FUNDAMENTA
MATHEMATICAE
174 (2002)

Stronger ideals over $\mathcal{P}_\kappa\lambda$

by

Yo Matsubara (Nagoya)

Abstract. In §1 we define some properties of ideals by using games. These properties strengthen precipitousness. We call these stronger ideals. In §2 we show some limitations on the existence of such ideals over $\mathcal{P}_\kappa\lambda$. We also present a consistency result concerning the existence of such ideals over $\mathcal{P}_\kappa\lambda$. In §3 we show that such ideals satisfy stronger normality. We show a cardinal arithmetical consequence of the existence of strongly normal ideals. In §4 we study some "large cardinal-like" consequences of stronger ideals.

1. Introduction. In this paper we investigate some properties of ideals over $\mathcal{P}_\kappa\lambda$ where κ is an uncountable regular cardinal and λ is a cardinal $\geq \kappa$. Since all of these properties imply precipitousness, they are large cardinal properties. Throughout this paper by an *ideal* over $\mathcal{P}_\kappa\lambda$ we mean a proper κ-complete ideal over $\mathcal{P}_\kappa\lambda$ whose dual filter is fine.

Let $\Gamma^{\phi}_{\omega}(I)$ denote the following two-player game of length ω. In $\Gamma^{\phi}_{\omega}(I)$, Player 1 opens the game by choosing X_1 from $I^+ = \{X \subseteq \mathcal{P}_\kappa\lambda : X \notin I\}$; then Player 2 chooses $X_2 \subseteq X_1$ from I^+; then Player 1 chooses $X_3 \subseteq X_2$ from I^+ etc. Two players alternately choose X_n for $n \in \omega - \{0\}$ to build a descending $\subseteq$-chain $\langle X_n : n \in \omega - \{0\}\rangle$ from I^+. Player 1 wins $\Gamma^{\phi}_{\omega}(I)$ if $\bigcap_{n\in\omega-\{0\}} X_n = \emptyset$. Otherwise Player 2 wins. In Galvin–Jech–Magidor [5] it is shown that "I is precipitous" is equivalent to "Player 1 has no winning strategy in $\Gamma^{\phi}_{\omega}(I)$".

We can also define the following two-player game $\Gamma_\omega(I)$. In $\Gamma_\omega(I)$ just as in $\Gamma^{\phi}_{\omega}(I)$ Player 1 and Player 2 alternately choose X_n from I^+ for $n \in \omega - \{0\}$ to build a descending $\subseteq$-chain $\langle X_n : n \in \omega - \{0\}\rangle$. Player 1 wins $\Gamma^{\phi}_{\omega}(I)$ if $\bigcap_{n\in\omega-\{0\}} X_n \in I$. Otherwise Player 2 wins.

For an ideal I we denote by $\mathbf{P}_I$ the poset of *I-positive sets*, i.e. members of I^+, ordered by inclusion. Jech [6] proved that "$\mathbf{P}_I$ is $\aleph_0$-distributive" is equivalent to "Player 1 has no winning strategy in $\Gamma_\omega(I)$". In this paper we say that an ideal I is $\aleph_0$-*distributive* if $\mathbf{P}_I$ is $\aleph_0$-distributive.

2000 *Mathematics Subject Classification*: Primary 03E55; Secondary 03E35.

The author was partially supported by Grant-in-Aid for Scientific Research (no. 11640112), Ministry of Education, Science and Culture.

Let δ be an infinite cardinal. Let $\Gamma_{<\delta}(I)$ denote the two-person game in which the players build a $\subseteq$-descending sequence $\langle X_\alpha : \alpha \in \delta - \{0\}\rangle$ from I^+, where Player 1 plays odd stages and Player 2 plays even and limit stages. Player 2 wins $\Gamma_{<\delta}(I)$ iff the game can be continued to build a descending $\subseteq$-chain $\langle X_\alpha : \alpha \in \delta - \{0\}\rangle$.

Let $\Gamma_\delta(I)$ be a variant of $\Gamma_{<\delta}(I)$ in which Player 2 wins iff the game can be continued to build a $\subseteq$-descending sequence $\langle X_\alpha : \alpha \in \delta - \{0\}\rangle$ such that $\bigcap_{\alpha\in\delta-\{0\}} X_\alpha \in I^+$. We follow the terminology used in Apter–Shelah [1].

Definition 1. We say that an ideal I is *δ-strategically closed* if Player 2 has a winning strategy in $\Gamma_\delta(I)$. We define I to be *$\prec\delta$-strategically closed* if Player 2 has a winning strategy in $\Gamma_{<\delta}(I)$.

It is clear that if an ideal is δ-strategically closed then it is δ-distributive. In this paper we investigate the properties of distributive ideals over $\mathcal{P}_\kappa\lambda$ and strategically closed ideals over $\mathcal{P}_\kappa\lambda$. We refer to these ideals as *stronger ideals* over $\mathcal{P}_\kappa\lambda$.

2. Limitations and a consistency result. The following theorem shows that $\mathcal{P}_{\aleph_1}\lambda$ cannot carry a stronger ideal. Part (ii) of Theorem 1 is due to Doug Burke.

