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On the uniqueness of the ergodic maximal function

by

Lasha Ephremidze (Tbilisi)

Abstract. It is proved that the ergodic maximal operator is one-to-one.

1. Introduction. Let $\mathbb{T}$ be any operator which maps one function space into another. The uniqueness theorem for $\mathbb{T}$, i.e., the injectivity property of $\mathbb{T}$ is often used as an important tool in studying some problems related to this operator. In many situations the uniqueness theorem is an easy consequence of more profound results concerning $\mathbb{T}$. For example, when $\mathbb{T}$ is the Fourier transform, then we have the inversion formula for both discrete and continuous cases and for the conjugate operator or the Hilbert transform $\mathbb{T}^2 f = -f$. These imply the uniqueness of $\mathbb{T}$. On the other hand there are some integral operators for which the uniqueness theorem fails to hold at all.

The situation is different for the Hardy–Littlewood maximal operator (considered on a set of positive functions). Here the uniqueness theorem cannot be easily deduced from the known results. For one-sided maximal functions the uniqueness theorem is proved in [2], while for non-one-sided maximal operators the problem is still open.

The ergodic maximal operator $f \mapsto f^*$ (see (2)) plays an important role in the theory of measure-preserving transformations, including the proof of the main result—Birkhoff Ergodic Theorem—which actually reduces to the Ergodic Maximal Theorem. Thus the problem naturally arises whether this operator has the uniqueness property. The present paper gives an affirmative answer to this question for finite measure spaces (see Theorem 1 in Section 2). The idea of the proof is borrowed from [2], though it needs some additional effort.

In Section 3 the uniqueness theorem is proved as an auxiliary statement for the discrete maximal operator (see Proposition 2).

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In Section 4 the infinite measure situation is studied thoroughly. In this case the uniqueness theorem holds for non-negative functions only.

Although we cannot provide further applications of the uniqueness theorems proved in the paper, in our opinion this does not diminish the independent interest of these results.

In this paper we deal only with the discrete case. The continuous parameter case is considered in [3].

2. Formulation of the main result; some auxiliary propositions. Let $(X, \mathbb{S}, \mu)$ be a finite measure space,

$$\mu(X) < \infty, \tag{1}$$

and let $T : X \to X$ be a measure-preserving ergodic transformation. For an integrable function f, $f \in L(X)$, the ergodic maximal function is denoted by f^*:

$$f^*(x) = \sup_n \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x), \qquad x \in X. \tag{2}$$

The main goal of this paper is to prove the following assertion.

THEOREM 1. *Let $f, g \in L(X)$ and*

$$f^* = g^* \qquad a.e. \tag{3}$$

Then

$$f(x) = g(x) \tag{4}$$

for a.a. $x \in X$ (with respect to the measure μ).

REMARK 1. Condition (1) is necessary for the theorem to be valid. Counter-examples will be given in Section 4.

REMARK 2. The theorem remains valid if we do not require the transformation T to be ergodic. By the standard discussion, the general situation can be reduced to the ergodic case.

First we prove some auxiliary statements.

LEMMA 1. *Let $f \in L(X)$. Then*

$$\operatorname{ess\,inf} f^* = \frac{1}{\mu(X)} \int_X f \, d\mu \equiv \lambda_0.$$

Proof. That $f^* \geq \lambda_0$ a.e. follows from the Individual Ergodic Theorem:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k = \lambda_0 \qquad \text{a.e.} \tag{5}$$

(see [4], [6]). On the other hand, the Maximal Ergodic Theorem asserts that

$$\mu(f^* > \lambda) \le \frac{1}{\lambda} \int_{(f^*>\lambda)} f\,d\mu, \qquad \lambda \ge \lambda_0 \tag{6}$$

(see [6, p. 30]), and if $\mu(f^* > \lambda) = \mu(X)$ for some $\lambda > \lambda_0$, we see from (6) that $\mu(X) \le \lambda^{-1} \int_X f\,d\mu$. This implies $\lambda \le \lambda_0$, which is a contradiction. ∎

The following proposition will be used in proving Theorem 1. It is not a new result (see [6, p. 84]; [7]) but a different proof by means of the filling scheme method apparently is of independent interest.

