FUNDAMENTA
MATHEMATICAE
174 (2002)

Hyperbolic components of the complex exponential family

by

Robert L. Devaney (Boston, MA), **Núria Fagella** (Barcelona) and **Xavier Jarque** (Barcelona)

Abstract. We describe the structure of the hyperbolic components of the parameter plane of the complex exponential family, as started in [1]. More precisely, we label each component with a *parameter plane kneading sequence*, and we prove the existence of a hyperbolic component for any given such sequence. We also compare these sequences with the more commonly used *dynamical kneading sequences*.

1. Introduction. Our goal in this paper is to describe the structure of the hyperbolic components in the parameter plane for the complex exponential family.

Let $E_\lambda(z) = \lambda e^z$ with $\lambda \in \mathbb{C}$. The map E_λ has a unique singular value at 0 (the omitted value). As is well known, the fate of the orbit of 0 determines much of the dynamical behavior of E_λ. For example, if E_λ admits an attracting cycle, then the orbit of 0 must tend to this cycle. As a consequence, E_λ has at most one attracting cycle.

The parameter space of the exponential family was first studied in [1] and [6, 7] and later on in [3–5] and [10].

Let $\mathcal{H}_n$ denote the set of λ-values for which E_λ admits an attracting cycle of period n. The connected components of $\mathcal{H}_n$ are called *hyperbolic components* and it is conjectured that they are dense in the parameter plane. As shown in [1] and [6], any hyperbolic component is simply connected and unbounded, with the exception of $\mathcal{H}_1$ which is a cardioid-shaped region containing 0. The region $\mathcal{H}_2$ consists of a single component which occupies

2000 *Mathematics Subject Classification*: Primary 37F10, 30D20.

Research of R. L. Devaney partially supported by NSF Grant 98-18666.

Research of N. Fagella supported by DGICYT Grant No PB96-1153, DGICYT Grant No BFM2000-0805-C02-01 and CIRIT Grant No 2001SGR-70.

Research of X. Jarque supported by DGICYT Grant No PB96-1153 and CIRIT Grant No SGR99-00349.

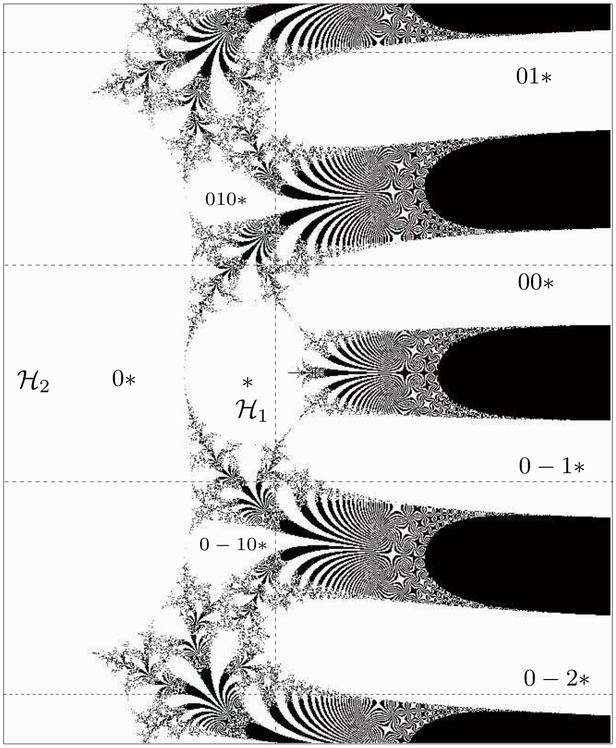

Fig. 1. The parameter plane of E_λ. White regions correspond to hyperbolic components. Black smooth regions are due to numerics. Dotted lines have been drawn on the imaginary axis and on the horizontal lines with imaginary parts $-3\pi, -\pi, \pi$ and 3π.

a large portion of the left half plane. Each $\mathcal{H}_n$ for $n > 2$ consists of infinitely many distinct components, each of which extends to ∞ in the right half plane.

The arrangement of these hyperbolic components in the λ-plane is quite complicated. A partial description can be found in [1] where the authors show the existence of infinitely many hyperbolic components of period n in between two hyperbolic components of period $n-1$. Our goal in this paper is to give a more precise description by using the dynamics of the corresponding maps. In particular we shall give a label to each of the components which will describe the dynamical behaviour of the critical orbit for those parameters in the given component. We shall see that this label also determines the position of the component in the right half plane. See Figure 1.

With this goal in mind, there is a choice to be made, for there are two ways to identify the various hyperbolic components in the λ-plane. Each of these involves the association of a *kneading sequence* to the component. This sequence is a string of $n-2$ integers. For technical reasons we precede the string with a 0 and end the string with a $*$. That is, a kneading sequence assumes the form $0s_1 \dots s_{n-2}*$ with $s_j \in \mathbb{Z}$. The $*$ denotes a "wild card" that will be described below.

One of the two kneading sequences is a *dynamical kneading sequence* (K-kneading sequence) which is useful mainly in the dynamical plane (see [2]), since it determines the topological structure of the Julia set of E_λ for any λ in the hyperbolic component. The other kneading sequence is a *parameter plane kneading sequence* (S-kneading sequence) and, as we shall see, is more useful for describing the structure of the λ-plane. The main result in this paper is as follows.

THEOREM A. *Fix $n \geq 3$ and let $s_1, \dots, s_{n-2} \in \mathbb{Z}$. There exists a hyperbolic component $\Omega_{0s_1\dots s_{n-2}*}$ that extends to ∞ in the right half plane and such that if $\lambda \in \Omega_{0s_1\dots s_{n-2}*}$, the map E_λ has an attracting cycle of period n with parameter plane kneading sequence $s = 0s_1 \dots s_{n-2}*$. Moreover, the components $\Omega_{0s_1\dots s_{n-2}*}$ are ordered lexicographically.*

From the proof of this theorem one obtains the following corollary (see Figure 2).

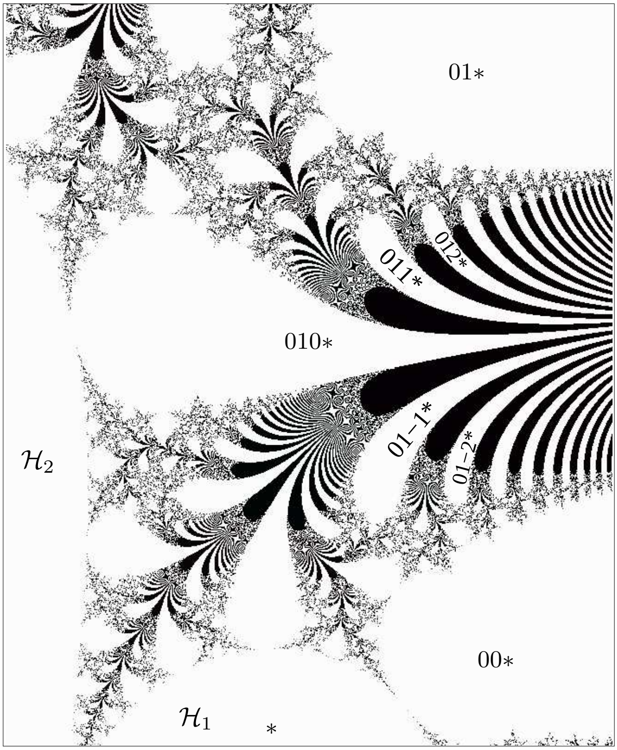

Fig. 2. Magnification of Figure 1 showing infinitely many period 4 components in between two period 3 components

Corollary B. *Let $\Omega_{0s_1\ldots s_{n-2}*}$ be as in Theorem* A. *Then between this hyperbolic component and the hyperbolic component $\Omega_{0s_1\ldots(s_{n-2}+1)*}$ there exist hyperbolic components $\Omega_{0s_1\ldots(s_{n-2}+1)k*}$ for each $k \in \mathbb{Z}$.*

In this statement the word "between" refers to the ordering given by the imaginary part, since all hyperbolic components extend to infinity in the right half plane.

These results give a description of the ordering of the hyperbolic components in the far right half plane as a function of their kneading sequence. Note that these are existence type results. Although uniqueness is most likely true, this fact does not follow directly from our work in this paper. D. Schleicher [10] has announced some results in this direction using the coding of hairs in parameter space.

In Section 2 below we define each of these kneading sequences and discuss several of their properties. We also derive an algorithm for obtaining one sequence given the other. In Section 3 we prove Theorem A; that is, we show the existence of hyperbolic components corresponding to any S-kneading sequence.

2. Kneading sequences. Let us consider a hyperbolic component Ω of period $n > 2$. The main goal of this section is to define two different kneading sequences associated to the parameter value $\lambda \in \Omega$. We shall also study the relation between the two sequences and give an algorithm that transforms one into the other.

We start by giving a topological description of the dynamical plane of $E_\lambda(z) = \lambda e^z$ that holds for any parameter λ in the hyperbolic component Ω.

