FUNDAMENTA
MATHEMATICAE
216 (2012)

Porcupine-like horseshoes: Transitivity, Lyapunov spectrum, and phase transitions

by

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Abstract. We study a partially hyperbolic and topologically transitive local diffeomorphism F that is a skew-product over a horseshoe map. This system is derived from a homoclinic class and contains infinitely many hyperbolic periodic points of different indices and hence is not hyperbolic. The associated transitive invariant set Λ possesses a very rich fiber structure, it contains uncountably many trivial and uncountably many non-trivial fibers. Moreover, the spectrum of the central Lyapunov exponents of $F|_\Lambda$ contains a gap and hence gives rise to a first order phase transition. A major part of the proofs relies on the analysis of an associated iterated function system that is genuinely non-contracting.

Contents

2010 *Mathematics Subject Classification*: 37D35, 37D25, 37E05, 37D30, 37C29.
Key words and phrases: homoclinic class, Lyapunov exponent, non-contracting iterated function system, partial hyperbolicity, phase transition, spectral gap, skew-product, transitive set.

DOI: 10.4064/fm216-1-2

1. Introduction. In this paper we provide examples of non-hyperbolic transitive sets that we call *porcupine-like horseshoes* or, briefly, *porcupines*, which show a rich dynamics although they admit a quite simple formulation. Their dynamics are conjugate to skew-products over a shift whose fiber dynamics are given by genuinely non-contracting iterated function systems (IFS) on the unit interval.

Naively, from a topological point of view, a porcupine is a transitive set that looks like a horseshoe with infinitely many spines attached at various levels and in a dense way. In terms of its hyperbolic-like structure, it is a partially hyperbolic set with a one-dimensional center, whose spectrum of central Lyapunov exponents contains an interval with negative and positive values which, in particular, illustrates that the porcupine is non-hyperbolic. Although the dynamics on the porcupine is transitive, its spectrum of central exponents has a gap and thus gives rise to a first order phase transition.

Our goal is to present these examples and to explore their dynamical properties. We are not aiming for the most general setting possible, but instead want to present the ideas behind our constructions. We think that these examples are representative models for a number of key properties of non-hyperbolic dynamics.

1.1. Non-contracting iterated function systems. The analysis in this paper is essentially built on properties of a certain class of non-contracting iterated function systems associated to the central dynamics of the porcupine.

We consider $f_0, f_1\colon [0,1] \to \mathbb{R}$ that are C^k smooth, $k \geq 1$, and satisfy

- f_0 is orientation preserving, has an expanding fixed point q, a contracting fixed point $p > q$, and no further fixed points in (q, p),
- f_1 is an orientation reversing contraction;

such maps form an open set in the corresponding product topology. And we study the codimension 1 submanifold of maps satisfying the (q,p)-*cycle condition*

$$f_1(p) = q$$

(compare Figure 2).

Now we consider compositions of the maps f_i, $i = 0, 1$. Given a sequence $\xi = (\ldots \xi_{-1}.\xi_0\xi_1\ldots) \in \Sigma_2 := \{0,1\}^{\mathbb{Z}}$, for a point $x \in [q,p]$, we define the (*forward*) *Lyapunov exponent* of the IFS generated by f_0, f_1 at (x,ξ) by

$$\chi(x,\xi) := \lim_{n\to\infty} \frac{1}{n} \log |(f_{[\xi_0\ldots\xi_n]})'(x)|, \quad \text{where} \quad f_{[\xi_0\ldots\xi_n]} := f_{\xi_n} \circ \cdots \circ f_{\xi_0}.$$

We restrict our considerations to points that remain in the interval $[q,p]$ under forward and backward iterations. For that we define the *admissible domain* $I_\xi \subset [q,p]$ by

$$I_\xi := \bigcap_{m\geq 1} (f_{\xi_{-1}} \circ \cdots \circ f_{\xi_{-m}})([q,p]).$$

We obtain the following auxiliary result, which is a one-dimensional version of the main result in this paper (Theorem 2).

Theorem 1. *For the IFS generated by maps f_0, f_1 as above satisfying the (q,p)-cycle condition, we have:*

(A) *There is an uncountable and dense set of sequences $\xi \in \Sigma_2$ such that the admissible domain I_ξ is non-trivial. There is a residual set of sequences $\xi \in \Sigma_2$ such that I_ξ contains a single point only.*

(B) *The points that are fixed with respect to $f_{[\xi_0\ldots\xi_m]}$ for certain $\xi \in \Sigma_2$ and $m \geq 0$ are dense in $[q,p]$. Moreover, there exist $x \in (q,p)$ and $\xi \in \Sigma_2$ such that $\{f_{[\xi_0\ldots\xi_m]}(x)\}_{m\geq 0}$ is dense in $[q,p]$.*

(C) *There exists $\rho \in (0, \log|f_0'(q)|)$ such that the spectrum of all possible Lyapunov exponents is contained in*

$$[\log|f_0'(p)|, \rho] \cup \{\log|f_0'(q)|\}.$$

The cycle condition seems to play a role similar to the Misiurewicz property in one-dimensional dynamics best illustrated by the behavior of the quadratic map $f(x) = 1 - 2x^2$ that has some alike features ([1]). However, we point out that in our case the breaking of hyperbolicity and the spectral gap are not caused by any critical behavior. Note also that in our case the spectrum is richer and contains a continuum with positive and negative values. In our case, the cycle condition and the fact that f_1 is orientation reversing allows transitivity. However, typical orbits only slowly approach the cycle points p (which corresponds to the critical point) and q (which corresponds to the post-critical point) giving rise to some transient behavior and hence to the gap in the spectrum.

