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Some questions of Arhangel'skii on rotoids

by

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Abstract. A *rotoid* is a space X with a special point $e \in X$ and a homeomorphism $F : X^2 \to X^2$ having $F(x,x) = (x,e)$ and $F(e,x) = (e,x)$ for every $x \in X$. If *any* point of X can be used as the point e, then X is called a *strong rotoid*. We study some general properties of rotoids and prove that the Sorgenfrey line is a strong rotoid, thereby answering several questions posed by A. V. Arhangel'skii, and we pose further questions.

1. Introduction. In his paper on spaces with a flexible diagonal [1], A. V. Arhangel'skii defined that a space X is a *rotoid* if there is a special point $e \in X$ and a homeomorphism F from X^2 onto itself with the properties:

(i) $F(x,x) = (x,e)$ for all $x \in X$, and
(ii) $F(e,x) = (e,x)$ for all $x \in X$.

One may think of the homeomorphism F as rotating the diagonal $\Delta = \{(x,x) : x \in X\}$ in X^2 onto the "horizontal" axis $X \times \{e\}$ and mapping the "vertical" axis $\{e\} \times X$ of X^2 identically onto itself. If it happens that every point of X can be used as the point e, then X is called a *strong rotoid*. Clearly, any homogeneous rotoid is a strong rotoid.

Rotoids are one of several recent generalizations to the category of topological spaces of properties of topological groups. It is easy to see that any topological group $(G,*)$ is a rotoid—let e be the multiplicative identity of G and define the function $F(x,y) = (x, x^{-1} * y)$—and previous research [5] has shown that theorems for topological groups can sometimes be proved for more general classes of spaces. For example, rectifiable spaces (defined in Section 3, below) are another recent generalization of topological groups and Gul'ko [5] has proved that two classical results from group theory—

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that first countable topological groups and topological groups of countable π-character are metrizable—must hold for rectifiable spaces. It is easy to see that any rectifiable space is a rotoid and, in [1], Arhangel'skii asked several questions about rotoids:

8.13: Is every (compact) strong rotoid rectifiable?
8.21: Is every first countable strong rotoid metrizable?
8.22: Is every strong rotoid with countable π-character metrizable?

In addition, suggesting that he saw the Sorgenfrey line as a test case for Questions 8.21 and 8.22, he asked

8.20: Is the Sorgenfrey line a strong rotoid?

In this paper we prove that the Sorgenfrey line S *is* a strong rotoid, thereby answering Question 8.20. Our positive answer to Question 8.20 provides a negative answer to the non-compact part of Question 8.13 and to Questions 8.21 and 8.22 as well. Question 8.13 remains open for the compact case.

The Sorgenfrey line is a particularly well-known example of a generalized ordered space. A *generalized ordered space* (GO-space) is a triple $(X, <, \mathcal{S})$ where $<$ is a linear ordering of X and where $\mathcal{S}$ is a Hausdorff topology on X that has a base of order-convex sets. If it happens that $\mathcal{S}$ has a base of open intervals, then $(X, <, \mathcal{S})$ is a *linearly ordered topological space* (LOTS).

The sets of irrational, rational, and real numbers will be denoted by $\mathbb{P}$, $\mathbb{Q}$, and $\mathbb{R}$ respectively, and $\mathbb{Z}$ denotes the set of all integers. In a linearly ordered set $(X, <)$ we will use $]a, b[$ to denote the set of points strictly between a and b, while $[a, b[= \{x \in X : a \leq x < b\}$, and $]a, b]$ is defined analogously. We adopted those somewhat infelicitous notations for intervals because many of our proofs involve simultaneous use of the ordered pair (a, b) and the interval from a to b, and readers of early drafts of our paper found this confusing. There can be no confusion if (a, b) denotes an ordered pair and $]a, b[$ an open interval.

2. A set of Sorgenfrey rectangles and some order-isomorphisms. The *Sorgenfrey line* is the set of real numbers topologized in such a way that for each number a, the collection $\{[a, a + \epsilon[: \epsilon > 0\}$ is a neighborhood base at a. A *Sorgenfrey rectangle* is any set of the form $[a, b[\times [c, d[$ where $a < b$ and $c < d$ are real numbers. By the *Euclidean closure* of such a rectangle we mean $[a, b] \times [c, d]$.

PROPOSITION 2.1. *There is a countable collection $\mathcal{T}$ of pairwise disjoint Sorgenfrey rectangles such that:*

(1) $\bigcup \mathcal{T} = [0, 1[^2 - \Delta$;
(2) *for each $T \in \mathcal{T}$, the Euclidean closure of T is disjoint from Δ;*

(3) *for each* $x \in [0,1[$ *the set* $\{T \in \mathcal{T} : T \cap (]x,1[\times \{x\}) \neq \emptyset\}$ *is infinite and can be indexed as* $\{T_n : n \geq 1\}$ *where points of* T_{j+1} *lie to the left of points of* T_j*;*
(4) *for each* $x \in [0,1[$ *the set* $\{T \in \mathcal{T} : T \cap (\{x\} \times]x,1[) \neq \emptyset\}$ *is infinite and can be indexed as* $\{T_m : m \geq 1\}$ *in such a way that points of* T_k *lie above points of* T_{k+1}.

Proof. We will show how to construct Sorgenfrey rectangles $[a,b[\times [c,d[$ that partition the half of $[0,1[\times [0,1[$ lying strictly above the diagonal; this will give half of the members of $\mathcal{T}$. The rectangles that partition the half of $[0,1[\times [0,1[$ below the diagonal are obtained in an analogous way, or could be obtained by reflecting the rectangles obtained below across the diagonal.

Let L_0 be the line $[0,1[\times \{1\}$ and for $k \geq 1$ let L_k be the straight line joining the points $(0,1/k)$ and $(1,1)$. For $k \geq 1$ there is a step function $S_k : [0,1[\to]0,1[$ satisfying:

- the graph of S_k lies strictly between the graphs of L_k and L_{k-1};
- the jump points of S_k occur at rational numbers and those jump points are an increasing sequence that converges to 1;
- for each $x \in [0,1[$, $S_k(x)$ is rational;
- the horizontal segments of the graph of S_k contain their left endpoints, but not their right endpoints.