PROPOSITION 1. *Let T be a measure-preserving ergodic transformation of a finite measure space $(X, \mathbb{S}, \mu)$ and let*

$$\int_X f\,d\mu = 0. \tag{7}$$

Then

$$\mu(E) = 0, \tag{8}$$

where $E = \{x \in X : S_n f(x) = \sum_{k=0}^{n} f(T^k x) < 0$ for all $n \ge 0\}$.

The filling scheme method (see [5], [6]) defines a sequence of functions as follows:

$$f_0 = f, \qquad f_{n+1} = -f_n^- + f_n^+ \circ T, \qquad n = 0, 1, \ldots,$$

where $f^+ = \max(0, f)$ and $f^- = \max(0, -f)$.

Note that, for $n = 0, 1, \ldots,$

$$f_n(x) \ge 0 \;\Rightarrow\; f_{n+1}(x) \ge 0, \tag{9}$$

$$f_{n+1}(x) > 0 \;\Rightarrow\; f_n(Tx) > 0, \tag{10}$$

$$f_{n+1}^-(x) \le f_n^-(x), \tag{11}$$

$$f_n(x) < 0,\; f_n(x) < f_{n+1}(x) \;\Rightarrow\; f_n(Tx) > 0. \tag{12}$$

LEMMA 2 (cf. [1, Lemma 1.1]). *For each $n \ge 0$, we have*

$$\Big\{x : \max_{0 \le m \le n} S_m f(x) \ge 0\Big\} = (f_n \ge 0).$$

Proof. We slightly modify the proof of Lemma 1.1 of [1]. Since

$$\sum_{k=0}^{m} f_{n+1} \circ T^k = \sum_{k=0}^{m} -f_n^- \circ T^k + \sum_{k=0}^{m} f_n^+ \circ T^{k+1} = -f_n^- + \sum_{k=1}^{m} f_n \circ T^k + f_n^+ \circ T^{m+1},$$

we have

$$f_n(T^{m+1}x) > 0 \;\Rightarrow\; \sum_{k=0}^{m} f_{n+1}(T^k x) \le \sum_{k=0}^{m+1} f_n(T^k x), \tag{13}$$

$$f_n(x) < 0 \ \Rightarrow \ \sum_{k=0}^{m} f_{n+1}(T^k x) \geq \sum_{k=0}^{m+1} f_n(T^k x), \tag{14}$$

for $n, m = 0, 1, \dots$

We shall show that:

(i) If $f_m(x) < 0$ for all $m < n$ and $f_n(x) \geq 0$, then $S_n f(x) \geq 0$.

(ii) If $S_m f(x) \geq 0$, then $f_m(x) \geq 0$ (consequently, $f_n(x) \geq 0$ for all $n \geq m$, see (9)).

Indeed, if the assumptions in (i) hold, then

$$0 < f_n(x) - f_{n-1}(x) = f_n(x) + f_{n-1}^-(x) = f_{n-1}^+(Tx),$$

and consequently $f_{n-1}(Tx) > 0$, $f_{n-2}(T^2 x) > 0, \dots, f(T^n x) > 0$ (see (10)). Thus, by (13) we have

$$0 \leq f_n(x) \leq f_{n-1}(x) + f_{n-1}(Tx) \leq \dots \leq \sum_{k=0}^{n} f(T^k x)$$

and (i) follows.

If now $S_m f(x) \geq 0$ and we assume $f_n(x) < 0$ for all $n < m$, then by (14) we have

$$\sum_{k=0}^{m} f(T^k x) \leq \sum_{k=0}^{m-1} f_1(T^k x) \leq \dots \leq f_m(x)$$

and (ii) is proved. ■

Proof of Proposition 1. By Lemma 2,

$$E = \{x \in X : f_n(x) < 0 \text{ for all } n \geq 0\}.$$

Since (7) holds and $\lim_{n\to\infty} \int_X f_n^+ \, d\mu = 0$ (see [1, (19)]), we have

$$\lim_{n\to\infty} \int_X f_n^- \, d\mu = 0.$$

Consequently,

$$\lim_{n\to\infty} f_n^-(x) = 0 \tag{15}$$

for a.a. $x \in X$, since $f_n^-(x) \downarrow$ (see (11)).