([1]) A differentiable interval map satisfies the *Misiurewicz condition* if the forward orbit of a critical point does not accumulate onto critical points. Note that f is conjugate to the tent map in $[-1,1]$ and that this conjugation is differentiable in $(-1,1)$. Thus, in particular, the spectrum of the Lyapunov exponents of f contains only the two values $2\log 2$ and $\log 2$.

It would be interesting to find other representative examples that have a gap in the Lyapunov spectrum and hence indicate the presence of a first order phase transition (the associated pressure function is not differentiable, see Proposition 5.6). We believe that this point deserves special attention. A collection of examples that exhibit phase transitions are provided in [19] in the case of interval maps and in [23] for abstract shift spaces.

1.2. Non-hyperbolic transitive homoclinic classes. We now put the above abstract results into the framework of local diffeomorphisms and return to the analysis of porcupines.

The porcupines considered in this paper are in fact homoclinic classes (see Definitions 1.2 and 1.3) that contain infinitely many saddles of different *indices* (dimension of the unstable direction) scattered throughout the class preventing hyperbolicity. Moreover, they exhibit a rich topological structure in their fibers (which are tangent to the central direction): there are uncountably many fibers whose intersections with the porcupine are continua and infinitely many fibers whose intersections with the porcupine are just points. Further, the spectrum of the central Lyapunov exponents of these sets has a gap and contains an interval (containing positive and negative values). These properties will be stated in Theorem 2 that is a higher-dimensional version of Theorem 1 for local diffeomorphisms.

We point out that porcupines also have strong indications to exhibit a lot of genuinely non-hyperbolic properties, to be explored elsewhere. For example, the transitive porcupines that we construct do not possess the shadowing property and, following the constructions in [17, 13, 5], one can show that they carry non-hyperbolic ergodic measures with large supports and, in view of [4], we expect that they also display robust heterodimensional cycles.

Let us point out some further motivation. Il'yashenko in his lecture [18] presented topological examples of fibered systems over a shift map that possess recurrent sets (which he calls *bony sets*) containing some fibers. Transitive porcupines provide examples of smooth realizations of such systems. Kudryashov [20, 21] recently obtained a quite general open class of smooth skew-product systems which exhibit bony attractors. Our examples are also motivated by the construction in [14] of bifurcating homoclinic classes and the subsequent study of their Lyapunov spectrum in [22]. These sets have indeed porcupine-like features but are "essentially hyperbolic" in the sense that all their ergodic measures are hyperbolic.

Let us now point out two topological properties of our examples (in fact also present in [14]):

(1) the porcupine is the homoclinic class $H(Q^*, F)$ of a saddle Q^* that contains two fixed points P and Q of different indices that are related by a heterodimensional cycle (see Definition 2.4),

(2) the saddle Q has the same index as Q^*, but is not homoclinically related to Q (compare Definition 1.3).

Comparing with the one-dimensional setting in Section 1.1 the points P and Q play the role of p and q, and the heterodimensional cycle corresponds to the cycle property. Our examples are "essentially non-hyperbolic": The porcupines contain infinitely many saddles of different indices. This property (and the fact that the porcupine is transitive) is the main reason that the central Lyapunov spectrum contains an interval with positive and negative values. We remark that (2) is also the main reason for the presence of a gap in the spectrum of the central Lyapunov exponents in [22]. In fact, the condition about the saddle Q in (2) is *necessary* to obtain such a gap (see Lemma 5.1). In our example the spectrum of the Lyapunov exponent associated to the central direction contains a continuum with positive and negative values and an isolated point. This is, as far as we know, the first example with a spectral gap that is essentially non-hyperbolic and is not related to the occurrence of critical points.

We are aware of the fact that sets that display properties (1) and (2) above are quite specific: By the Kupka–Smale theorem saddles of generic diffeomorphisms are not related by heterodimensional cycles, and by [2] property (2) is C^1 non-generic. But this clearly does not imply that the classes discussed here are not representative.

Finally, we would like to mention that this paper provides a systematic study of non-hyperbolic homoclinic classes. Besides the above references, we would like to mention [1], [13], and [5] where ergodic properties (related to the Lyapunov spectrum) of homoclinic classes are stated.

Before presenting our results, let us state precisely the main objects we are going to study.

Definition 1.1 (Partial hyperbolicity). An F-invariant compact set Λ is said to be *partially hyperbolic* if there is a dF-invariant dominated splitting $E^s \oplus E^c \oplus E^u$ where $dF_{|E^s}$ is uniformly contracting, $dF_{|E^u}$ is uniformly expanding, and E^c is non-trivial and non-hyperbolic. We say that E^c is the *central bundle.* The set Λ is called *strongly partially hyperbolic* if the three bundles E^s, E^c, and E^u are non-trivial. See [7, Definition B.1] for more details.

