

Infinite paths and cliques in random graphs

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Abstract. We study the thresholds for the emergence of various properties in random subgraphs of $(\mathbb{N}, <)$. In particular, we give sharp sufficient conditions for the existence of (finite or infinite) cliques and paths in a random subgraph. No specific assumption on the probability is made. The main tools are a topological version of Ramsey theory, exchangeability theory and elementary ergodic theory.

1. Introduction. In this paper we introduce a new method in order to deal with some combinatorial problems in random graphs, originally proposed in [EH:64]. Some of these questions have been successfully addressed in [FT:85], using different techniques. We obtain new and self-contained proofs of some of the results in [FT:85]; moreover with this method we expect to be able to treat similar problems in more general random graphs.

Let $G = (\mathbb{N}, \mathbb{N}^{(2)})$ be the directed graph over \mathbb{N} with set of edges $\mathbb{N}^{(2)} := \{(i, j) \in \mathbb{N}^2 : i < j\}$. Let us randomly choose some of the edges of G , that is, we associate to the edge $(i, j) \in \mathbb{N}^{(2)}$ a measurable set $\mathbb{X}_{i,j} \subseteq \Omega$, where $(\Omega, \mathcal{A}, \mu)$ is a base probability space. Assuming $\mu(\mathbb{X}_{i,j}) \geq \lambda$ for each (i, j) , we then ask whether the resulting random subgraph \mathbb{X} of $(\mathbb{N}, \mathbb{N}^{(2)})$ contains an infinite path:

PROBLEM 1. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Let $\lambda > 0$, and for all $(i, j) \in \mathbb{N}^{(2)}$, let $\mathbb{X}_{i,j}$ be a measurable subset of Ω with $\mu(\mathbb{X}_{i,j}) \geq \lambda$. Is there an infinite increasing sequence $\{n_i\}_{i \in \mathbb{N}}$ such that $\bigcap_{i \in \mathbb{N}} \mathbb{X}_{n_i, n_{i+1}}$ is non-empty?

More formally, a *random subgraph* \mathbb{X} of a directed graph $G = (V_G, E_G)$ (with set of edges $E_G \subset V_G \times V_G$) is a measurable function $\mathbb{X} : \Omega \rightarrow 2^{E_G}$ where $(\Omega, \mathcal{A}, \mu)$ is a probability space, and 2^{E_G} is the power set of E_G , identified with the set of all functions from E_G to $\{0, 1\}$ (with the product

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topology and the σ -algebra of its Borel sets). For each $x \in \Omega$, we identify $\mathbb{X}(x)$ with the subgraph of G with vertices V_G and edges $\mathbb{X}(x)$. Given $e \in E_G$, the set $\mathbb{X}_e := \{x \in \Omega : e \in \mathbb{X}(x)\}$ represents the event that the random graph \mathbb{X} contains the edge $e \in E_G$. The family $(\mathbb{X}_e)_{e \in E_G}$ determines \mathbb{X} by putting $\mathbb{X}(x) = \{e \in E_G : x \in \mathbb{X}_e\}$. So a random subgraph of G can be equivalently defined as a function from E_G to 2^Ω assigning to each $e \in E_G$ a measurable subset \mathbb{X}_e of Ω .

As in classical percolation theory, we wish to estimate the probability that \mathbb{X} contains an infinite path, in terms of a parameter λ that bounds from below the probability $\mu(\mathbb{X}_e)$ that an edge e belongs to \mathbb{X} . Note that it is not a priori obvious that the existence of an infinite path has a well-defined probability, since it corresponds to the uncountable union of the sets $\bigcap_{k \in \mathbb{N}} \mathbb{X}_{i_k, i_{k+1}}$ over all strictly increasing sequences $i : \mathbb{N} \rightarrow \mathbb{N}$. However, it turns out that it belongs to the μ -completion of the σ -algebra generated by the $\mathbb{X}_{i,j}$. One has to notice that the analogy with classical bond percolation is only formal, the main difference being that in the usual percolation models (see for instance [G:99]) the events $\mathbb{X}_{i,j}$ are supposed *independent*, whereas in the present case the probability distribution is completely general, i.e. we do not impose any restriction on the events $\mathbb{X}_{i,j}$, and on the probability space Ω .

Problem 1 has been originally proposed by P. Erdős and A. Hajnal in [EH:64], and an answer was given by D. H. Fremlin and M. Talagrand in [FT:85], where other related and more general problems are also considered. In particular [FT:85] shows that the threshold for the existence of infinite paths is $\lambda = 1/2$, under the assumption that the probability space $(\Omega, \mathcal{A}, \mu)$ is $[0, 1]$ equipped with the Lebesgue measure. We point out that our result holds for any probability space $(\Omega, \mathcal{A}, \mu)$. One of the main goals of this paper is to present a general method, different from the one in [FT:85], which in particular allows us to recover the same result as in [FT:85] (see Theorem 4.5). Our approach relies on the reduction to the following dual problem:

PROBLEM 2. Given a directed graph F , determine the minimal λ_c such that, whenever $\inf_{e \in \mathbb{N}^{(2)}} \mu(\mathbb{X}_e) > \lambda_c$, there is a graph morphism $f : \mathbb{X}(x) \rightarrow F$ for some $x \in \Omega$.

Problem 1 can be reformulated in this setting by letting F be the graph $(\omega_1, >)$ where ω_1 is the first uncountable ordinal. This depends on the fact that a subgraph H of $(\mathbb{N}, \mathbb{N}^{(2)})$ does not contain an infinite path if and only if it admits a rank function with values in ω_1 . Therefore, if a random subgraph \mathbb{X} of $(\mathbb{N}, \mathbb{N}^{(2)})$ has no infinite paths, it yields a μ -measurable map $\varphi : \Omega \rightarrow \omega_1^{\mathbb{N}}$ where $\varphi(x)(i)$ is the rank of the vertex $i \in \mathbb{N}$ in the graph $\mathbb{X}(x)$. It turns out that $\phi_{\#}(\mu)$ is a compactly supported Borel measure on $\omega_1^{\mathbb{N}}$, and that $\phi(\mathbb{X}_{i,j}) \subseteq A_{i,j} := \{x \in \omega_1^{\mathbb{N}} : x_i > x_j\}$. As a consequence, in the

determination of the threshold for existence of infinite paths

(1.1)

$$\lambda_c := \sup \left\{ \inf_{(i,j) \in \mathbb{N}^{(2)}} \mu(\mathbb{X}_{i,j}) : \mathbb{X} \text{ a random graph without infinite paths} \right\}$$

we can set $\Omega = \omega_1^{\mathbb{N}}$, $\mathbb{X}_{i,j} = A_{i,j}$, and reduce to the variational problem on the convex set $\mathcal{M}_c^1(\omega_1^{\mathbb{N}})$ of compactly supported probability measures on $\omega_1^{\mathbb{N}}$:

$$(1.2) \quad \lambda_c = \sup_{m \in \mathcal{M}_c^1(\omega_1^{\mathbb{N}})} \inf_{(i,j) \in \mathbb{N}^{(2)}} m(A_{i,j}).$$

As a next step, we show that in (1.2) we can equivalently take the supremum in the smaller class of all compactly supported *exchangeable measures* on $\omega_1^{\mathbb{N}}$ (see Appendix 6 and references therein for a precise definition). Thanks to this reduction, we can explicitly compute $\lambda_c = 1/2$ (Theorem 4.5). We note that the supremum in (1.2) is not attained, which implies that for $\mu(\mathbb{X}_{i,j}) \geq 1/2$ infinite paths occur with positive probability.

In Section 5, we consider Problem 2 again and we give a complete solution when F is a finite graph, showing in particular that

$$\lambda_c = \sup_{\lambda \in \Sigma_F} \sum_{(a,b) \in E_F} \lambda_a \lambda_b$$

where Σ_F is the set of all sequences $\{\lambda_a\}_{a \in V_F}$ with values in $[0, 1]$ and such that $\sum_{a \in V_F} \lambda_a = 1$. By the appropriate choice of F we can determine the thresholds for the existence of paths of a given finite length (Section 3 and Remark 5.2), or for the property of having chromatic number $\geq n$ (Section 6).

We can consider Problems 1 and 2 for a random subgraph \mathbb{X} of an arbitrary directed graph G , not necessarily equal to $(\mathbb{N}, \mathbb{N}^{(2)})$. However, it can be shown that, if we replace $(\mathbb{N}, \mathbb{N}^{(2)})$ with a finitely branching graph G (such as a finite-dimensional network), the probability that \mathbb{X} has an infinite path may be zero even if $\inf_{e \in E_G} \mu(\mathbb{X}_e)$ is arbitrarily close to 1 (Proposition 4.8). Another variant is to consider subgraphs of $\mathbb{R}^{(2)}$ rather than $\mathbb{N}^{(2)}$ but it turns out that this makes no difference in terms of the threshold for having infinite paths in random subgraphs (Remark 4.9).

In Section 6 we again fix $G = (\mathbb{N}, \mathbb{N}^{(2)})$ and we ask if a random subgraph \mathbb{X} of G contains an infinite clique, i.e. a copy of G itself. More generally we consider the following problem.

PROBLEM 3. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Let $\lambda > 0$ and, for all $(i_1, \dots, i_k) \in \mathbb{N}^{(k)}$, let $\mathbb{X}_{i_1, \dots, i_k}$ be a measurable subset of X with $\mu(\mathbb{X}_{i_1, \dots, i_k}) \geq \lambda$. Is there an infinite set $J \subset \mathbb{N}$ such that $\bigcap_{(i_1, \dots, i_k) \in J^{(k)}} \mathbb{X}_{i_1, \dots, i_k}$ is non-empty?

This problem is a random version of the classical Ramsey theorem [R:30] (we refer to [GP:73, DP:05], and references therein, for various generalizations of the Ramsey theorem). Clearly the Ramsey theorem implies that the answer to Problem 3 is positive when Ω is finite. Moreover it can be shown that the answer remains positive when Ω is countable (Example 6.3). However when $\Omega = [0, 1]$ (with the Lebesgue measure), the probability that \mathbb{X} contains an infinite clique may be zero even if $\inf_{e \in E_G} \mu(\mathbb{X}_e)$ is arbitrarily close to 1 (see Example 6.2). We will show that Problem 3 has a positive answer if the indicator functions of the sets $\mathbb{X}_{i_1, \dots, i_k}$ all belong to a compact subset of $L^1(\Omega, \mu)$ (see Theorem 6.5).

Our original motivation for the above problems came from the following situation. Suppose we are given a space E and a certain family Ω of sequences on E (e.g., minimizing sequences of a functional, or orbits of a discrete dynamical system, etc.). A typical general problem asks for existence of a sequence in the family Ω that admits a subsequence with a prescribed property. One approach to this problem is by means of measure theory. The archetypal situation here comes from recurrence theorems: one may ask if there exists a subsequence which belongs frequently to a given subset C of the “phase” space Ω (we refer to such sequences as “ C -recurrent orbits”). If we consider the set $\mathbb{X}_i := \{x \in \Omega : x_i \in C\}$, then a standard sufficient condition for the existence of C -recurrent orbits is $\mu(\mathbb{X}_i) \geq \lambda > 0$ for some probability measure μ on Ω . In fact it is easy to check that the set of C -recurrent orbits has measure at least λ by an elementary version of a Borel–Cantelli lemma (see Proposition 6.1). This is indeed the existence argument in the Poincaré recurrence theorem for measure preserving transformations. A more subtle question arises when one looks for a subsequence satisfying a given relation between two successive (or possibly more) terms: given a subset R of $E \times E$ we look for a subsequence x_{i_k} such that $(x_{i_k}, x_{i_{k+1}}) \in R$ for all $k \in \mathbb{N}$. As before, we may consider the subset of Ω , with double indices $i < j$, $\mathbb{X}_{i,j} := \{x \in \Omega : (x_i, x_j) \in R\}$ and we are then led to Problem 1.

2. Notation. We follow the set-theoretical convention of identifying a natural number p with the set $\{0, 1, \dots, p-1\}$ of its predecessors. More generally an ordinal number α coincides with the set of its predecessors. With these conventions the set of natural numbers \mathbb{N} coincides with the least infinite ordinal ω . As usual ω_1 denotes the first uncountable ordinal, namely the set of all countable ordinals.