Theorem 1. *Suppose I is an ideal over $\mathcal{P}_{\aleph_1}\lambda$. Then*

(i) *I is not $\aleph_0$-distributive.*

(ii) *Furthermore if I is normal, then Player 2 cannot have a winning strategy in $\Gamma^\phi_\omega(I)$.*

Proof. (i) Suppose I is an $\aleph_0$-distributive ideal over $\mathcal{P}_{\aleph_1}\lambda$. Clearly I is precipitous. Let G be a $\mathbf{P}_I$-generic filter over V. Let $j : V \to M$ be the corresponding generic elementary embedding. Since $\aleph_1^V$ is the critical point of j, we know $\aleph_1^V$ is collapsed in $V[G]$, contradicting the $\aleph_0$-distributivity of $\mathbf{P}_I$. Therefore I cannot be $\aleph_0$-distributive.

(ii) Suppose σ is a winning strategy for Player 2 in $\Gamma^\phi_\omega(I)$. For each $s \in \mathcal{P}_{\aleph_1}\lambda$, fix an enumeration $\langle \alpha^s_n : n \in \omega\rangle$ of the members of s. Define $F : \mathcal{P}_{\aleph_1}\lambda \times \omega \to \lambda$ by $F(s,n) = \alpha^s_n$. Let θ be a regular cardinal sufficiently larger than λ. Let M be a countable elementary substructure of H_θ such that $\mathcal{P}_{\aleph_1}\lambda$, σ, F and I belong to M. Let $\langle \beta_i : i \in \omega\rangle$ enumerate the members of $M \cap \lambda$. Note that there exists some X in M such that $X \in I^+$ and $M \cap \lambda \notin X$. Now work in M. Since $F(s,0) \in s$ for every s in X, there is some $\gamma_0 < \lambda$ such that $\{s \in X : F(s,0) = \gamma_0\} \in I^+$. Let Player 1 play $X_1 = \{s \in X : F(s,0) = \gamma_0\} \cap \{s \in \mathcal{P}_{\aleph_1}\lambda : \beta_0 \in s\}$ as her first move. Then let Player 2 play according to σ. Inductively we will define Player 1's move as follows: Suppose Player 2 plays $X_{2n} = \sigma(\langle X_1, X_3, \ldots, X_{2n-1}\rangle)$. Then there exists some $\gamma_n < \lambda$ such that $\{s \in X_{2n} : F(s,n) = \gamma_n\} \in I^+$. Let Player 1 play $X_{2n+1} = \{s \in X_{2n} : F(s,n) = \gamma_n\} \cap \{s \in \mathcal{P}_{\aleph_1}\lambda : \beta_n \in s\}$.

Now let us argue in V. By the way Player 1 played the game, it is clear that $\bigcap_{m\in\omega-\{0\}} X_m \subseteq \{M\cap\lambda\}$. Since Player 2 played according to σ, we must have $\bigcap_{m\in\omega-\{0\}} X_m = \{M\cap\lambda\}$. But this contradicts $M\cap\lambda \notin X_1$. ∎

The next theorem also gives a restriction on when we can have an ω-strategically closed ideal over $\mathcal{P}_\kappa\lambda$.

THEOREM 2. *If κ is the successor cardinal of a singular cardinal, then $\mathcal{P}_\kappa\lambda$ cannot carry an ω-strategically closed ideal.*

Proof. Let $\kappa = \delta^+$ where δ is a singular cardinal. Suppose I is an ω-strategically closed ideal over $\mathcal{P}_\kappa\lambda$. It is well known that every ω-strategically closed partial order is proper. We will derive a contradiction by showing that $\mathbf{P}_I$ cannot be proper.

Let $\{A_\alpha : \alpha < \delta\}$ be a family of pairwise disjoint stationary subsets of $\{\beta<\kappa : \mathrm{cf}(\beta)=\omega\}$. Let $f:\mathcal{P}_\kappa\lambda\to\kappa$ be a function such that $X \Vdash_{\mathbf{P}_I} [\check{f}] = \check{\kappa}$ for some $X\in\mathbf{P}_I$, i.e. f represents κ in the generic ultrapower if we force with the condition X.

If $\mathbf{P}_I$ is proper, then we must have $\emptyset \Vdash_{\mathbf{P}_I} \mathrm{cf}(\check{\kappa}) > \omega$. So without loss of generality we may assume that $\mathrm{cf}(f(s)) > \omega$ for each $s\in X$. Since $\kappa=\delta^+$ and δ is singular, we know that $\mathrm{cf}(f(s)) < \delta$ for every s in X. Therefore for each s in X, there exists some $\alpha_s<\delta$ such that $A_{\alpha_s}\cap f(s)$ is non-stationary in $f(s)$. By the δ^+-completeness of I, for some $\alpha^*<\delta$ we have $\{s\in X : \alpha_s=\alpha^*\}\notin I$. Let $Y=\{s\in X:\alpha_s=\alpha^*\}$. If we force with the condition Y, then in the generic ultrapower the function defined by $s\mapsto A_{\alpha_s}\cap f(s)$ represents a subset of κ containig A_{α^*}. Therefore we know that $Y\Vdash_{\mathbf{P}_I}$ "$\check{A}_{\alpha^*}$ is a non-stationary subset of $\check{\kappa}$". Using the fact that every proper partial order preserves stationary subsets of $\{\beta<\kappa:\mathrm{cf}(\beta)=\omega\}$, we conclude that $\mathbf{P}_I$ is not proper. ∎

We note that the proof of Theorem 2 shows that if I is an ideal over $\mathcal{P}_\kappa\lambda$ where κ is the successor cardinal of a singular cardinal, then $\mathbf{P}_I$ cannot be proper. This slightly extends a result obtained in Matsubara–Shelah [9].