Definition 1.2 (Porcupines). We call a compact F-invariant set Λ of a (local) diffeomorphism F a *porcupine-like horseshoe* or just a *porcupine* if

- Λ is the maximal invariant set in some neighborhood, *transitive* (existence of a dense orbit), and strongly partially hyperbolic with one-dimensional central bundle,

- there is a subshift of finite type $\sigma\colon \Sigma \to \Sigma$ and a semiconjugation $\pi\colon \Lambda \to \Sigma$ such that $\sigma \circ \pi = \pi \circ F$, $\pi^{-1}(\xi)$ contains a continuum for uncountably many $\xi \in \Sigma$ and is a single point for uncountably many $\xi \in \Sigma$.

We call $\pi^{-1}(\xi)$ a *spine* and say that it is *non-trivial* if it contains a continuum. The spine of a point $X \in \Lambda$ is the set $\pi^{-1}(\pi(X))$.

For examples resembling porcupines but having non-trivial spines only we refer to [3, 15, 16]. Concerning bony attractors, according to the definition in [20] such an attractor may have only non-trivial fibers. Furthermore, there are quite interesting examples in [20, 21] of bony attractors where the trivial fibers form a "graph" of a continuous function over a subset of the shift space.

We are in particular interested in the case where σ is the full shift defined on $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ and Λ is a homoclinic class. More precisely, we have the following standard definition:

Definition 1.3 (Homoclinic class). Given a diffeomorphism F, the *homoclinic class* $H(P,F)$ of a saddle point P of F is defined to be the closure of the transverse intersections of the stable and unstable manifolds of the orbit of P. Two saddle points P and Q are said to be *homoclinically related* if the invariant manifolds of their orbits meet cyclically and transversely. We say that a homoclinic class is *non-trivial* if it contains at least two different orbits.

Given a neighborhood U of the orbit of P, we call the closure of the set of points R that are in the transverse intersections of the stable and unstable manifolds of the orbit of P and have an orbit entirely contained in U the *homoclinic class relative to* U. We denote this set by $H_U(P,F)$.

Remark 1.4. Homoclinically related saddles have the same index. We also remark that the homoclinic class of a saddle may contain periodic points that are not homoclinically related to it. Indeed, this is the situation analyzed in this paper. Finally, observe that the homoclinic class $H(P,F)$ coincides with the closure of all saddle points that are homoclinically related to P. Moreover, a homoclinic class is always transitive. Finally, a non-trivial homoclinic class is always uncountable.

Definition 1.5 (Lyapunov spectrum and gaps). Consider a compact invariant set Λ of a diffeomorphism F with a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$. Given a Lyapunov regular point $S \in \Lambda$, its *Lyapunov exponent associated to the central direction* E^c is

$$\chi_c(S) := \lim_{n\to\infty} \frac{1}{n} \log\|dF^n|_{E^c_S}\|. \tag{1.1}$$

We consider the *spectrum of central Lyapunov exponents* of Λ defined by

$$\mathcal{I}^c_{\mathrm{reg}}(\Lambda) := \{\chi_c(S)\colon S \in \Lambda \text{ and } S \text{ is Lyapunov regular}\}.$$

We say that (ρ, ρ') is a *gap* of the spectrum of Λ if there are numbers $\lambda \le \rho < \rho' \le \beta$ such that

$$(\rho, \rho') \cap \mathcal{I}^c_{\mathrm{reg}}(\Lambda) = \emptyset \quad \text{and} \quad \lambda, \beta \in \mathcal{I}^c_{\mathrm{reg}}(\Lambda).$$

The following is our main result.

THEOREM 2. *There are C^1 local diffeomorphisms F having a porcupine Λ_F with the following properties:*

(A) *There is a continuous semiconjugation $\varpi\colon \Lambda_F \to \Sigma_2$, $\sigma \circ \varpi = \varpi \circ F$, such that*

  (a) *There is an uncountable and dense subset of sequences $\xi \in \Sigma_2$ such that $\varpi^{-1}(\xi)$ is non-trivial. There is a residual subset of sequences $\xi \in \Sigma_2$ such that $\varpi^{-1}(\xi)$ is trivial.*

  (b) *There is an uncountable and dense subset of Λ_F with non-trivial spines.*

(B) *The subset of Λ_F of saddles of index $u+1$ is dense in Λ_F. Moreover, Λ_F contains also infinitely many saddles of index u and thus is not uniformly hyperbolic.*

(C) *There are numbers $0 < \rho < \rho'$ such that (ρ, ρ') is a gap of the spectrum of central Lyapunov exponents of Λ_F. Moreover, this spectrum contains an interval with negative and positive values. Furthermore, the pressure function $t \mapsto P(-t \log \|dF|_{E^c}\|)$ is not differentiable at some point, that is, has a first order phase transition.*

Furthermore, there is an open set U such that Λ_F is the (relative) homoclinic class $H = H_U(R, F)$ of a saddle R of index $u + 1 \ge 2$ satisfying:

(D) *The set $H = \Lambda_F$ is the locally maximal invariant set in U. Moreover, this class contains the (relative) non-trivial homoclinic class of a saddle of index u. Further, there is a saddle $Q \in H$ of index $u + 1$ such that $H_U(Q, F) = \{Q\} \subset H$.*

Our examples are associated to step skew-product diffeomorphisms, locally we have

$$F(\widehat{x}, x) = (\Phi(\widehat{x}), f_{\widehat{x}}(x)), \quad \widehat{x} \in [0,1]^n,\ x \in [0,1],$$

where Φ is a horseshoe map and $f_{\widehat{x}} = f_0$ or f_1 for some injective maps f_0 and f_1 of $[0,1]$. We observe that for our analysis we require only C^1 smoothness, that is, less than the often required $C^{1+\varepsilon}$ hypothesis, and we base our proofs on a tempered distortion argument. Any step skew-product diffeomorphism with C^∞ fiber maps f_0, f_1 and with the properties stated below will provide an example for Theorem 2.

Concerning the structure of spines, we find that non-trivial spines are tangent to the central direction E^c and that spines of saddles of index $u+1$ are non-trivial and dense in Λ_F. We also observe that, given any periodic sequence $\xi \in \Sigma_2$, there is a periodic point P_ξ in $\varpi^{-1}(\xi)$. Under some mild additional Kupka–Smale-like hypothesis, the spine of any saddle of index $u+1$ also contains saddles of index u. In fact, in this case, for every periodic sequence ξ there is a saddle $S \in \Lambda_F$ of index u projecting to ξ, $\varpi(S) = \xi$ (see Theorem 4.16).

Question 1.6. *Do there exist examples of porcupine-like transitive sets such that $\varpi^{-1}(\xi)$ contains a continuum for an "even larger" subset of Σ_2? Here larger could mean, for instance, a residual subset of Σ_2 or a set of large dimension and we would like to state this question in a quite vague sense.*

Let us observe that in the examples in [20] the set with non-trivial fibers is "small", though the setting is slightly different from ours.

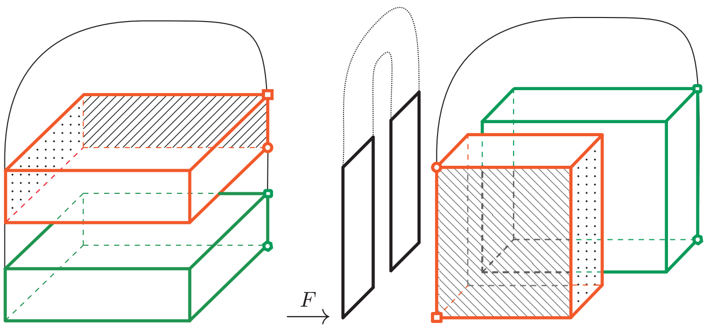

Fig. 1. Construction of a porcupine

This paper is organized as follows. In Section 2 we describe the construction of our examples and derive first preliminary properties. In Section 3 we collect properties of the IFS generated by the interval maps f_0 and f_1. In Section 4 we prove that the porcupine is a (relative) homoclinic class of a saddle of index $u+1$ and that it contains a non-trivial homoclinic class of a saddle of index u. This implies that the porcupine is a transitive non-hyperbolic set containing infinitely many saddles of both types of indices. In that section we will systematically use the results in Section 3. The skew-product structure allows us to translate properties of the IFS to the global dynamics. We will also study some particular cases that imply stronger properties. In Section 5 we finally study the Lyapunov exponents that are associated to the central direction. Note that our methods of proof in Sections 4 and 5 are based on those used previously in studying heterodimensional cycles and homoclinic classes (see for example [10, 2, 4, 14]). We conclude the proof of Theorem 2 in Section 6.

2. Examples of porcupine-like homoclinic classes. In this section we are going to construct examples of porcupine-like homoclinic classes with the properties claimed in Theorem 2.

Consider $s, u \in \mathbb{N}$, the cube $\widehat{\mathbf{C}} = [0,1]^{s+u}$, and a diffeomorphism Φ defined on $\mathbb{R}^{s+u}$ having a horseshoe Γ in $\widehat{\mathbf{C}}$ conjugate to the full shift σ on two symbols and whose stable bundle has dimension s and whose unstable bundle has dimension u. Denote by $\varpi\colon \Gamma \to \Sigma_2$ the conjugation map, $\varpi \circ \Phi = \sigma \circ \varpi$. We consider the subcubes $\widehat{\mathbf{C}}_0$ and $\widehat{\mathbf{C}}_1$ of $\widehat{\mathbf{C}}$ such that Φ maps each $\widehat{\mathbf{C}}_i$ in a Markovian way into $\widehat{\mathbf{C}}$, where $\widehat{\mathbf{C}}_i$ contains all the points X of the horseshoe whose 0-coordinate $(\varpi(X))_0$ is i. In order to produce the simplest possible example, we will assume that Φ is affine in $\widehat{\mathbf{C}}_0$ and $\widehat{\mathbf{C}}_1$.