Given two sets X, Y we denote by X^Y the set of all functions from Y to X . If X, Y are linearly ordered we denote by $X^{(Y)}$ the set of all increasing functions from Y to X . In particular $\mathbb{N}^{(p)}$ (with $p \in \mathbb{N}$) is the set of all increasing p -tuples from \mathbb{N} , where a p -tuple $\mathbf{i} = (i_0, \dots, i_{p-1})$ is a function

$i: p \rightarrow \mathbb{N}$. The case $p = 2$, with the obvious identifications, takes the form $\mathbb{N}^{(2)} = \{(i, j) \in \mathbb{N}^2 : i < j\}$.

Any function $f: X \rightarrow X$ induces $f_*: X^Y \rightarrow X^Y$ by $f(u) = f \circ u$. On the other hand a function $f: Y \rightarrow Z$ induces $f^*: X^Z \rightarrow X^Y$ by $f^*(u) = u \circ f$. In particular, if $S: \mathbb{N} \rightarrow \mathbb{N}$ is the successor function, then $S^*: X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$ is the *shift map*.

We let $\mathfrak{S}_c(\mathbb{N}), \text{Inj}(\mathbb{N}), \text{Incr}(\mathbb{N}) \subset \mathbb{N}^{\mathbb{N}}$ be the families of maps $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ which are compactly supported permutations (i.e. they fix all but finitely many points), injective functions and strictly increasing functions, respectively. Note that with the above conventions $\text{Incr}(\mathbb{N}) = \mathbb{N}^{(\omega)}$.

Given a measurable function $\psi: X \rightarrow Y$ between two measurable spaces and given a measure m on X , we denote as usual by $\psi_{\#}(m)$ the induced measure on Y .

Given a compact metric space Λ , the space $\mathcal{M}(\Lambda^{\mathbb{N}})$ of signed Borel measures on $\Lambda^{\mathbb{N}}$ can be identified with $C(\Lambda^{\mathbb{N}})^*$, i.e. the dual of the Banach space of all continuous functions on $\Lambda^{\mathbb{N}}$. By the Banach–Alaoglu theorem the subset $\mathcal{M}^1(\Lambda^{\mathbb{N}}) \subset \mathcal{M}(\Lambda^{\mathbb{N}})$ of probability measures is a compact (metrizable) subspace of $C(\Lambda^{\mathbb{N}})^*$ endowed with the weak* topology.

Given $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ we have $\sigma^*: \Lambda^{\mathbb{N}} \rightarrow \Lambda^{\mathbb{N}}$ and $\sigma_{\#}^*: \mathcal{M}^1(\Lambda^{\mathbb{N}}) \rightarrow \mathcal{M}^1(\Lambda^{\mathbb{N}})$. To simplify notation we also write $\sigma \cdot m$ for $\sigma_{\#}^* m$. Note the contravariance of this action:

$$(2.1) \quad \theta \cdot \sigma \cdot m = (\sigma \circ \theta) \cdot m.$$

Similarly, given $r \in \mathbb{N}$ and $\iota \in \mathbb{N}^{(r)}$, we have $\iota_{\#}^*: \mathcal{M}^1(\Lambda^{\mathbb{N}}) \rightarrow \mathcal{M}^1(\Lambda^r)$ and we define $\iota \cdot m = \iota_{\#}^*(m)$.

Given a family $\mathcal{F} \subset \mathbb{N}^{\mathbb{N}}$, we say that m is \mathcal{F} -invariant if $\sigma \cdot m = m$ for all $\sigma \in \mathcal{F}$.

3. Finite paths in random subgraphs. As a preparation for the study of infinite paths (Problem 1) we first consider the case of finite paths. The following example shows that there are random subgraphs \mathbb{X} of $(\mathbb{N}, \mathbb{N}^{(2)})$ such that $\inf_{e \in \mathbb{N}^{(2)}} \mathbb{X}_e$ is arbitrarily close to $1/2$, and yet \mathbb{X} has probability zero of having infinite paths.

EXAMPLE 3.1. Let $p \in \mathbb{N}$ and let $\Omega = p^{\mathbb{N}}$ with the Bernoulli probability measure $\mu = B_{(1/p, \dots, 1/p)}$. For $i < j$ in \mathbb{N} let $\mathbb{X}_{i,j} = \{x \in p^{\mathbb{N}} : x_i > x_j\}$. Then $\mu(\mathbb{X}_{i,j}) = \frac{1}{2}(1 - 1/p)$ for all $(i, j) \in \mathbb{N}^{(2)}$ and yet for each $x \in \Omega$ the graph $\mathbb{X}(x) = \{(i, j) \in \mathbb{N}^{(2)} : x_i > x_j\}$ has no paths of length $\geq p$ (where the *length* of a path is the number of its edges).

We will next show that the bounds in Example 3.1 are optimal. We need:

LEMMA 3.2. *Let $p \in \mathbb{N}$ and let $m \in \mathcal{M}^1(p^{\mathbb{N}})$. Let*

$$(3.1) \quad A_{i,j} := \{x \in p^{\mathbb{N}} : x_i > x_j\}.$$

Then

$$(3.2) \quad \inf_{(i,j) \in \mathbb{N}^{(2)}} m(A_{i,j}) \leq \frac{1}{2} \left(1 - \frac{1}{p}\right).$$

Proof. The proof is a reduction to the case of *exchangeable measures* (see Appendix 6). Note that if $\sigma \in \text{Incr}(\mathbb{N})$, then $(\sigma \cdot m)(A_{i,j}) = m(A_{\sigma(i),\sigma(j)})$. Hence, replacing m with $\sigma \cdot m$ in (3.2) can only increase the infimum, as it is equivalent to the infimum of $m(A_{i,j})$ over a subset of $\mathbb{N}^{(2)}$. By Theorem B.8 we can then assume that m is asymptotically exchangeable, so that in particular the sequence $m_k = \mathbf{S}^k \cdot m$ converges, in the weak* topology, to an exchangeable measure $m' \in \mathcal{M}^1(p^{\mathbb{N}})$. Since p is finite, the sets $A_{i,j}$ are clopen, and therefore $\lim_{k \rightarrow \infty} m_k(A_{i,j}) = m'(A_{i,j}) = m'(A_{0,1})$. Noting that $m_k(A_{i,j}) = m(A_{i+k,j+k})$, we deduce that

$$(3.3) \quad \begin{aligned} \inf_{(i,j) \in \mathbb{N}^{(2)}} m(A_{i,j}) &\leq \lim_{k \rightarrow \infty} m_k(A_{0,1}) = m'(A_{0,1}) \\ &= \frac{1}{2}(1 - m'\{x : x_0 = x_1\}) \leq \frac{1}{2} \left(1 - \frac{1}{p}\right) \end{aligned}$$

where the last inequality follows from Corollary B.11. ■

THEOREM 3.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\mathbb{X} : \Omega \rightarrow 2^{E_G}$ be a random subgraph of $G := (\mathbb{N}, \mathbb{N}^{(2)})$. Consider the set*

$$P := \{x \in \Omega : \mathbb{X}(x) \text{ has a path of length } \geq p\}.$$

Assume $\inf_{e \in \mathbb{N}^{(2)}} \mu(\mathbb{X}_e) > \frac{1}{2}(1 - 1/p)$. Then $\mu(P) > 0$.

Proof. Suppose for a contradiction that $\mu(P) = 0$. We can then assume $P = \emptyset$ (otherwise replace Ω with $\Omega - P$). For $x \in \Omega$ let $\varphi(x) : \mathbb{N} \rightarrow p$ assign to each $i \in \mathbb{N}$ the length of the longest path starting from i in $\mathbb{X}(x)$. We thus obtain a function $\varphi : \Omega \rightarrow p^{\mathbb{N}}$ which is easily seen to be measurable (this is a special case of Lemma 4.3). Let $m = \varphi_{\#}(\mu) \in \mathcal{M}^1(p^{\mathbb{N}})$. Since $\varphi(\mathbb{X}_{i,j}) \subset A_{i,j}$, we have $m(A_{i,j}) \geq \mu(\mathbb{X}_{i,j}) > \frac{1}{2}(1 - 1/p)$ for all i, j , which contradicts Lemma 3.2. ■

A different proof of this result has been given in [FT:85, 3F] (when the probability space Ω is $[0, 1]$ equipped with the Lebesgue measure).

Having determined the critical threshold $\lambda_p = \frac{1}{2}(1 - 1/p)$, we can see that if $\inf_{e \in \mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq \lambda \geq \lambda_p$, then the lower bound for $\mu(P)$ grows linearly with λ . More precisely we have:

COROLLARY 3.4. *In the setting of Theorem 3.3, let $\lambda \in [0, 1]$ and suppose that $\inf_{e \in \mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq \lambda$. Then*

$$\mu(P) \geq \frac{\lambda - \lambda_p}{1 - \lambda_p} \quad \text{where} \quad \lambda_p = \frac{1}{2} \left(1 - \frac{1}{p} \right).$$

Proof. Suppose $\inf_{e \in \mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq \lambda$. Consider the conditional probability $\mu(\cdot \mid \Omega - P) \in \mathcal{M}^1(\Omega)$. We have

$$(3.4) \quad \mu(\mathbb{X}_e \mid \Omega - P) \geq \frac{\mu(\mathbb{X}_e) - \mu(P)}{1 - \mu(P)} \geq \frac{\lambda - \mu(P)}{1 - \mu(P)}.$$

Clearly $\mu(P \mid \Omega - P) = 0$. Applying Theorem 3.3 to $\mu(\cdot \mid \Omega - P)$ then shows that

$$\frac{\lambda - \mu(P)}{1 - \mu(P)} \leq \lambda_p, \quad \text{or equivalently} \quad \mu(P) \geq \frac{\lambda - \lambda_p}{1 - \lambda_p}. \quad \blacksquare$$

4. Infinite paths. By Theorem 3.3, if $\inf_{e \in \mathbb{N}^{(2)}} \mu(\mathbb{X}_{i,j}) \geq 1/2$, then the random subgraph \mathbb{X} of $(\mathbb{N}, \mathbb{N}^{(2)})$ has arbitrarily long finite paths, namely for each p there is $x \in \Omega$ (depending on p) such that $\mathbb{X}(x)$ has a path of length $\geq p$. We want to show that for some $x \in \Omega$, $\mathbb{X}(x)$ has an infinite path. To this end it is not enough to find a single x that works for all p . Indeed, $\mathbb{X}(x)$ could have arbitrarily long finite paths without having an infinite path. The existence of infinite paths can be neatly expressed in terms of the following definition.

DEFINITION 4.1. Let G be a countable directed graph and let ω_1 be the first uncountable ordinal. We recall that the *rank function* $\phi_G: V_G \rightarrow \omega_1 \cup \{\infty\}$ of G is defined as follows. For $i \in V_G$,

$$\phi_G(i) = \sup_{j: (i,j) \in E_G} (\phi_G(j) + 1).$$

This is a well-defined countable ordinal if G has no infinite paths starting at i . In the opposite case we set

$$\phi_G(i) = \infty$$

where ∞ is a conventional value greater than all the countable ordinals. For notational convenience we will take $\infty = \omega_1$ so that $\omega_1 \cup \{\infty\} = \omega_1 \cup \{\omega_1\} = \omega_1 + 1$. Note that if i is a leaf, then $\phi_G(i) = 0$. Also note that G has an infinite path if and only if ϕ_G assumes the value ∞ .

Given a random subgraph $\mathbb{X}: \Omega \rightarrow 2^{E_G}$ of G , we let $\phi_{\mathbb{X}}(x) = \phi_{\mathbb{X}(x)}$, namely $\phi_{\mathbb{X}}(x)(i)$ is the rank of the vertex i in the graph $\mathbb{X}(x)$. So $\phi_{\mathbb{X}}$ is a map from Ω to $(\omega_1 + 1)^{V_G}$. It can also be considered as a map from $\Omega \times V_G$ to $\omega_1 + 1$ by writing $\phi_{\mathbb{X}}(x, i)$ instead of $\phi_{\mathbb{X}}(x)(i)$.

REMARK 4.2. We have $\phi_{\mathbb{X}}(x, i) = \phi_{\omega_1}(x, i)$ where $\phi_\alpha: \Omega \rightarrow (\omega_1 + 1)^{V_G}$ is the truncation $\phi_\alpha := \min(\phi, \alpha)$, that we can equivalently define by induction

on $\alpha \leq \omega_1$ as follows:

$$\begin{aligned} \phi_0(x, i) &= 0, \\ \phi_\alpha(x, i) &= \sup\{\phi_\beta(x, j) + 1 : \beta < \alpha, (i, j) \in \mathbb{X}(x)\}. \end{aligned}$$

The above representation will be of use in the following lemma in connection with measurability properties of the map $\phi_{\mathbb{X}}$.