Set

$$l_n = \mu\{x \in E : f_{n+1}^-(x) < f_n^-(x)\}, \qquad n = 0, 1, \dots$$

If

$$\mu(E) > 0, \tag{16}$$

then

$$\sum_{n=0}^{\infty} l_n = \infty, \tag{17}$$

because of (15) (otherwise the sequence $f_n^-(x)$ would change its value only finitely many times for a.a. $x \in E$). Since $\{x \in X : -f_n^-(x) < -f_{n+1}^-(x) < 0\} \subset \{x \in X : f_n(Tx) > 0\} \setminus (f_{n+1} > 0) = T^{-1}(f_n > 0) \setminus (f_{n+1} > 0)$ (see (12)), we have

$$l_n \leq \mu(T^{-1}(f_n > 0)) - \mu(f_{n+1} > 0) = \mu(f_n > 0) - \mu(f_{n+1} > 0).$$

Hence

$$\sum_{n=0}^{\infty} l_n \leq \sum_{n=0}^{\infty} (\mu(f_n > 0) - \mu(f_{n+1} > 0)) = \mu(f > 0),$$

which contradicts (17). Consequently, (16) is not valid and (8) holds. ∎

Corollary 1. *Let $f \in L(X)$ and let*

$$F_f = \left\{ x \in X : f^*(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \text{ for some } n > 0 \right\}. \tag{18}$$

Then

$$\mu(F_f) = \mu(X), \tag{19}$$

and, *consequently*,

$$\mu\{x \in X : T^n x \in F_f \text{ for all } n \geq 0\} = \mu(X). \tag{20}$$

Proof. Lemma 1 implies that

$$\mu(f^* \geq \lambda_0) = \mu(X).$$

It follows from the Individual Ergodic Theorem (see (5)) that almost all $x \in (f^* > \lambda_0)$ belong to F_f. Obviously, $(f - \lambda)^* = f^* - \lambda$, and if we apply Proposition 1 to the function $f - \lambda_0$, we find that almost all $x \in (f^* = \lambda_0)$ belong to F_f. Thus (19) holds.

Since $\{x \in X : T^n x \in F_f \text{ for all } n \geq 0\} = \bigcap_{n=0}^{\infty} T^{-n}(F_f)$, (20) holds as well. ∎

3. Discrete maximal operator; the proof of the main result. Let Γ denote the set of all sequences of real numbers indexed by $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The discrete maximal operator M is defined by

$$M\alpha(n) = \sup_{m \geq n} \frac{1}{m - n + 1} \sum_{k=n}^{m} \alpha(k), \qquad n \in \mathbb{N}_0,\ \alpha \in \Gamma.$$

Thus, if $\alpha(n) = f(T^n x)$, then

$$M\alpha(n) = f^*(T^n x). \tag{21}$$

Let us use the brief notations $(M\alpha > \lambda) = \{n \in \mathbb{N}_0 : M\alpha(n) > \lambda\}$, $I_{p,q} = \{p, p+1, \dots, q\}$ and

$$A_{p,q} = \frac{1}{q-p+1} \sum_{k=p}^{q} \alpha(k), \qquad 0 \le p \le q < \infty.$$

We say that $I_{p,q}$, $0 \le p \le q < \infty$, is a *finite connected component* of $N \subset \mathbb{N}_0$ if $I_{p,q} \subset N$ and $p-1, q+1$ do not belong to N.

LEMMA 3. *Let $\alpha \in \Gamma$. If*

$$M\alpha(n) = \frac{1}{q-n+1} \sum_{k=n}^{q} \alpha(k), \tag{22}$$

then $A_{p,q} \ge M\alpha(n)$ for each $p \in I_{n,q}$.