We will show how to construct S_3 between the straight lines L_3 and L_2. The other constructions are analogous. Drawing pictures makes this construction clearer. Let v_1 be the average of $L_2(0)$ and $L_3(0)$ (so $v_1 = 5/12$). Find $a_1 \in [0,1[$ with $L_3(a_1) = v_1$ and let v_2 be the average of the numbers $v_1 = L_3(a_1)$ and $L_2(a_1)$. Find a_2 with $L_3(a_2) = v_2$ and let v_3 be the average of the numbers $v_2 = L_3(a_2)$ and $L_2(a_2)$. In general, given $a_1, \ldots, a_n$ and $v_1, \ldots, v_n$, let v_{n+1} be the average of the numbers $v_n = L_3(a_n)$ and $L_2(a_n)$ and find a_{n+1} so that $L_3(a_{n+1}) = v_{n+1}$. This recursion gives points a_n (which will be called the *jump points* of S_2) and v_n which will be the set of values of S_3. Note that each $v_n \in \mathbb{Q}$. For $0 \leq x < a_1$ we define $S_3(x) = v_1$ and for $n \geq 2$ and $a_{n-1} \leq x < a_n$ we define $S_3(x) = v_n$. Notice that because the graph of S_3 lies between the lines L_3 and L_2, while the graph of S_2 is constructed between the lines L_2 and L_1, we have $S_3(x) < S_2(x)$ for all $x \in [0,1[$.

Once we have the step functions S_j for all $j \geq 1$, we will use the graphs of S_j and S_{j+1} and their jump points to describe the edges of the Sorgenfrey rectangles that we will put into $\mathcal{T}$. Once again, drawing pictures will help. The top tier of rectangles is described as follows, using the step function S_1: List the jump points of S_1 as $a_0 = 0 < a_1 < a_2 < \cdots$ and use the Sorgenfrey rectangles $[0,a_1[\times [S_1(0),1[$ and in general, $[a_j,a_{j+1}[\times [S_1(a_j),1[$.

The next tier of rectangles is defined using S_1 and S_2. List the jump points of S_2 as $b_0 = 0 < b_1 < b_2 < \cdots$. If there is no jump point of S_1 in the interval $[b_0, b_1[$, then use the Sorgenfrey rectangle $[b_0, b_1[\times [S_2(0), S_1(0)[$. If there are jump points of S_1 in the interval $[b_0, b_1[$ list them as $b_0 < c_1 < \cdots < c_k < b_1$ and use the rectangles $[b_0, c_1[\times [S_2(0), S_1(0)[, [c_1, c_2[\times [S_2(c_1), S_1(c_1)[, \ldots, [c_k, b_1[\times [S_2(c_k), S_1(c_k)[$. This process is repeated in each interval $[b_j, b_{j+1}[$ of consecutive jump points of S_2, and then repeated using each pair S_k, S_{k+1} of consecutive step functions. The resulting collection of Sorgenfrey rectangles satisfies the parts of (1) through (4) above that deal with points above the diagonal. ■

LEMMA 2.2. *Given real numbers $a < b$ and $c < d$ the function*

$$h_{abcd}(x) = c + \frac{d-c}{b-a}(x-a)$$

is an order-isomorphism from $[a, b[$ onto $[c, d[$, and the inverse of h_{abcd} is h_{cdab}. If a, b, c, d are rational numbers, then h_{abcd} maps $[a, b[\cap \mathbb{Q}$ and $[a, b[\cap \mathbb{P}$ onto the rational and irrational numbers (respectively) in $[c, d[$.

3. The Sorgenfrey line is a strong rotoid. This section is devoted to a rather technical proof of our claim that the Sorgenfrey line is a rotoid. Because the Sorgenfrey line is homogeneous, it will follow that it is a strong rotoid. This answers several questions from [1] as noted in the Introduction.

We know that the Sorgenfrey line is homeomorphic to $[0, 1[$ with the topology in which $\{[a, a+\epsilon[: \epsilon > 0\}$ is a neighborhood base at each point $a \in [0, 1[$, so we will write $S = [0, 1[$. In the rotoid definition, we will choose $e = 0$ and construct a homeomorphism $F : S^2 \to S^2$ that has $F(x, x) = (x, 0)$ and $F(0, y) = (0, y)$ for all $x, y \in S$.

PROPOSITION 3.1. *The Sorgenfrey line is a rotoid.*

Proof. We will show that S is a rotoid by defining a continuous function $F : S^2 \to S^2$ that satisfies $F(x, x) = (x, 0)$, $F(0, x) = (x, x)$, and $F(0, y) = (0, y)$, and where $F(F(x, y)) = (x, y)$ for all $x, y \in [0, 1[$. To that end, we will describe some special notation, then define the function F in steps D-1 through D-6, then prove that F is self-inverse in steps SI-1 through SI-3, and finally prove that F is continuous in steps C-1 through C-6. Once that is done, we will know that F is the homeomorphism needed to prove that S is a rotoid.

Special notation. For $k \geq 1$ let $D(k) := [1/2^k, 1/2^{k-1}[\times [1/2^k, 1/2^{k-1}[$. The sets $D(k)$ will be called the *basic diamonds* of S^2.

Define two step functions σ and τ, as follows, both having domain $]0, 1[$ and range $[0, 1]$. For each $x \in]0, 1[$ there is a unique $k \geq 1$ such that $1/2^k \leq x < 1/2^{k-1}$, and we define $\sigma(x) = 1/2^k$ and $\tau(x) = 1/2^{k-1}$. The horizontal pieces of the graphs of σ and τ are, respectively, the bottom and top of the basic diamonds $D(k)$. It will be important to note that each horizontal segment of the graph of σ contains a left endpoint, but not a right endpoint, and the same is true of τ.

For each $x \in [0,1[$ let $B(x) := \{x\} \times [0,\sigma(x)[$, and for $n \geq 1$, subdivide $B(x)$ into subsegments $B(x,n) := \{x\} \times [\sigma(x)/2^n, \sigma(x)/2^{n-1}[$. Further subdivide each $B(x,n)$ into two halves using the midpoint M of the vertical component of $B(x,n)$. We denote the lower half by $B^L(x,n) = \{x\} \times [\sigma(x)/2^n, M[$ and the upper half by $B^U(x,n) = \{x\} \times [M, \sigma(x)/2^{n-1}[$.