LEMMA 4.3. *Let G be a countable directed graph, let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\mathbb{X} : \Omega \rightarrow 2^{E_G}$ be a random subgraph of G .*

- (1) *For all $\alpha < \omega_1$ and $i \in V_G$, the set $\{x \in \Omega : \phi_{\mathbb{X}}(x, i) = \alpha\}$ belongs to the σ -algebra \mathcal{A} .*
- (2) *The set $P := \{x \in \Omega : \mathbb{X}(x) \text{ has an infinite path}\}$ is μ -measurable, that is, it is measurable in the μ -completion of the σ -algebra \mathcal{A} .*
- (3) *$\varphi_{\mathbb{X}} : \Omega \rightarrow (\omega_1 + 1)^{V_G}$ is μ -measurable and its restriction to $\Omega - P$ is essentially bounded, namely it takes values in $\alpha_0^{V_G}$ for some $\alpha_0 < \omega_1$, off a μ -null set.*

Proof. Since taking the supremum over a countable set preserves measurability, from Remark 4.2 it follows that for all $i \in V_G$ and $\alpha < \omega_1$ the sets $\{x : \phi_{\mathbb{X}}(x, i) = \alpha\}$ are measurable. We will show that $\{x : \phi_{\mathbb{X}}(x, i) = \omega_1\}$ is μ -measurable. Fix $i \in V_G$. The sequence of values $\mu(\{x : \phi_{\mathbb{X}}(x, i) \leq \beta\})$ is increasing with respect to the countable ordinal β and uniformly bounded by $1 = \mu(\Omega)$, therefore it is stationary at some finite value. So there is $\alpha_0 < \omega_1$ such that

$$(4.1) \quad \mu(\{x \in \Omega : \phi_{\mathbb{X}}(x, i) = \beta\}) = 0 \quad \text{for } \alpha_0 \leq \beta < \omega_1.$$

Notice that

$$P = \{x : \phi_{\mathbb{X}}(x) = \omega_1\} = (\Omega - \{x : \phi_{\mathbb{X}}(x) < \alpha_0\}) - \{x : \alpha_0 \leq \phi_{\mathbb{X}}(x) < \omega_1\}.$$

Since

$$\{x : \alpha_0 \leq \phi_{\mathbb{X}}(x) < \omega_1\} \subseteq \bigcup_{i \in V_G} \{x \in \Omega : \phi_{\mathbb{X}}(x, i) = \alpha_0\}$$

and, by (4.1),

$$\mu\left(\bigcup_{i \in V_G} \{x \in \Omega : \phi_{\mathbb{X}}(x, i) = \alpha_0\}\right) = 0,$$

it follows that P is μ -measurable and so is $\phi_{\mathbb{X}}$. ■

Notice that the set P is universally measurable with respect to \mathcal{A} , that is, it is measurable in the completion of any measure μ defined on the σ -algebra \mathcal{A} .

Given an ordinal α , we put on α the topology generated by the open intervals. Note that a non-zero ordinal is compact if and only if it is a successor ordinal, and it is metrizable if and only if it is countable. Let $\mathcal{M}_c(\omega_1^{\mathbb{N}})$ be the set of compactly supported Borel measures on $\omega_1^{\mathbb{N}}$, i.e. measures with

support in $\alpha_0^{\mathbb{N}}$ for some $\alpha_0 < \omega_1$. The following lemma reduces to Lemma 3.2 if α_0 is finite.

LEMMA 4.4. *Let $m \in \mathcal{M}_c(\omega_1^{\mathbb{N}})$ be a non-zero measure with compact support. Let*

$$(4.2) \quad A_{i,j} := \{x \in \omega_1^{\mathbb{N}} : x_i > x_j\}.$$

Then

$$(4.3) \quad \inf_{(i,j) \in \mathbb{N}^{(2)}} m(A_{i,j}) < \frac{m(\omega_1^{\mathbb{N}})}{2}.$$

Proof. With no loss of generality we can assume that $m \in \mathcal{M}^1(\omega_1^{\mathbb{N}})$, i.e. $m(\omega_1^{\mathbb{N}}) = 1$. We divide the proof into four steps.

STEP 1. Letting $\partial\omega_1$ be the derived set of ω_1 , that is, the subset of all countable limit ordinals, we can assume that

$$m(\{x : x_i \in \partial\omega_1\}) = 0 \quad \forall i \in \mathbb{N}.$$

Indeed, it is enough to observe that the left-hand side of (4.3) can only increase if we replace m with $s_{\#}(m)$, where $s : \omega_1 \rightarrow \omega_1 \setminus \partial\omega_1$ is the successor map sending $\alpha < \omega_1$ to $\alpha + 1$, and $s_{\#}(m) = (s_*)_{\#}$, namely $s_{\#}(m)(X) := m(\{x \in \omega_1^{\mathbb{N}} : s \circ x \in X\})$.

STEP 2. Since the support of m is contained in $\alpha_0^{\mathbb{N}}$ for some ordinal $\alpha_0 < \omega_1$, thanks to Theorem B.8 we can assume that m is asymptotically exchangeable, i.e. the sequence $m_k = S^k \cdot \theta \cdot m$ converges, in the weak* topology, to an exchangeable measure $m' \in \mathcal{M}^1(\omega_1^{\mathbb{N}})$, with support in $\alpha_0^{\mathbb{N}}$, for all $\theta \in \omega^{(\omega)}$. Note however that, unless α_0 is finite, we cannot conclude that $\lim_{k \rightarrow \infty} m_k(A_{i,j}) = m'(A_{i,j})$ since the sets $A_{i,j} = \{x \in \omega_1^{\mathbb{N}} : x_i > x_j\}$ are not clopen.

STEP 3. We shall prove by induction on $\alpha < \omega_1$ that

$$(4.4) \quad \liminf_{(i,j) \rightarrow +\infty} m(\{x : x_j < x_i \leq \alpha\}) \leq m'(\{x : x_1 < x_0 \leq \alpha\}).$$

For $\alpha = 0$ we have $\{x : x_j < x_i \leq 0\} = \emptyset$, and (4.4) holds.

At the inductive step, let us assume that (4.4) holds for all $\alpha < \beta < \omega_1$; we distinguish whether β is a successor or a limit ordinal.

In the former case let $\beta = \alpha + 1$. For $(i, j) \rightarrow +\infty$ (with $i < j$) we have

$$\begin{aligned} m(\{x_j < x_i \leq \beta\}) &= m(\{x_j < x_i \leq \alpha\}) + m(\{x_j \leq \alpha, x_i = \beta\}) \\ &\leq m'(\{x_1 < x_0 \leq \alpha\}) + m'(\{x_1 \leq \alpha, x_0 = \beta\}) + o(1) \\ &= m'(\{x_1 < x_0 \leq \beta\}) + o(1), \end{aligned}$$

where we used the induction hypothesis, and the fact that $\{x_j \leq \alpha, x_i = \beta\}$ is clopen.

Let us now assume that β is a limit ordinal. For all $i \in \mathbb{N}$ we have

$$(4.5) \quad \bigcap_{\alpha < \beta} \{x : \alpha < x_i < \beta\} = \emptyset.$$

In particular, for all $\varepsilon > 0$ there exists $\alpha < \beta$ such that

$$m'(\{\alpha < x_0 < \beta\}) < \varepsilon.$$

Since m' is exchangeable, we also have

$$m'(\{\alpha < x_i < \beta\}) < \varepsilon$$

for all $i \in \mathbb{N}$. Moreover by assumption $m(\{x_i = \beta\}) = 0$ for every $i \in \mathbb{N}$. Hence, again by (4.5), for all $i \in \mathbb{N}$ there exists $\alpha \leq \alpha_i < \beta$ such that

$$m(\{\alpha_i \leq x_i \leq \beta\}) < \varepsilon.$$

Given $i < j$, distinguishing the relative positions of x_i, x_j with respect to α and α_i , we have

$$\begin{aligned} \{x_j < x_i \leq \beta\} &\subseteq \{x_j < x_i \leq \alpha\} \cup \{x_j \leq \alpha < x_i \leq \beta\} \\ &\cup \{\alpha < x_j \leq \alpha_i\} \cup \{\alpha_i < x_i \leq \beta\}, \end{aligned}$$

which gives

$$(4.6) \quad \begin{aligned} m(\{x_j < x_i \leq \beta\}) &\leq m(\{x_j < x_i \leq \alpha\}) + m(\{x_j \leq \alpha < x_i \leq \beta\}) \\ &\quad + m(\{\alpha < x_j \leq \alpha_i\}) + m(\{\alpha_i < x_i \leq \beta\}). \end{aligned}$$

Since $\{x_j \leq \alpha < x_i \leq \beta\}$ and $\{\alpha < x_j \leq \alpha_i\}$ are both clopen, we can approximate their m -measure by their m' -measure. So we have

$$m\{x_j \leq \alpha < x_i \leq \beta\} = m'(\{x_1 \leq \alpha < x_0 \leq \beta\}) + o(1) \quad \text{for } (i, j) \rightarrow \infty$$

and

$$m(\{\alpha < x_j \leq \alpha_i\}) = m'(\{\alpha < x_1 \leq \alpha_i\}) + o(1) \quad \text{for } j \rightarrow \infty,$$

where we used Remark B.7 to allow $j \rightarrow \infty$ keeping i fixed.

Note that, by the choice of α , we have $m'(\{\alpha < x_1 \leq \alpha_i\}) < \varepsilon$, and by induction hypothesis $\liminf_{(i,j) \rightarrow +\infty} m(\{x_j < x_i \leq \alpha\}) < m'(\{x_1 < x_0 \leq \beta\})$. Hence, from (4.6) we obtain

$$\begin{aligned} \liminf_{(i,j) \rightarrow +\infty} m(\{x_j < x_i \leq \beta\}) \\ \leq m'(\{x_1 < x_0 \leq \alpha\}) + m'(\{x_1 \leq \alpha < x_0 \leq \beta\}) + \varepsilon + \varepsilon. \end{aligned}$$

Therefore,

$$\liminf_{(i,j) \rightarrow +\infty} m(\{x_j < x_i \leq \beta\}) \leq m'(\{x_1 < x_0 \leq \beta\}) + 2\varepsilon.$$

Inequality (4.4) is thus proved for all $\alpha < \omega_1$.

STEP 4. We now conclude the proof of the lemma. From (4.4) it follows that

$$(4.7) \quad \inf_{(i,j) \in \mathbb{N}^{(2)}} m(A_{i,j}) \leq m'(\{x : x_1 < x_0\}) \\ = \frac{1}{2}(1 - m'(\{x : x_1 = x_0\})) < \frac{1}{2},$$

where we used the fact that m' is exchangeable, and Corollary B.10. ■

THEOREM 4.5. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\mathbb{X} : \Omega \rightarrow 2^{E_G}$ be a random subgraph of $G := (\mathbb{N}, \mathbb{N}^{(2)})$. Consider the set*

$$P := \{x \in \Omega : \mathbb{X}(x) \text{ has an infinite path}\}.$$

Assume $\inf_{e \in \mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq 1/2$. Then $\mu(P) > 0$.

As observed in the Introduction, this result follows from [FT:85, 4D] when $\Omega = [0, 1]$ with the Lebesgue measure.

Proof of Theorem 4.5. Suppose for a contradiction $\mu(P) = 0$. We can then assume $P = \emptyset$ (replacing Ω with $\Omega - P$). Hence the rank function $\varphi := \varphi_{\mathbb{X}} : \Omega \rightarrow (\omega_1 + 1)^{\mathbb{N}}$ takes values in $\omega_1^{\mathbb{N}}$. Let $m = \varphi_{\#}(\mu) \in \mathcal{M}^1(\omega_1^{\mathbb{N}})$. Note that $\varphi(\mathbb{X}_{i,j}) \subset A_{i,j} := \{x \in p^{\mathbb{N}} : x_i > x_j\}$. Hence $m(A_{i,j}) \geq \mu(\mathbb{X}_{i,j}) \geq 1/2$ for all $(i, j) \in \mathbb{N}^{(2)}$. This contradicts Lemma 4.4. ■

REMARK 4.6. Note that the bound $1/2$ is optimal by Example 3.1.