2.1. *The fingers and the glove..* If $\lambda \in \Omega$, the map $E_\lambda(z) = \lambda \exp(z)$ has an attracting periodic orbit of period $n > 2$. This orbit varies analytically with λ as long as λ lies in the hyperbolic component. Let $z_0(\lambda), z_1(\lambda) = E_\lambda(z_0), \ldots, z_{n-1}(\lambda) = E_\lambda(z_{n-2})$ be the points of the periodic orbit. To simplify notation we will omit the dependence on λ if it does not lead to confusion.

Let A^* denote the immediate basin of attraction of the periodic orbit and, for $0 \le i \le n-1$, define $A^*(z_i)$ to be the connected component of A^* which contains z_i. We name the points in the orbit so that the asymptotic value 0 belongs to $A^*(z_0)$.

We now construct geometrically and define what we call *fingers*. More details can be found in [2]. For $\nu \in \mathbb{R}$, let $H_\nu = \{z \mid \operatorname{Re} z > \nu\}$.

Definition. An unbounded simply connected $F \in \mathbb{C}$ is called a *finger* of *width* c if

(a) F is bounded by a single simple curve $\gamma \subset \mathbb{C}$.

(b) There exists ν such that $F \cap H_\nu$ is simply connected, extends to infinity, and satisfies

$$F \cap H_\nu \subset \{z \mid \operatorname{Im} z \in [a - d/2, a + d/2]\} \quad \text{for some } a \in \mathbb{R},$$

and c is the infimum value for d.

Observe that the preimage of any finger which does not contain 0 consists of infinitely many fingers of width smaller than 2π which are $2\pi i$-translates of each other.

We begin the construction by choosing $B = B(\lambda)$ to be a topological disc in $A^*(z_0)$ that contains both 0 and z_0, and having the property that B is mapped strictly inside itself under E_λ^n. This set can be defined precisely using linearizing coordinates, and one can show that it moves holomorphically with λ. Although this is not crucial for this work, we have included the details in the Appendix.

We now take successive preimages of the disc B. More precisely, let B_{n-1} be the open set in $\mathbb{C}$ which is mapped to B. Note that, since $0 \in B$, it follows that B_{n-1} has a single connected component which contains a left half plane, and whose image under E_λ wraps infinitely many times over $B \setminus \{0\}$. Note that the point z_{n-1} belongs to the set B_{n-1}, which lies inside $A^*(z_{n-1})$.

We now consider the preimage of B_{n-1}. It is easy to check (by looking at the image of vertical lines with increasing real part) that this preimage consists of infinitely many disjoint fingers of width less than 2π which are $2\pi i$-translates of each other. We define $B_{n-2} \subset A^*(z_{n-2})$ to be the connected component for which $z_{n-2} \in B_{n-2}$. The map E_λ takes B_{n-2} conformally onto B_{n-1}.

Similarly, we define the sets $B_{n-3}, \ldots, B_0$, by setting B_i to be the connected component of $E_\lambda^{-1}(B_{i+1})$ that contains the point z_i. These inverses are all well defined and the map E_λ sends B_i conformally onto B_{i+1}. Each B_i belongs to the immediate basin $A^*(z_i)$. The following characterization of the sets B_i, $i = 0, \ldots, n-2$, is proved in [2].

PROPOSITION 2.1. *Let $n > 2$. For $i = 0, \ldots, n-2$, B_i is a finger of width $c_i < 2\pi$.*

It follows immediately from the above construction that the width of the finger B_{n-2} that is mapped by E_λ conformally onto B_{n-1} is π, while the widths of the other fingers are 0. So we will refer to B_{n-2} as the *big finger*.

We proceed to the final step, by defining the set

$$G = \{z \in \mathbb{C} \mid E_\lambda(z) \in B_0\},$$

which we call the *glove*. We observe from the above construction that G is a connected set and $B_{n-1} \subset G \subset A^*(z_{n-1})$. See Figure 3. Moreover, the complement of G consists of infinitely many fingers, all of which are

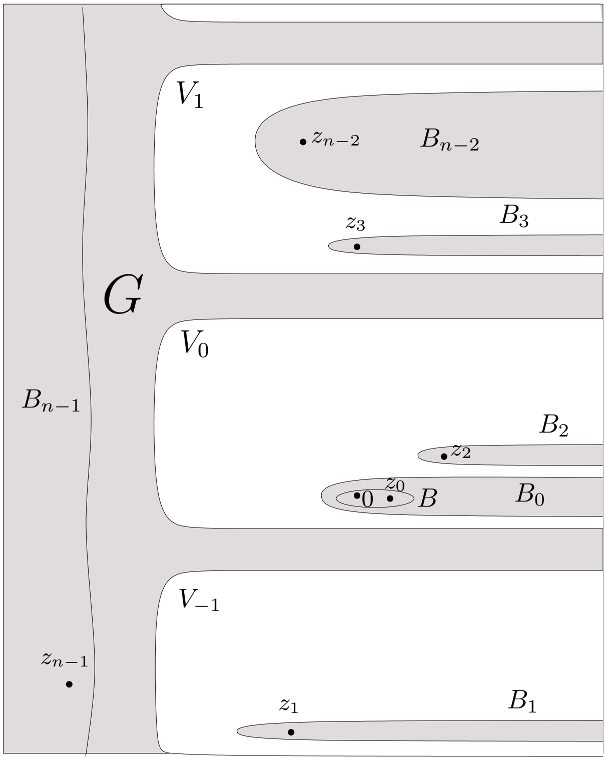

Fig. 3. Sketch of the sets B_0 to B_{n-1}, G and V_j for $j \in \mathbb{Z}$. Points in grey belong to the basin of attraction of the periodic orbit.

$2\pi i$-translates of each other. We index these infinitely many connected components by V_j, $j \in \mathbb{Z}$, so that $2\pi i j \in V_j$.

In fact, these V_j form a set of fundamental domains for the Julia set of E_λ in the following sense:

- $J(E_\lambda) \subset \bigcup_{j \in \mathbb{Z}} V_j$.
- E_λ maps each V_j conformally onto $\mathbb{C} \setminus B_0$, and so $E_\lambda(V_j) \supset J(E_\lambda)$.

Hence, for each $j \in \mathbb{Z}$ we have a well defined inverse branch of E_λ:

$$L_j = L_{\lambda,j} : \mathbb{C} \setminus B_0 \to V_j.$$

Note that B_0 lies inside V_0 since $0 \in B_0$. The other fingers $B_1, \ldots, B_{n-2}$ may lie inside any of the fundamental domains V_j, depending on the value of λ. In particular, several B_i may lie in the same V_j.

2.2. *K-kneading sequences and S-kneading sequences..* We first introduce the kneading sequence given by the fundamental domains V_j. We define the *K-kneading sequence* of $\lambda \in \Omega$ to be

$$K(\lambda) = 0\, k_1\, k_2\, k_3\, \ldots\, k_{n-2} *$$

where $B_j \subset V_{k_j}$ for all $1 \leq j \leq n-2$. We use $*$ for the position of the point z_{n-1}, since this point does not belong to any of the V_j. We claim that this kneading sequence is constant throughout the entire hyperbolic component Ω. To see this we first notice that the function $K : \Omega \to \Sigma$, where Σ denotes the set of all sequences with integer terms, is locally constant. Hence, since Ω is connected, K must be constant through the entire hyperbolic component. An alternative proof of this fact can be deduced from the Appendix.

We define the *K-itinerary* of any point $z \in J(E_\lambda)$ to be

$$K(z) = k_0\, k_1\, k_2\, k_3\, \ldots$$

where $E_\lambda^j(z) \in V_{k_j}$ for any $j \geq 0$.

One can then use these itineraries together with the kneading sequence to give a complete description of the structure of the Julia set for E_λ in terms of symbolic dynamics. See [2].

We now define the *S-kneading sequence* of a value $\lambda \in \Omega$. This sequence has been independently introduced in [10], in the same context as ours. If we look at the dynamical plane very far to the right, we see that any finger is basically a straight horizontal band; therefore it makes sense to define the order of fingers in terms of their imaginary part. In this fashion, we can speak about fingers sitting *above* or *below* each other. Likewise, we can talk about the *upper boundary* and the *lower boundary* of a finger, as long as we look in the far right half plane.

Consider the half plane $H_\mu = \{z \in \mathbb{C} \mid \operatorname{Re} z > \mu\}$ for a fixed μ large enough. Define the family of fingers F_j, $j \in \mathbb{Z}$, to be the infinitely many connected components of the preimage of B_{n-1}. We observe that the fingers F_j are the $2k\pi i$-translates of the big finger for any $k \in \mathbb{Z}$. We index these sets consecutively so that F_0 is the one immediately above B_0. For any $j \in \mathbb{Z}$, let T_j be the region in H_μ that lies between the upper boundaries of F_{j-1} and F_j (so, we have $F_j \cap H_\mu \subset T_j$). See Figure 4.

Finally, we define the *S-kneading sequence* of $\lambda \in \Omega$ to be

$$S(\lambda) = 0\, s_1\, s_2\, s_3\, \ldots\, s_{n-2}\, *$$

where $B_j \cap H_\mu \subset T_{s_j}$ for all $1 \leq j \leq n-2$. Equivalently, B_j tends to infinity between F_{s_j} and $F_{s_j - 1}$. It is clear that this definition does not depend on the choice of μ. See Figure 4. Moreover, since $S(\lambda)$ is again the same for all $\lambda \in \Omega$, we will use the notation $\Omega_{0s_1\ldots s_{n-2}*}$ to label the hyperbolic component with such dynamical behavior with respect to the T_j's.