The next theorem shows that it is consistent to have a strategically closed normal ideal over $\mathcal{P}_\kappa\lambda$ for κ the successor cardinal of an uncountable regular cardinal assuming the consistency of a supercompact cardinal. This theorem is proved in the same way as Theorem 4 of Galvin–Jech–Magidor [5].

THEOREM 3. *Suppose δ is an uncountable regular cardinal and κ is a supercompact cardinal $>\delta$. Then $\Vdash_{\mathrm{Coll}(\delta,<\kappa)}$ "For every $\lambda\ge\delta^+$, there is a $\prec\delta$-strategically closed normal ideal over $\mathcal{P}_{\delta^+}\lambda$" where* $\mathrm{Coll}(\delta,<\kappa)$ *is the Levy collapse making κ to be δ^+.*

Proof. Let $\mathbf{P}$ denote $\mathrm{Coll}(\delta,<\kappa)$. For $\alpha<\kappa$, let $\mathbf{P}_\alpha=\{p\in\mathbf{P} : \mathrm{dom}(p)\subseteq \delta\times\alpha\}$ and $\mathbf{P}^\alpha=\{p\in\mathbf{P}:\mathrm{dom}(p)\subseteq\delta\times(\kappa-\alpha)\}$. Clearly $\mathbf{P}\cong\mathbf{P}_\alpha\times\mathbf{P}^\alpha$. Let π_α and π^α be the natural projections of $\mathbf{P}$ to $\mathbf{P}_\alpha$ and $\mathbf{P}^\alpha$. Let $\mathcal{U}$ be a supercompact filter over $\mathcal{P}_\kappa\lambda$. Let G be a $\mathbf{P}$-generic filter over V. In $V[G]$

we define an ideal I over $\mathcal{P}_{\delta^+}\lambda$ by $X \in I$ iff $\exists Y \in \mathcal{U}$, $Y \cap X = \emptyset$. It is not difficult to check that I is a normal ideal on $\mathcal{P}_{\delta^+}\lambda$. We will define a strategy σ for Player 2 in $\Gamma_{<\delta}(I)$ to prove that I is $\prec\delta$-strategically closed. First we need the following claim:

Claim 1. *If τ is a name for an I-positive subset of $\mathcal{P}_\kappa\lambda$ then there exists p in G such that $\{s \in (\mathcal{P}_\kappa\lambda)^V : s \cap \kappa \in \kappa \wedge \exists q \in \mathbf{P}^{s\cap\kappa},\ p \cup q \Vdash \check{s} \in \dot{\tau}\} \in \mathcal{U}$.*

Proof. Suppose $q \Vdash_{\mathbf{P}}$ "$\dot{\tau} \subseteq \mathcal{P}_\kappa\lambda \wedge \dot{\tau} \notin \dot{I}$". We will show that there exists some $r \leq q$ such that $\{s \in (\mathcal{P}_\kappa\lambda)^V : s\cap\kappa \in \kappa \wedge \exists q \in \mathbf{P}^{s\cap\kappa},\ r\cup q \Vdash \check{s} \in \dot{\tau}\} \in \mathcal{U}$. Let $D = \{s \in (\mathcal{P}_\kappa\lambda)^V : \exists p \leq q,\ p \Vdash \check{s} \in \dot{\tau}\}$. Since $q \Vdash$ "$\dot{\tau} \cap ((\mathcal{P}_\kappa\lambda)^V - D) = \check{\phi} \wedge \dot{\tau} \notin \dot{I}$", we must have $D \in \mathcal{U}$. For each $s \in D$, fix $p_s \leq q$ such that $p_s \Vdash \check{s} \in \dot{\tau}$. By the normality of $\mathcal{U}$, there exists $E \in \mathcal{U}$ with $E \subseteq D$ and $p \in \mathbf{P}$ such that $\pi_{s\cap\kappa}(p_s) = p$ for every $s \in E$. If $s \in E$ and $q \in \mathbf{P}_{s\cap\kappa}$, then $\pi_{s\cap\kappa}(p_s) \leq q$. So $p \leq q$. Therefore $p \cup \pi^{s\cap\kappa}(p_s) \Vdash \check{s} \in \dot{\tau}$ for every $s \in E$. $\blacksquare_{\text{Claim 1}}$

Player 2's strategy σ, which we now define, is positional (i.e. it depends only on the last move of Player 1). Suppose Player 1's last move happens to be $X \subseteq \mathcal{P}_\kappa\lambda$. Let Player 2 pick a name $\dot{\tau}$ for X in $V[G]$. By Claim 1 there exists $p^* \in G$ such that $E = \{s \in (\mathcal{P}_\kappa\lambda)^V : s \cap \kappa \in \kappa \wedge \exists q \in \mathbf{P}^{s\cap\kappa}$, $p^* \cup q \Vdash \check{s} \in \dot{\tau}\} \in \mathcal{U}$. In the ground model V for each $s \in E$ choose $q^s \in \mathbf{P}^{s\cap\kappa}$ such that $p^* \cup q^s \Vdash \check{s} \in \dot{\tau}$. Let f be the function on E given by $f(s) = q^s$. Let Player 2 play $\sigma(X) = \{s \in X \cap E : q^s \in G\}$.