For each $x \in S$ with $x \neq 0$, there is a unique diamond $D(k)$ containing (x,x). Note that (x,x) might be the southwest corner point of $D(k)$, but it cannot be the northeast corner point of $D(k)$. Define $HL(x) =]x, 1/2^{k-1}[\times \{x\}$ and $VL(x) = \{x\} \times]x, 1/2^k[$. Let $L(x) = VL(x) \cup \{(x,x)\} \cup HL(x)$. The set $L(x)$ is an L-shaped subset of the basic diamond $D(k)$, and $HL(x)$ and $VL(x)$ are, respectively, the horizontal and vertical segments of $L(x)$ (each excluding (x,x)).

For each $x \in]0,1[$ we subdivide $HL(x)$ as follows. Find the unique k so that $(x,x) \in D(k) = [1/2^k, 1/2^{k-1}[^2$. Recalling Proposition 2.1, we list all members of the collection $\mathcal{T}$ that intersect $HL(x)$ as $T_1, T_2, \ldots$ where each point of T_{j+1} lies to the left of each point of T_j. We can write each $T_j = [a_j, b_j[\times [c_j, d_j[$, and then we have $\cdots < b_3 = a_2 < b_2 = a_1 < 1/2^{k-1} \leq b_1$. Define $HL(x,n) := HL(x) \cap T_n = [a_n, b_n[\times \{x\}$ for each $n \geq 2$, and let $HL(x,1) = [a_1, 1/2^{k-1}[\times \{x\}$.

Analogously, for each $x \in]0,1[$ we subdivide the vertical segment $VL(x)$ using the list of all members of $\mathcal{T}$ that intersect $VL(x)$, obtaining vertical sub-segments $VL(x,n)$ for $n \geq 1$.

Definition of $F(x,y)$. The definition of $F(x,y)$ has six parts, called D-1 through D-6, depending on the location of (x,y) in S^2. We proceed by cases.

D-1. Let $F(x,x) = (x,0)$ and $F(x,0) = (x,x)$ for each $x \in S$.

D-2. Define $F(0,y) = (0,y)$ for each $y \in S$.

D-3. If $(x,y) \in S^2$ has $\tau(x) \leq y$ then let $F(x,y) = (x,y)$. In other words, H is the identity map above the basic diamonds $D(k)$.

D-4. If $(x,y) \in D(k)$ for some $k \geq 1$ and $y < x$ (so that (x,y) lies below the diagonal, directly to the right of the diagonal point (y,y)), then (x,y) is in the horizontal line $HL(y)$ through (y,y). Find the unique n with $(x,y) \in HL(y,n)$. The set $HL(y,n)$ has the form $[a,b[\times \{y\}$. Consider the vertical segment $B(y) = \{y\} \times [0,\sigma(y)[$ which is subdivided into pairwise disjoint subsegments $B(y,j)$. As in the "Special notation" section of our proof, the subsegment $B(y,n)$ is divided into an upper and lower half using the midpoint of the set of its second coordinates, and the upper half $B^U(y,n)$ has the form $\{y\} \times [c,d[$. Consequently, Lemma 2.2 gives us an order isomorphism $h_{abcd} : [a,b[\to [c,d[$ and we define $F(x,y) = (y, h_{abcd}(x))$.

D-5. If $(x,y) \in D(k)$ for some $k \geq 1$ and $y > x$ (so that (x,y) lies directly above the point (x,x) on the diagonal), then (x,y) is in the vertical line $VL(x)$ and therefore in a unique subsegment $VL(x,m)$ which has the form $VL(x,m) = \{x\} \times [p,q[$. The vertical segment $B(x) = \{x\} \times [0, \sigma(x)[$ contains the subsegment $B(x,m)$ which is divided into two halves $B^U(x,m)$ and $B^L(x,m)$ as in the "Special notation" section. The set $B^L(x,m)$ has the form $B^L(x,m) = \{x\} \times [r,s[$ so that Lemma 2.2 gives us an order isomorphism $h_{pqrs} : [p,q[\to [r,s[$. Now define $F(x,y) = (x, h_{pqrs}(y))$.

D-6. If (x,y) has $0 < y < \sigma(x)$, then (x,y) is below the basic diamonds $B(k)$, and $(x,y) \in B(x) = \{x\} \times [0, \sigma(x)[$. Then there is a unique subsegment of $B(x,k)$ of $B(x)$ that contains (x,y) so that $(x,y) \in B^U(x,k)$ or $(x,y) \in B^L(x,k)$. Consider the case where $(x,y) \in B^L(x,k)$. We will map (x,y) into the vertical line $VL(x)$ in the following way. The set $B^L(x,k)$ has the form $B^L(x,k) = \{x\} \times [t,u[$ and the subsegment $VL(x,k)$ of $VL(x)$ has the form $VL(x,k) = \{x\} \times [v,w[$. From Lemma 2.2 we have the order-isomorphism $h_{tuvw} : [t,u[\to [v,w[$ and we define $F(x,y) = (x, h_{tuvw}(y))$. Next consider the case where $(x,y) \in B^U(x,k)$. Write $B^U(x,k) = \{x\} \times [e,f[$. The set $HL(x,k)$ has the form $HL(x,k) = [g,h[\times \{x\}$. From Lemma 2.2 we have a function $h_{efgh} : [e,f[\to [g,h[$ and we define $F(x,y) = (h_{efgh}(y), x)$. The other case, where $(x,y) \in B^U(x,k)$, is analogous, except that $F(x,y)$ will belong to $HL(x,k)$.

We now have a function $F : S^2 \to S^2$.

The function F is self-inverse. We will show that $F(F(x,y)) = (x,y)$ for all $(x,y) \in S^2$. There are several cases, called SI-1, SI-2, and SI-3, to consider.