Definition 2.1 (The map F). Let $\mathbf{C} = \widehat{\mathbf{C}} \times [0,1]$. Given a point $X \in \mathbf{C}$, we write $X = (\widehat{x}, x)$, where $\widehat{x} \in \widehat{\mathbf{C}}$ and $x \in [0,1]$. We consider the map

$$F\colon \widehat{\mathbf{C}} \times [0,1] \to \mathbb{R}^{s+u} \times \mathbb{R}$$

given by

$$F(\widehat{x}, x) := \begin{cases} (\Phi(\widehat{x}), f_0(x)) & \text{if } X \in \widehat{\mathbf{C}}_0 \times [0,1], \\ (\Phi(\widehat{x}), f_1(x)) & \text{if } X \in \widehat{\mathbf{C}}_1 \times [0,1], \end{cases}$$

where $f_0, f_1\colon [0,1] \to [0,1]$ are assumed to be C^1 injective interval maps satisfying the following properties (see Figure 2):

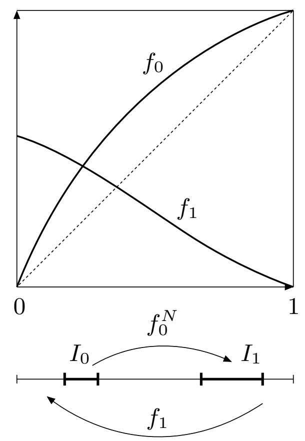

Fig. 2. Iterated function system satisfying (F0), (F1), and (F01)

(F0.i) The map f_0 is increasing and has exactly two hyperbolic fixed points, 0 (repelling) and 1 (attracting). Let $f_0'(0) = \beta > 1$ and $f_0'(1) = \lambda \in (0,1)$. Moreover, $\lambda \le f_0'(x) \le \beta$ for all $x \in [0,1]$.

(F0.ii) There are fundamental domains $I_0 = [a_0, b_0] \subset (0,1)$, $b_0 = f_0(a_0)$, and $I_1 = [a_1, b_1]$, $b_1 = f_0(a_1)$, of the map f_0 together with numbers $\alpha > 1$ and $N \geq 1$ such that

$$f_0^N(I_0) = I_1 \quad \text{and} \quad \lambda \cdot (f_0^N)'(x) > \alpha > 1 \quad \text{for all } x \in I_0.$$

Moreover, f_0 is expanding in $[0, b_0]$ and contracting in $[a_1, 1]$.

(F1.i) The map f_1 is a decreasing contraction satisfying

$$\gamma' := \min\{|f_1'(x)| \colon x \in [0,1]\} \leq \gamma := \max\{|f_1'(x)| \colon x \in [0,1]\} < 1.$$

(F1.ii) We have

$$|f_1'(x)| \geq \overline{\alpha} > 1/\alpha \quad \text{for all } x \in [f_1^2(a_1), a_1].$$

(F01) The following conditions are satisfied:

(1) $f_1(1) = 0$,
(2) $f_1([a_1, 1]) \subset [0, a_0)$,
(3) $[0, f_0^{-2}(b_0)) \subset f_1([0,1])$.

Note that in order to get the conditions above we need to require that

$$\lambda \frac{1-\lambda}{1-\beta^{-1}} > 1.$$

The *maximal invariant set* of F in the cube $\mathbf{C}$ is defined by

$$\Lambda_F := \Lambda_F^+ \cap \Lambda_F^-, \quad \text{where} \quad \Lambda_F^\pm := \bigcap_{i \in \mathbb{N}} F^{\pm i}(\mathbf{C}). \tag{2.1}$$

Remark 2.2. We point out that we restrict our analysis to the dynamics within the cube $\mathbf{C}$. Notice that the usual definition of a (locally) maximal invariant set Λ with respect to F requires that F is well-defined in some neighborhood U of Λ and that $\Lambda = \bigcap_{i \in \mathbb{Z}} F^i(U)$. Observe that in our case we can consider an extension of the local diffeomorphism F to some neighborhood of $\mathbf{C}$ such that Λ_F is the locally maximal invariant set with respect to such an extension. Indeed, this can be done since the extremal points P and Q are hyperbolic.

From now on we restrict our considerations to the dynamics in $\mathbf{C}$. In particular, we consider relative homoclinic classes in $\mathbf{C}$. For notational simplicity, we suppress the dependence on $\mathbf{C}$ and simply write $H(R, F)$.

Note that, by construction, for any saddle $Q^* \in \mathbf{C}$ the homoclinic class $H(Q^*, F)$ is contained in Λ_F but, in principle, may be different from Λ_F. The analysis of the dynamics of $F|_{\Lambda_F}$ will be completed in Section 4.

For simplicity, we assume that the rate of expansion of the horseshoe is stronger than any expansion of f_0 and f_1, that is, in particular, stronger than β, and that the rate of contraction of the horseshoe is stronger than any contraction of f_0 and f_1, that is, in particular, stronger than $\min\{\lambda, \gamma'\}$.

In this way the DF-invariant splitting $E^{ss} \oplus E^c \oplus E^{uu}$ defined over Λ_F and given by

$$E^{ss} := \mathbb{R}^s \times \{0^u, 0\}, \quad E^c := \{0^s, 0^u\} \times \mathbb{R}, \quad E^{uu} := \{0^s\} \times \mathbb{R}^u \times \{0\} \tag{2.2}$$

is dominated. Note that this splitting is DF-invariant because of the skew-product structure of F.