We observe that the regions T_i do not define a family of fundamental domains in the sense described above. Consequently, the S-itinerary (defined in the obvious way) is not well defined for all points in the Julia set, but only for those whose orbits have sufficiently large real part. However, E_λ preserves the orientation in each T_i, a feature that will prove to be useful

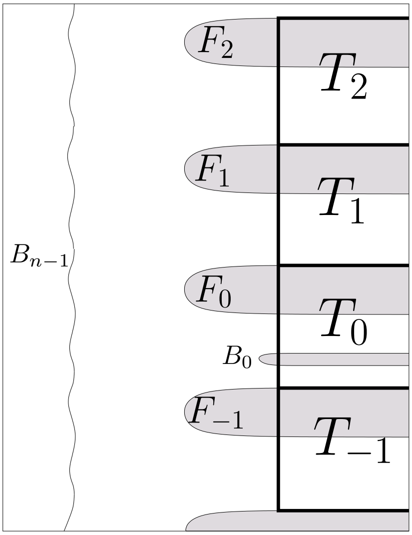

Fig. 4. The families F_i and T_i

later on. The S-kneading sequences and itineraries are not suitable for use in the dynamical plane, but we shall see that they are very convenient when the parameter plane is considered. Therefore, it is of interest to be able to use both of these kneading sequences.

2.3. *Translation algorithm..* In this section we describe an algorithm that relates the K- and S-kneading sequences. Let us denote the S-kneading sequence of E_λ by

$$S = 0s_1 \dots s_{n-2}*.$$

We will show how to compute the K-kneading sequence

$$K = 0k_1 \dots k_{n-2}*$$

associated to λ.

The algorithm consists of two steps. The first step is to attach a sign ($+$ or $-$) to each of the zero entries of S (with the exception of the first entry of the sequence that will remain as 0). This sign indicates that the corresponding B_i is above (0^+) or below (0^-) B_0, at least far to the right.

The second step will determine each of the k_i based on s_i and s_{i+1}, except for the last entry k_{n-2} which will be determined by s_{n-2} and s_1.

Step 1: *Deciding on 0^+ or 0^-.* Let $s_i = 0$. Then $B_i \subset T_0$ and B_i lies either above or below B_0 in the far right half plane. We will attach the superscript $+$ or $-$ to 0 depending on whether B_i is above (0^+) or below (0^-) B_0.

To determine the sign, consider the words $s_1 s_2 \dots$ and $s_{i+1} s_{i+2} \dots$ Compare these two words until the minimal $j \geq 1$ is found such that $s_j \neq s_{i+j}$.

Then set

$$s_i = \begin{cases} 0^+ & \text{if } s_j < s_{i+j}, \\ 0^- & \text{if } s_j > s_{i+j}. \end{cases}$$

We write $* = \infty$ for ordering purposes in the above criterion.

We now show that this rule gives the correct superscript. Since $s_i = 0$, B_i meets T_0 as does B_0. If $s_1 > s_{i+1}$ (resp. $s_1 < s_{i+1}$) then B_{i+1} is below (resp. above) B_1. Since the order is preserved inside each of the T_k's we deduce that B_i is below (resp. above) B_0. Hence $s_i = 0^-$ (resp. 0^+). Observe that having defined $* = \infty$ takes care of the case $s_{i+1} = *$, i.e., the case of the big finger.

Now we use induction. Let us assume $s_j = s_{i+j}$ for $j = 1, \ldots, k$ but $s_{k+1} \neq s_{i+k+1}$. Then B_j and B_{i+j} are contained in T_{s_j}, $j = 1, \ldots, k$, and hence, their relative order can be decided by looking at their respective images B_{k+1} and B_{i+k+1}. There are two cases.

If $s_{k+1} > s_{i+k+1}$ then B_{i+k+1} is below B_{k+1}, and consequently, B_{i+j} is below B_i for all $j = 1, \ldots, k$. Therefore B_i is below B_0 and so $s_i = 0^-$. If $s_{k+1} < s_{i+k+1}$ we substitute "above" for "below" in the previous paragraph and conclude that $s_i = 0^+$.

In particular, we remark that there are two cases that never occur: (a) $s_i = 0^+$ and $s_{i+1} \leq 0^-$ in the case $s_1 \geq 0^+$, and (b) $s_i = 0^-$ and $s_{i+1} \geq 0^+$ in the case $s_1 \leq 0^-$. More generally, by arguments similar to the above, any 0^+ (respectively, 0^-) must be followed by entries larger than or equal to (respectively, less than or equal to) s_1.

STEP 2: *Obtaining k_i.* Let S be a signed S-kneading sequence obtained by replacing each 0 with the corresponding 0^+ and 0^- symbols. There are two completely symmetric cases: $s_1 \geq 0^+$ and $s_1 \leq 0^-$. We adopt the conventions that $1-1 = 0^+$ and $-1+1 = 0^-$. Now, for any i with $1 \leq i \leq n-2$,

(a) If $s_1 \geq 0^+$ then $\quad k_i = \begin{cases} s_i & \text{if } i = n-2 \text{ or } s_{i+1} \geq 0^+, \\ s_i - 1 & \text{if } s_{i+1} \leq 0^-. \end{cases}$

(b) If $s_1 \leq 0^-$ then $\quad k_i = \begin{cases} s_i + 1 & \text{if } i = n-2 \text{ or } s_{i+1} \geq 0^+, \\ s_i & \text{if } s_{i+1} \leq 0^-. \end{cases}$

We now prove that for a given $\lambda \in \Omega$ the above rule translates any signed S to a unique K. We consider the case $s_1 \geq 0^+$, the other case being symmetric.

We denote by g_i the piece of the glove G that falls into the region T_i. Since $s_1 \geq 0^+$, B_1 is above B_0 and hence the piece of the glove g_0 must be below B_0.

This implies that V_0 is the fundamental domain between the pieces g_0 and g_1 and, in general, each V_i lies between g_i and g_{i+1}, in particular including F_i. This last remark implies that the last digit of the sequence will not change. That is, $k_{n-2} = s_{n-2}$.

Consider s_i for $1 \leq i < n-2$. Hence B_i lies in T_{s_i}. By the observations above, either (see Figure 5)

1. B_i lies in V_{s_i} because the piece g_{s_i} lies below B_i (case $k_i = s_i$), or
2. B_i lies in V_{s_i-1} because the piece g_{s_i} lies above B_i (case $k_i = s_i - 1$).

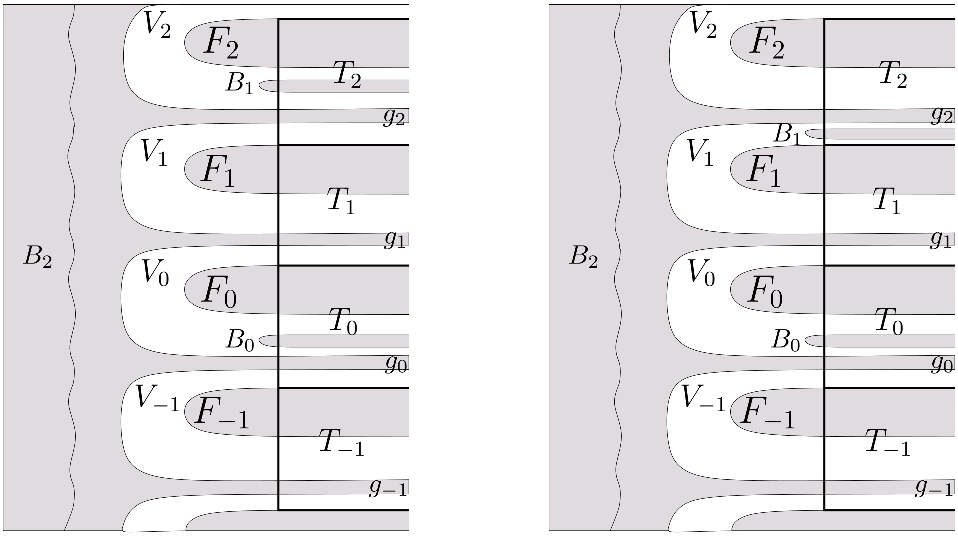

Fig. 5. Example of the two possibilities: the S-kneading sequence $02*$ translating either into $02*$ or into $01*$

It is straightforward to check that the first case occurs if and only if B_{i+1} is above B_0, i.e., $s_{i+1} \geq 0^+$. The second case occurs if and only if B_{i+1} is below B_0, i.e., $s_{i+1} \leq 0^-$.

As an example, consider the S-kneading sequence

$$S = 0\ {-2}\ 0\ 0\ {-1}\ 2\ 3\ 0\ 0\ {-1}\ 2\ 0\ 0\ \ *\,.$$

After the first step we have

$$S = 0\ {-2}\ 0^+\ 0^+\ {-1}\ 2\ 3\ 0^+\ 0^+\ {-1}\ 2\ 0^+\ 0^+\ *,$$

and after the second step the corresponding K-kneading sequence is

$$K = 0\ {-1}\ 1\ 0^+\ 0^-\ 3\ 4\ 1\ 0^+\ 0^-\ 3\ 1\ 1\ \ *\,.$$

We finally observe that the above 2-step algorithm can also be used in the reverse direction, that is, for a given K with the symbols 0^+ and 0^- we obtain, via the inverse algorithm, a unique S. The next section will refer to this point when taking into account the admissibility of the given sequence.