Claim 2. *$\sigma(X) \notin I$.*

Proof. Assume otherwise. Then there exist $q \in G$ and $Y \in \mathcal{U}$ such that $q \Vdash \sigma(\dot{\tau}) \cap \check{Y} = \emptyset$. We may assume that $q \leq p^*$. Pick $s^* \in E \cap Y$ such that $q \in \mathbf{P}_{s^*\cap\kappa}$. Therefore $q \cup q^{s^*}$ is a condition extending $p^* \cup q^{s^*}$. Thus $q \cup q^{s^*} \Vdash \check{s}^* \in \sigma(\dot{\tau}) \cap \check{Y}$ contradicting $q \Vdash \sigma(\dot{\tau}) \cap \check{Y} = \emptyset$. $\blacksquare_{\text{Claim 2}}$

At the limit stage the move according to σ is to play the intersection of the $\subseteq$-descending chain constructed thus far.

Claim 3. *σ is a winning strategy for Player 2 in $\Gamma_{<\delta}(I)$.*

Proof. Since Player 2 cannot lose at a successor stage by Claim 2, we will concentrate on the limit stages. Suppose $\langle X_\alpha : 1 \leq \alpha < \mu\rangle$, where μ is a limit ordinal $< \delta$, is a run of our game in which Player 2 played according to σ. We need to show that $\bigcap_{\alpha<\mu} X_\alpha \notin I$. For each odd ordinal $\beta < \mu$, at the $\beta + 1$ stage for Player 2 to play $\sigma(X_\beta)$, she needed to pick $p^*_\beta \in G$, τ_β (a name for X_β), $E_\beta \in \mathcal{U}$, and a function f_β on E_β defined by $f_\beta(s) = q^s_\beta$ where $p^*_\beta \cup q^s_\beta \Vdash \check{s} \in \dot{\tau}_\beta$. Let A be the set of odd ordinals $< \mu$. Since $\mathbf{P}$ is $<\delta$-closed, $\langle p^*_\beta : \beta \in A\rangle$, $\langle \tau_\beta : \beta \in A\rangle$, $\langle E_\beta : \beta \in A\rangle$, and $\langle f_\beta : \beta \in A\rangle$ all belong to the ground model V.

Subclaim 1. *For every $\beta, \gamma \in A$ with $\beta < \gamma$, there exists some $\alpha < \kappa$ such that if $s \in E_\beta \cap E_\gamma$ and $s \cap \kappa > \alpha$ then q^s_β and q^s_γ are compatible.*

Proof. There exists some $r \in G$ such that $r \Vdash \dot{\tau}_\gamma \subseteq \sigma(\dot{\tau}_\beta)$. We may assume that $r \leq p^*_\beta$ and $r \leq p^*_\gamma$. Let α be an ordinal $< \kappa$ such that $r \in \mathbf{P}^\alpha$. Suppose $s \in E_\beta \cap E_\gamma$ and $s \cap \kappa > \alpha$. Then $r \cup q^s_\gamma \Vdash \check{s} \in \dot{\tau}_\gamma \subseteq \sigma(\dot{\tau}_\beta)$. By the definition of σ, $r \cup q^s_\gamma \Vdash q^s_\beta \in \dot{G}$. Therefore q^s_β and q^s_γ are compatible. $\blacksquare_{\text{Subclaim 1}}$

For each $\beta, \gamma \in A$ with $\beta < \gamma$, let $\alpha_{\beta\gamma}$ be the least ordinal α such that the statement of Subclaim 1 holds. Let $E^* = \{s \in \bigcap_{\xi \in A} E_\xi : s \cap \kappa > \sup_{\beta<\gamma<\mu} \alpha_{\beta\gamma}\}$. Clearly $E^* \in \mathcal{U}$. Let $p = \bigcup_{\xi \in A} p^*_\xi$. So $p \in G$. Define a function f on E^* by $f(s) = \bigcup_{\xi \in A} q^s_\xi$. Subclaim 1 guarantees that $f(s) \in \mathbf{P}^{s \cap \kappa}$ for every $s \in E^*$.

Now we are ready to show $\bigcap_{\xi \in A} X_\xi \notin I$. Suppose $\bigcap_{\xi \in A} X_\xi \in I$. Thus there exist $r \in G$ and $Y \in \mathcal{U}$ such that $r \Vdash \bigcap_{\xi \in A} \dot{\tau}_\xi \cap \check{Y} = \emptyset$. We may assume that $r \leq p$. Let $\beta < \kappa$ be large enough so that $r \in \mathbf{P}_\beta$. Choose $s \in E^* \cap Y$ with $s \cap \kappa > \beta$. Note that $r \cup f(s) \in \mathbf{P}$ and $r \cup f(s) \leq p \cup q^s_\xi$ for every $\xi \in A$. Since $p \cup q^s_\xi \leq p^*_\xi \cup q^s_\xi$ and $p^*_\xi \cup q^s_\xi \Vdash \check{s} \in \dot{\tau}_\xi$, we have $r \cup f(s) \Vdash \check{s} \in \bigcap_{\xi \in A} \dot{\tau}_\xi$. Thus $r \cup f(s) \Vdash \check{s} \in \bigcap_{\xi \in A} \dot{\tau}_\xi \cap \check{Y}$ contradicting $r \Vdash \bigcap_{\xi \in A} \dot{\tau}_\xi \cap \check{Y} = \emptyset$. Therefore we conclude that $\bigcap_{\xi < \mu} X_\mu = \bigcap_{\xi \in A} X \notin I$. This proves that σ is a winning strategy for Player 2 in $\Gamma_{<\kappa}(I)$. $\blacksquare_{\text{Claim 3 \& Theorem 3}}$

3. Stronger normality. In the last section we proved that $\mathcal{P}_{\aleph_1}\lambda$ cannot carry an $\aleph_0$-distributive ideal. It turns out that if κ is the successor cardinal of a singular cardinal of cofinality $\aleph_0$ then $\mathcal{P}_\kappa\lambda$ cannot carry an $\aleph_0$-distributive ideal. In order to prove this result, we need to introduce the following definition.