Proof. If $p = n$, there is nothing to prove. If $A_{p,q} < M\alpha(n)$ for some $p \in I_{n+1,q}$, then taking into account that $A_{n,p-1} \le M\alpha(n)$, we have

$$\begin{aligned} \sum_{k=n}^{q} \alpha(k) &= \sum_{k=n}^{p-1} \alpha(k) + \sum_{k=p}^{q} \alpha(k) = (p-n)A_{n,p-1} + (q-p+1)A_{p,q} \\ &< (p-n)M\alpha(n) + (q-p+1)M\alpha(n) = (q-n+1)M\alpha(n). \end{aligned}$$

This contradicts (22). ■

COROLLARY 2. *If $n < m$ and $M\alpha(n) > M\alpha(m)$, then (22) holds for some $q \in I_{n,m-1}$.*

Proof. We have $M\alpha(n) \ne \limsup_{q\to\infty} A_{n,q}$, since otherwise $M\alpha(m)$ would have the same value as $M\alpha(n)$. Hence (22) holds for some $q \ge n$ and it follows from the lemma that $q < m$. ■

LEMMA 4. *Let $\alpha \in \Gamma$. If $I_{p,q}$ is a finite connected component of the set $(M\alpha > \lambda)$, then for all $n \in I_{p,q}$:*

$$\text{(i)} \quad M\alpha(n) = \frac{1}{m-n+1} \sum_{k=n}^{m} \alpha(k)$$

for some $m \in I_{n,q}$, and

$$\text{(ii)} \quad \frac{1}{q-n+1} \sum_{k=n}^{q} \alpha(k) > \lambda. \tag{23}$$

Proof. Since $M\alpha(n) > \lambda \ge M\alpha(q+1)$, Corollary 2 implies that (i) holds. It follows from (i) that $\sup\{m \le q : A_{n,m} > \lambda\} = q$. Hence (23) holds. ■

For $\alpha \in \Gamma$, let $N_\alpha \subset \mathbb{N}_0$ be the set of integers for which the supremum is achieved after finitely many steps, i.e., $n \in N_\alpha$ if and only if $M\alpha(n) = A_{n,m}$ for some $m \ge n$.

REMARK 3. Observe that if

$$\alpha(n) = f(T^n x), \tag{24}$$

then $n \in N_\alpha \Leftrightarrow T^n x \in F_f$ (see Corollary 1). Hence if $f \in L(X)$, then for a.a. $x \in X$ the sequence (24) has the property that $n \in N_\alpha$ for all $n \geq 0$ (see (20)).

LEMMA 5. *Let $M\alpha(n) = \lambda$ and $n \in N_\alpha$.*

(i) *If $M\alpha(n+1) \leq \lambda$, then*

$$\alpha(n) = \lambda. \tag{25}$$

(ii) *If $I_{n+1,m}$ is a finite connected component of $(M\alpha > \lambda)$, then*

$$\frac{1}{m-n+1}\sum_{k=n}^{m} \alpha(k) = \lambda. \tag{26}$$

Proof. Obviously, $\alpha(n) \leq \lambda$ and

$$A_{n,m} \leq \lambda. \tag{27}$$

Let $q \geq n$ be the integer for which

$$A_{n,q} = \lambda. \tag{28}$$

(i) If $\alpha(n) < \lambda$, then the inequality $A_{n+1,q} \leq M\alpha(n+1) \leq \lambda$ implies

$$\sum_{k=n}^{q} \alpha(k) = \alpha(n) + (q-n)A_{n+1,q} < \lambda + (q-n)\lambda = (q-n+1)\lambda,$$

which contradicts (28). Thus (25) holds.

(ii) Let us first show that q cannot be less than m. Indeed, if $q < m$, then $A_{q+1,m} > \lambda$, by Lemma 4(ii), and

$$\sum_{k=n}^{m} \alpha(k) = \sum_{k=n}^{q} \alpha(k) + \sum_{k=q+1}^{m} \alpha(k) = (q-n+1)\lambda + (m-q)A_{q+1,m} > (m-n+1)\lambda,$$

which contradicts (27). Assume now that $q > m$. If $A_{n,m} < \lambda$, then the inequality $A_{m+1,q} \leq M\alpha(m+1) \leq \lambda$ yields

$$\sum_{k=n}^{q} \alpha(k) = \sum_{k=n}^{m} \alpha(k) + \sum_{k=m+1}^{q} \alpha(k) < \lambda(m-n+1) + \lambda(q-m) = \lambda(q-n+1),$$

which contradicts (28). Thus $q = m$ and (26) holds. ∎

The following lemma is crucial in proving Theorem 1.