SI-1. If $(x,y) = (0,y)$ of if $(x,y) = (x,x)$, or if $(x,y) = (x,0)$, or if $\tau(x) \leq y < 1$, it is immediate from the definition of F that $F(F(x,y)) = (x,y)$.

SI-2. Consider the case where $(x,y) \in D(n)$ with $y < x$ (so that (x,y) lies below the diagonal). Then $(x,y) \in HL(y)$ and there is a unique k with $(x,y) \in HL(y,k)$ and $HL(y,k)$ has the form $HL(y,k) = [a,b[\times \{y\}$. The subsegment $B(y,k)$ has two halves and we consider $B^U(y,k)$, which has the form $B^U(y,k) = \{y\} \times [c,d[$. Then, using the order isomorphism $h_{abcd} : [a,b[\to [c,d[$, we have $F(x,y) = (y, h_{abcd}(x))$. To simplify notation, write $u = h_{abcd}(x)$. We must compute $F(y,u)$. Because $(y,u) \in B^U(y,k) = \{y\} \times [c,d[$ we know that $F(y,u)$ will lie in $HL(y,k) = [a,b[\times \{y\}$ and will be given by $F(y,u) = (h_{cdab}(u), y)$. But $h_{cdab}(u) = h_{cdab}(h_{abcd}(x)) = x$ because of the way our order isomorphisms were chosen in Lemma 2.2. Consequently, $F(F(x,y)) = (x,y)$ as claimed. The case where $(x,y) \in D(n)$ with $x < y$ is analogous.

SI-3. Finally consider the case where the point (x,y) has $0 < y < \sigma(x)$, i.e., where (x,y) lies below one of the basic diamonds $D(n)$. Then there is a unique k with $(x,y) \in B(x,k)$ and so $(x,y) \in B^L(x,k)$ or $(x,y) \in B^U(x,k)$. Consider the case where $(x,y) \in B^L(x,k)$ and suppose $B^L(x,k) = \{x\} \times [t,u[$. Then we map (x,y) into $VL(x,k) = \{x\} \times [v,w)$ by the rule $F(x,y) = (x, h_{tuvw}(y))$. To simplify notation, write $z = h_{tuvw}(y)$. Because $(x,z) \in VL(x,k) = \{x\} \times [v,w[$ we know that $F(x,z)$ will be a point of $B^L(x,k) = \{x\} \times [t,u[$ and so we have $F(x,z) = (x, h_{vwtu}(z))$. But $h_{vwtu}(z) = h_{vwtu}(h_{tuvw}(y)) = y$ because of Lemma 2.2. Consequently, $F(F(x,y)) = (x,y)$ in this case. The remaining possibility, where $(x,y) \in B^U(x,k)$, is analogous.

At this stage we know that $F(F(x,y)) = (x,y)$ for all $(x,y) \in S^2$.

The function F is continuous. To prove that F is continuous at (x,y) requires different arguments for points in different parts of S^2. We consider six separate cases that we will call C-1 through C-6.

C-1. First, suppose $(x,y) \in S^2 - \{(0,0)\}$ has $0 < x$ and $\tau(x) \leq y < 1$, or $x = 0$ and $0 < y$. Then $F(x,y) = (x,y)$ and the set of all points of this type is an open subset of S^2, so that F is continuous at each such point.

C-2. Second, consider the point $(x,y) = (0,0)$ and suppose a sequence $\langle (x_n, y_n) \rangle$ converges to $(0,0)$. We may assume $(x_n, y_n) \neq (0,0)$ for each n. Separately consider two subsequences, namely, those with $\tau(x_n) \leq y_n$ and those with $y_n < \tau(x_n)$. If there are infinitely many points of the first type, then their images certainly converge to $(0,0)$ because for such a point $F(x_n, y_n) = (x_n, y_n)$. Notice that every point (x_n, y_n) of the second type has the property that $F(x_n, y_n)$ lies below the graph of τ and therefore the subsequence of such points (if infinite) will converge to $(0,0)$. Therefore $\lim\langle F(x_n, y_n) : n \geq 1 \rangle = (0,0)$ as required.

C-3. Third, consider a point $(x,y) = (x,x)$ on the diagonal, with $x \neq 0$. There is a unique diamond $D(k)$ with $(x,x) \in D(k)$ and we defined $F(x,x) = (x,0)$. Let $\epsilon > 0$ and let $V = [x, x+\epsilon[\times [0,\epsilon[$ be any neighborhood of $F(x,x)$. We may assume that $\epsilon < \sigma(x)$ and that for each $x' \in [x, x+\epsilon[$ we have $\sigma(x') = \sigma(x)$. (This is possible because no horizontal segment of the graph of σ can contain its right endpoint.) We will find a $\delta > 0$ so small that the set $U = [x, x+\delta[^2$ has $F[U] \subseteq V$. We do this in two steps. First we find $\delta_1 > 0$ such that if $(x', y') \in [x, x+\delta_1[^2$ and if (x', y') lies below the diagonal, then $F(x', y') \in U$. Analogously we can find $\delta_2 > 0$ such that if $(x', y') \in [x, x+\delta_2[^2$ lies above the diagonal, then $F(x', y') \in V$. Then we let $\delta = \min(\delta_1, \delta_2)$. Because these two steps are so similar, we describe only the first.

Because $\sigma(x) = \sigma(x')$ whenever $x \le x' < x + \epsilon$ we know that $B(x) = \{x\} \times [0, \sigma(x)[$ and $B(x') = \{x'\} \times [0, \sigma(x')[$ have exactly the same set of second coordinates. Therefore, if we write π_2 for the second coordinate projection, we have $\pi_2[B(x)] = \pi_2[B(x')]$ for all $x' \in [x, x + \epsilon[$, and we also have $\pi_2[B(x, k)] = \pi_2[B(x', k)]$ and $\pi_2[B^U(x, k)] = \pi_2[B^U(x', k)]$ for all $x' \in [x, x + \epsilon[$. Because $\lim_{j\to\infty} \sigma(x)/2^j = 0$ and the segment $B(x, j)$ lies below $y = \sigma(x)/2^j$, we may choose N so large that $\bigcup\{B(x, j) : j \ge N\} \subseteq \{x\} \times [0, \epsilon[\subseteq V$. Then for each $x' \in [x, x+\epsilon[$ we have $\bigcup\{B(x', j) : j \ge N\} \subseteq \{x'\} \times [0, \epsilon[\subseteq V$ so that

$$(*) \qquad \bigcup\{B(x', j) : x \le x' < x + \epsilon,\ j \ge N\} \subseteq [x, x + \epsilon[\times [0, \epsilon[= V.$$

