thin lines mark the movements of those points during the action of the homeomorphism f. First they move horizontally under the action of F_2, so the diagonal deforms as in (c). Then they move vertically

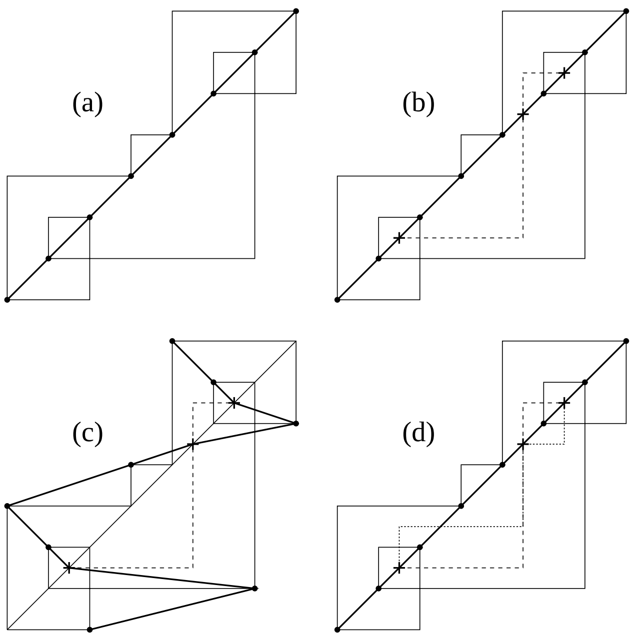

Fig. 2. A positive permutation cyclic braid with fixed points.

under the action of F_3, so the diagonal gets back to itself, but folded at several places.

Additionally in Figure 2 fixed points are marked. In this example there are three of them. They are located on the diagonal between each pair of adjacent points of P such that one of the points moves to the left, while the other one moves to the right. We connected adjacent fixed points with a curve (a dashed line in (b)) that goes below the diagonal if the points of P between them move to the left, and above the diagonal if those points move to the right. In such a way this curve does not intersect the horizontal parts of the lines marking movements of the points of P. When drawing the image of the diagonal after the first part of the movement in (c), we took into account that the fixed points do not move. During the second part of the movement, the points of P move vertically, and they will displace the curves joining fixed points. Their images are shown in (d) by dotted lines.

Thus, each vertical solid line intersecting the dashed line pushes the curve joining two adjacent fixed points x, y (that is, the dashed line) to the other side of one point of P. This increases or decreases (depending on the direction of movement of the points of P between x and y) the lapse of (x, y) by 1. This means that $|\ell(x, y)|$ is equal to the number of the vertical solid lines intersecting the dashed line from x to y. It remains to interpret this number in more manageable terms and to find the sign of $\ell(x, y)$.

We will say that there is a *turn at* $z \in P$ if z and $f^{-1}(z)$ move in the opposite directions; in other words, when we follow the solid line then the left-right direction changes at z. Observe that the total number of turns is equal to twice the first turning number of the braid corresponding to P.

LEMMA 6.1. *For adjacent fixed points x, y the number $|\ell(x, y)|$ is equal to the number of turns at points of P between x and y.*

Proof. All points of P between x and y move in the same direction; we may assume without losing generality that it is to the left. Let z be a point of P between x and y. A turn at z occurs if and only if $f^{-1}(z)$ moves to the right, that is, when the vertical solid line from z goes down. However, if this line goes down then it has to intersect the dashed line from x to y. Otherwise it ends at a vertical level between x and y, so the horizontal solid line from this end gets to the diagonal between x and y. This means that $f^{-1}(z)$ lies between x and y, so it has to move to the left, while we know that it moves to the right, a contradiction. Therefore the vertical solid line from z intersects the dashed line from x to y if and only if there is a turn at z. As we already know, the number of those intersections is $|\ell(x, y)|$. ∎

Determining the sign of $\ell(x, y)$ is simple. Assume $x < y$. If the points of P between x and y move to the left, then the curve from x to y goes below the diagonal and parts of its image may be pushed above the diagonal. Therefore the loop created this way is followed in the counterclockwise direction, so $\ell(x,y) \geq 0$. Similarly, if the points of P between x and y move to the right then $\ell(x,y) \leq 0$. We can reinterpret this in terms of the indices of x and y.

The following lemma follows immediately from Lemma 5.3.