LEMMA 6. *Let $\alpha \in \Gamma$ and let $I_{p,q}$ be a finite connected component of $(M\alpha > \lambda)$ for some λ. Then the values $M\alpha(n)$, $n \in I_{p,q}$, uniquely define the values $\alpha(n)$, $n \in I_{p,q}$. Thus if $M\alpha(n) = M\beta(n)$, $n \geq 0$, for some $\beta \in \Gamma$, then $\alpha(n) = \beta(n)$ for $n \in I_{p,q}$.*

Proof. Note that $n \in N_\alpha$ for each $n \in I_{p,q}$, by Lemma 4(i).

Arrange the values $M\alpha(n)$, $n \in I_{p,q}$, in descending order, i.e., assume $\lambda_1 > \ldots > \lambda_j > \lambda$, where

$$I_i = \{n \in I_{p,q} : M\alpha(n) = \lambda_i\} \neq \emptyset \quad \text{and} \quad \bigcup_{i=1}^{j} I_i = I_{p,q}.$$

Define the values $\alpha(n)$ by induction with respect to i. For $i = 1$, one can readily say that α is equal to λ_1 on I_1, i.e., $\alpha(n) = \lambda_1$ for all $n \in I_1$. Indeed, since

$$\lambda_1 = \max_{k \in I_{p,q}} M\alpha(k),$$

if $\alpha(n) < \lambda_1$, then $A_{n,m} < \lambda_1$ for all $m \leq q$ and $M\alpha(n)$ cannot be equal to λ_1 by Lemma 4(i).

Assume now that α is already defined on $I_1 \cup \ldots \cup I_i$, $i < j$; we will define it on I_{i+1}. Fix $n \in I_{i+1}$ (so that $M\alpha(n) = \lambda_{i+1}$) and consider two cases:

(i) $M\alpha(n+1) \leq \lambda_{i+1}$. Then, by Lemma 5(i),

$$\alpha(n) = \lambda_{i+1}.$$

(ii) $M\alpha(n+1) > \lambda_{i+1}$. Then there exists $m \leq q$ such that $I_{n+1,m}$ is a finite connected component of $(M\alpha > \lambda_{i+1})$, and since α is already defined on $I_{n+1,m} \subset (M\alpha > \lambda_{i+1}) \cap I_{p,q} = \bigcup_{k=1}^{i} I_k$ by induction, we can apply formula (26) to define $\alpha(n)$:

$$\alpha(n) = \lambda_{i+1}(m - n + 1) - \sum_{k=n+1}^{m} \alpha(k). \ \blacksquare$$

COROLLARY 3. *Let $\alpha \in \Gamma$ and let $M\alpha(0) > M\alpha(m)$ for some $m \geq 0$. Then the values $M\alpha(n)$, $n \in I_{0,m}$, uniquely define the value $\alpha(0)$. Thus if $M\alpha(n) = M\beta(n)$, $n \geq 0$, for some $\beta \in \Gamma$, then $\alpha(0) = \beta(0)$.*

Proof. If we take $\lambda \in (M\alpha(m), M\alpha(0))$, then $m \notin (M\alpha > \lambda)$ and 0 belongs to the finite connected component of $(M\alpha > \lambda)$. $\blacksquare$

LEMMA 7. *Let $\alpha, \beta \in \Gamma$ be sequences such that $M\alpha = M\beta$. If $n \in N_\alpha \cap N_\beta$ and $M\alpha(n) = M\beta(n) \geq M\alpha(m) = M\beta(m)$ for some $m > n$, then*

$$\alpha(n) = \beta(n). \tag{29}$$

Proof. Let $M\alpha(n) = \lambda = M\beta(n)$. If $\lambda \geq M\alpha(n+1)$, then Lemma 5(i) shows that $\alpha(n) = \lambda = \beta(n)$. Thus (29) holds.