Consider the horizontal line $HL(x) =]x, 1/2^{k-1}[\times \{x\}$, which lies inside of the diamond $D(k)$. Using Proposition 2.1 we can list all members of $\mathcal{T}$ that intersect $HL(x)$ as $T_1, T_2, \ldots$ where each point of T_{j+1} lies to the left of each point of T_j. The rectangle T_j has the form $T_j = [a_j, b_j[\times [c_j, d_j[$ and we have $x < \cdots < b_3 = a_2 < b_2 = a_1 < b_1$. In addition, we have $\lim_{j\to\infty} a_j = x$ so we may choose $M \ge N$ with the property that $a_j < x + \epsilon$ whenever $j \ge M$.

Recall that $T_j = [a_j, b_j[\times [c_j, d_j[$ and that each T_j intersects $HL(x)$, with $HL(x, j) = T_j \cap HL(x) = [a_j, b_j[\times \{x\}$. Because no set $T_1, \ldots, T_M$ contains its top edge, there is some $\eta > 0$ with the property that $[a_i, b_i[\times [x, x + \eta[\subseteq T_i$ for $1 \le i \le M$. Consider any $x' \in [x, x + \eta[$ and let T'_n be the listing in decreasing order of all members of $\mathcal{T}$ that intersect $HL(x')$ as in Proposition 2.1. The collection $\mathcal{T}$ is pairwise disjoint, so that because $HL(x') \cap T_1 \neq \emptyset$, we conclude that $T'_1 = T_1$. Similarly $T'_i = T_i$ for $1 \le i \le M$ so that if $j \ge M$ then $HL(x', j) \subseteq]x', a_M[\times \{x'\} \subseteq [x, a_M[\times [x, x + \eta[$.

Let $\delta_1 = \min(\epsilon, \eta, a_M - x)$ and note that if $x' \in [x, x + \delta_1[$ then $x \le x' < x + \eta$ so that the subset $]x', a_M[\times \{x'\}$ of $HL(x')$ has

$$(**) \qquad]x', a_M[\times \{x'\} \subseteq \bigcup\{H(x', j) : j \ge M\}.$$

Now consider any $(x_1, y_1) \in [x, x + \delta_1[^2$ that lies below the diagonal (i.e., $y_1 < x_1$). Then $(y_1, y_1) \in \Delta$ and $(x_1, y_1) \in HL(y_1)$. Because $x \le y_1 < x + \delta_1 \le x + \eta$ and $x \le y_1 < x_1 < x + \delta_1 \le a_M$, equation $(**)$ gives

$$(***) \qquad (x_1, y_1) \in]y_1, a_M[\times \{y_1\} \subseteq \bigcup\{HL(y_1, j) : j \ge M\}.$$

Consequently, the unique k such that $(x_1, y_1) \in HL(y_1, k)$ must have $k \ge M$ so we know that $F(x_1, y_1) \in B(y_1, k)$. Because $M \ge N$, equation $(*)$ now gives $F(x_1, y_1) \in B(y_1, k) \subseteq V$. As noted above, an analogous argument provides a $\delta_2 > 0$ such that if $(x_2, y_2) \in [x, x+\delta_2[^2$ and (x_2, y_2) lies above the diagonal, then $F(x_2, y_2) \in V$. If we let $\delta = \min(\delta_1, \delta_2)$ then $F[[x, x+\delta[^2] \subseteq V$ so we have proved that F is continuous at each point $(x, x) \in S^2$.

C-4. Fourth, for any $k \geq 1$, consider any point $(x, y) \in D(k) - \Delta$. To show that F is continuous at (x, y), there are two cases to consider, depending upon whether (x, y) lies above or below the diagonal. Because the two cases are analogous, we consider only the first, where $x < y$. Recall how $F(x, y)$ was defined. We have $(x, y) \in VL(x)$. List all members of $\mathcal{T}$ that intersect $VL(x)$ as $T_1, T_2, \ldots$ where the points of T_j lie above the points of T_{j+1}. Write $T_j = [a_j, b_j[\times [c_j, d_j[$. These sets subdivide $VL(x)$ into vertical segments $VL(x, j) = VL(x) \cap T_j = \{x\} \times [c_j, d_j[$. Choose the unique N with $(x, y) \in VL(x, N)$. For notational convenience, write $VL(x, N) = \{x\} \times [c, d[$. Next consider the set $B(x)$, which is divided into subsegments $B(x, k)$, each of which is, in turn, divided in half by $B^L(x, k)$ and $B^U(x, k)$. Use $B^L(x, N)$, which has the form $B^L(x, N) = \{x\} \times [e, f[$. Then $F(x, y) = (x, h_{cdef}(y))$ where $h_{cdef} : [c, d[\to [e, f[$ is the order-isomorphism found in Lemma 2.2.

Write $p = h_{cdef}(y)$ and for $\epsilon > 0$ consider any neighborhood V of $F(x, y) = (x, p)$ having the form $V = [x, x + \epsilon[\times [p, p + \epsilon[$. We may assume that ϵ is so small that $p + \epsilon < \sigma(x)$ and that if $x \leq x' < x + \epsilon$, then $\sigma(x') = \sigma(x)$. It follows that $\pi_2[B(x, j)] = \pi_2[B(x', j)]$ for every j (where π_2 is the second coordinate projection), provided $x' \in [x, x + \epsilon[$. In particular, for $x \leq x' < x + \epsilon$ we have $B^L(x', N) = \{x'\} \times [e, f[$.