2.4. *Properties..* Why are we working with two distinct kneading sequences? The answer to this question is based on the fact that the two sequences have different properties and consequently each is suitable in different circumstances.

More precisely, the K-kneading sequences work well when studying the dynamical plane since they are defined using fundamental domains. These domains work for all points of the Julia set and give rise to good symbolic dynamics and consequently to a complete description of the Julia set (see [2]). In contrast, when working in the parameter plane, one can find many different hyperbolic components sharing the same K-kneading sequence. For instance, for any $n \in \mathbb{N}$, all hyperbolic components of period n bifurcating from the main cardioid have their K-kneading sequence given by $K = 0000\ldots0$. To fix this uniqueness problem we might consider the symbols 0^+ and 0^- as before. But then an admissibility problem arises, without an obvious way to decide if a sequence is admissible or not (except, of course, by going through the inverse algorithm to check if the resulting sequence is possible).

The S-kneading sequences do not involve fundamental domains and hence they are not as useful as the K-kneading sequences when working in the dynamical plane. However, we prove in the next section that all sequences are admissible; that is, we can find a hyperbolic component Ω corresponding to any given sequence of integers. Moreover, these sequences give a significant amount of information about the location of the periodic orbit.

We remark that the uniqueness of hyperbolic components having a given S-kneading sequence would seem to be a natural result but it is not a straightforward deduction from the construction below.

3. Hyperbolic components. Proof of the main result. Our goal in this section is to construct a parameter value λ for which E_λ has an attracting cycle with any given S-kneading sequence. We first consider the special case where the S-kneading sequence consists of a single digit; the proof in this case makes use of many of the ideas of the general case, but in a simpler setting.

3.1. *The case* $0k*$.. This result follows from the next two propositions. The first proposition can be found in [1], but we give a proof adapted to the general case.

PROPOSITION 3.1. *Fix $k \in \mathbb{Z}$. For $a \in \mathbb{R}$, let $\lambda_a = a + (2k+1)\pi i$. Then, for sufficiently large values of a, the map E_{λ_a} has an attracting cycle of period* 3.

Proof. We assume throughout that $a \geq |2k+1|\pi$, so that $|\mathrm{Arg}(\lambda_a)| \leq \pi/4$, where Arg denotes the principal branch of the argument. Then $\lambda_a = E_{\lambda_a}(0)$ lies in the right half plane, but $E^2_{\lambda_a}(0) = \lambda_a \exp(\lambda_a)$ lies in the left half plane since $E^2_{\lambda_a}(0) = -e^a\lambda_a$. Choosing a large enough, we may assume that $a < |\lambda_a| \leq a+1$. Since

$$3\pi/4 \leq |\operatorname{Arg} E^2_{\lambda_a}(0)| \leq \pi$$

it follows that

$$\operatorname{Re} E^2_{\lambda_a}(0) = |\lambda_a| e^a \cos(\operatorname{Arg} E^2_{\lambda_a}(0)) \leq -\frac{|\lambda_a|}{\sqrt{2}}\, e^a < -ae^a/\sqrt{2}.$$

Let U_2 be the ball of radius 1 about $E^2_{\lambda_a}(0)$. The preimage of U_2 containing λ_a is an open set U_1 which is mapped univalently onto U_2 by E_{λ_a}, and the preimage of U_1 containing 0 is another open set, say U_0, which is mapped univalently onto U_2 by $E^2_{\lambda_a}$. We claim that there is an attracting cycle of period 3 whose orbit under E_{λ_a} lies in U_0, U_1, and U_2. Let F denote the appropriate branch of the inverse of $E^2_{\lambda_a}$ that takes U_2 univalently into U_0. See Figure 6. Define the absolute constant $C_a = e^{-a}/(4(a+1)^2)$. By the Koebe 1/4 Theorem, we have

$$\operatorname{dist}(0, \partial U_0) \geq \frac{1}{4}|F'(E^2_{\lambda_a}(0))| = \frac{1}{4} \cdot \left|\frac{1}{\lambda_a}\right| \cdot \left|\frac{1}{\lambda_a e^{\lambda_a}}\right| = \frac{e^{-a}}{4|\lambda_a|^2} \geq C_a.$$

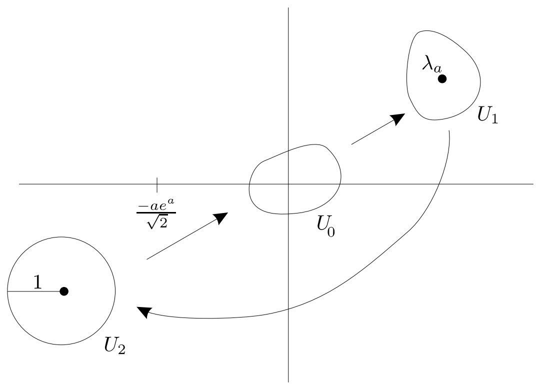

Fig. 6. The sets U_0, U_1 and U_2 in the proof of Proposition 3.1

Now

$$|E^3_{\lambda_a}(0)| = |\lambda_a| \exp(\operatorname{Re} E^2_{\lambda_a}(0)) \leq (a+1)\exp(-ae^a/\sqrt{2}) \ll C_a$$

for large a. Hence $E^3_{\lambda_a}(0)$ is contained in U_0. Moreover, if $w \in U_2$, then

$$\begin{aligned}|E_{\lambda_a}(w) - E^3_{\lambda_a}(0)| &\leq \max_{z \in U_2} |E'_{\lambda_a}(z)| \leq |\lambda_a \exp(\operatorname{Re} E^2_{\lambda_a}(0) + 1)| \\ &\leq (a+1)e\exp(-ae^a/\sqrt{2}) \ll C_a\end{aligned}$$

as before. Hence,

$$\operatorname{dist}(0, \partial E^3_{\lambda_a}(U_0)) \leq (a+1)(e+1)\exp(-ae^a/\sqrt{2}) \ll C_a,$$

and it follows that $E^3_{\lambda_a}(U_0)$ is properly contained in U_0. Thus we have an attracting cycle whose orbit visits U_0, U_1 and U_2. ∎

Before proceeding, we observe that the above estimates guarantee that the entire half plane $\operatorname{Re} z \leq \operatorname{Re} E^2_{\lambda_a}(0) + 1$ is contained in the basin of the cycle.

We now claim that the S-kneading sequence of λ_a is $0k*$.

PROPOSITION 3.2. *Let $k \in \mathbb{Z}$ and set $\lambda_a = a + (2k+1)\pi i$. Then for sufficiently large a, E_{λ_a} has an attracting 3-cycle with $S(\lambda_a) = 0k*$.*

Proof. Let $\gamma(t) = t + (2k+1)\pi i$ for $t \geq a$. Then $E_{\lambda_a}(\gamma(t))$ is a straight line which lies to the left of $E^2_{\lambda_a}(0)$. By the above observation, $E_{\lambda_a}(\gamma(t))$ lies in the connected component of the immediate basin of attraction which contains $E^2_{\lambda_a}(0)$. Hence $\gamma(t)$ lies in the component of the immediate basin which contains λ_a.

Let S be the strip $\{z \mid |\operatorname{Im} z| \leq \pi\}$. There is a preimage of $\gamma(t)$ contained in the interior of S, at least for t large. We claim that the entire preimage of $\gamma(t)$ lies in S. The preimage of $\gamma(t)$ can never meet the boundary of S, for E_{λ_a} maps the boundary of S into the left half plane, far from $\gamma(t)$. Hence the preimage of $\gamma(t)$ lying in S must be the preimage that contains 0.

We then consider the set B as above so that B contains $E^3_{\lambda_a}(0)$. It then follows that B_2 contains $E^2_{\lambda_a}(0)$ and $E_{\lambda_a}(\gamma(t))$. After taking one more preimage, the big finger B_1 contains λ_a and $\gamma(t)$, and its translations contain the half lines $\{t + (2j+1)\pi \mid t \geq a\}$. Moreover, the finger B_0 contains 0 and the preimage of $\gamma(t)$ in S. It follows then that the fingers are indexed so that $B_1 = F_k$ and hence $S(\lambda_a) = 0k*$. ■

3.2. *The general case..* Now we proceed to the general case. For the remainder of this section we fix a kneading sequence $s = 0s_1 \ldots s_{n-2}*$. Let $\widehat{s} = \max |s_i|$ and define $M = (2\widehat{s}+1)\pi$. We assume throughout that $a > M$. Let $H(a)$ denote the closed half strip

$$H(a) = \{z \mid \operatorname{Re} z \geq a,\ |\operatorname{Im}(z)| \leq M\}.$$

We let $L(a)$ denote the left boundary of $H(a)$. We will prove:

THEOREM 3.3. *For each sufficiently large a, there is $\lambda_a \in L(a)$ for which E_{λ_a} has an attracting n-cycle with $S(\lambda_a) = s$.*

If we denote the first n points on the orbit of 0 by w_i, so $w_0 = 0$, $w_1 = \lambda_a$, $\ldots$, $w_n = E^n_{\lambda_a}(0)$, as in the previous special case, we will construct λ_a so that the orbit of 0 under E_{λ_a} has the following properties:

1. $w_i \in H(a)$ for $i = 1, \ldots, n-2$ and $\operatorname{Re} w_{i+1} \gg \operatorname{Re} w_i$ for $i = 0, \ldots, n-3$.
2. w_{n-1} lies in the left half plane and

$$|\operatorname{Re} w_{n-1}| \gg \operatorname{Re} w_{n-2}.$$

3. w_n lies close to 0 and, as in the period 3 case, there is an attracting cycle of period n lying close to $w_0, \ldots, w_{n-1}$.