If $\lambda < M\alpha(n)$, then there exist $q \leq m$ such that $I_{n+1,q}$ is a finite connected component of $(M\alpha > \lambda)$. Hence $\alpha(k) = \beta(k)$ for all $k \in I_{n+1,q}$, by Lemma 6, and

$$\sum_{k=n}^{q} \alpha(k) = \lambda(q - n + 1) = \sum_{k=n}^{q} \beta(k),$$

by Lemma 5(ii). Consequently, (29) holds. $\blacksquare$

PROPOSITION 2. *Let* $\alpha, \beta \in \Gamma$, $N_\alpha = N_\beta = \mathbb{N}_0$ *and*

$$M\alpha(n) = M\beta(n), \qquad n \geq 0.$$

Then

$$\alpha(n) = \beta(n), \qquad n \geq 0.$$

REMARK 4. If $\alpha(n) = 0$, $n \geq 0$, and $\beta(n_0) = -1$, $\beta(n) = 0$ whenever $n \neq n_0$, then $M\alpha(n) = M\beta(n) = 0$, $n \geq 0$. Thus the requirement that at each point the supremum be achieved after finitely many steps is necessary.

Proof of Proposition 2. By Lemma 7 it is enough to show that each $\alpha \in \Gamma$ such that $N_\alpha = \mathbb{N}_0$ has the following property: For each $n \in \mathbb{N}_0$ there exist $m > n$ such that

$$M\alpha(n) \geq M\alpha(m). \tag{30}$$

Obviously, $\limsup_{m\to\infty} A_{0,m} = \limsup_{m\to\infty} A_{n,m} \equiv \lambda_0$ and $M\alpha(n) \geq \lambda_0$ for each $n \geq 0$. Since $n \in N_\alpha$, we have $\lambda_0 < \infty$.

If $M\alpha(n) = \lambda > \lambda_0$ and $M\alpha(m) \geq \lambda$ for each $m > n$, then we can construct an increasing sequence of non-negative integers $n = n_0 < n_1 < \ldots$ such that $A_{n_{k-1},n_k-1} \geq \lambda$, $k = 1, 2, \ldots$ Thus, $\limsup_{k\to\infty} A_{n,n_k-1} \geq \lambda$. This is a contradiction and therefore (30) holds for some $m > n$.

If $M\alpha(n) = \lambda_0 = A_{n,m}$, then $A_{m+1,q}$ cannot be greater than λ_0 for any $q > m$. Otherwise

$$\begin{aligned}\sum_{k=n}^{q} \alpha(k) &= \sum_{k=n}^{m} \alpha(k) + \sum_{k=m+1}^{q} \alpha(k) \\ &= \lambda_0(m - n + 1) + (q - m)A_{m+1,q} > \lambda_0(q - n + 1),\end{aligned}$$

which is a contradiction. Thus $M\alpha(m + 1) = \lambda_0$. ∎

Proof of Theorem 1. Equality (3) implies that

$$M\alpha_x(n) = M\beta_x(n), \qquad n \geq 0,$$

for a.a. $x \in X$ (more exactly for all $x \notin \bigcap_{n=0}^{\infty} T^{-n}(f^* \neq g^*)$), where

$$\alpha_x(n) = f(T^n x) \quad \text{and} \quad \beta_x(n) = g(T^n x) \tag{31}$$

(see (21)). Thus the sequences (31) satisfy the conditions of Proposition 2 for a.a. $x \in X$ (see Remark 3). Consequently, $f(x) = \alpha_x(0) = \beta_x(0) = g(x)$ and thus (4) holds for a.a. $x \in X$. ∎

4. Infinite measure case. In this section we consider a situation when the measure of the space is infinite,

$$\mu(X) = \infty. \tag{32}$$

Then for each integrable f we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = 0 \tag{33}$$

for a.a. $x \in X$ (see [4]) and consequently $f^* \geq 0$ a.e.

The uniqueness theorem is no longer valid in this case. However we claim that the following theorem is true. The first part contains a positive statement of the theorem, while the second part provides a great variety of counter-examples showing that the uniqueness fails to hold.

THEOREM 2. *Let T be a measure-preserving ergodic transformation of a σ-finite measure space $(X, \mathbb{S}, \mu)$ with $\mu(X) = \infty$.*

(i) *If $f \in L$ and*

$$f^* = g^* \quad \text{a.e. on } X, \tag{34}$$

then $f = g$ a.e. on $(f^* > 0)$.