DEFINITION 2. Let δ be an infinite cardinal $< \kappa$. An ideal I on $\mathcal{P}_\kappa\lambda$ is said to be δ-*normal* if the following holds: $\{s \in \mathcal{P}_\kappa\lambda : f(s) \in {}^\delta s\} \notin I$ implies that $\{s \in \mathcal{P}_\kappa\lambda : f(s) = \vec{a}\} \notin I$ for some $\vec{a} \in {}^\delta\lambda$.

Throughout this section we let δ represent an infinite cardinal $< \kappa$. It is clear that δ-normality strengthens regular normality. It turns out that distributive ideals satisfy this stronger normality.

THEOREM 4. *If I is a normal ideal over $\mathcal{P}_\kappa\lambda$ such that $\mathbf{P}_I$ is δ-distributive, then I is δ-normal.*

Proof. Let I be such an ideal and $\{s \in \mathcal{P}_\kappa\lambda : f(s) \in {}^\delta s\} \notin I$. Suppose G is a $\mathbf{P}_I$-generic filter over V such that $\{s \in \mathcal{P}_\kappa\lambda : f(s) \in {}^\delta s\} \in G$. Let $j : V \to M$ denote the corresponding generic elementary embedding. Then $M \vDash [f] \in {}^\delta j''\lambda$. Let $\vec{b} \in {}^\delta\lambda$ be defined by $\vec{b}(\alpha) = j^{-1}([f](\alpha))$. By δ-distributivity of $\mathbf{P}_I$ we have $\vec{b} \in V$. Since $[f] = j(\vec{b})$ we conclude $\{s \in \mathcal{P}_\kappa\lambda : f(s) = \vec{b}\} \notin I$. $\blacksquare$

The existence of a δ-normal ideal and a certain cardinal arithmetic condition are equivalent.

Theorem 5. *The following are equivalent*:

(i) *There exists a δ-normal ideal over $\mathcal{P}_\kappa\lambda$.*
(ii) *For every $\alpha < \kappa$, $\alpha^\delta < \kappa$.*

Proof. (i)⇒(ii). Suppose (ii) fails. Say α is an ordinal $< \kappa$ such that $\alpha^\delta \geq \kappa$. Suppose I is an ideal on $\mathcal{P}_\kappa\lambda$. Let $\langle X_{\vec{a}} : \vec{a} \in {}^\delta\alpha\rangle$ be a pairwise disjoint partition of $\mathcal{P}_\kappa\lambda$ such that $X_{\vec{a}} \in I$ for each $\vec{a} \in {}^\delta\alpha$. Define a function f on $\{s \in \mathcal{P}_\kappa\lambda : \alpha \subseteq s\}$ by $f(s) = \vec{a}$ where $s \in X_{\vec{a}}$. This f shows that our ideal I is not δ-normal.

(ii)⇒(i). Assume that $\alpha^\delta < \kappa$ for every $\alpha < \kappa$. Define $\mathcal{F} \subseteq \mathcal{P}(\mathcal{P}_\kappa\lambda)$ by $X \in \mathcal{F}$ iff there exists some $f : {}^\delta\lambda \to \mathcal{P}_\kappa\lambda$ such that $\{s \in \mathcal{P}_\kappa\lambda : \forall\vec{a} \in {}^\delta s,\ f(\vec{a}) \subseteq s\} \subseteq X$. It is clear that $\mathcal{F}$ is a κ-complete fine filter over $\mathcal{P}_\kappa\lambda$. Let I be the dual ideal of $\mathcal{F}$. To prove that I is δ-normal, it is enough to prove the following: if $\{X_{\vec{a}} : \vec{a} \in {}^\delta\lambda\} \subseteq \mathcal{F}$, then $\{s \in \mathcal{P}_\kappa\lambda : \forall\vec{b} \in {}^\delta s,\ s \in X_{\vec{b}}\} \in \mathcal{F}$. For each $\vec{b} \in {}^\delta\lambda$, fix a function $f_{\vec{b}} : {}^\delta\lambda \to \mathcal{P}_\kappa\lambda$ such that $\{s \in \mathcal{P}_\kappa\lambda : \forall\vec{a} \in {}^\delta s,\ f_{\vec{b}}(\vec{a}) \subseteq s\} \subseteq X_{\vec{b}}$. Now fix a bijection l from $\{0,1\} \times \delta$ to δ. For each $\vec{a} \in {}^\delta\lambda$, define $\vec{a}_0, \vec{a}_1 \in {}^\delta\lambda$ by $\vec{a}_i(\alpha) = \vec{a}(l(i,\alpha))$.