Reasoning as in Corollary 3.4 we obtain:

COROLLARY 4.7. *Let $0 \leq \lambda < 1$. If $\inf_{e \in \mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq \lambda$, then*

$$\mu(P) > \frac{\lambda - 1/2}{1 - 1/2}.$$

Note that if we replace $(\mathbb{N}, \mathbb{N}^{(2)})$ with a finitely branching countable graph G , then the threshold for the existence of infinite paths becomes 1, namely we cannot ensure the existence of infinite paths even if each edge of G belongs to the random subgraph \mathbb{X} with probability very close to 1. In fact, the following more general result holds:

PROPOSITION 4.8. *Let $G = (V_G, E_G)$ be a graph admitting a colouring function $c : E_G \rightarrow \mathbb{N}$ such that each infinite path in G meets all but finitely many colours (it is easy to see, considering the distance from a fixed vertex in each connected component, that a finitely branching countable graph G has this property). Then for every $\varepsilon > 0$ there is a probability space $(\Omega, \mathcal{A}, \mu)$ and a random subgraph $\mathbb{X} : \Omega \rightarrow 2^{E_G}$ of G such that for all $x \in \Omega$, $\mathbb{X}(x)$ has no infinite paths, and yet $\mu(\mathbb{X}_e) > 1 - \varepsilon$ for all $e \in E_G$.*

Proof. Let μ be a probability measure on $\Omega := \mathbb{N}$ with $\mu(\{n\}) < \varepsilon$ for every n . Given $n \in \Omega$ let $\mathbb{X}(n)$ be the subgraph of G (with vertices V_G) containing all edges $e \in E_G$ of colour $c(e) \neq n$. Given $e \in E_G$ there is at

most one n such that $c(e) \in Z_n$. Hence clearly $\mu(\mathbb{X}_e) \geq 1 - \varepsilon$, and yet $\mathbb{X}(n)$ has no infinite paths for any $n \in \Omega$. ■

REMARK 4.9. It is natural to ask whether the answer to Problem 1 changes if we substitute \mathbb{N} with the set of the real numbers. Since $\mathbb{N} \subset \mathbb{R}$, the probability threshold for the existence of infinite paths can only decrease, but the following example shows that it still equals $1/2$. Let $\Omega = [0, 1]^{\mathbb{R}}$ equipped with the product Lebesgue measure \mathcal{L} , let $\varepsilon > 0$, and let

$$\mathbb{X}_{i,j} := \{x \in \Omega : x_i > x_j + \varepsilon\}$$

for all $i < j \in \mathbb{R}$. The assertion follows by observing that $\mathcal{L}(\mathbb{X}_{i,j}) = (1 - \varepsilon)^2/2$ for all $i < j \in \mathbb{R}$, and

$$\bigcap_{i \in \{1, \dots, N\}} \mathbb{X}_{n_i, n_{i+1}} = \emptyset$$

whenever n_i is a strictly increasing sequence of real numbers, and $N > 1/\varepsilon$.

5. Threshold functions for graph morphisms

DEFINITION 5.1. Let F and G be directed graphs. A *graph morphism* $\varphi: G \rightarrow F$ is a map $\varphi: V_G \rightarrow V_F$ such that $(\varphi(a), \varphi(b)) \in E_F$ for all $(a, b) \in E_G$. We write $G \rightarrow F$ if there is a graph morphism from G to F .

The results of the previous sections were implicitly based on the following observation:

REMARK 5.2. Let G be a directed graph.

- (1) G has a path of length $\geq p$ if and only if $G \rightarrow (p, p^{(2)})$.
- (2) G has an infinite path if and only if $G \rightarrow (\omega_1, \omega_1^{(2)})$.

This suggests generalizing the above results by considering other properties of graphs that can be expressed in terms of non-existence of graph morphisms. Let us give the relevant definitions.

DEFINITION 5.3. Given two directed graphs F, G and given $i, j \in V_G$ let

$$(5.1) \quad A_{i,j}(F, G) := \{u \in V_F^{V_G} : (u(i), u(j)) \in E_F\}$$

and define the *relative capacity* of F with respect to G as

$$(5.2) \quad c(F, G) := \sup_{m \in \mathcal{M}^1(V_F^{V_G})} \inf_{(i,j) \in E_G} m(A_{i,j}(F, G)) \in [0, 1].$$

Theorems 3.3 and 4.5 have the following counterpart.

THEOREM 5.4. *Let F and G be directed countable graphs, let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\mathbb{X}: \Omega \rightarrow 2^{E_G}$ be a random subgraph of G . Let $P := \{x \in \Omega : \mathbb{X}(x) \rightarrow F\}$. Assume $\inf_{e \in E_G} \mu(\mathbb{X}_e) > c(F, G)$. Then $\mu(P) > 0$. Moreover there are examples in which P is empty and $\inf_{e \in E_G} \mu(\mathbb{X}_e)$ is as*

close to $c(F, G)$ as required. So $c(F, G)$ is the threshold for non-existence of graph morphisms $f: \mathbb{X}(x) \rightarrow F$.

Proof. Suppose for a contradiction $\mu(P) = 0$. We can then assume $P = \emptyset$ (replacing Ω with $\Omega - P$). Hence for each $x \in \Omega$ there is a graph morphism $\varphi(x): \mathbb{X}(x) \rightarrow F$, which can be seen as an element of $V_F^{V_G}$. We thus obtain a map $\varphi: \Omega \rightarrow V_F^{V_G}$. By Lemma 5.7 below, φ can be chosen to be μ -measurable. Since $x \in \mathbb{X}_{i,j}$ implies $(\phi(x)(i), \phi(x)(j)) \in E_F$, we have $\varphi(\mathbb{X}_{i,j}) \subset A_{i,j}(F, G)$ for all $(i, j) \in E_G$. Let $m := \varphi_{\#}(\mu) \in \mathcal{M}^1(V_F^{V_G})$. Then $m(A_{i,j}(F, G)) \geq \mu(\mathbb{X}_{i,j}) > c(F, G)$. This is absurd by the definition of $c(F, G)$. We have thus proved $\mu(P) > 0$. To prove the second part it suffices to take $\Omega = V_F^{V_G}$ and $\mathbb{X}_{i,j} = A_{i,j}(F, G)$. ■

Reasoning as in Corollary 3.4 we obtain:

COROLLARY 5.5. *Suppose $c(F, G) < 1$. If $\inf_{e \in \mathbb{N}^{(2)}} \mu(\mathbb{X}_e) \geq \lambda$, then*

$$\mu(P) \geq \frac{\lambda - c(F, G)}{1 - c(F, G)}.$$

REMARK 5.6. If the sup in the definition of $c(F, G)$ is not reached, it suffices to have the weak inequality $\inf_{e \in E_G} \mu(\mathbb{X}_e) \geq c(F, G)$ in order to have $\mu(P) > 0$ (this is indeed the case of Theorem 4.5).

It remains to show that the map $\varphi: \Omega \rightarrow V_F^{V_G}$ in the proof of Theorem 5.4 can be taken to be μ -measurable.

LEMMA 5.7. *Let F, G be countable directed graphs, let $(\Omega, \mathcal{A}, \mu)$ be a probability space, and let $\mathbb{X}: \Omega \rightarrow 2^{E_G}$ be a random subgraph of G .*

- (1) *The set $\Omega_0 := \{x \in \Omega : \mathbb{X}(x) \rightarrow F\}$ is μ -measurable (i.e. measurable with respect to the μ -completion of \mathcal{A}).*
- (2) *There is a μ -measurable function $\varphi: \Omega_0 \rightarrow V_F^{V_G}$ that selects, for each $x \in \Omega_0$, a graph morphism $\phi(x): \mathbb{X}(x) \rightarrow F$.*
- (3) *If F is finite, then Ω_0 is measurable and φ can be chosen measurable.*

Proof. Given a function $f: V_G \rightarrow V_F$, we have $f: \mathbb{X}(x) \rightarrow F$ (i.e., f is a graph morphism from $\mathbb{X}(x)$ to F) if and only if $x \in \bigcap_{(i,j) \in V_G} \bigcup_{(a,b) \in V_F} B_{i,j,a,b}$, where $x \in B_{i,j,a,b}$ says that $f(i) = a$, $f(j) = b$ and $x \in \mathbb{X}_{i,j}$. This shows that $B := \{(x, f) : f: \mathbb{X}(x) \rightarrow F\}$ is a measurable subset of $\Omega \times V_F^{V_G}$. We are looking for a (μ -)measurable function $\varphi: \pi_X(B) \rightarrow V_F^{V_G}$ whose graph is contained in B .

Special case: Let us first assume that Ω is a Polish space (i.e., a complete separable metric space) with its algebra \mathcal{A} of Borel sets. By the Jankov–von Neumann uniformization theorem (see [K:95, Thm. 29.9]), if X, Y are Polish

spaces and $Q \subset X \times Y$ is a Borel set, then the projection $\pi_X(Q) \subset X$ is universally measurable (i.e. it is m -measurable for every σ -finite Borel measure m on X), and there is a universally measurable function $f: \pi_X(Q) \rightarrow Y$ whose graph is contained in Q . We can apply this to $X = \Omega$, $Y = V_F^{V_G}$ and $Q = B$ to obtain (1) and (2). It remains to show that if F is finite then $\pi_X(Q)$ and f can be chosen to be Borel measurable. To this end it suffices to use the following uniformization theorem of Arsenin–Kunugui (see [K:95, Thm. 35.46]): if X, Y, Q are as above and each section $Q_x = \{y \in Y : (x, y) \in Q\}$ is a countable union of compact sets, then $p_X(Q)$ is Borel and there is a Borel measurable function $f: \pi_X(Q) \rightarrow Y$ whose graph is contained in Q .

General case: We reduce the problem to the special case as follows. Let $X = 2^{V_G}$, $Y = V_F^{V_G}$ and consider the set $B' \subset X \times Y$ consisting of those pairs (H, f) such that H is a subgraph of G (with the same vertices) and $f: H \rightarrow F$ is a graph morphism. Consider the pushforward measure $m = \mathbb{X}_\#(\mu)$ defined on the Borel algebra of 2^{V_G} . By the special case there is an $(m$ -)measurable function $\psi: \pi_X(B') \rightarrow V_F^{V_G}$ whose graph is contained in B' . To conclude it suffices to take $\varphi := \psi \circ \mathbb{X}$. ■

We now show how to compute the relative capacity $c(F, (\mathbb{N}, \mathbb{N}^{(2)}))$ (see Definition 5.3) for any finite graph F . The following invariant of directed graphs has been studied in [R:82] and [FT:85, Section 3].

DEFINITION 5.8. Given a directed graph F , we define the *capacity* of F as

$$(5.3) \quad c_0(F) := \sup_{\lambda \in \Sigma_F} \sum_{(a,b) \in E_F} \lambda_a \lambda_b \in [0, 1],$$

where Σ_F is the simplex of all sequences $\{\lambda_a\}_{a \in V_F}$ of real numbers such that $\lambda_a \geq 0$ and $\sum_{a \in V_F} \lambda_a = 1$.

PROPOSITION 5.9. *If F is a finite directed graph, then*

$$(5.4) \quad c(F, (\mathbb{N}, \mathbb{N}^{(2)})) = c_0(F).$$

Proof. Let $G = (\mathbb{N}, \mathbb{N}^{(2)})$. The proof is a series of reductions.

STEP 1. Note that if $\sigma \in \text{Incr}(\mathbb{N})$, then $\sigma \cdot m(A_{i,j}(F, G)) = m(A_{\sigma(i), \sigma(j)})$. Hence the infimum in (5.2) can only increase when m is replaced with $\sigma_\#^*(m)$. By Theorem B.8 there is $\sigma \in \text{Incr}(\mathbb{N})$ such that $\sigma \cdot m$ is asymptotically exchangeable. It then follows that we can equivalently take the supremum in (5.2) over the measures $m \in \mathcal{M}^1(V_F^{\mathbb{N}})$ which are asymptotically exchangeable.

STEP 2. By definition, if m is asymptotically exchangeable, there is an exchangeable measure m' such that $\lim_{k \rightarrow \infty} m_k = m'$, where $m_k = \mathbf{S}^k \cdot m$.