(ii) *If $f \in L$ and $\mu(f^* = 0) > 0$, then (34) holds for each $g \in L$ such that $g = f$ on $(f^* > 0)$ and $g \leq f$ on $(f^* = 0)$.*

The lemma below which is trivial for conservative ergodic transformations needs a little proof in the ergodic case.

LEMMA 8. *Let T be a measure-preserving ergodic transformation of a σ-finite measure space $(X, \mathbb{S}, \mu)$. If $\mu(Q_0) < \infty$ and $Q_0^c = X \setminus Q_0$, then*

$$\mu\Big(Q_0 \setminus \bigcup_{m=1}^{\infty} T^{-m}(Q_0^c)\Big) = 0.$$

Proof. Note first that for each measurable Q such that $0 < \mu(Q) < \infty$ we have

$$\mu(Q \cap T^{-1}(Q^c)) > 0.$$

Indeed, otherwise $\mu(Q \cap T^{-1}(Q)) = \mu(Q \cap T^{-1}(X)) = \mu(Q)$, which contradicts the ergodicity of T.

Let now $Q = Q_0 \setminus \bigcup_{m=1}^{\infty} T^{-m}(Q_0^c)$. Since $Q^c = \bigcup_{m=0}^{\infty} T^{-m}(Q_0^c)$ we have $T^{-1}(Q^c) \subset \bigcup_{m=1}^{\infty} T^{-m}(Q_0^c)$. Hence

$$\mu(Q \cap T^{-1}(Q^c)) = \mu(\emptyset) = 0$$

and this implies that $\mu(Q) = 0$. ∎

Proof of Theorem 2. (i) We may assume that the equality (34) holds everywhere.

For each $\lambda > 0$ we have $\mu(f^* > \lambda) < \infty$ (see (6); we can assume that $\lambda_0 = 0$ when (32) holds). By Lemma 8,

$$\mu\Big((f^* > \lambda) \setminus \bigcup_{m=1}^{\infty} T^{-m}(f^* \le \lambda)\Big) = 0.$$

Thus, for a.a. $x \in (f^* > \lambda)$ there exists $m = m(x)$ such that $f^*(T^m x) \le \lambda < f^*(x)$. Since $\lambda > 0$ is arbitrary, we can conclude that for a.a. $x \in (f^* > 0)$ there exists $m = m(x)$ such that $f^*(T^m x) < f^*(x)$. For each such x, if we apply Corollary 3 to the sequence $\alpha(n) = f(T^n x)$, $n = 0, 1, \ldots$, the assertion shows that $f(x) = \alpha(0)$ is uniquely determined. Hence part (i) follows.

(ii) Obviously, $g^* \le f^*$ and since $g^* \ge 0$, we have $g^* = 0$ a.e. on $(f^* = 0)$. It remains to show that for a.a. $x \in (f^* > 0)$ we have

$$f^*(x) \le g^*(x). \tag{35}$$

If $x \in F_f$ (see (18); observe that a.a. $x \in (f^* > 0)$ belongs to F_f by (33)) and

$$f^*(x) = \frac{1}{q+1} \sum_{k=0}^{q} f(T^k x) > 0,$$

then $f^*(T^k x) \ge f^*(x) > 0$ for each $k \in I_{0,q}$, by Lemma 3. Thus $f(T^k x) = g(T^k x)$ for each $k \in I_{0,q}$, by hypothesis. Consequently,

$$\frac{1}{q+1} \sum_{k=0}^{q} g(T^k x) = \frac{1}{q+1} \sum_{k=0}^{q} f(T^k x)$$

and (35) holds. ∎

It follows from Theorem 2 that for non-negative functions the uniqueness theorem is always true.

COROLLARY 4. *Let $0 \le f, g \in L$ and $f^* = g^*$ a.e. Then $f = g$ a.e.*

Proof. $f = g$ a.e. on $(f^* > 0)$ by Theorem 2, while a.e. on $(f^* = 0)$ both f and g are 0. ∎

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A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St.
Tbilisi 380093, Georgia
E-mail: lasha@rmi.acnet.ge

Current address:
Mathematical Institute AS ČR
Žitna 25
11567 Praha 1, Czech Republic
E-mail: lasha@math.cas.cz

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