We will find a neighborhood $U = [x, x+\delta[\times [y, y+\delta[$ such that $F[U] \subseteq V$. We find δ in two steps. First note that h_{cdef} is continuous at $y \in [c, d[$ and is order-preserving, so that there is a $\delta_1 > 0$ such that $y+\delta_1 < d$ and if $y \leq y' < y + \delta_1$, then $p = h_{cdef}(y) \leq h_{cdef}(y') < h_{cdef}(y) + \epsilon = p + \epsilon$. Next, consider the sets $T_1, \ldots, T_N$ where $T_j = [a_j, b_j[\times [c_j, d_j[$. Because none of these sets contains its right edge, there is an $\eta > 0$ such that $[x, x + \eta[\times [c_j, d_j[\subseteq T_j$ for $1 \leq j \leq N$. Shrinking η if necessary, we may assume $\eta < \epsilon$. Then for any $x' \in [x, x + \eta[$ it follows that $VL(x', j) = \{x'\} \times [c_j, d_j[$ for $1 \leq j \leq N$.

Recall that we are writing $[c_N, d_N[= [c, d[$. Consequently, for $x \leq x' < x + \eta$ we have $VL(x', N) = \{x'\} \times [c, d[$ and for any $(x', y') \in VL(x', N)$ we have $F(x', y') = (x, h_{cdef}(y'))$.

Now let $\delta = \min(\eta, \delta_1)$ and consider $U = [x, x + \delta[\times [y, y + \delta[$. For any $(x', y') \in U$ we have $F(x', y') \in [x, x + \epsilon[\times [p, p + \epsilon[= V$, as required to prove continuity of F at any point $(x, y) \in D(k) - \Delta$.

C-5. Fifth, consider any point $(x, y) \in S^2$ with $x > 0$ and $0 < y < \sigma(x)$. Then $(x, y) \in B(x)$ and there is a unique n with $(x, y) \in B(x, n)$ and hence either $(x, y) \in B^L(x, n)$ or $(x, y) \in B^U(x, n)$. The two cases are analogous, so we describe only the second. Write $B^U(x, n) = \{x\} \times [r, s[$. Because $(x, y) \in B^U(x, n)$ we know that there is some basic diamond $D(k)$ with $F(x, y) \in HL(x, n) \subseteq D(k)$. In order to write down the formula for $F(x, y)$

we list all members of $\mathcal{T}$ that intersect the horizontal segment $HL(x)$ as $T_1, T_2, \ldots$ in such a way that points of T_{j+1} lie to the left of points of T_j. Write $T_j = [t_j, u_j[\times[v_j, w_j[$, so that $HL(x,n) = [t_n, u_n[\times\{x\}$. For notational simplicity, we write $t = t_n$ and $u = u_n$. Then $F(x,y) = (h_{rstu}(y), x)$ where h_{rstu} is the order-isomorphism given by Lemma 2.2. For any $\epsilon > 0$ consider the neighborhood $V = [h_{rstu}(y), h_{rstu}(y) + \epsilon[\times[x, x+\epsilon[$ of $F(x,y)$.

We will find $\delta > 0$ so that if $U = [x, x+\delta[\times[y, y+\delta[$ then $f[U] \subseteq V$. Our first step is to find a $\delta_1 > 0$ so that $y + \delta_1 < \sigma(x)$ and for each $x' \in [x, x+\delta_1[$ we have $\sigma(x') = \sigma(x)$. Then for each $x' \in [x, x+\delta_1[$ we have $B^U(x', n) = \{x'\} \times [r, s[$.

The function $h_{rstu} : [r, s[\to [t, u[$ is continuous and order-preserving, and $y \in [r, s[$. Therefore, corresponding to the $\epsilon > 0$ given above (in C-5), there is a $\delta_2 > 0$ such that if $y \leq y' < y + \delta_2$ then $h_{rstu}(y) \leq h_{rstu}(y') < h_{rstu}(y) + \epsilon$.

Let $\delta = \min(\delta_1, \delta_2, \eta)$ and suppose $(x', y') \in U := [x, x+\delta[\times[y, y+\delta[$. Then $\sigma(x') = \sigma(x)$ so that $\pi_2[B^U(x', n)] = \pi_2[B^U(x, n)] = [r, s[$. Next consider the horizontal line $HL(x')$. List the members of $\mathcal{T}$ that intersect $HL(x')$ as $T'_1, T'_2, \ldots$ so that the points of T'_{j+1} lie to the left of the points of T'_j for all $j \geq 1$. The fact that $y \leq y' < y + \delta \leq y + \eta$ and the special properties of η guarantee that $T'_1 = T_1, \ldots, T'_n = T_n$ so that $HL(x', n) = [t_n, u_n[\times\{x'\} = [t, u[\times\{x'\}$. Therefore $F(x', y') \in [h_{rstu}(y), h_{rstu}(y) + \epsilon[\times[x, x+\epsilon[= V$ as required to complete the proof of the fifth case.

C-6. In the sixth (and final) case, we show that F is continuous at $(x, 0) \in S^2$ provided $x > 0$. We know that $F(x, 0) = (x, x)$ belongs to some basic diamond $D(k)$. For $\epsilon > 0$ consider any basic neighborhood $V = [x, x+\epsilon[^2 \subseteq D(k)$ of (x, x). We will find a neighborhood U_1 of $(x, 0)$ such that if $(x', y') \in U_1$ and $(x', y') \in B^U(x', j)$ for some j, then $F(x', y') \in V$. Analogously, there is a neighborhood U_2 of $(x, 0)$ such that if $(x', y') \in U_2$ and $(x', y') \in B^L(x, j)$ for some j, then $F(x', y') \in V$. Then we let $U = U_1 \cap U_2$.

To find U_1 we begin by listing all members of $\mathcal{T}$ that intersect the horizontal line $HL(x) \subseteq D(k)$ as $T_1, T_2, \ldots$ where the points of T_{j+1} lie to the left of the points in T_j. Write $T_j = [a_j, b_j[\times[c_j, d_j[$. There is some N with $x < b_N < x + \epsilon$ and then there is some $\eta > 0$ with the property that $[a_j, b_j[\times[x, x+\eta[\subseteq T_j$ for $1 \leq j \leq N$. We may assume that $\eta < \epsilon$. Then if $x \leq x' < x + \eta$, we have $\pi_1[HL(x', j)] = \pi_1[HL(x, j)] = [a_j, b_j[$ for $1 \leq j \leq N$ (where π_1 is first coordinate projection) so that

(∗∗∗∗) if $x \leq x' < x + \eta$ and $j \geq N$, then

$$HL(x', j) \subseteq]x', b_N] \times \{x'\} \subseteq [x, x+\epsilon[^2 = V.$$

The vertical segment $B(x) = \{x\} \times [0, \sigma(x)[$ is partitioned by the sets $B(x, j)$. Let d_N be the top point of $B(x, N)$ and let $\delta = \min(\delta_1, d_N, \eta, \epsilon)$.