We will divide the proof into three parts, namely Propositions 3.5–3.7 (Proposition 3.4 is auxiliary). Afterwards we will see how Theorem A (see Section 1) follows.

Let $\nu = \nu(a) = |a+(2\widehat{s}+1)\pi i| = \max_{z\in L(a)} |z|$, and note that $\nu(a)-a \to 0$ as $a \to \infty$.

For $-\widehat{s} \le i \le \widehat{s}$, let $H_i(a)$ be the substrip of $H(a)$ given by

$$H_i(a) = \{z \in H(a) \mid \operatorname{Re} z \ge a,\ (2i-1)\pi \le \operatorname{Im} z \le (2i+1)\pi\}.$$

See Figure 7.

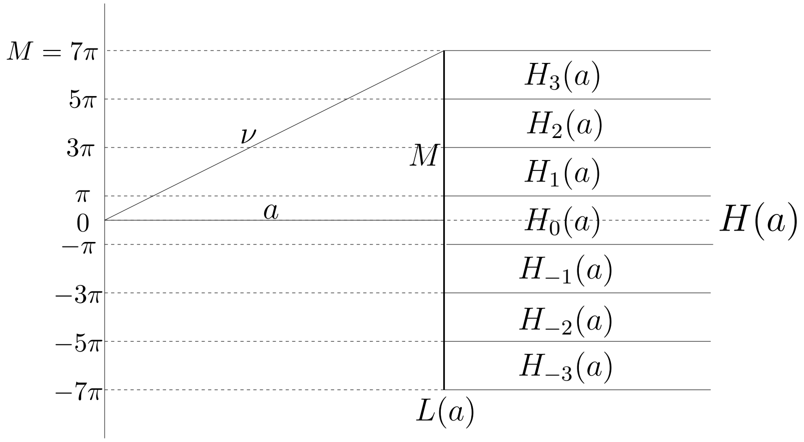

Fig. 7. The sets $H(a)$, $L(a)$ and the substrips $H_i(a)$ for the case $\widehat{s} = 3$

For $j = 1, \ldots, n-2$, define $w_j(\lambda) = E_\lambda^j(0)$. Note that each w_j is a function of the parameter λ and is analytic. For example, $w_1(\lambda) = \lambda$ and $w_2(\lambda) = \lambda e^\lambda$.

For $j = 1, \ldots, n-2$, define

$$I_{s_1\ldots s_j}(a) = \{\lambda \in L(a) \,|\, w_i(\lambda) \in H_{s_i}(a) \text{ for } i = 1, \ldots, j\}.$$

Note that $I_{s_1}(a) = L(a) \cap H_{s_1}(a)$ and that the $I_{s_1\ldots s_j}$ are nested, provided they are nonempty. The following proposition shows that each of the $I_{s_1\ldots s_j}$ consists of a single vertical segment.

We say that a smooth curve $\mu(t)$ in $H_{s_i}(a)$ is a *vertical curve* if the curve connects the upper and lower boundaries of $H_{s_i}(a)$.

PROPOSITION 3.4. *For each sufficiently large a and $1 \le j \le n-2$, the set $\{w_j(\lambda) \mid \lambda \in I_{s_1\ldots s_j}(a)\}$ consists of a single vertical curve in $H_{s_j}(a)$. Hence $I_{s_1\ldots s_j}$ is a single vertical segment.*

Proof. If $j = 1$, there is nothing to prove since $w_1(\lambda) = \lambda$ and $I_{s_1}(a) = L(a) \cap H_{s_1}(a)$. Let $j > 1$. We parametrize the segment $I_{s_1}(a)$ as $\lambda(t) =$

$a + (2s_1\pi + t)i$ for $t \in (-\pi, \pi)$ and consider the set

$$J_{s_1s_2}(a) = \{\lambda \in I_{s_1}(a) \mid w_2(t) \subset H(a)\},$$

where $w_2(t) = w_2(\lambda(t)) = \lambda(t)e^{\lambda(t)}$. We will show that, given any $\varepsilon > 0$ and taking a large enough,

$$|\operatorname{Arg} w_2'(t) - \pi/2| < \varepsilon \tag{1}$$

for any t such that $\lambda(t) \in J_{s_1s_2}(a)$. This implies that, when t runs from $-\pi$ to π, as the curve $w_2(t)$ crosses the strip $H(a)$, its tangent vector points upwards and it is almost vertical. It follows that the imaginary part of $w_2(t)$ is an increasing function of t, and hence the curve crosses the strip only once. We now proceed to show (1).

Set a_0 large enough so that

$$|\operatorname{Arg} \lambda(t)| < \frac{\varepsilon}{3(n-3)} = \varepsilon'$$

for all $t \in (-\pi, \pi)$.

The tangent vector to $w_2(t)$ is

$$w_2'(t) = \lambda'(t)e^{\lambda(t)}(1 + \lambda(t)) = ie^{\lambda(t)}(1 + \lambda(t))$$

and thus

$$|\operatorname{Arg} w_2'(t) - \pi/2| = |\operatorname{Arg} e^{\lambda(t)} + \operatorname{Arg}(1 + \lambda(t))| \le |\operatorname{Arg} e^{\lambda(t)}| + \varepsilon'.$$

If $\lambda(t) \in J_{s_1s_2}$, it is clear that $|w_2(t)| > |\lambda(t)|$. Since both are inside the strip $H(a)$, we have $\varepsilon' > |\operatorname{Arg} w_2(t)| = |\operatorname{Arg} \lambda(t) + \operatorname{Arg} e^{\lambda(t)}|$. It is then easy to see that $|\operatorname{Arg} e^{\lambda(t)}| < 2\varepsilon'$. Plugging this in the expression above, we obtain

$$|\operatorname{Arg} w_2'(t) - \pi/2| < 3\varepsilon' = \frac{\varepsilon}{n-3} < \varepsilon$$

as required.

We now proceed to investigate $I_{s_1s_2s_3}$ which will illustrate the general case. As above, consider

$$J_{s_1s_2s_3}(a) = \{\lambda \in I_{s_1s_2}(a) \mid w_3(t) \subset H(a)\},$$

where $w_3(t) = w_3(\lambda(t)) = \lambda(t)e^{w_2(t)}$ and $w_2(t) \in H_{s_2}(a)$. For these values of t, we will show that

$$|\operatorname{Arg} w_3'(t) - \pi/2| < \varepsilon. \tag{2}$$

Now the tangent vector to $w_3(t)$ is

$$w_3'(t) = e^{w_2(t)}(\lambda'(t) + \lambda(t)w_2'(t)) = e^{w_2(t)}(i + \lambda(t)w_2'(t))$$

and thus

$$\operatorname{Arg} w_3'(t) = \operatorname{Arg} e^{w_2(t)} + \operatorname{Arg}(i + \lambda(t)w_2'(t)).$$

We claim that

$$\pi/2 - 4\varepsilon' < \operatorname{Arg}(i + \lambda(t)w_2'(t)) < \pi/2 + 4\varepsilon'.$$

Indeed, we showed above that

$$\pi/2 - 3\varepsilon' < \operatorname{Arg} w_2'(t) < \pi/2 + 3\varepsilon'.$$

Moreover, since $|\operatorname{Arg} \lambda(t)| < \varepsilon'$, we obtain

$$\pi/2 - 4\varepsilon' < \operatorname{Arg}(\lambda(t) w_2'(t)) < \pi/2 + 4\varepsilon'.$$

Finally, it remains to add the vector i to this expression, which makes the argument even closer to $\pi/2$.

To finish the proof of (2) observe that, by the same argument as in the first case, $\operatorname{Arg} w_3(t) = \operatorname{Arg}(\lambda(t) e^{w_2(t)}) < \varepsilon'$ and hence $|\operatorname{Arg} e^{w_2(t)}| < 2\varepsilon'$. Putting all this together we have

$$\pi/2 - 6\varepsilon' < \operatorname{Arg} w_3'(t) < \pi/2 + 6\varepsilon'$$

as we wanted to prove.