Clearly

$$\inf_{(i,j) \in E_G} m(A_{i,j}(F, G)) \leq \lim_{k \rightarrow \infty} m_k(A_{0,1}(F, G)) = m'(A_{0,1}(F, G)).$$

So the supremum in (5.2) coincides with $\sup_m m(A_{0,1}(F, G))$, for m ranging over the exchangeable measures.

STEP 3. By (B.11), every exchangeable measure is a convex integral combination of Bernoulli measures B_λ , with $\lambda \in \Sigma_F$. It follows that it is sufficient to compute the supremum over the Bernoulli measures B_λ . We have

$$\begin{aligned} B_\lambda(\{x \in V_F^{\mathbb{N}} : (x_0, x_1) \in E_F\}) &= \sum_{(a,b) \in E_F} B_\lambda(\{x : x_0 = a, x_1 = b\}) \\ &= \sum_{(a,b) \in E_F} \lambda_a \lambda_b, \end{aligned}$$

so that (5.2) reduces to (5.3). ■

Notice that if there is a morphism of graphs from G to F , then $c_0(G) \leq c_0(F)$. Also note that $c_0(F) = 1$ if there is some $a \in V_F$ with $(a, a) \in E_F$. Recall that F is said to be: *irreflexive* if $(a, a) \notin E_F$ for all $a \in V_F$; *symmetric* if $(a, b) \in E_F \Leftrightarrow (b, a) \in E_F$ for all $a, b \in V_F$; *anti-symmetric* if $(a, b) \in E_F \Rightarrow (b, a) \notin E_F$ for all $a, b \in V_F$.

The *clique number* $\text{cl}(F)$ of F is defined as the largest integer n such that there is a subset $S \subset V_F$ of size n which forms a clique, i.e. $(a, b) \in E_F$ or $(b, a) \in E_F$ for all $a, b \in S$.

PROPOSITION 5.10 (see also [FT:85, Section 3]). *Let F be a finite irreflexive directed graph. If F is anti-symmetric, then*

$$(5.5) \quad c_0(F) = \frac{1}{2} \left(1 - \frac{1}{\text{cl}(F)} \right).$$

If F is symmetric, then

$$(5.6) \quad c_0(F) = 1 - \frac{1}{\text{cl}(F)}.$$

In particular $c_0(K_p) = 1 - 1/p$.

Proof. The anti-symmetric case follows from the symmetric one by taking the symmetric closure. So we can assume that F is symmetric. Let $\lambda \in \Sigma_F$ be a maximizing distribution, meaning that $c_0(F) = \sum_{(a,b) \in E_F} \lambda_a \lambda_b$, and let S_λ be the subgraph of F spanned by the support of λ , that is, $V_{S_\lambda} = \{a \in V_F : \lambda_a > 0\}$. Given $a \in S_\lambda$ note that

$$\frac{\partial}{\partial \lambda_a} \sum_{(u,v) \in E_F} \lambda_u \lambda_v = 2 \sum_{b \in V_F : (a,b) \in E_F} \lambda_b.$$

From Lagrange’s multiplier theorem it then follows that $\sum_{b \in V_F: (a,b) \in E_F} \lambda_b$ is constant, namely it does not depend on the choice of $a \in S_\lambda$. Since $\sum_{a \in S_\lambda} (\sum_{b: (a,b) \in E_F} \lambda_a) = c_0(F)$, it follows that for each $a \in S_\lambda$ we have

$$(5.7) \quad \sum_{b \in V_F: (a,b) \in E_F} \lambda_b = c_0(F).$$

If $c, c' \in V_{S_\lambda}$, we can consider the distribution $\lambda' \in \Sigma_F$ such that $\lambda'_c = 0$, $\lambda'_{c'} = \lambda_c + \lambda_{c'}$, and $\lambda'_b = \lambda_b$ for all $b \in V_F \setminus \{c, c'\}$. From (5.7) it then follows that λ' is also a maximizing distribution whenever $(c, c') \dashrightarrow E_F$. (In fact $\sum_{(a,b) \in E_F} \lambda'_a \lambda'_b = \sum_{(a,b) \in E_F} \lambda_a \lambda_b - \lambda_c \sum_{b: (c,b) \in E_F} \lambda_b + \lambda_c \sum_{b: (c',b) \in E_F} \lambda_b = c_0(F) - \lambda_c c_0(F) + \lambda_c c_0(F)$.)

As a first consequence, S_λ is a clique whenever λ is a maximizing distribution with minimal support. Indeed, let K be a maximal clique contained in S_λ , and assume for contradiction that there exists $a \in V_{S_\lambda} \setminus V_K$. Letting $a' \in V_K$ be a vertex of F independent of a (such an element exists since K is a maximal clique), and letting $\lambda' \in \Sigma_F$ be as above, we have $c_0(F) = \sum_{(a,b) \in E_F} \lambda'_a \lambda'_b$, contradicting the minimality of V_{S_λ} .

Once we know that S_λ is a clique, again from (5.7) we deduce that λ is a uniform distribution, that is, $\lambda_a = \lambda_b$ for all $a, b \in V_{S_\lambda}$. It follows that

$$c_0(F) = 1 - \frac{1}{|S_\lambda|} \leq 1 - \frac{1}{\text{cl}(F)},$$

which in turn implies (5.5), the opposite inequality being realized by a uniform distribution on a maximal clique. ■

Notice that the proof of Proposition 5.10 shows that there exists a maximizing $\lambda \in \Sigma_F$ whose support is a clique (not necessarily of maximal order).

5.1. Chromatic number. We will apply the results of the previous section to study the chromatic number of a random subgraph of $(\mathbb{N}, \mathbb{N}^{(2)})$. We point out that an alternative proof of this result follows from [EH:64, Theorem 1].

We recall that the *chromatic number* $\chi(G)$ of a directed graph G is the smallest n such that there is a colouring of the vertices of G with n colours in such a way that $a, b \in V_G$ have different colours whenever $(a, b) \in E_G$ (see [B:79]).

For $p \in \mathbb{N}$, let K_p be the complete graph on p vertices, namely K_p has set of vertices $p = \{0, 1, \dots, p - 1\}$ and set of edges $\{(x, y) \in p^2 : x \neq y\}$. Clearly $\chi(K_p) = p$. Note also that

$$(5.8) \quad G \rightarrow K_p \Leftrightarrow \chi(G) \leq p.$$

Now let (Ω, \mathcal{A}, m) be a probability space, and let $\mathbb{X}: \Omega \rightarrow 2^{E_G}$ be a random subgraph of $G = (\mathbb{N}, \mathbb{N}^{(2)})$. Let $P = \{x \in \Omega : \chi(\mathbb{X}(x)) \geq p\}$. By (5.8) and the results of the previous section, if $\inf_{e \in \mu(\mathbb{X}_e)} > c(K_p, (\mathbb{N}, \mathbb{N}^{(2)}))$, then

$\mu(P) > 0$. This however does not say much unless we manage to determine $c(K_p, (\mathbb{N}, \mathbb{N}^{(2)}))$. We will show that $c(K_p, (\mathbb{N}, \mathbb{N}^{(2)})) = 1 - 1/p$, so we have:

THEOREM 5.11. *Let (Ω, \mathcal{A}, m) be a probability space, and $\mathbb{X}: \Omega \rightarrow 2^{E_G}$ be a random subgraph of $(\mathbb{N}, \mathbb{N}^{(2)})$. If $\inf_{e \in \mu(\mathbb{X}_e)} > 1 - 1/p$, then*

$$\mu(\{x \in \Omega : \chi(\mathbb{X}(x)) \geq p + 1\}) > 0.$$

6. Infinite cliques. We recall the following standard Borel–Cantelli type result, which shows that Problem 3 has a positive answer for $k = 1$.

PROPOSITION 6.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Let $\lambda > 0$ and for each $i \in \mathbb{N}$ let $X_i \subseteq \Omega$ be a measurable set such that $\mu(X_i) \geq \lambda$. Then there is an infinite set $J \subset \mathbb{N}$ such that*

$$\bigcap_{i \in J} X_i \neq \emptyset.$$

Proof. The set $Y := \bigcap_n \bigcup_{i > n} X_i$ is a decreasing intersection of sets of (finite) measure greater than $\lambda > 0$, hence $\mu(Y) \geq \lambda$ and, in particular, Y is non-empty. Now it suffices to note that any element x of Y belongs to infinitely many X_i 's. ■

Proposition 6.1 has the following interpretation: if we choose each element of \mathbb{N} with probability at least λ , we obtain an infinite subset with probability at least λ .

The following example shows that Problem 3 has in general a negative answer for $k > 1$.

EXAMPLE 6.2. Let $p \in \mathbb{N}$ and consider the Cantor space $\Omega = p^{\mathbb{N}}$, equipped with the Bernoulli measure $B_{(1/p, \dots, 1/p)}$, and let $\mathbb{X}_{i,j} := \{x \in \Omega : x_i \neq x_j\}$. Then each $\mathbb{X}_{i,j}$ has measure $\lambda = 1 - 1/p$, and for all $x \in X$ the graph $\mathbb{X}(x) := \{(i, j) \in \mathbb{N}^{(2)} : x \in \mathbb{X}_{i,j}\}$ does not contain cliques (i.e. complete subgraphs) of cardinality $p + 1$.

In view of Example 6.2, we need further assumptions in order to get a positive answer to Problem 3.

EXAMPLE 6.3. By the Ramsey theorem, Problem 3 has a positive answer if there is a finite set $S \subset \Omega$ such that each X_{i_1, \dots, i_k} has a non-empty intersection with S . In particular, this is the case if Ω is countable.

PROPOSITION 6.4. *Let $r > 0$. Assume that Ω is a compact metric space and each set $\mathbb{X}_{i_1, \dots, i_k}$ contains a ball B_{i_1, \dots, i_k} of radius $r > 0$. Then Problem 3 has a positive answer.*

Proof. Applying Lemma A.1 to the centres of the balls B_{i_1, \dots, i_k} shows that for all $0 < r' < r$ there exists an infinite set J and a ball B of radius r'

such that

$$B \subset \bigcap_{(j_1, \dots, j_k) \in J^{(k)}} X_{j_1, \dots, j_k} \cdot \blacksquare$$

We now give a sufficient condition for a positive answer to Problem 3.

THEOREM 6.5. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Let $\lambda > 0$ and assume that $\mu(\mathbb{X}_{i_1, \dots, i_k}) \geq \lambda$ for each $(i_1, \dots, i_k) \in \mathbb{N}^{(k)}$. Assume further that the indicator functions of $\mathbb{X}_{i_1, \dots, i_k}$ belong to a compact subset \mathcal{K} of $L^1(\Omega, \mu)$. Then for any $\varepsilon > 0$ there exists an infinite set $J \subset \mathbb{N}$ such that*

$$\mu\left(\bigcap_{(i_1, \dots, i_k) \in J^{(k)}} X_{i_1, \dots, i_k}\right) \geq \lambda - \varepsilon.$$

Proof. Consider first the case $k = 1$. By compactness of \mathcal{K} , for all $\varepsilon > 0$ there exist an increasing sequence $\{i_n\}$ and a set $X_\infty \subset X$, with $\mu(X_\infty) \geq \lambda$, such that

$$\mu(X_\infty \Delta X_{i_n}) \leq \frac{\varepsilon}{2^n} \quad \forall n \in \mathbb{N}.$$

As a consequence, letting $J := \{i_n : n \in \mathbb{N}\}$ we have

$$\mu\left(\bigcap_{n \in \mathbb{N}} X_{i_n}\right) \geq \mu\left(X_\infty \cap \bigcap_{n \in \mathbb{N}} X_{i_n}\right) \geq \mu(X_\infty) - \sum_{n \in \mathbb{N}} \mu(X_\infty \Delta X_{i_n}) \geq \lambda - \varepsilon.$$

For $k > 1$, we apply Lemma A.1 with

$$M = \mathcal{K} \subset L^1(\Omega, \mu), \quad f(i_1, \dots, i_k) = \chi_{X_{i_1 \dots i_k}} \in L^1(\Omega, \mu).$$