The following is a key result in our constructions. Its proof will be completed in Section 6.

Proposition 2.3. *There is a periodic point $Q^* \in \Lambda_F$ of index $u+1$ whose homoclinic class $H(Q^*, F)$ is a porcupine-like set having all the properties claimed in Theorem* 2.

Let us now introduce some more notation and derive some simple properties that can be obtained from the above definitions.

Notation. We equip the sequence space $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ with the usual metric $d(\xi,\eta) = \sum_{i\in\mathbb{Z}} 2^{-|i|}|\xi_i - \eta_i|$ for $\xi = (\ldots\xi_{-1}.\xi_0\xi_1\ldots)$, $\eta = (\ldots\eta_{-1}.\eta_0\eta_1\ldots) \in \Sigma_2$. We denote by $\xi = (\xi_0 \ldots \xi_{m-1})^{\mathbb{Z}}$ the periodic sequence of period m such that $\xi_i = \xi_{i+m}$ for all i. We will always refer to the least period of a sequence. The zero sequence with $\xi_i = 0$ for all i is denoted by $0^{\mathbb{Z}}$. Further, we denote by $\xi = (0^{\mathbb{N}}.10^{\mathbb{N}})$ the sequence with $\xi_0 = 1$, $\xi_{\pm i} = 0$ for all $i \neq 0$.

Let $\theta = \varpi^{-1}(0^{\mathbb{Z}})$ be the fixed point of Φ which corresponds to the zero sequence $0^{\mathbb{Z}}$. Note that $\theta = 0^{s+u}$. Simplifying the representation, we also assume that $[0,1]^s \times \{0^u\} = W^s_{\mathrm{loc}}(\theta, \Phi)$ and $\{0^s\} \times [0,1]^u = W^u_{\mathrm{loc}}(\theta, \Phi)$. Set

$$P := (\theta, 1) \quad \text{and} \quad Q := (\theta, 0). \tag{2.3}$$

These saddles have indices u and $u+1$, respectively. The previous assumptions and the choice of f_0 imply immediately that

$$\begin{aligned} [0,1]^s \times \{0^u\} \times (0,1] &\subset W^s(P,F), \\ \{0^s\} \times [0,1]^u \times \{1\} &\subset W^u(P,F), \\ [0,1]^s \times \{0^u\} \times \{0\} &\subset W^s(Q,F), \\ \{0^s\} \times [0,1]^u \times [0,1) &\subset W^u(Q,F). \end{aligned} \tag{2.4}$$

In what follows we write

$$W^s_{\mathrm{loc}}(Q,F) = [0,1]^s \times \{(0^s, 0)\} \quad \text{and} \quad W^u_{\mathrm{loc}}(P,F) = \{0^s\} \times [0,1]^u \times \{1\}.$$

Definition 2.4 (Heterodimensional cycle). A diffeomorphism F is said to have a *heterodimensional cycle* associated to saddle points P and Q of different indices if their invariant manifolds *intersect cyclically*, that is, if $W^s(P,F) \cap W^u(Q,F) \neq \emptyset$ and $W^u(P,F) \cap W^s(Q,F) \neq \emptyset$. Here we denote by $W^s(P,F)$ (resp. $W^u(P,F)$) the stable (resp. unstable) manifold of the orbit of P with respect to F.

The definition of F immediately implies the following fact.

LEMMA 2.5 (Heterodimensional cycle). *The points P and Q defined in* (2.3) *are saddle fixed points with indices u and $u+1$, respectively, that are related by a heterodimensional cycle.*

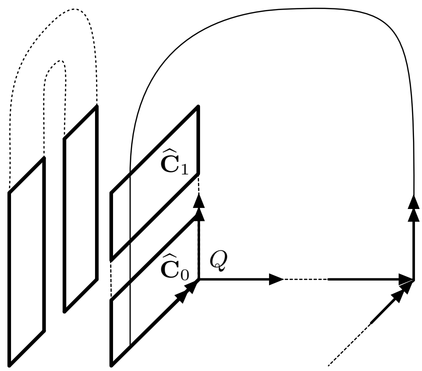

Fig. 3. Heterodimensional cycle

Proof. Note that by (2.4) we have

$$\{(0^s,0^u)\}\times(0,1)\subset W^s(P,F)\cap W^u(Q,F).$$

On the other hand, as $f_1(1)=0$, we have

$$\begin{aligned}F(W^u_{\mathrm{loc}}(P,F)) &= F((\{0^s\}\times[0,1]^u\times\{1\})\cap\mathbf{C}_1)\\ &= \{\theta^s\}\times[0,1]^u\times\{0\}\subset W^u(P,F),\end{aligned}$$

where $\theta^s=\varpi^{-1}(0^{\mathbb{N}}.10^{\mathbb{N}})$ and $(\theta^s,0^u,0)\in W^s(Q,F)$, and hence $W^s(Q,F)\cap W^u(P,F)\neq\emptyset$. This gives a heterodimensional cycle associated to P and Q, proving the lemma. ∎

We will now derive some properties of the homoclinic class $H(P,F)$.