It is easy to check that we may iterate this procedure to deduce that, for $j = 2, \ldots, n-2$ and for all t such that $\lambda(t) \in J_{s_1 \ldots s_j}(a)$,

$$|\operatorname{Arg} w_j'(t) - \pi/2| < 3(j-1)\varepsilon' = (j-1)\,\frac{\varepsilon}{n-3} \le \varepsilon,$$

which concludes the proof of the proposition. ∎

PROPOSITION 3.5. *Let $\varepsilon > 0$. For each sufficiently large a, there is $\lambda_a \in L(a)$ satisfying*

1. *$w_i(\lambda_a) \in H_{s_i}(a)$ for $i = 1, \ldots, n-2$.*
2. *$\operatorname{Im} w_{n-2}(\lambda_a) = (2s_{n-2} + 1)\pi$.*
3. *$E_{a-\varepsilon}^{j-1}(a-\varepsilon) \le \operatorname{Re} w_j(\lambda_a) \le |w_j(\lambda_a)| \le E_{a+\varepsilon}^{j-1}(a+\varepsilon)$ for $j = 1, \ldots, n-2$, where E_b is the real exponential $E_b(x) = be^x$.*

Proof. By the previous proposition, if $\lambda \in I_{s_1 \ldots s_j}(a)$, then the curve $\lambda \mapsto w_j(\lambda)$ is a vertical curve in $H_{s_j}(a)$. We will show that, moreover,

$$E_{a-\varepsilon}^{j-1}(a-\varepsilon) \le \operatorname{Re} w_j(\lambda) \le E_{a+\varepsilon}^{j-1}(a+\varepsilon)$$

for each j. Then λ_a will be defined as the upper endpoint of $I_{s_1 \ldots s_{n-2}}(a)$.

If $\lambda \in V_{s_1}(a)$, then $\exp(\lambda)$ lies on a circle of radius e^a centered at 0. Hence $\lambda \mapsto w_2(\lambda) = \lambda e^\lambda$ is a nearly circular arc contained in the annulus

$$E_a(a) \le |z| \le E_\nu(\nu) \tag{3}$$

where we recall that $\nu = \max_{z \in L(a)} |z|$. This arc crosses $H_{s_2}(a)$ in a single vertical curve η_2, provided a is sufficiently large.

Given $\varepsilon > 0$, we claim we may choose a large enough so that if $\lambda \in I_{s_1 s_2}(a)$ then

$$E_{a-\varepsilon}(a-\varepsilon) \le \operatorname{Re} w_2(\lambda) \le |w_2(\lambda)| \le E_{a+\varepsilon}(a+\varepsilon). \tag{4}$$

Indeed, both estimates are deduced from (3). The lower estimate holds since the circle of radius $E_a(a)$ meets $H(a)$ in a nearly vertical arc. The upper estimate follows since $\nu(a) - a \to 0$ as $a \to \infty$ and hence we may choose a so that $\nu < a + \varepsilon$.

Now we exponentiate points on η_2. The result is a curve whose endpoints lie in $\mathbb{R}^-$. Multiplication of this curve by the appropriate $\lambda \in I_{s_1 s_2}(a)$ expands this curve, but the image must cross $H_{s_3}(a)$ in a single vertical curve which we denote by η_3.

As above, we claim that by choosing a large enough we have, for $\lambda \in I_{s_1 s_2 s_3}(a)$,

$$E^2_{a-\varepsilon}(a-\varepsilon) \le \operatorname{Re} w_3(\lambda) \le |w_3(\lambda)| \le E^2_{a+\varepsilon}(a+\varepsilon). \tag{5}$$

The upper estimate holds since

$$|w_3(\lambda)| = |\lambda| \exp(\operatorname{Re} w_2(\lambda)) \le \nu \exp(E_{a+\varepsilon}(a+\varepsilon)) \le E^2_{a+\varepsilon}(a+\varepsilon).$$

To obtain the lower estimate, first set $R_{a,\varepsilon} = a\exp(E_{a-\varepsilon}(a-\varepsilon))$ and observe that, by (4),

$$|w_3(\lambda)| = |\lambda| e^{\operatorname{Re} w_2(\lambda)} \ge R_{a,\varepsilon}.$$

By a simple trigonometric argument (see Figure 8) one can see that

$$\operatorname{Re} w_3(\lambda) \ge \sqrt{R_{a,\varepsilon} - M^2}. \tag{6}$$

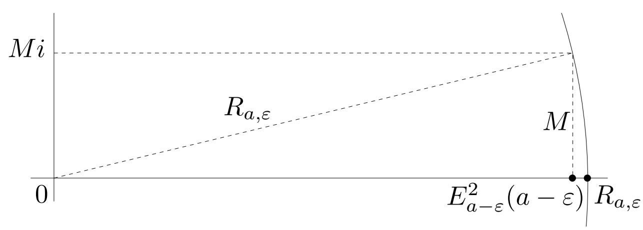

Fig. 8. Sketch of the construction above

We then have, on the one hand,

$$R_{a,\varepsilon} - \sqrt{R_{a,\varepsilon} - M^2} \underset{a\to\infty}{\longrightarrow} 0,$$

and, on the other hand,

$$R_{a,\varepsilon} - E^2_{a-\varepsilon}(a-\varepsilon) = \varepsilon \exp(E_{a-\varepsilon}(a-\varepsilon)) \underset{a\to\infty}{\longrightarrow} \infty.$$

Putting everything together, we obtain the lower estimate in (5).

It is now clear that continuing in the same fashion we obtain the required $I_{s_1\ldots s_j}(a)$. Note that, by construction, if λ is the upper endpoint of $I_{s_1\ldots s_j}(a)$, then $z_j(\lambda) \in \partial H_{s_j}(a)$. Hence, pick λ to be the upper endpoint of $I_{s_1\ldots s_{n-2}}(a)$ and then $\operatorname{Im} w_{n-2}(\lambda_a) = (2s_{n-2}+1)\pi$. ∎

Proposition 3.6. *Choose λ_a as in Proposition 3.5. Then E_{λ_a} has an attracting cycle of period n.*

Proof. By the same arguments as in Proposition 3.5, it is clear that

$$E_{a-\varepsilon}^{n-2}(a-\varepsilon) \leq |w_{n-1}(\lambda)| \leq E_{a+\varepsilon}^{n-2}(a+\varepsilon).$$

We know that $\operatorname{Im} w_{n-2}(\lambda_a) = (2s_{n-2}+1)\pi$, and hence

$$\operatorname{Re} w_{n-1}(\lambda_a) \leq -E_{a-\varepsilon}^{n-2}(a-\varepsilon)\cos(\operatorname{Arg}\lambda_a)$$

since $\operatorname{Arg} w_{n-1}\lambda_a = \operatorname{Arg}\lambda_a + \pi$. Now $|\operatorname{Arg}\lambda_z| \leq \pi/4$ so that

$$\operatorname{Re} w_{n-1}(\lambda_a) \leq -(E_a^{n-2}(a)-1)/\sqrt{2}.$$

Let B be an open ball of radius 1 about $w_{n-1}(\lambda_a)$. The preimages of B containing $w_j(\lambda_a)$ for $j = 1, \ldots, n-2$ are open sets, and $E_{\lambda_a}^{n-1-j}$ maps them univalently onto B. Let U be the preimage of B containing 0. Then $E_{\lambda_a}^{n-1}$ maps U univalently onto B.

Let $F : B \to U$ denote the appropriate branch of the inverse of $E_{\lambda_a}^{n-1}$ taking $w_{n-1}(\lambda_a)$ to 0. We have

$$\begin{aligned}|F'(w_{n-1}(\lambda_a))| &= \left|\frac{1}{\prod_{j=0}^{n-2} E'_{\lambda_a}(w_j(\lambda_a))}\right| = \frac{1}{\prod_{j=1}^{n-1}|w_j(\lambda_a)|} \\ &\geq \frac{1}{\prod_{j=1}^{n-1} E_{a+\varepsilon}^{j-1}(a+\varepsilon)}\end{aligned}$$

by Proposition 3.5. By the Koebe 1/4 Theorem we have

$$\operatorname{dist}(0,\partial U) \geq \frac{1}{4}|F'(w_{n-1}(\lambda_a))| \geq \frac{1}{4}\frac{1}{\prod_{j=1}^{n-1} E_{a+\varepsilon}^{j-1}(a+\varepsilon)}.$$

Now consider $w_n(\lambda_a)$. We have

$$\begin{aligned}|w_n(\lambda_a)| &= |E_{\lambda_a}(w_{n-1}(\lambda_a))| = |\lambda_a|\exp(\operatorname{Re} w_{n-1}(\lambda_a)) \\ &\leq (a+\varepsilon)\exp\left(-\frac{1}{\sqrt{2}}\,E_{a-\varepsilon}^{n-2}(a-\varepsilon)\right) \ll \frac{1}{4}\frac{1}{\prod_{j=0}^{n-1}(E_{a+\varepsilon}^{j-1}(a+\varepsilon))}.\end{aligned}$$

The last inequality follows (for a large enough and ε small enough) since the expression for $|E_{\lambda_a}(w_{n-1}(\lambda_a))|$ contains one higher iterate of E_a. Hence $w_n(\lambda_a)$ lies well within U. We claim that $E_{\lambda_a}(B) \subset U$ as well. Indeed, for $w \in B$, we have

$$\begin{aligned}|E'_{\lambda_a}(w)| &\leq |E'_{\lambda_a}(w_{n-1}(\lambda_a)+1)| = |\lambda_a|\exp(\operatorname{Re} w_{n-1}(\lambda_a)+1) \\ &\leq (a+\varepsilon)\exp\left(-\frac{1}{\sqrt{2}}\,E_{a-\varepsilon}^{n-2}(a-\varepsilon)+1\right) \ll \frac{1}{4}\frac{1}{\prod_{j=1}^{n-1} E_{a+\varepsilon}^{j-1}(a+\varepsilon)}\end{aligned}$$

as above. This shows that $E_{\lambda_a}(B)$ lies well within U since

$$|E_{\lambda_a}(w) - E_{\lambda_a}(w_{n-1}(\lambda_a))| \leq \max_{w\in B}|E'_\lambda(w)|.$$

It follows that E_{λ_a} has an attracting cycle of period n that lies close to $w_j(\lambda_a)$ for $j = 0, \ldots, n-1$. ∎

The following proposition completes the proof of Theorem 3.3.