Define $U = [x, x+\delta[\times [0, \delta[$ and suppose $(x', y') \in U$ has $(x', y') \in B^U(x', j)$ for some j. Then $y' \in \pi_2[B^U(x', j)] = \pi_2[B^U(x, j)]$ so that $y' < \delta \leq d_N$ gives $j \geq N$. Therefore $F(x', y') \in HL(x', j)$ and $HL(x', j) \subseteq V$ by $(****)$, and this completes the continuity proof in the sixth and final case.

At this stage we know that F is continuous and self-inverse, and therefore is the required homeomorphism. ∎

It is natural to ask which subspaces of the Sorgenfrey line are rotoids in their subspace topologies.

PROPOSITION 3.2. *Let X be either the set of rational numbers, or the set of irrational numbers, topologized as a subspace of the Sorgenfrey line S. Then X is a rotoid.*

Proof. If $X = \mathbb{Q}$, then X with the Sorgenfrey topology is homeomorphic to $\mathbb{Q}$ with its usual metric topology, and $(\mathbb{Q}, +)$ is a topological group and therefore a rotoid.

The case where $X = \mathbb{P}$ (which we will call the *irrational Sorgenfrey line*) is more complicated because the irrational Sorgenfrey line is not homeomorphic to any topological group, being first-countable and not metrizable. Note that the irrational Sorgenfrey line is homeomorphic to its subspace $\mathbb{P} \cap [\sqrt{2}, \sqrt{2}+1[$. If we use $e = \sqrt{2}$ as the special point of $\mathbb{P} \cap [\sqrt{2}, \sqrt{2}+1[$, then the proof of Proposition 3.1 can be modified to apply to $(\mathbb{P} \cap [\sqrt{2}, \sqrt{2}+1[)^2$ rather than $[0,1[^2$ provided one is careful to make sure that all mappings involved will take the irrational numbers onto themselves. The easily defined order-isomorphisms $h_{abcd} : [a, b[\to [c, d[$ in Lemma 2.2 will suffice provided the points $a, b, c, d \in \mathbb{Q}$. Alternatively, use a recursion to find an order-isomorphism from $\mathbb{Q} \cap [a, b[$ onto $\mathbb{Q} \cap [c, d[$ and then extend that mapping to $\mathbb{P} \cap [a, b[$ by taking suprema. The resulting function k_{abcd} has the required properties. ∎

Up to this point, our results are ZFC results. Enhancing ZFC with additional axioms gives other types of subspaces of S that are rotoids. For example, one consequence of the Proper Forcing Axiom (PFA) is that if X and Y are any $\aleph_1$-dense (1) subsets of $\mathbb{R}$, then X and Y are order-isomorphic [6].

PROPOSITION 3.3. *Assume PFA. Then any $\aleph_1$-dense subset of $\mathbb{R}$ is a rotoid when it carries the Sorgenfrey topology.*

Proof. Suppose X is $\aleph_1$-dense in $\mathbb{R}$. Then PFA allows us to get order-isomorphisms $h_{abcd} :]a, b[\cap X \to]c, d[\cap X$ for any $a < b$ and $c < d$. We can modify the step functions S_k used in the proof of Proposition 2.1 to guarantee that the values of the functions S_k are not in X, i.e., $\{S_k(x) :$

(1) A subspace $X \subseteq \mathbb{R}$ is $\aleph_1$-*dense* if $X \cap]a, b[$ has cardinality $\aleph_1$ for each open interval $]a, b[\subseteq \mathbb{R}$.

$x \in X$, $k \geq 1\} \cap X = \emptyset$. Now the proof of Proposition 3.1 can be used to show that if X carries the Sorgenfrey topology, then X is a rotoid. ∎

Before leaving the Sorgenfrey line, let us note that there is a property that is stronger than being a rotoid, namely, the property of being a rectifiable space. As originally defined, a space X is *rectifiable* [5] if there is a point $e \in X$ and a homeomorphism G from X^2 onto itself with two properties:

(i) $G(x,x) = (x,e)$ for all $x \in X$, and
(ii) for any $x, y \in X$, $G(x,y) \in \mathrm{Vert}(x) := \{x\} \times X$.

Proposition 8.12 in [1] shows that a space X is rectifiable if and only if there is a homeomorphism H from X^2 onto itself that has three properties:

(i) $H(x,x) = (x,e)$ for all $x \in X$,
(ii) for any $x, y \in X, G$ maps $\mathrm{Vert}(x)$ onto itself, and
(iii) for any $x \in X$, $H(e,x) = (e,x)$.

Therefore (as Arhangel'skii points out) every rectifiable space is a rotoid. The rotoid homeomorphism F in the proof of Proposition 3.1 does not satisfy (ii) above because it maps part of each set $\mathrm{Vert}(x)$ into the horizontal part of the L-shaped set $L(x)$. Might there be another rotoid homeomorphism for the Sorgenfrey line that satisfies (i)–(iii)? The answer is "No" and this resolves Arhangel'skii's Question 8.13.

EXAMPLE 3.4. *The Sorgenfrey line is a strong rotoid that is not rectifiable.*

Proof. The Sorgenfrey line is a strong rotoid but cannot be rectifiable because A. Gul'ko proved in [5] that any first-countable rectifiable space that is at least T_0 must be metrizable. ∎

4. Other examples and questions. The real line $\mathbb{R}$, being a topological group, must be a rotoid. Hence so is any open interval $]a,b[\subseteq \mathbb{R}$. Slightly less obvious is that any closed interval $[a,b] \subseteq \mathbb{R}$ is also a rotoid. That follows from our next proposition.