In particular, recalling Remark A.4, for all $\varepsilon > 0$ there exist $J = \sigma(\mathbb{N})$, $X_\infty \subset \Omega$, and $X_{i_1, \dots, i_m} \subset X$, for all $(i_1, \dots, i_m) \in J^{(m)}$ with $1 \leq m < k$, such that $\mu(X_\infty) \geq \lambda$ and for all $(i_1, \dots, i_k) \in J^{(k)}$ we have

$$\mu(X_\infty \Delta X_{i_1}) \leq \frac{\varepsilon}{2^{\sigma^{-1}(i_1)}}, \quad \mu(X_{i_1 \dots i_m} \Delta X_{i_1, \dots, i_{m+1}}) \leq \frac{\varepsilon}{2^{\sigma^{-1}(i_{m+1})}}.$$

Reasoning as above, we find that

$$\begin{aligned} \mu\left(X_\infty \Delta \bigcap_{(i_1, \dots, i_k) \in J^{(k)}} X_{i_1, \dots, i_k}\right) &\leq \sum_{i_1 \in \mathbb{N}} \mu(X_\infty \Delta X_{i_1}) + \sum_{i_1 < i_2} \mu(X_{i_1} \Delta X_{i_1, i_2}) \\ &\quad + \dots + \sum_{i_1 < \dots < i_k} \mu(X_{i_1, \dots, i_{k-1}} \Delta X_{i_1, \dots, i_k}) \leq C(k)\varepsilon, \end{aligned}$$

where $C(k) > 0$ is a constant depending only on k . Therefore

$$\begin{aligned} \mu\left(\bigcap_{(i_1, \dots, i_k) \in J^{(k)}} X_{i_1, \dots, i_k}\right) &\geq \mu\left(X_\infty \cap \bigcap_{(i_1, \dots, i_k) \in J^{(k)}} X_{i_1, \dots, i_k}\right) \\ &\geq \mu(X_\infty) - \mu\left(X_\infty \Delta \bigcap_{(i_1, \dots, i_k) \in J^{(k)}} X_{i_1, \dots, i_k}\right) \\ &\geq \lambda - C(k)\varepsilon. \quad \blacksquare \end{aligned}$$

Notice that from Theorem 6.5 it follows that Problem 3 has a positive answer if there exist an infinite $J \subseteq \mathbb{N}$ and sets $\tilde{X}_{i_1, \dots, i_k} \subseteq X_{i_1, \dots, i_k}$ with $(i_1, \dots, i_k) \in J^{(k)}$ such that $\mu(\tilde{X}_{i_1, \dots, i_k}) \geq \lambda$ for some $\lambda > 0$, and the indicator functions of $\tilde{X}_{i_1, \dots, i_k}$ belong to a compact subset of $L^1(\Omega, \mu)$.

REMARK 6.6. We recall that, when Ω is a compact subset of \mathbb{R}^n and the perimeters of the sets X_{i_1, \dots, i_k} are uniformly bounded, then the family $\chi_{X_{i_1, \dots, i_k}}$ has compact closure in $L^1(\Omega, \mu)$ (see for instance [AFP:00, Thm. 3.23]). In particular, if the sets X_{i_1, \dots, i_k} have equibounded Cheeger constant, i.e. if there exists $C > 0$ such that

$$\min_{E \subset X_{i_1, \dots, i_k}} \frac{\text{Per}(E)}{|E|} \leq C \quad \forall (i_1, \dots, i_k) \in \mathbb{N}^{(k)},$$

then Problem 3 has a positive answer.

Appendix A. A topological Ramsey theorem. The following metric version of the Ramsey theorem reduces to the classical Ramsey theorem when M is finite.

LEMMA A.1. *Let M be a compact metric space, let $k \in \mathbb{N}$, and let $f : \mathbb{N}^{(k)} \rightarrow M$. Then there exists an infinite set $J \subset \mathbb{N}$ such that the limit*

$$\lim_{\substack{(i_1, \dots, i_k) \rightarrow +\infty \\ (i_1, \dots, i_k) \in J^{(k)}}} f(i_1, \dots, i_k)$$

exists.

Proof. Notice first that the assertion is trivial for $k = 1$, since M is compact. Assume that the assertion holds for some $k \in \mathbb{N}$. Let $f : \mathbb{N}^{(k+1)} \rightarrow M$. By inductive assumption, for all $j \in \mathbb{N}$ there exist an infinite set $J_j \subset \mathbb{N}$ and a point $x_j \in M$ such that $x_j = \lim_{i_1, \dots, i_k \rightarrow \infty} f(j, i_1, \dots, i_k)$, with $(i_1, \dots, i_k) \in J_j^{(k)}$. Possibly extracting further subsequences we can also assume that

$$(A.1) \quad d(x_j, f(j, i_1, \dots, i_k)) \leq 1/2^j$$

for all $(i_1, \dots, i_k) \in J_j^{(k)}$. Moreover, by a recursive construction, we can assume that $J_{j+1} \subseteq J_j$. Now define $\tau \in \text{Incr}(\mathbb{N})$ by choosing $\tau(0) \in \mathbb{N}$ and inductively $\tau(n+1) \in J_{\tau(n)}$. Since $J_{j+1} \subset J_j$ for all j , this implies $\tau(m) \in J_{\tau(n)}$ for all $m > n$. By compactness of M , there exists $\lambda \in \text{Incr}(\mathbb{N})$ and a point $x \in M$ such that $x_{\tau(\lambda(n))} \rightarrow x$ for $n \rightarrow \infty$. Take $J = \text{Im}(\tau \circ \lambda)$. The result follows from the triangle inequality $d(x, f(j, i_1, \dots, i_k)) \leq d(x, x_j) + d(x_j, f(j, i_1, \dots, i_k))$, noting that if $j < i_1 < \dots < i_k$ are in J , then $i_1, \dots, i_k \in J_j$, and inequality (A.1) applies. ■

Note that in Lemma A.1, the condition $(i_1, \dots, i_k) \rightarrow +\infty$ is equivalent to $i_1 \rightarrow \infty$ (since $i_1 < \dots < i_k$). We would like to strengthen Lemma A.1

by requiring the existence of all the partial limits

$$x = \lim_{i_{j(1)} \rightarrow \infty} \lim_{i_{j(2)} \rightarrow \infty} \dots \lim_{i_{j(r)} \rightarrow \infty} x_{i_1, \dots, i_k}$$

where $1 \leq r \leq k$ and $(i_{j(1)}, \dots, i_{j(r)}) \in J^{(r)}$ is a subsequence of the (finite) sequence $(i_1, \dots, i_k) \in J^{(k)}$. Note that the existence of all these 2^{k-1} partial limits does not follow directly from Lemma A.1.

To prove the desired strengthening it is convenient to introduce some terminology. Let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of \mathbb{N} . Given a distance δ on \mathbb{N} , we consider on $\mathbb{N}^{(k)}$ the induced metric

$$\delta_k((n_1, \dots, n_k), (m_1, \dots, m_k)) := \max_i \delta(n_i, m_i).$$

Given $\sigma \in \text{Incr}(\mathbb{N})$, let $\sigma_* : \mathbb{N}^{(k)} \rightarrow \mathbb{N}^{(k)}$ be the induced map defined by $\sigma_*(n_1, \dots, n_k) := (\sigma(n_1), \dots, \sigma(n_k))$. Given $f : \mathbb{N}^{(k)} \rightarrow M$, by the following theorem there is an infinite $J \subset \mathbb{N}$ such that all the partial limits of $f \upharpoonright_{J^{(k)}}$ exist. Moreover the arbitrariness of δ shows that we can impose an arbitrary modulus of convergence on all the partial limits of $f \circ \sigma_*$, where $\sigma \in \text{Incr}(\mathbb{N})$ is an increasing enumeration of J .

THEOREM A.2. *Let M be a compact metric space, let $k \in \mathbb{N}$, and let $f : \mathbb{N}^{(k)} \rightarrow M$. Then for any distance δ on $\bar{\mathbb{N}}$ there exists $\sigma \in \text{Incr}(\mathbb{N})$ such that $f \circ \sigma_* : \mathbb{N}^{(k)} \rightarrow M$ is 1-Lipschitz, and as a consequence, it can be extended to a 1-Lipschitz function on the closure of $\mathbb{N}^{(k)}$ in $\bar{\mathbb{N}}^k$.*

LEMMA A.3. *Let δ be a metric on $\bar{\mathbb{N}}$. Then there is another metric δ^* on $\bar{\mathbb{N}}$ such that*

- (1) $\delta^*(x, y) \leq \delta(x, y)$ for all x, y .
- (2) δ^* is monotone in the following sense: $\delta^*(x', y') \leq \delta^*(x, y)$ for all x, x, y, y' provided $x < \min(y, x', y')$.
- (3) $\varepsilon^*(x) \geq \varepsilon^*(y)$ for all $x \leq y$, where

$$(A.2) \quad \varepsilon^*(x) := \min_{y \geq x+1} \delta^*(x, y).$$

Proof. We shall define a distance of the form $\delta^*(x, y) = \delta(\psi(x), \psi(y))$ for a suitable strictly increasing function $\psi : \bar{\mathbb{N}} \rightarrow \bar{\mathbb{N}}$. To this end, let us consider, for any $x \in \bar{\mathbb{N}}$, the diameter of the interval $[x, \infty] \cap \bar{\mathbb{N}}$,

$$(A.3) \quad \eta(x) := \max_{x \leq y \leq z} \delta(y, z),$$

and the point-set distance from x to the interval $[x + 1, \infty] \cap \bar{\mathbb{N}}$,

$$(A.4) \quad \varepsilon(x) := \min_{y \geq x+1} \delta(x, y).$$

Since $\varepsilon(x) > 0$ for all $x < \infty$ and $\eta(x) = o(1)$ as $x \rightarrow \infty$, there exists a recursively defined, strictly increasing function $\psi: \overline{\mathbb{N}} \rightarrow \overline{\mathbb{N}}$ such that for any $x \in \mathbb{N}$,

$$(A.5) \quad \eta(\psi(x)) \leq \varepsilon(x), \quad \eta(\psi(x+1)) \leq \varepsilon(\psi(x)).$$

As a consequence, the distance

$$\delta^*(x, y) := \delta(\psi(x), \psi(y))$$

satisfies, for all $x < y \leq \infty$,

$$\delta^*(x, y) = \delta(\psi(x), \psi(y)) \leq \eta(\psi(x)) \leq \varepsilon(x) \leq \delta(x, y),$$

and, assuming also $x < x' \leq \infty$ and $x < y' \leq \infty$,

$$\begin{aligned} \delta^*(x', y') &= \delta(\psi(x'), \psi(y')) \leq \eta(\psi(x')) \leq \eta(\psi(x+1)) \\ &\leq \varepsilon(\psi(x)) \leq \delta(\psi(x), \psi(y)) = \delta^*(x, y). \end{aligned}$$

To prove the last statement we observe that

$$\varepsilon^*(x) \geq \varepsilon(\psi(x)) \geq \eta(\psi(x+1)) \geq \varepsilon^*(x+1). \quad \blacksquare$$

Proof of Theorem A.2. By Lemma A.3 we can assume that δ is monotone in the sense of part (2) of that lemma.

We proceed by induction on k . When $k = 1$, consider the function $\varepsilon(n) := \min_{m \geq n+1} \delta(n, m)$ as in (A.2). By compactness of M there exist $x \in M$ and a subsequence $f \circ \sigma$ of f converging to x with the property

$$(A.6) \quad d_M(f(\sigma n), x) \leq \varepsilon(n)/2.$$

Recalling Lemma A.3(3), for $n \neq m$ we have

$$(A.7) \quad d_M(f(\sigma n), f(\sigma m)) \leq (\varepsilon(n) + \varepsilon(m))/2 \leq \delta(n, m).$$

So $f \circ \sigma$ is 1-Lipschitz.

Now assume inductively that the assertion holds for some $k \in \mathbb{N}$. Let $f: \mathbb{N}^{(k+1)} \rightarrow M$. We need to prove the existence of $\sigma \in \text{Incr}(\mathbb{N})$ such that

$$(A.8) \quad d_M(f(\sigma_*(n, \mathbf{m})), f(\sigma_*(n', \mathbf{m}'))) \leq \delta_{k+1}((n, \mathbf{m}), (n', \mathbf{m}'))$$

for all $(n, \mathbf{m}) \in \mathbb{N}^{(k+1)}$ and $(n', \mathbf{m}') \in \mathbb{N}^{(k+1)}$, where $\mathbf{m} = (m_1, \dots, m_k)$ and $\mathbf{m}' = (m'_1, \dots, m'_k)$.