PROPOSITION 3.7. *Choose* λ_a *as in Proposition* 3.5. *Then* $S(\lambda_a) = 0s_1 \dots s_{n-2}*$.

Proof. Let $\gamma(t) = t + (2s_{n-2} + 1)\pi i$ with $t \geq \operatorname{Re} w_{n-2}(\lambda_a)$ so $w_{n-2}(\lambda_a)$ is the left endpoint of this horizontal line. We claim that $\gamma(t)$ belongs to the basin of attraction of the attracting cycle. Indeed, $E_{\lambda_a}(\gamma(t))$ is a straight line lying to the left of $w_{n-1}(\lambda_a)$. Hence $|E^2_{\lambda_a}(\gamma(t))| \leq |w_n(\lambda_a)|$ and it follows that this line lies in the immediate basin containing $w_{n-1}(\lambda_a)$.

For any $\varepsilon > 0$ we let $\tau = \varepsilon/n$. Then for a sufficiently large we have $|\operatorname{Arg} w_j(\lambda_a)| \leq \tau$ for $j = 1, \dots, n-2$. This follows since $|\operatorname{Arg} w_j(\lambda_a)| \leq |\operatorname{Arg}(a + (2\widehat{s} + 1)\pi i)|$, which may be made arbitrarily small as a increases.

Now let $\mu_j(t)$ denote the curve that contains $w_{n-2-j}(\lambda_a)$ and satisfies $E^j_{\lambda_a}(\mu_j(t)) = \gamma(t)$ for $t \geq \operatorname{Re} w_{n-2}(\lambda_a)$ and $j = 1, \dots, n-2$. So $\mu_1(t)$ contains $w_{n-3}(\lambda_a)$ while $\mu_{n-2}(t)$ contains 0. By construction, each μ_j is in a different component of the immediate basin of the attracting cycle. To prove the result, we will show that $\mu_j(t) \subset H_{s_{n-2-j}}(a)$ for each $j \leq n-3$ and $|\operatorname{Im} \mu_{n-2}(t)| < \pi$.

Consider $\mu_1(t)$. We have $E_{\lambda_a}(\mu_1(t)) = \gamma(t)$ so that

$$E'_{\lambda_a}(\mu_1(t)) \cdot \mu'_1(t) = \gamma'(t).$$

Therefore

$$\operatorname{Arg} E'_{\lambda_a}(\mu_1(t)) + \operatorname{Arg} \mu'_1(t) = \operatorname{Arg} \gamma'(t) = 0$$

and consequently

$$|\operatorname{Arg} \mu'_1(t)| = |\operatorname{Arg} E'_{\lambda_a}(\mu_1(t))| = |\operatorname{Arg} E_{\lambda_a}(\mu_1(t))| = |\operatorname{Arg} \gamma(t)| \leq \tau.$$

In particular, this implies $\mu_1(t)$ lies to the right of its endpoint, $w_{n-3}(\lambda_a)$, for $t > \operatorname{Re} w_{n-2}(\lambda_a)$.

Continuing inductively, we find that

$$|\operatorname{Arg} \mu'_j(t)| \leq \tau j$$

so that $|\operatorname{Arg} \mu'_j(t)| \leq \varepsilon$ for all j, and that each $\mu_j(t)$ lies to the right of its endpoint, $w_{n-z-j}(\lambda_a)$.

Now suppose that $\operatorname{Im} \mu_j(t_0) = (2k+1)\pi$ for some $k \in \mathbb{Z}$. It follows that $E_{\lambda_a}(\mu_j(t_0))$ lies in the left half plane. But $E_{\lambda_a}(\mu_j(t)) = \mu_{j-1}(t)$ if $j > 1$ and $E_{\lambda_a}(\mu_1(t)) = \gamma(t)$. This contradicts the fact that $\mu_{j-1}(t_0)$ lies to the right of the endpoint of μ_{j-1}. Hence each μ_j must lie in a horizontal strip of width at most 2π and contained between the translates of $\gamma(t)$. This implies that $\mu_j(t) \subset H_{s_{n-2-j}}(a)$, and the result follows. ∎

This concludes the proof of Theorem 3.3. To complete the proof of Theorem A, observe that the result holds for any a larger than a certain value a_0. Following the construction, we then see that we have constructed a curve of λ_a-values, one for each sufficiently large $a \in \mathbb{R}$, having the property that $\operatorname{Re}\lambda_a = a$ and $S(\lambda) = s$. Note that λ_a lies in the intervals $I_{s_1\ldots s_{n-2}}(a)$ and, by construction, we have $\operatorname{Im} I_{s_1\ldots s_{n-3}\alpha}(a) < \operatorname{Im} I_{s_1\ldots s_{n-3}\beta}(a)$ if and only if $\alpha < \beta$. Thus, the hyperbolic components *of the same period* are ordered lexicographically. The following corollary shows how the components of period $n+1$ may be inserted between the components of period n.

COROLLARY 3.8. *Let λ_a and $\widetilde{\lambda}_a$ have kneading sequences $0s_1\ldots s_{n-2}*$ and $0s_1\ldots(s_{n-2}+1)*$ for a sufficiently large. Then, given any $k \in \mathbb{Z}$, there exists $\lambda_a(k)$ with $\operatorname{Re}\lambda_a(k) = a$ and $S(\lambda_a(k)) = 0s_1\ldots(s_{n-2}+1)\,k*$.*

Proof. By construction, the λ-values in the vertical segment in between λ_a and $\widetilde{\lambda}_a$ are exactly those belonging to $I_{s_1\ldots(s_{n-2}+1)}(a)$. Hence, if we iterate the process one step further to obtain $\lambda_a(k)$ such that $S(\lambda_a(k)) = 0s_1\ldots(s_{n-2}+1)\,k*$, we must iterate once more for values of λ in this segment. Hence each of the $\lambda(k)$ belongs to $I_{s_1\ldots(s_{n-2}+1)}(a)$. ■

Appendix. Let E_λ, Ω, n, $z_0(\lambda),\ldots,z_{n-1}(\lambda)$, A^* and $A^*(z_i)$ be as in Section 2.1. Our goal in this section is to show how the disc B_λ in the construction of the kneading sequence may be defined for any $\lambda \in \Omega$ so that it varies holomorphically with respect to λ. Although this is not crucial in this paper, we believe it is interesting in itself.

More precisely, our goal is to prove the following proposition.

PROPOSITION A.1. *For any $\lambda \in \Omega$, there exists a topological disc B_λ such that*

(a) *∂B_λ is a simple closed curve in $\mathbb{C}$*;
(b) $0, z_0 \in B_\lambda$;
(c) $\overline{E_\lambda^n(B_\lambda)} \subset B_\lambda$;
(d) $B_\lambda \subset A^*(z_0)$;
(e) *B_λ depends holomorphically on the parameter λ. More precisely, the boundary of B_λ is defined by a map*

$$\gamma : \Omega \times \mathbb{T} \to \partial(B_\lambda) \subset \mathbb{C}, \qquad (\lambda, t) \mapsto \gamma(\lambda, t),$$

satisfying

(1) *for a fixed $t \in \mathbb{T}$, the map $\lambda \mapsto \gamma_\lambda(t) = \gamma(\lambda, t)$ from Ω to $A^*(z_0)$ is holomorphic*;

(2) *for a fixed $\lambda \in \Omega$, the map $t \mapsto \gamma_\lambda(t)$ is an injection* (*and hence γ_λ is a simple closed curve*).

REMARK A.2. Observe that conditions (1) and (2) imply that the map γ (appropriately rewritten after choosing a basepoint $\lambda_0 \in \Omega$) defines a holomorphic motion (see [9]) of ∂B_{λ_0}. Then we can deduce from the λ-lemma that the map γ is jointly continuous.

To prove Proposition A.1 we shall study how the linearizing coordinates of the attracting cycle behave in its basin of attraction, and use them to define precisely the boundary of the set B_λ. From the construction it will become clear how this curve depends on the parameter.