PROPOSITION 4.1. *The closed interval* $[-1,1] \subseteq \mathbb{R}$ *is a rotoid.*

Proof. Consider the linear homeomorphism $f(x,y) = (x, x+y)$ from $\mathbb{R}^2$ onto itself. This function maps the x-axis onto the diagonal with $f(x,0) = (x,x)$ and maps the y-axis onto itself with $f(0,y) = (0,y)$. Let $X = [-1,1]$. The image of the square X^2 under f is a parallelogram with vertices $(1,2)$, $(1,0)$, $(-1,-2)$ and $(-1,0)$, and there is a homeomorphism g from that parallelogram onto X^2 that sends each point p of the parallelogram to itself provided p lies on the x-axis, on the y-axis, or on the diagonal. (Drawing a picture of the parallelogram and of X^2 on the same set of axes makes this clear.) Then the composite $h := g \circ f|_{X^2}$ is a homeomorphism from

X^2 onto itself that has $h(x,0) = (x,x)$ and $h(0,y) = (0,y)$ for each $x, y \in [-1,1]$ so that $h^{-1} : X^2 \to h^2$ is the homeomorphism required by the rotoid definition. ∎

Our next result can be used to show that many spaces cannot be rotoids. This result is due to Arhangel'skii (see Theorem 2.4 in [1]).

PROPOSITION 4.2. *Suppose X is a rotoid with at least one isolated point. Then X is discrete.*

Proof. Suppose X is a rotoid with special point e and homeomorphism $F : X^2 \to X^2$. In case the special point e is not an isolated point of X, choose any isolated point $p \in X$. Then the homeomorphism F sends the isolated point (p,p) of X^2 to the non-isolated point (p,e) of X^2, and that is impossible. In case the special point e is isolated in X, the line $L = X \times \{e\}$ is a clopen subset of X^2. Then $F^{-1}[L] = \Delta$ is also clopen in X^2, and that makes X a discrete space. ∎

COROLLARY 4.3. *Let M be the Michael line* (²). *Then M is not a rotoid.*

COROLLARY 4.4. *Let X be a stationary subset of a regular initial ordinal. Then X is not a rotoid.*

Recall that a generalized ordered space (GO-space) X fails to be paracompact if and only if X contains a closed copy of a stationary set in some regular uncountable cardinal [4] and that the same is true for the much larger class of monotonically normal spaces [2]. Therefore, Corollary 4.4 combined with the theorem of Balogh and Rudin [2] suggests our next question:

PROBLEM 4.5. *Suppose X is a GO-space that is a rotoid. Is X paracompact? More generally, must Y be paracompact if Y is monotonically normal and a rotoid?*

A partial answer to that question is a corollary to our next result, which applies to any rotoid (not just to GO-spaces). It generalizes Proposition 5.1 in [1], which is stated for the more restrictive class of diagonal resolvable spaces.

PROPOSITION 4.6. *If X is a rotoid in which the special point e is a G_δ, then X has a G_δ-diagonal. In particular, any first-countable rotoid has a G_δ-diagonal.*

Proof. Suppose e is the special point and $F : X^2 \to X^2$ is the homeomorphism for the rotoid X. Let $\{V(n) : n \geq 1\}$ be a sequence of open sets with $\{e\} = \bigcap\{V(n) : n \geq 1\}$ and let $W(n) = V(n) \times \{e\}$ for each n. Recall that $\Delta = F^{-1}[X \times \{e\}]$. Then $\Delta = \bigcap\{F^{-1}[W(n)] : n \geq 1\}$, as required. ∎

(²) The *Michael line* is the set of real numbers topologized in such a way that rational numbers have their usual open-interval neighborhoods and irrational numbers are isolated.

Corollary 4.7. *Any first-countable LOTS that is a rotoid must be metrizable, and any first-countable monotonically normal space that is a rotoid must be hereditarily paracompact.*

Proof. Any such space has a G_δ-diagonal. That is enough to make a LOTS metrizable. If X is monotonically normal with a G_δ-diagonal, then any subspace of X also has a G_δ-diagonal, so that no subspace of X can be homeomorphic to a stationary subset of a regular uncountable cardinal. Now use the characterization of paracompactness in monotonically normal spaces given by [2]. ∎

Experience shows that many theorems that are false for GO-spaces are true for the more restrictive class of LOTS. Therefore, even though Proposition 3.1 shows that there are non-metrizable GO-spaces that are rotoids, it is fair to wonder whether a LOTS that is a rotoid, or that is rectifiable, must be metrizable. Corollary 4.7 does not settle the question because, as our next example shows, there is a space X that is a LOTS and is rectifiable (and hence a rotoid) but is not first-countable.

Example 4.8. *There is a non-first-countable linearly ordered topological space* (*LOTS*) *that is a rectifiable space in the sense of Gul'ko* [5] (*the definition appears at the end of Section* 3) *and hence also a rotoid.*

Proof. As noted in [3] there is an η_1-set $(X, \prec)$ that is a linearly ordered topological field under the interval topology of $\prec$. Let e be the zero element of the field and define $G : X^2 \to X^2$ by $G(x, y) = (x, y - x)$. This G shows that X is rectifiable and is a rotoid. ∎

The referee pointed out that examples of non-metrizable LOTS that are topological groups can be obtained by taking the G_δ-topology on large σ-products of the two-point group $D = \{0, 1\}$.

Problem 4.9. *Which subspaces of the Sorgenfrey line are rotoids?* (*See Propositions* 3.2 *and* 3.3.)

Example 4.10. *The topological sum of two rotoids may fail to be a rotoid.*

Proof. Let A be the set of negative irrationals and let B be the set of positive rationals, each topologized as a subspace of the Michael line M. The subspace A is discrete and has cardinality $\mathfrak{c}$, so that A is homeomorphic to G where G is the free Abelian group with $\mathfrak{c}$ generators with the discrete topology. The subspace B is homeomorphic to the usual group of rational numbers. Hence, each of A and B is a rotoid. But Proposition 4.2 shows that the subspace $X = A \cup B$ of M is not a rotoid, even though X is the topological sum of the rotoids A and B. ∎

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