Given $n \in \mathbb{N}$ define $f_n: \mathbb{N}^{(k)} \rightarrow M$ by

$$(A.9) \quad f_n(\mathbf{m}) := \begin{cases} f(n, \mathbf{m}) & \text{if } n < m_1, \\ \perp & \text{if } n \geq m_1, \end{cases}$$

where \perp is an arbitrary element of M . Note that the condition $n < m_1$ is equivalent to $(n, \mathbf{m}) \in \mathbb{N}^{(k+1)}$.

By inductive assumption, for all $n \in \mathbb{N}$ there exists $\theta_n \in \text{Incr}(\mathbb{N})$ such that $f_n \circ \theta_{n*}: \mathbb{N}^{(k)} \rightarrow M$ is 1-Lipschitz. By a recursive construction, we can

also assume that θ_{n+1} is a subsequence of θ_n , namely $\theta_{n+1} = \theta_n \circ \gamma_n$ for some $\gamma_n \in \text{Incr}(\mathbb{N})$. Indeed to obtain θ_{n+1} as desired it suffices to apply the induction hypothesis to $f_{n+1} \circ \theta_{n*}: \mathbb{N}^{(k)} \rightarrow M$ rather than directly to f_{n+1} .

Since $f_n \circ \theta_{n*}$ is 1-Lipschitz, the limit

$$g(n) := \lim_{\min(\mathbf{m}) \rightarrow \infty} f(n, \theta_{n*}(\mathbf{m}))$$

exists. Passing to a subsequence we can further assume that all the values of $f_n \circ \theta_n$ are within distance $\varepsilon(n)/4$ of its limit, that is,

$$(A.10) \quad d_M(g(n), f(n, \theta_n(\mathbf{m}))) < \varepsilon(n)/4.$$

Let $J_n := \theta_n(\mathbb{N}) \subset \mathbb{N}$ and let $\tau \in \text{Incr}(\mathbb{N})$ be such that

$$(A.11) \quad \tau(n+1) \in J_{\tau(n)},$$

It then follows that

$$(A.12) \quad \forall n, m \in \tau(\mathbb{N}) \quad m > n \Rightarrow m \in J_n.$$

For later purposes we need to define $\tau(n+1)$ as an element of $J_{\tau(n)}$ bigger than its $(n+1)$ th element, namely $\tau(n+1) > \theta_{\tau(n)}(n+1)$. So, for definiteness, we define inductively $\tau(0) := 0$ and $\tau(n+1) := \theta_{\tau(n)}(n+2)$. It then follows that

$$(A.13) \quad \forall i, j \in \tau(\mathbb{N}) \quad \forall k \in \mathbb{N} \quad j > i, j \geq k \Rightarrow \tau(j) > \theta_{\tau(i)}(k).$$

Reasoning as in the case $k=1$, we find $\lambda \in \text{Incr}(\mathbb{N})$ and $x_\infty \in M$ such that

$$(A.14) \quad d_M(g(\tau(\lambda(n))), x_\infty) < \varepsilon(n)/4.$$

Now define $\sigma := \tau \circ \lambda \in \text{Incr}(\mathbb{N})$. Note that $\sigma(\mathbb{N}) \subset \tau(\mathbb{N})$ so (A.12) and (A.13) continue to hold with σ instead of τ . We claim that $f \circ \sigma_*: \mathbb{N}^{(k+1)} \rightarrow M$ is 1-Lipschitz.

As a first step we show that

$$(A.15) \quad \exists \mathbf{k} > \mathbf{m} \quad (f \circ \sigma_*)(n, \mathbf{m}) = (f_{\sigma(n)} \circ \theta_{\sigma(n)})(n, \mathbf{k})$$

where $\mathbf{k} > \mathbf{m}$ means that $k_i > m_i$ for all respective components. To prove (A.15) recall that $(f \circ \sigma_*)(n, \mathbf{m}) = f(\sigma(n), \sigma(m_1), \dots, \sigma(m_k))$. Since $n < \min(\mathbf{m})$, by (A.12) the elements $\sigma(m_1), \dots, \sigma(m_k)$ are in the image of $\theta_{\sigma(n)}$, namely for each i we have $\sigma(m_i) = \theta_{\sigma(n)}(k_i)$ for some $k_i \in \mathbb{N}$. Moreover applying (A.13) we must have $k_i > m_i$. The proof of (A.15) is thus complete.

It follows from (A.15) and (A.10) that $(f \circ \sigma_*)(n, \mathbf{m})$ is within distance $\varepsilon(\sigma(n))/4$ of its limit $g(\sigma(n))$, which in turn is within distance $\varepsilon(n)/4$ of its limit x_∞ by (A.14). We have thus proved

$$(A.16) \quad d_M(f(\sigma_*(n, \mathbf{m})), x_\infty) < \frac{1}{4}\varepsilon(\sigma(n)) + \frac{1}{4}\varepsilon(n).$$

Recalling that for $x \neq y$ we have $\varepsilon(x) + \varepsilon(y) \leq 2\delta(x, y)$, we see that for $n \neq n'$ the left-hand side of (A.8) is bounded by $[\delta(\sigma(n), \sigma(n')) + \delta(n, n')]/2$, which in turn is $\leq \delta(n, n')$ by monotonicity of δ .

It remains to prove (A.8) in the case $n = n'$. Given \mathbf{m}, \mathbf{m}' as in (A.8), we apply (A.15) to get $\mathbf{k} > \mathbf{m}$ and $\mathbf{k}' > \mathbf{m}'$ with $(f \circ \sigma_*)(n, \mathbf{m}) = (f_{\sigma(n)} \circ \theta_{\sigma(n)})(n, \mathbf{k})$ and $(f \circ \sigma_*)(n, \mathbf{m}') = (f_{\sigma(n)} \circ \theta_{\sigma(n)})(n, \mathbf{k}')$.

Using the monotonicity of δ and the fact that $f_{\sigma(n)} \circ \theta_{\sigma(n)}$ is 1-Lipschitz, we conclude that

$$(A.17) \quad d_M(f(\sigma_*(n, \mathbf{m})), f(\sigma_*(n, \mathbf{m}'))) \leq \delta_k(\mathbf{k}, \mathbf{k}') \leq \delta_k(\mathbf{m}, \mathbf{m}'). \quad \blacksquare$$

REMARK A.4. Theorem A.2 implies that there exists an infinite set $J = \sigma(\mathbb{N}) \subset \mathbb{N}$ such that, for all $0 \leq m < k$ and $(i_1, \dots, i_m) \in J^{(m)}$, there are limit points $x_{i_1, \dots, i_m} \in M$ with the property

$$x_{i_1, \dots, i_m} = \lim_{\substack{(i_{m+1}, \dots, i_k) \rightarrow \infty \\ (i_1, \dots, i_k) \in J^{(k)}}} x_{i_1, \dots, i_k},$$

where we set $x_{i_1, \dots, i_k} := f(i_1, \dots, i_k)$. Moreover, by choosing the distance $\delta(n, m) = \varepsilon|2^{-n} - 2^{-m}|$, we may also require

$$d_M(x_{i_1, \dots, i_m}, x_{i_1, \dots, i_k}) \leq \frac{\varepsilon}{2^{\sigma^{-1}(i_{m+1})}} \quad \forall (i_1, \dots, i_k) \in J^{(k)}.$$

Appendix B. Exchangeable measures. Let Λ be a compact metric space. We recall a classical notion of *exchangeable measure* due to De Finetti [DF:74], showing some equivalent conditions.

PROPOSITION B.1. *Given $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$, the following conditions are equivalent:*

- (a) m is $\mathfrak{S}_c(\mathbb{N})$ -invariant;
- (b) m is $\text{Inj}(\mathbb{N})$ -invariant;
- (c) m is $\text{Incr}(\mathbb{N})$ -invariant.

DEFINITION B.2. If m satisfies one of these equivalent conditions we say that m is *exchangeable*.

Notice that an exchangeable measure is always shift-invariant, while there are shift-invariant measures which are not exchangeable. To prove Proposition B.1 we need some preliminary results concerning measures satisfying condition (c).

DEFINITION B.3. Given $m \in \mathcal{M}(\Lambda^{\mathbb{N}})$ and $f \in L^p(\Lambda^{\mathbb{N}})$, with $p \in [1, +\infty]$, we let

$$\tilde{f} = E(f | \mathcal{A}_s) \in L^p(\Lambda^{\mathbb{N}})$$

be the conditional probability of f with respect to the σ -algebra \mathcal{A}_s of the shift-invariant Borel subsets of $\Lambda^{\mathbb{N}}$. In particular, \tilde{f} is shift-invariant, and by

Birkhoff’s theorem (see for instance [P:81]) we have

$$\tilde{f} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ S^{*k},$$

where the limit holds almost everywhere and in the strong topology of $L^1(\Lambda^{\mathbb{N}})$.

LEMMA B.4. *Assume that $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ is $\text{Incr}(\mathbb{N})$ -invariant. Then for all $f \in L^\infty(\Lambda^{\mathbb{N}}, m)$ we have*

$$(B.1) \quad \tilde{f} = \lim_{n \rightarrow \infty} f \circ S^{*n},$$

where the limit is taken in the weak* topology of $L^\infty(\Lambda^{\mathbb{N}})$, that is, for every $g \in L^1(\Lambda^{\mathbb{N}}, m)$ we have

$$(B.2) \quad \lim_{n \rightarrow \infty} \int_{\Lambda^{\mathbb{N}}} g(f \circ S^{*n}) dm = \int_{\Lambda^{\mathbb{N}}} g \tilde{f} dm.$$

Proof. It suffices to prove that $\lim_{n \rightarrow \infty} f \circ S^{*n}$ exists, since it is then necessarily equal to the (weak*) limit of the arithmetic means $\frac{1}{n} \sum_{k=0}^{n-1} f \circ S^{*k}$, and therefore to \tilde{f} (since $\tilde{f} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ S^{*k}$ in an even stronger topology). Since the sequence $f \circ S^{*n}$ is equibounded in $L^\infty(\Lambda^{\mathbb{N}}, m)$, it is enough to prove (B.2) for all g in a dense subset D of $L^1(\Lambda^{\mathbb{N}})$. We can take D to be the set of those functions $g \in L^1(\Lambda^{\mathbb{N}}, m)$ that depend on finitely many coordinates (that is, $g(x) = h(x_1, \dots, x_r)$ for some $r \in \mathbb{N}$ and some $h \in L^1(\Lambda^r, m)$). The convergence of (B.2) for $g(x) = h(x_1, \dots, x_r)$ follows at once from the fact that $\sigma \cdot m = m$ for all $\sigma \in \text{Incr}(\mathbb{N})$, which implies that the quantity in (B.2) is constant for all $n > r$. Indeed to prove that $\int_{\Lambda^{\mathbb{N}}} g(f \circ S^{*n}) dm = \int_{\Lambda^{\mathbb{N}}} g(f \circ S^{*n+l}) dm$ it suffices to consider the function $\sigma \in \text{Incr}(\mathbb{N})$ which fixes $0, \dots, r-1$ and sends i to $i+l$ for $i \geq r$. ■

We are now ready to prove the equivalence of the conditions in the definition of exchangeable measure.

Proof of Proposition B.1. Since $\mathfrak{S}_c(\mathbb{N}) \subset \text{Inj}(\mathbb{N})$ and $\text{Incr}(\mathbb{N}) \subset \text{Inj}(\mathbb{N})$, the implications (b) \Rightarrow (a) and (b) \Rightarrow (c) are obvious. The implication (a) \Rightarrow (b) is also evident since it is true on the Borel subsets of $\Lambda^{\mathbb{N}}$ of the form $\{x \in \Lambda^{\mathbb{N}} : x_{i_1} \in A_1, \dots, x_{i_r} \in A_r\}$, which generate the whole Borel σ -algebra of $\Lambda^{\mathbb{N}}$.