Define a function $f : {}^\delta\lambda \to \mathcal{P}_\kappa\lambda$ by $f(\vec{a}) = f_{\vec{a}_0}(\vec{a}_1)$. It is easy to see that $\{s \in \mathcal{P}_\kappa\lambda : \forall\vec{a} \in {}^\delta s,\ f(\vec{a}) \subseteq s\} \subseteq \{s \in \mathcal{P}_\kappa\lambda : \forall\vec{b} \in {}^\delta s,\ s \in X_{\vec{b}}\}$. ■

Corollary 1. *If there exists a normal ideal I over $\mathcal{P}_\kappa\lambda$ such that Player* 1 *does not have a winning strategy in $\Gamma_\delta(I)$, then $\alpha^\delta < \kappa$ for every $\alpha < \kappa$.*

Proof. Let I be such an ideal. Since Player 1 does not have a winning strategy $\mathbf{P}_I$ must be δ-distributive. Therefore, by Theorem 4, I is δ-normal. So the conclusion follows from Theorem 5. ■

We explicitly state two more corollaries of Theorem 5.

Corollary 2. (i) *If κ is the successor cardinal of a singular cardinal of cofinality $\aleph_0$, then $\mathcal{P}_\kappa\lambda$ cannot carry an $\aleph_0$-distributive ideal.*

(ii) *If $\mathcal{P}_{\aleph_2}\lambda$ carries an $\aleph_0$-distributive ideal*, *then the continuum hypotesis holds.*

In [7] the following proposition is used.

Proposition. *If I is a δ-normal $(\lambda^\delta)^+$-saturated ideal over $\mathcal{P}_\kappa\lambda$, then I is precipitous.*

We can improve this result.

Theorem 6. *If I is a δ-normal $(\lambda^\delta)^+$-saturated ideal over $\mathcal{P}_\kappa\lambda$, then I is δ-distributive.*

Proof. Let $\langle D_\alpha : \alpha < \delta\rangle$ be a sequence of open dense subsets of $\mathbf{P}_I$ and X be any I-positive set. We will show that there is some I-positive $Y \subseteq X$ such that $Y \in \bigcap_{\alpha<\delta} D_\alpha$.

For each $\alpha < \delta$, let A_α be a maximal antichain $\subseteq D_\alpha$. Since $\mathbf{P}_I$ is $(\lambda^\delta)^+$-saturated, $|A_\alpha| \le \lambda^\delta$ for each $\alpha < \delta$. For each $\alpha < \delta$, label the elements of A_α with elements of ${}^\delta\lambda$ so that $A_\alpha = \{W^\alpha_{\vec{a}} : \vec{a} \in {}^\delta\lambda\}$. Without loss of generality we may assume that if $s \in W^\alpha_{\vec{a}}$ then $\vec{a} \in {}^\delta s$. By shrinking X if necessary, we may assume that $X \subseteq \bigcup_{\vec{a}\in{}^\delta\lambda} W^\alpha_{\vec{a}}$ for each $\alpha < \delta$. For each $\alpha < \delta$, define a function $f_\alpha : X \to {}^\delta\lambda$ such that $s \in W^\alpha_{f_\alpha(s)}$ for each $s \in X$. By applying the δ-normality of I to a function obtained by weaving f_α's, we have an I-positive $Y \subseteq X$ such that f_α is constant on Y for each $\alpha < \delta$. Therefore $Y \in \bigcap_{\alpha<\delta} D_\alpha$. ∎

The above proof of Theorem 6 is due to Yasuo Yoshinobu. We originally proved Theorem 6 using a generic elementary embedding.

4. Some consequences of stronger ideals. Theorem 3 gave us an upper bound for the consistency strength of stronger ideals. In this section we want to investigate the consequences of stronger ideals. Some of these consequences will provide us with lower bounds for stronger ideals.

First the following result shows us that ω-strategically closed normal ideals are rather strong in consistency.

Theorem 7. *If there exists an ω-strategically closed normal ideal over $\mathcal{P}_\kappa\lambda$, then for every regular cardinal δ such that $\kappa \le \delta \le \lambda$ every stationary subset of $\{\alpha < \delta : \mathrm{cf}(\alpha) = \omega\}$ reflects, so $\square_\mu$ fails for $\kappa \le \mu < \lambda$.*

Proof. Suppose $A \subseteq \{\alpha < \delta : \mathrm{cf}(\alpha) = \omega\}$ is stationary. We want to show that there is some $\beta < \delta$ with $\mathrm{cf}(\beta) > \omega$ such that $A \cap \beta$ is stationary in β. Let I be an ω-strategically closed normal ideal on $\mathcal{P}_\kappa\lambda$ and G be a $\mathbf{P}_I$-generic filter over V. Let $j : V \to M$ be the corresponding generic elementary embedding.

Now we want to show that $V[G] \models$ "A is stationary in δ". Let $\bar{A} = \{s \in (\mathcal{P}_{\aleph_1}\delta)^V : \sup s \in A\}$. Since every ω-strategically closed poset is proper, we know $V[G] \models$ "$\bar{A}$ is stationary in $\mathcal{P}_{\aleph_1}\delta$". Now work in $V[G]$. Note that if $X \subseteq \mathcal{P}_{\aleph_1}\delta$ is stationary, then $\{\sup s : s \in X\}$ is stationary in δ. So $A = \{\sup s : s \in \bar{A}\}$ is stationary in δ.