For any $\lambda \in \Omega$, let $\varrho_\lambda = (E_\lambda^n)'(z_0(\lambda))$ be the multiplier of the periodic orbit, and let ϕ_λ be the linearizing coordinates defined on a neighborhood U_λ of z_0 and conjugating E_λ^n to multiplication by ϱ_λ. That is,

$$\phi_\lambda \circ E_\lambda^n(z) = \varrho_\lambda \cdot \phi_\lambda(z). \tag{7}$$

Lemma A.3. *The linearizing coordinates ϕ_λ and the neighborhood U_λ may be chosen such that $0 \in \partial U_\lambda$, $\phi_\lambda(U_\lambda) = \mathbb{D}$, and $\phi_\lambda(0) = 1$.*

Proof. Let $\widetilde{\phi}_\lambda$ be some linearizing coordinates on a neighborhood $\widetilde{U}_\lambda$, i.e., satisfying

$$\widetilde{\phi}_\lambda \circ E_\lambda^n(z) = \varrho_\lambda \cdot \widetilde{\phi}_\lambda(z) \tag{8}$$

for all $z \in \widetilde{U}_\lambda$. We may assume we have restricted $\widetilde{U}_\lambda$ so that it is mapped by $\widetilde{\phi}_\lambda$ to a round disc centered at 0.

By construction we know that $0 \in A^*(z_0)$. If $0 \in \widetilde{U}_\lambda$, we are done by further restricting $\widetilde{U}_\lambda$ and composing with a rotation. So we assume this is not the case. Hence there exists $p \in \mathbb{N}$ such that $E_\lambda^{np}(0) \in \widetilde{U}_\lambda$. Let $\widetilde{\omega} = \widetilde{\phi}_\lambda(E_\lambda^{np}(0))$. Then the preimage of the disc $D(0, |\widetilde{\omega}|)$ under $\widetilde{\phi}_\lambda$ is a neighborhood of z_0 contained inside $\widetilde{U}_\lambda$, which we denote by $\widetilde{V}$. Since $\widetilde{V}$ does not contain 0, we may pull it back by the branch of $(E_\lambda^n)^{-1}$ that maps z_0 to itself, and obtain a new neighborhood of z_0 that strictly contains $\widetilde{V}$. The map $\widetilde{\phi}_\lambda$ can be extended to this new domain by using the functional equation (8) (see Figure 9).

We may repeat this process exactly p times to obtain a nested sequence of (bounded) neighborhoods of z_0, where the map $\widetilde{\phi}_\lambda$ is well defined and whose image is a nested sequence of discs of radii $|\widetilde{\omega}|, |\widetilde{\omega}/\varrho_\lambda|, |\widetilde{\omega}/\varrho_\lambda^2|, \ldots, |\widetilde{\omega}/\varrho_\lambda^p|$ respectively.

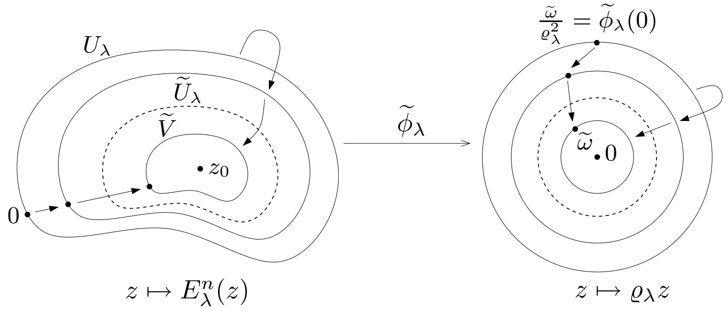

Fig. 9. Sketch of the construction in the proof of Lemma A.3 for $p = 2$. Both maps are univalent on these domains.

We denote the largest neighborhood of z_0 in the process by U_λ. Observe that, by construction, U_λ contains 0 in its boundary and it is mapped by $\widetilde{\phi}_\lambda$ to a round disc of radius $|\widetilde{\omega}/\varrho_\lambda^p| = |\widetilde{\phi}_\lambda(0)|$.

Since the linearizing coordinates are defined uniquely up to multiplication by a nonzero scale factor, the map defined as

$$\phi_\lambda(z) = \widetilde{\phi}_\lambda(z)/\widetilde{\phi}_\lambda(0)$$

on the domain U_λ has the required properties. ∎

Let $V_0 = U_\lambda$. The boundary of V_0 is a (real-analytic) closed simple curve containing 0 in its boundary. Hence, we may still take one further preimage of U_λ under E_λ^n (taking the appropriate branch of the inverse), and obtain a finger V_1 (open, simply connected, unbounded to the right) that strictly contains V_0 and is mapped one-to-one onto V_0 by E_λ^n. See Figure 10. To see this, one can check that the set of preimages of V_0 under E_λ is a collection of disjoint fingers unbounded in the left half plane, which are $2\pi i$-translations of each other and map univalently onto V_0. Only one of them contains the point z_{n-1} and hence this finger is contained in $A^*(z_{n-1})$. We now pull back this finger along the periodic orbit and obtain V_1. The linearizing map ϕ_λ sends V_1 onto the disc $D(0, |\varrho_\lambda|^{-1})$ univalently.

We are now ready to define the set B_λ in Proposition A.1. We observe that, by construction, any disc of radius in between 1 and $|\varrho_\lambda|^{-1}$ has a preimage under ϕ_λ which contains 0, and is mapped under E_λ^n strictly inside itself. In particular, if we take for example

$$\gamma_\lambda(t) = \phi_\lambda^{-1}(e^{2\pi i t}\varrho_\lambda^{-1/2})$$

for $t \in \mathbb{T}$, and we define B_λ to be the open set bounded by γ_λ, this set is a topological disc in $A^*(z_0)$ which contains 0 and z_0 and maps one-to-one strictly inside itself under E_λ^n.

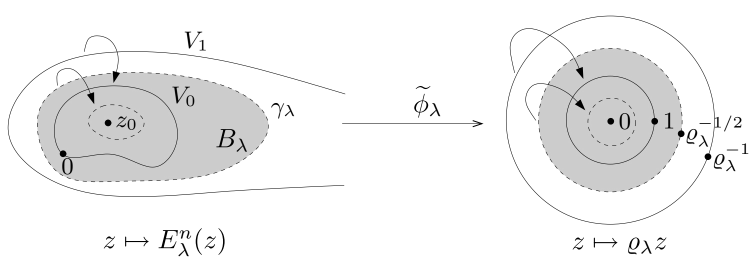

Fig. 10. The unbounded domain V_1 where ϕ_λ is univalent and the construction of the curve γ_λ, boundary of B_λ

It remains to be checked that the curve γ_λ depends holomorphically on the parameter λ. More precisely we want to show that the map

$$\gamma : \Omega \times \mathbb{T} \to \partial(B_\lambda) \subset \mathbb{C}, \qquad (\lambda, t) \mapsto \phi_\lambda^{-1}(e^{2\pi i t}\varrho_\lambda^{-1/2}),$$

is well defined and holomorphic in λ. To see this we just need to recall that the multiplier function

$$\varrho : \Omega \to \mathbb{D}^*, \qquad \lambda \mapsto \varrho_\lambda,$$

is a universal covering map [8]. Hence no closed loop in Ω can ever map to a closed loop in $\mathbb{D}$ surrounding the point 0. This implies that the principal square root $\varrho_\lambda^{1/2}$ is well defined and holomorphic in λ. Since ϕ_λ is biholomorphic in z and λ we conclude that γ is holomorphic in λ.

This concludes the proof of Proposition A.1.

References

[1] I. Baker and J. Rippon, *Iteration of exponential functions*, Ann. Acad. Sci. Fenn. Ser. 1A Math. 9 (1984), 49–77.

[2] R. Bhattacharjee and R. L. Devaney, *Tying hairs for structurally stable exponentials*, Ergodic Theory Dynam. Systems 20 (2000), 1603–1617.

[3] C. Bodelón, R. L. Devaney, M. Hayes, G. Roberts. L. Goldberg and J. Hubbard, *Hairs for the complex exponential family*, Internat. J. Bifur. Chaos 9 (1999), 1517–1534.

[4] —, —, —, —, —, —, *Dynamical convergence of polynomials to the exponential*, J. Differ. Equations Appl. 6 (2000), 275–307.

[5] R. L. Devaney, e^z: *Dynamics and bifurcations*, Internat. J. Bifur. Chaos 1 (1991), 287–308.

[6] A. Eremenko and M. Lyubich, *Iterates of entire functions*, preprint no. 6-84, Fiz.-Tekhn. Inst. Nizkikh Temperatur, Akad. Nauk Ukrain. SSR, Kharkov, 1984 (in Russian).

[7] —, —, *Iterates of entire functions*, Dokl. Akad. Nauk SSSR 279 (1984), 25–27 (in Russian); English transl.: Soviet Math. Dokl. 30 (1984), 592–594.

[8] —, —, *Structural stability in some family of entire functions*, Funktsional. Anal. i Prilozhen. 19 (1985), no. 4, 86–87 (in Russian).

[9] C. T. McMullen, *Complex Dynamics and Renormalization*, Ann. of Math. Stud. 135, Princeton Univ. Press, 1994.

[10] D. Schleicher, *Attracting dynamics of exponential maps*, prepint, 2000.

Department of Mathematics
Boston University
Boston, MA 02215, U.S.A.
E-mail: bob@bu.edu

Dep. Mat. Apl. i Anàlisi
Univ. de Barcelona
08071 Barcelona, Spain
E-mail: fagella@maia.ub.es

Dep. d'Economia i d'Hist. Econ.
Escola d'Empresarials de Sabadell
Univ. Autònoma de Barcelona
08193 Bellaterra, Spain
E-mail: xavier.jarque@uab.es

Received 2 February 2000;
in revised form 8 September 2001