LEMMA 2.6. *The homoclinic class $H(P,F)$ contains the saddle Q. Therefore, this class is non-trivial and non-hyperbolic. Moreover, there are points in $\{(0^s,0^u)\}\times(0,1)$ that are contained in $H(P,F)$.*

Proof. Let $\widehat{x}=(x^s,0^u)=\varpi^{-1}(0^{\mathbb{N}}.10^{\mathbb{N}})$ and $\Phi(\widehat{x})=(y^s,0^u)$. Note that by Definition 2.1 we have

$$F((\{x^s\}\times[0,1]^u\times\{0\})\cap\mathbf{C}_1)=\{y^s\}\times[0,1]^u\times\{f_1(0)\}\subset W^u(P,F), \tag{2.5}$$

and therefore, by (2.4) and since $f_1(0)\in(0,1)$, $(y^s,0^u,f_1(0))$ is a transverse homoclinic point of P. This implies that $H(P,F)$ is non-trivial. Moreover, it implies that $W^u(P,F)$ accumulates at $W^u_{\mathrm{loc}}(P,F)$ from the left and thus $X=(x^s,0^u,0)$ is a limit point of a sequence $X_i:=(x^s_i,0^u,x_i)$, $x_i=f_1(f_0^{N+i}(f_1(0)))>0$, of transverse homoclinic points of P from the right. Finally, as $X\in W^s(Q,F)$ and $H(P,F)$ is invariant, we have $Q\in H(P,F)$.

To prove that $\{(0^s,0^u)\}\times(0,1)$ contains points of $H(P,F)$, for each $\delta>0$ and each (closed) fundamental domain D of f_0 in $(0,1)$ consider the disk

$$D_\delta := [0,\delta]^s\times\{0^u\}\times D\subset W^s(P,F).$$

Note that for large j the set $F^{-j}(D_\delta)$ contains some point X_i. Therefore, D_δ contains a transverse homoclinic point of P. As this holds for any $\delta>0$ and since $H(P,F)$ is a closed set, the set $\{(0^s,0^u)\}\times D$ intersects $H(P,F)$ and the claimed property follows. ∎

Let us consider the attracting fixed point $\widehat{p}=f_1(\widehat{p})\in(0,1)$ and denote by $(\widehat{p}^s,\widehat{p}^u)=\varpi^{-1}(1^{\mathbb{Z}})$ the corresponding fixed point for the horseshoe map Φ. Note that $\widehat{P}=(\widehat{p}^s,\widehat{p}^u,\widehat{p})$ is fixed with respect to F and has index u.

Lemma 2.7. *The saddles $\widehat{P}$ and P are homoclinically related.*

Proof. Let $\widehat{P}=(\widehat{p}^s,\widehat{p}^u,\widehat{p})$ and note that $\widehat{p}^s\times[0,1]^u\times\widehat{p}\subset W^u(\widehat{P},F)$. Thus, together with $\widehat{p}\in(0,1)$ and $[0,1]^s\times\{0^u\}\times(0,1]\subset W^s(P,F)$, we see that $W^u(\widehat{P},F)$ and $W^s(P,F)$ meet transversely.

Let us now show that $W^s(\widehat{P},F)$ and $W^u(P,F)$ also meet transversely. First note that $[0,1]\subset W^s(\widehat{p},f_1)$ and therefore $[0,1]^s\times\{\widehat{p}^u\}\times[0,1]\subset W^s(\widehat{P},F)$. Also note that by (2.5) we have $\{y^s\}\times[0,1]^u\times\{f_1(0)\}\subset W^u(P,F)$, where $f_1(0)$ in $(0,1)$. Thus $W^s(\widehat{P},F)\pitchfork W^u(P,F)\neq\emptyset$, proving the assertion. ∎

Remark 2.8. The construction in the proof of Lemma 2.7 also shows that any saddle $\overline{P}\in\Lambda_F$ of F of index u satisfies $W^u(\overline{P},F)\pitchfork W^s(P,F)\neq\emptyset$ and $W^u(\overline{P},F)\pitchfork W^s(\widehat{P},F)\neq\emptyset$.

As, by construction, $W^s_{\mathrm{loc}}(Q,F)$ does not intersect $W^u(Q,F)\setminus\{Q\}$, we immediately obtain the following fact.

Lemma 2.9. *We have $H(Q,F)=\{Q\}$.*

3. One-dimensional central dynamics. In this section we are going to derive properties of the abstract iterated function system generated by the interval maps f_0 and f_1 introduced in Section 2. These properties will carry over immediately to corresponding properties of the spines. We point out that, in contrast to other commonly studied IFSs, in our case the system is genuinely non-contracting and, in particular, in Section 3.2 we will study *expanding itineraries* of these IFSs.