LEMMA 6.2. *Let x be a fixed point between two adjacent points of P. If the right point moves to the left and the left one to the right, then* $\mathrm{Ind}(f,x) = 1$. *If the right point moves to the right and the left one to the left then* $\mathrm{Ind}(f,x) = -1$.

LEMMA 6.3. *When looking at the fixed points from left to right, their indices alternate, beginning and ending with* 1.

Proof. When we look at the points of P from left to right, the first one moves to the right, then there is a change of directions, then maybe more changes, and the last one moves to the left. By Lemma 6.2, when the direction changes from right to left, the index of the fixed point situated there is 1, and when the direction changes from left to right, the index is -1. This proves the statement. ∎

Now we can state the main result of this section. It follows immediately from what we already proved.

THEOREM 6.4. *Let f be an orientation preserving disk homeomorphism with a periodic orbit P whose braid type can be represented as a ppc braid. Let $x_0, \ldots, x_k$ be the fixed points of f on the diagonal of the corresponding diagram, ordered from left to right. Then k is even. For $i = 1, \ldots, k$, let n_i be the number of points of P between x_{i-1} and x_i at which there is a turn. Then $\ell(x_{i-1}, x_i) = n_i$ if i is odd and $\ell(x_{i-1}, x_i) = -n_i$ if i is even.*

Now, from the information gathered through Theorem 6.4, we can easily build the level index function. We put 1 at 0, then -1 at $-n_1$, then 1 at $-n_1 + n_2$, then -1 at $-n_1 + n_2 - n_3$, etc. The value of the level index at j will be the sum of the numbers put at j.

If between x_{i-1} and x_i there is no point of P at which there is a turn (call such a point a *turning point*), then $\ell(x_{i-1}, x_i) = n_i = 0$. Thus, when building the level index function, we put ± 1 at $-n_1 + \cdots \pm n_{i-1}$ and then ∓ 1 at $-n_1 + \cdots \mp n_i = -n_1 + \cdots \pm n_{i-1}$. That is, effectively we do not put anything anywhere in those two moves. This means that we can remove the points x_{i-1} and x_i from our list of fixed points. Of course, we have to be careful, and if we remove x_{i-1} and x_i then we cannot remove x_i and x_{i+1} in the next step, because x_i is not on our list of fixed points any more.

Now we observe that between two adjacent turning points where the turn occurs in the same direction there are an even number of fixed points, while between two adjacent turning points where the turn occurs in the opposite directions there are an odd number of fixed points. Thus, after all possible removals of fixed points, we are left with exactly one fixed point between adjacent turning points where the turn occurs in the opposite directions and no fixed points anywhere else.

Because of this, our algorithm for finding the level index function can be described as follows (remember that it is defined up to the shift by an integer). For the consecutive (in space) turning points (assume there are s of them) define numbers $a_1, \dots, a_s$ by $a_i = +1$ if at the ith turning point the incoming and outgoing arrows are from/to the right and $a_i = -1$ if they are from/to the left. Note that $a_1 = +1$ and $a_s = -1$. Then for $i = 2, \dots, s$ we put $(a_i - a_{i-1})/2$ at $a_1 + \dots + a_{i-1}$ and, as before, the value of the level index at j will be the sum of the numbers put at j.

7. An example. In [10] we studied a simple conjugacy invariant of braids, which we called turning numbers. As we mentioned in the introduction, their simplest interpretation is as linking numbers of components of the nth power of the braid (where n is the number of strands of the braid).

Let B be a braid with n strands and permutation τ (that is, the ith strand joins i in the bottom to $\tau(i)$ at the top; we assume that the strands go up—this is because of the suspension model). For each crossing of two strands we define its *sign* in a standard way: it is ± 1 depending on whether the left strand goes over the right one or vice versa (we assume that in the former case it is $+1$, and in the latter case -1).

We defined the *kth turning number of B* as

$$\mathrm{TN}_k(B) = \frac{1}{2}\sum_{i=1}^{n} T_k(i),$$

where $T_k(i)$ for $k = 1, \dots, n-1$ is the sum of the signs of the crossings between the ith and $\tau^k(i)$th strands.

Although for ppc braids with seven strands or less, two braids with the same turning numbers are always conjugate, the two braids with eight strands corresponding to the periodic orbits from Figure 3 have the same turning numbers: $3, 1, 2, 1, 2, 1, 3$, but they are not conjugate.