We will show that $M \models$ "$j''A$ is stationary in $\sup j''\delta$". Let $C \in M$ be a subset of $\sup j''\delta$ such that $M \models$ "C is a club subset of $\sup j''\delta$". Let $D = \{\alpha < \delta : j(\alpha) \in C\}$. It is easy to see that $V[G] \models$ "D is ω-closed and unbounded in δ". So $A \cap D \neq \emptyset$. Say $\alpha \in A \cap D$. So $j(\alpha) \in j''A \cap C$. Therefore $j''A \cap C \neq \emptyset$, showing $M \models$ "$j''A$ is stationary in $\sup j''\delta$".

Since $j''A \subseteq j(A) \cap \sup j''\delta$, $\sup j''\delta < j(\delta)$, and $\mathrm{cf}^M(\sup j''\delta) > \omega$, we have $M \models \exists\gamma < j(\delta)\ (j(A) \cap \gamma$ is stationary in γ and $\mathrm{cf}(\gamma) > \omega)$. So by the elementarity of j, there exists some $\gamma < \delta$ such that $A \cap \gamma$ is stationary in γ with $\mathrm{cf}(\gamma) > \omega$. ∎

Todorčević [11] proved that the singular cardinal hypothesis, reflection of stationary sets and Chang's conjecture follow from Rado's conjecture.

He also proved in [10] that Rado's conjecture is equivalent to the following statement about trees:

Rado's conjecture (tree version). *A tree T is special* (*i.e. the union of countably many antichains*) *if and only if every subtree of T of size $\aleph_1$ is special.*

Furthermore in [10] Todorčević proved that Rado's conjecture holds in the model obtained by Levy collapsing a supercompact cardinal to $\aleph_2$. It is not difficult to see that the same proof shows that Rado's conjecture follows from the existence of ω-strategically closed ideals over $\mathcal{P}_{\aleph_2}\lambda$ for every λ.

Theorem 8. *If for every $\lambda \geq \aleph_2$ there exists an ω-strategically closed normal ideal over $\mathcal{P}_{\aleph_2}\lambda$, then Rado's conjecture holds.*

Proof. Let T be a non-special tree on λ. We will show that T has a non-special subset of size $\aleph_1$. If the height of T is $> \aleph_1$, then the existence of such a subset is clear. Therefore we may assume that the height of T equals $\aleph_1$.

Let I be an ω-strategically closed ideal over $\mathcal{P}_{\aleph_2}\lambda$. Let G be a $\mathbf{P}_I$-generic filter over V and $j : V \to M$ be the corresponding generic elementary embedding. Define $h : \mathcal{P}_{\aleph_2}\lambda \to V$ by $h(s) = T \cap {}^{<\omega_1}s$. Denote $[h]$ by T^*. Then $M \vDash$ "T^* is a subtree of $j(T)$ such that $|T^*| = \aleph_1$".

Claim 4. *$M \vDash$ "T^* is non-special".*

Once this claim is proved, then we know $M \vDash$ "$j(T)$ has a non-special subset of size $\aleph_1$". So by the elementarity of j, we conclude that T has a non-special subset of size $\aleph_1$.

Proof of Claim 4. Foreman [3] proved that if $\mathbf{P}$ is ω-strategically closed and $|\mathbf{P}| \leq \aleph_1$, then $\mathbf{P}$ has a $<\omega_1$-closed dense subset. Therefore if $\mathbf{P}$ is ω-strategically closed, then there is a poset $\mathbf{Q}$ such that $\mathbf{P} * \mathbf{Q}$ has a $<\omega_1$-closed dense subset. Let D be a $<\omega_1$-closed dense subset of $\mathbf{P}_I * \mathbf{Q}$.

Since T^* is isomorphic to T, it is enough to prove that $1 \Vdash_{\mathbf{P}_I}$ "T is non-special". Therefore it is enough to show that $1 \Vdash_{\mathbf{P}_I * \mathbf{Q}}$ "T is non-special". Suppose otherwise. There are some p in $\mathbf{P}_I * \mathbf{Q}$ and a $\mathbf{P}_I * \mathbf{Q}$-name $\dot{f}$ such that $p \Vdash_{\mathbf{P}_I * \mathbf{Q}}$ "$\dot{f} : \check{T} \to \omega$ and $f^{-1}(n)$ is an antichain for every $n \in \omega_1$". For each node a of T define $p_a \in D$ and $n_a \in \omega$ by induction on the $\mathrm{rank}_T(a)$ satisfying the following conditions:

(i) $p_a \Vdash_{\mathbf{P}_I * \mathbf{Q}} \dot{f}(\check{a}) = \check{n}_a$,
(ii) $b <_T a \Rightarrow p_a \leq p_b \leq p$.

We can carry out this construction using the fact that D is a $<\omega_1$-closed dense subset of $\mathbf{P}_I * \mathbf{Q}$.

Now let $h : T \to \omega$ be defined by $h(a) = n_a$. Since this h is defined in the ground model, there must be some $n \in \omega$ such that $h^{-1}(n)$ is not an antichain. So there must be some $b, a \in h^{-1}(n)$ such that $b <_T a$. Then

$p_a \Vdash_{\mathbf{P}_I * \mathbf{Q}} \dot{f}(\check{a}) = \dot{f}(\check{b}) = \check{n}$, which contradicts $p_a \Vdash_{\mathbf{P}_I * \mathbf{Q}}$ "$f^{-1}(\check{n})$ is an antichain". $\blacksquare_{\text{Claim 4 \& Theorem 8}}$

Seeing Theorems 7 and 8 it is natural to ask the following question:

QUESTION. Can the existence of an ω-strategically closed normal ideal over $\mathcal{P}_\kappa\lambda$ imply a local version of the singular cardinal hypothesis?