Let $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ be $\text{Incr}(\mathbb{N})$ -invariant, and let us prove that m is $\text{Inj}(\mathbb{N})$ -invariant. So let $\sigma \in \text{Inj}(\mathbb{N})$. We must show that

$$(B.3) \quad \int_{\Lambda^{\mathbb{N}}} g dm = \int_{\Lambda^{\mathbb{N}}} g \circ \sigma^* dm$$

for all $g \in C(\Lambda^{\mathbb{N}})$. It suffices to prove (B.3) for g in a dense subset D of $C(\Lambda^{\mathbb{N}})$. So we can assume that $g(x)$ has the form $g_0(x_0) \cdots g_r(x_r)$ for

some $r \in \mathbb{N}$ and $g_1, \dots, g_r \in C(\Lambda)$. Note that $g_i(x_i) = (g_i \circ P_i)(x)$ where $P_i: \Lambda^{\mathbb{N}} \rightarrow \Lambda$ is the projection on the i th coordinate. Since $P_i = P_0 \circ S^*$ where S^* is the shift, we can apply Lemma B.4 to obtain

$$\int_{\Lambda^{\mathbb{N}}} g \, dm = \int_{\Lambda^{\mathbb{N}}} \widetilde{g_1 \circ P_1} \cdots \widetilde{g_r \circ P_1} \, dm.$$

Reasoning in the same way for the function $g \circ \sigma^*$, we finally get

$$\int_{\Lambda^{\mathbb{N}}} g \circ \sigma^* \, dm = \int_{\Lambda^{\mathbb{N}}} \widetilde{g_1 \circ P_1} \cdots \widetilde{g_r \circ P_1} \, dm = \int_{\Lambda^{\mathbb{N}}} g \, dm. \quad \blacksquare$$

DEFINITION B.5. We say that $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ is *asymptotically exchangeable* if the limit

$$m' = \lim_{\substack{\min \theta \rightarrow \infty \\ \theta \in \text{Incr}(\mathbb{N})}} \theta \cdot m$$

exists in $\mathcal{M}^1(\Lambda^{\mathbb{N}})$ and is an exchangeable measure.

REMARK B.6. Note that if m is asymptotically exchangeable, then

$$m' := \lim_{\substack{\min \theta \rightarrow \infty \\ \theta \in \text{Incr}(\mathbb{N})}} \theta \cdot m = \lim_{k \rightarrow \infty} S^k \cdot m.$$

However it is possible that $\lim_{k \rightarrow \infty} S^k \cdot m$ exists and is exchangeable, and yet m is not asymptotically exchangeable. As an example one may start with the Bernoulli probability measure μ on $2^{\mathbb{N}}$ with $\mu(\{x_i = 0\}) = 1/2$ and then consider the conditional probability $m(\cdot) = \mu(\cdot | A)$ where $A \subset 2^{\mathbb{N}}$ is the set of those sequences $x \in 2^{\mathbb{N}}$ satisfying $x_{(n+1)^2} = 1 - x_{n^2}$ for all n .

REMARK B.7. If m is asymptotically exchangeable and if $m' = \lim_{k \rightarrow \infty} S^k \cdot m$, then for all $r \in \mathbb{N}$ and $g_1, \dots, g_r \in C(\Lambda)$ we have

$$(B.4) \quad \lim_{\substack{i_1 \rightarrow +\infty \\ (i_1, \dots, i_r) \in \mathbb{N}^{(r)}}} \int_{\Lambda^{\mathbb{N}}} g_1(x_{i_1}) \cdots g_r(x_{i_r}) \, dm = \int_{\Lambda^{\mathbb{N}}} g_1(x_1) \cdots g_r(x_r) \, dm'.$$

THEOREM B.8. Given $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ there is $\sigma \in \omega^{(\omega)}$ such that $\sigma \cdot m$ is asymptotically exchangeable.

Proof. Fix $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$. Given $r \in \omega$ consider $f: \omega^{(r)} \rightarrow \mathcal{M}^1(\Lambda^r)$ sending ι to $\iota \cdot m \in \mathcal{M}^1(\Lambda^r)$. By Lemma A.1 there is an infinite set $J_r \subset \omega$ such that

$$(B.5) \quad \lim_{\substack{\min(\iota) \rightarrow \infty \\ \iota \in J_r^{(r)}}} \iota \cdot m$$

exists in $\mathcal{M}^1(\Lambda^r)$. By a diagonal argument we choose the same set $J = J_r$ for all r . Let $\sigma \in \text{Incr}(\mathbb{N})$ be such that $\sigma(\mathbb{N}) = J$. We claim that $\sigma \cdot m$ is asymptotically exchangeable. To this end consider $m_k := S^k \cdot \sigma \cdot m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$.

By compactness there is an accumulation point $m' \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ of $\{m_k\}_{k \in \mathbb{N}}$. We claim that

$$(B.6) \quad \lim_{\substack{\min(\theta) \rightarrow \infty \\ \theta \in J(\omega)}} \theta \cdot \sigma \cdot m = m',$$

hence in particular $m_k \rightarrow m'$ (taking $\theta = S^k$). Note that the claim also implies that m' is exchangeable. Indeed, given an increasing function $\gamma: \mathbb{N} \rightarrow \mathbb{N}$, to show $\gamma \cdot m' = m'$ it suffices to replace θ with $\theta \circ \gamma$ in (B.6). Since the subset of $C(\Lambda^{\mathbb{N}})$ consisting of the functions depending on finitely many coordinates is dense, it suffices to prove that for all $r \in \mathbb{N}$ and $\iota \in \mathbb{N}^{(r)}$ the limit

$$(B.7) \quad \lim_{\substack{\min(\theta) \rightarrow \infty \\ \theta \in J(\omega)}} \iota \cdot \theta \cdot \sigma \cdot m$$

exists in $\mathcal{M}^1(\Lambda^r)$ (the limit being necessarily $\iota \cdot m'$). This is however just a special case of (B.5). ■

We give below some representation results for exchangeable measures. First note that if Λ is countable, then a measure $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ is determined by the values it takes on the sets of the form $\{x : x_{i_1} = a_1, \dots, x_{i_r} = a_r\}$.

LEMMA B.9. *If Λ is countable, a measure $m \in \mathcal{M}(\Lambda^{\mathbb{N}})$ is exchangeable if and only if it admits a representation of the following form. There is a probability space (Ω, μ) (which in fact can be taken to be $(\Lambda^{\mathbb{N}}, m)$) and a family $\{\psi_a\}_{a \in \Lambda}$ in $L^\infty(\Omega, \mu)$ such that for all $i_1 < \dots < i_r$ in \mathbb{N} we have*

$$(B.8) \quad m(\{x : x_{i_1} = a_1, \dots, x_{i_r} = a_r\}) = \int_{\Omega} \psi_{a_1} \cdots \psi_{a_r} d\mu.$$

Proof. Since the right-hand side of the equation does not depend on i_1, \dots, i_r , a measure $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ admitting the above representation is clearly exchangeable. Conversely, if m is exchangeable then it suffices to take $\psi_a = \widetilde{\chi}_a$ where χ_a is the characteristic function of the set $\{x : x_0 = a\}$. We can in fact obtain the desired result by a repeated application of (B.2) after observing that the characteristic function $\chi_{\{x: x_{i_1}=a_1, \dots, x_{i_r}=a_r\}}$ is the product $\chi_{\{x_{i_1}=a_1\}} \cdots \chi_{\{x_{i_r}=a_r\}}$ and that $\chi_{\{x_i=a\}} = \chi_a \circ (S^*)^i$. ■

COROLLARY B.10. *If Λ is countable and $m \in \mathcal{M}^1(\Lambda^{\mathbb{N}})$ is exchangeable, then $m(\{x \in \Lambda^{\mathbb{N}} : x_0 = x_1\}) \neq 0$.*

Proof. By (B.8), $m(\{x \in \Lambda^{\mathbb{N}} : x_0 = x_1\}) = \sum_{a \in \Lambda} \int \psi_a^2 d\mu \neq 0$. ■

COROLLARY B.11. *If $p \in \mathbb{N}$ and $m \in \mathcal{M}^1(p^{\mathbb{N}})$ is exchangeable, then $m(\{x \in \Lambda^{\mathbb{N}} : x_0 = x_1\}) \geq 1/p$.*

Proof. Write $m(\{x \in A^{\mathbb{N}} : x_0 = x_1\}) = \sum_{a \in A} \int_{\Omega} \psi_a^2$ and apply the Cauchy–Schwarz inequality to the linear operator $\sum \int$ on $p \times \Omega$ to obtain

$$(B.9) \quad \left(\sum_{a < p} \int_{\Omega} \psi_a^2 d\mu \right) \cdot \left(\sum_{a < p} \int_{\Omega} 1 d\mu \right) \geq \left(\sum_{a < p} \int_{\Omega} \psi_a d\mu \right)^2,$$

which gives the desired result. ■

Thanks to a theorem of De Finetti, suitably extended in [HS:55], there is an integral representation *à la Choquet* for exchangeable measures on $A^{\mathbb{N}}$, where A is a compact metric space. More precisely, in [HS:55] it is shown that the extremal points of the (compact) convex set of all exchangeable measures are given by the product measures $\sigma^{\mathbb{N}}$, with $\sigma \in \mathcal{M}^1(A)$. As a consequence, Choquet’s theorem [C:69] provides an integral representation for any exchangeable measure m on $A^{\mathbb{N}}$, i.e. there is a probability measure $\mu \in \mathcal{M}^1(A)$ such that

$$(B.10) \quad m = \int_{\mathcal{M}^1(A)} \sigma^{\mathbb{N}} d\mu(\sigma).$$

When A is finite, i.e. $A = p = \{0, \dots, p-1\}$ for some $p \in \mathbb{N}$, we can identify $\mathcal{M}^1(A)$ with the simplex Σ_p of all $\lambda \in [0, 1]^p$ such that $\sum_{i=0}^{p-1} \lambda_i = 1$. Given $\lambda \in \Sigma_p$, we denote by B_{λ} the product measure on $p^{\mathbb{N}}$, that is, the unique measure making all the events $\{x : x_i = a\}$ independent with measure $B_{\lambda}(\{x : x_i = a\}) = \lambda_a$. In this case, (B.10) becomes

$$(B.11) \quad m = \int_{\Sigma_p} B_{\lambda} d\mu(\lambda),$$

where μ is a probability measure on Σ_p .

We finish this excursus on exchangeable measures with the following result:

PROPOSITION B.12. *Let $m \in \mathcal{M}^1(A^{\mathbb{N}})$ be exchangeable. Then for all $f \in L^1(A^{\mathbb{N}})$ the following conditions are equivalent:*

- (a) f is $\mathfrak{S}_c(\mathbb{N})$ -invariant;
- (b) f is $\text{Inj}(\mathbb{N})$ -invariant;
- (c) f is shift-invariant.

Proof. Since $\mathfrak{S}_c(\mathbb{N}) \subset \text{Inj}(\mathbb{N})$ and $s \in \text{Inj}(\mathbb{N})$, the implications (b) \Rightarrow (a) and (b) \Rightarrow (c) are obvious.

In order to prove that (a) \Rightarrow (b), we let $\mathcal{F} = \{\sigma \in \text{Inj}(\mathbb{N}) : f = f \circ \sigma^*\}$, which is a closed subset of $\text{Inj}(\mathbb{N})$ containing $\mathfrak{S}_c(\mathbb{N})$. Then it is enough to observe that $\mathfrak{S}_c(\mathbb{N})$ is a dense subset of $\overline{\text{Inj}(\mathbb{N})} \subset \mathbb{N}^{\mathbb{N}}$, with respect to the product topology of $\mathbb{N}^{\mathbb{N}}$, so that $\mathcal{F} = \overline{\mathfrak{S}_c(\mathbb{N})} = \text{Inj}(\mathbb{N})$.

Let us prove that (c) \Rightarrow (a). Let $\sigma \in \mathfrak{S}_c(\mathbb{N})$ and let n be such that $\sigma(i) = i$ for all $i \geq n$. It follows that $S^{*k} \circ \sigma^* = S^k$ for all $k \geq n$. As a consequence,

for m -almost every $x \in A^{\mathbb{N}}$ we have

$$f \circ \sigma^*(x) = f \circ S^{*n} \circ \sigma^*(x) = f \circ S^{*n}(x) = f(x),$$

where the first equality holds since the measure m is $\mathfrak{S}_c(\mathbb{N})$ -invariant. ■

Notice that from Proposition B.12 it follows that \tilde{f} is $\text{Inj}(\mathbb{N})$ -invariant for all $f \in L^1(A^{\mathbb{N}})$. In particular, for an exchangeable measure, the σ -algebra of shift-invariant sets coincides with the (a priori smaller) σ -algebra of $\text{Inj}(\mathbb{N})$ -invariant sets.

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