In [10] we showed that they are not conjugate by computing their pseudo-Anosov representations, using the methods of [1] or [5] (see also [6]). Corresponding Markov partitions give transition matrices that allowed us to estimate their entropies, which turned out to be different. This procedure is in general long and difficult. Other possible methods of distinguishing between them are (as pointed to us by the referee) by direct knot theory comparison, and by the use of invariants related to the Alexander polynomial.

However, the fastest way of distinguishing between these two braids may be to show that they have different level index functions. For the first braid we have $a_1 = a_2 = a_3 = +1$, $a_4 = a_5 = a_6 = -1$, and this gives us $L(3) = -1$

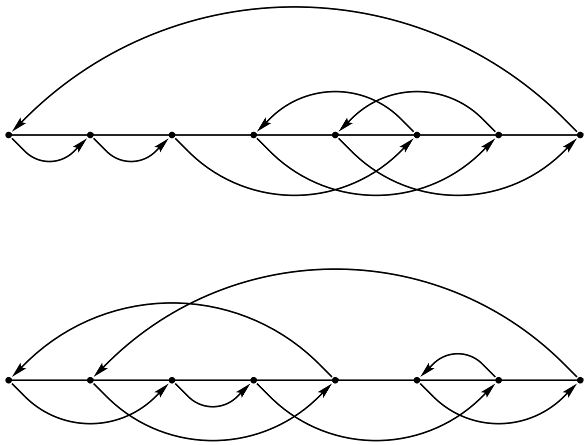

Fig. 3. Two orbits of period 8 with different braid types which have the same turning numbers but different level index functions.

and $L(j) = 0$ for all other j. For the second braid we get $a_1 = a_2 = +1$, $a_3 = -1$, $a_4 = +1$, $a_5 = a_6 = -1$, which gives us $L(1) = 1$, $L(2) = -2$, and $L(j) = 0$ for all other j.

Acknowledgements. The research of AR was supported by the Swedish Research Council (VR Grant 2010/5905), with additional funds coming from the Göran Gustafsson Foundation UU/KTH.

References

[1] M. Bestvina and M. Handel, *Train-tracks for surface homeomorphisms*, Topology 34 (1995), 109–140.

[2] P. Boyland, *Topological methods in surface dynamics*, Topology Appl. 58 (1994), 223–298.

[3] O. Davey, E. Hart and K. Trapp, *Computation of Nielsen numbers for maps of closed surfaces*, Trans. Amer. Math. Soc. 348 (1996), 3245–3266.

[4] E. A. Elrifai and H. R. Morton, *Algorithms for positive braids*, Quart. J. Math. Oxford Ser. (2) 45 (1994), 479–497.

[5] J. Franks and M. Misiurewicz, *Cycles for disk homeomorphisms and thick trees*, in: Contemp. Math. 152, Amer. Math. Soc., Providence, RI, 1993, 69–139.

[6] —, —, *Topological methods in dynamics*, in: Handbook of Dynamical Systems, Vol. 1A, North-Holland, Amsterdam, 2002, 547–598.

[7] J. Guaschi, *Nielsen theory, braids and fixed points of surface homeomorphisms*, Topology Appl. 117 (2002), 199–230.

[8] J. Jezierski, *The least number of coincidence points on surfaces*, J. Austral. Math. Soc. Ser. A 58 (1995), 27–38.

[9] B. J. Jiang, *Lectures on Nielsen Fixed Point Theory*, Contemp. Math. 14, Amer. Math. Soc., Providence, RI, 1983.

[10] M. Misiurewicz and A. Rodrigues, *Simple conjugacy invariants for braids*, preprint CMUP 2010-10.

[11] H. R. Morton and R. J. Hadji, *Conjugacy for positive permutation braids*, Fund. Math. 188 (2005), 155–166.

[12] U. K. Scholz, *The Nielsen fixed point theory for noncompact spaces*, Rocky Mountain J. Math. 4 (1974), 81–87.

[13] H. Shiraki, *On braid type of fixed points of homeomorphisms defined on the torus*, Mem. Fac. Sci. Kochi Univ. Ser. A Math. 20 (1999), 113–122.

Michał Misiurewicz
Department of Mathematical Sciences
IUPUI
402 N. Blackford Street
Indianapolis, IN 46202-3216, U.S.A.
E-mail: mmisiure@math.iupui.edu

Ana Rodrigues
Department of Mathematics
KTH
SE-100 44 Stockholm, Sweden
E-mail: amdsar@kth.se

Received 30 January 2011;
in revised form 5 November 2011