The next theorem gives a positive answer to this question. The proof of Theorem 9 was inspired by a proof in Foreman [4].

THEOREM 9. *If there is an ω-strategically closed normal ideal over $\mathcal{P}_\kappa\lambda$, then the singular cardinal hypothesis holds between κ and λ.*

Proof. It is enough to show that $\delta^{\aleph_0} = \delta$ for every regular cardinal δ such that $\kappa \le \delta \le \lambda$. Let I be an ω-strategically closed normal ideal over $\mathcal{P}_\kappa\lambda$. Let $A = \langle A_\alpha : \alpha < \delta\rangle$ be a partition of $\{\beta < \delta : \mathrm{cf}(\beta) = \omega\}$ such that each A_α is stationary.

CLAIM 5. *For every regular cardinal δ where $\kappa \le \delta \le \lambda$, $X = \{s \in \mathcal{P}_\kappa\lambda : \forall\alpha < \delta,\ \alpha \in s \cap \delta \leftrightarrow A_\alpha \cap \sup(s\cap\delta)$ is stationary in $\sup(s\cap\delta)\}$ is in the dual filter of I.*

Proof. We will show that X belongs to every $\mathbf{P}_I$-generic filter. Let G be a $\mathbf{P}_I$-generic filter over V. Let $j : V \to M$ be the corresponding generic elementary embedding. To prove that $X \in G$, it is enough to show that in M, $\forall\alpha < j(\delta)$ ($\alpha \in j''\delta \leftrightarrow j(A)_\alpha \cap \gamma$ is stationary in γ), where $\gamma = \sup j''\delta$. Now work in $V[G]$. Assume $\alpha \in j''\delta$. Thus there is some $\beta \in \delta$ such that $j(\beta) = \alpha$. Let C be a club subset of γ. Let $C^* = \{\xi < \delta : j(\xi) \in C\}$. So C^* is an unbounded subset of δ which is closed under ω-increasing sequences. Note that A_β remains stationary in $V[G]$ since I is ω-strategically closed. Therefore $A_\beta \cap C^* \neq \emptyset$. So $j(A)_{j(\beta)} \cap C \neq \emptyset$. This shows that $j(A)_\alpha \cap \gamma$ is stationary in γ. Now conversely assume that $j(A)_\alpha \cap \gamma$ is stationary in γ for some $\alpha < j(\delta)$. Note that $j''\delta$ is an unbounded subset of γ which is closed under ω-increasing sequences. So $j(A)_\alpha \cap \gamma \cap j''\delta \neq \emptyset$. Thus there is some $\xi \in \delta$ such that $j(\xi) \in j(A)_\alpha \cap \gamma \cap j''\delta$. There is a unique $\overline{\alpha} < \delta$ such that $\xi \in A_{\overline{\alpha}}$. So $j(\xi) \in j(A)_{j(\overline{\alpha})}$. This implies $\alpha = j(\overline{\alpha})$ showing $\alpha \in j''\delta$. This completes the proof of $X \in G$. $\blacksquare_{\text{Claim 5}}$

From Claim 5 we know that X is stationary. Let $Y = \{s \cap \delta : s \in X\}$. Then Y is a stationary subset of $\mathcal{P}_\kappa\delta$ and the sup function restricted to Y is one-to-one, so $|Y| \le \delta$. Note $|\mathcal{P}_{\aleph_1}\delta| = |\bigcup_{s\in Y}\mathcal{P}_{\aleph_1}s| \le |Y|\kappa^{\aleph_0} = \delta\cdot\kappa^{\aleph_0}$. But since I is an ω-normal ideal on $\mathcal{P}_\kappa\lambda$, by Theorem 5 we have $\kappa^{\aleph_0} = \kappa$. So $|\mathcal{P}_{\aleph_1}\delta| = \delta$, i.e. $\delta^{\aleph_0} = \delta$. Therefore the singular cardinal hypothesis holds between κ and λ. $\blacksquare_{\text{Theorem 9}}$

Recently Q. Feng [2] proved that the presaturation of $\mathrm{NS}_{\aleph_1}$, the non-stationary ideal on $\aleph_1$, follows from Rado's conjecture. In [8] we proved the following result:

Theorem. *Let λ be a cardinal $\geq 2^{2^{2^{\aleph_0}}}$. If there is an ω-strategically closed normal ideal over $\mathcal{P}_{\aleph_2}\lambda$, then $\mathrm{NS}_{\aleph_1}$ is precipitous. Furthermore if $2^{\aleph_1} = \aleph_2$ then the existence of such an ideal implies that $\mathrm{NS}_{\aleph_1}$ is presaturated.*

We conclude this paper with the following conjecture which seems reasonable in light of Feng's theorem and Theorem 8.

Conjecture. The existence of an ω-strategically closed normal ideal over $\mathcal{P}_{\aleph_2}\lambda$ for sufficiently large λ implies the presaturation of $\mathrm{NS}_{\aleph_1}$.

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Nagoya University
Chikusa-ku, Nagoya, 464-8601, Japan
E-mail: yom@math.nagoya-u.ac.jp

Received 24 May 2001;
in revised form 22 March 2002