3.1. Iterated function system. Let us start with some notation.

Notation 3.1. Slightly abusing notation, for a given *finite* sequence $\xi=(\xi_0\ldots\xi_m)$, $\xi_i\in\{0,1\}$, let

$$f_{[\xi]}=f_{[\xi_0\ldots\xi_m]}:=f_{\xi_m}\circ\cdots\circ f_{\xi_1}\circ f_{\xi_0}\colon[0,1]\to[0,1].$$

Moreover, let

$$\mathbf{C}_{[\xi]} = \mathbf{C}_{[\xi_0\ldots\xi_m]} := \{X \in \mathbf{C}\colon F^i(X) \in \mathbf{C}_{\xi_i} \text{ for all } i = 0, \ldots, m\}.$$

Given any set $K \subset \mathbf{C}$, let

$$K_{[\xi]} = K_{[\xi_0\ldots\xi_m]} := K \cap \mathbf{C}_{[\xi_0\ldots\xi_m]}.$$

Given a finite sequence $\xi = (\xi_{-m}\ldots\xi_{-1})$, $\xi_i \in \{0,1\}$, we denote

$$f_{[\xi.]} = f_{[\xi_{-m}\ldots\xi_{-1}.]} := (f_{\xi_{-1}} \circ \cdots \circ f_{\xi_{-m}})^{-1}.$$

Given a finite sequence $\xi = (\xi_{-m}\ldots\xi_{-1}.\xi_0\ldots\xi_n)$, $\xi_i \in \{0,1\}$, let

$$f_{[\xi]} = f_{[\xi_{-m}\ldots\xi_{-1}.\xi_0\ldots\xi_n]} := f_{[\xi_0\ldots\xi_n]} \circ f_{[\xi_{-m}\ldots\xi_{-1}.]}.$$

Note that these maps are only defined on a closed subinterval of $[0,1]$. A sequence $\xi = (\ldots\xi_{-1}.\xi_0\xi_1\ldots) \in \Sigma_2$ is said to be *admissible* for a point $x \in [0,1]$ if the map $f_{[\xi_{-m}\ldots\xi_{-1}.]}$ is well-defined at x for all $m \geq 1$. Note that admissibility of a sequence ξ does not depend on the symbols $(\xi_0\xi_1\ldots)$.

3.2. Expanding itineraries. We now start investigating expanding behavior of the iterated function system.

Recall that $I_0 = [a_0, b_0] = [f_0^{-1}(b_0), b_0]$. Given a closed interval $J \subset [f_0^{-2}(b_0), b_0]$, we start by localizing an itinerary for which the iterated function system is expanding. We will always assume that closed intervals are non-trivial. Recall the definition of a_1 in (F0.ii) and define

$$n(J) := \min\{n \geq 1\colon f_0^n(J) \subset [a_1, 1)\}.$$

Now let $J' := f_{[0^{n(J)}1]}(J)$ and observe that by (F01)(2) this interval is contained in $(0, a_0)$. Let

$$m(J) := \min\{m \geq 0\colon f_0^m(J') \cap I_0 \neq \emptyset\}.$$

Note that, by our choice of fundamental domains, we have $m(J) \geq 1$ and either $n(J) = N$ or $N+1$ with N given in (F0.ii).

Lemma 3.1 (Expanding itineraries). *There is a constant $\kappa > 1$ such that for every closed interval $J \subset [f_0^{-2}(b_0), b_0]$ and every $x \in J$ we have*

$$|(f_{[0^{n(J)}1\,0^{m(J)}]})'(x)| \geq \kappa.$$

Proof. Recall that $n(J)$ is N or $N+1$ and that $m(J) \geq 1$. Observe that the hypotheses (F0.i), (F0.ii), and (F1.ii) imply that for any $x \in J$ we have

$$|(f_{[0^{n(J)}1\,0^{m(J)}]})'(x)| = |(f_0^{m(J)} \circ f_1 \circ f_0^{n(J)})'(x)| \geq \overline{\alpha}\frac{\alpha}{\lambda}\lambda > 1,$$

using the fact that $f_0^{m(J)}$ is applied to points in an interval $[0, a_0]$ where $f_0' > 1$. Taking $\kappa := \overline{\alpha}\alpha$ proves the lemma. ∎

DEFINITION 3.2 (Expanding successor). To every interval $J \subset [f_0^{-2}(b_0), b_0]$, we associate the finite sequence $\xi(J)$ given by

$$\xi_0 = \cdots = \xi_{n(J)-1} = 0, \quad \xi_{n(J)} = 1, \quad \xi_{n(J)+1} = \cdots = \xi_{n(J)+m(J)} = 0,$$

where $n(J)$ and $m(J)$ are defined as above. In view of Lemma 3.1, we call $(\xi_0 \dots \xi_{n(J)+m(J)})$ the *expanding itinerary* of J. We call the interval

$$f_{[\xi_0 \dots \xi_{n(J)+m(J)}]}(J) \subset (0,1)$$

the *expanded successor* of J. We say that an interval J'' is the *ith expanded successor of* J if there is a sequence of intervals $J_0 = J, J_1, \dots, J_{i-1}, J_i = J''$ such that for all $j = 0, \dots, i-1$, we have

$$J_j \subset [f_0^{-2}(b_0), b_0] \quad \text{and} \quad J_{j+1} \text{ is the expanded successor of } J_j.$$

We denote the ith expanded successor of J by $J_{\langle i \rangle}$. Using this notation we define the finite sequence $\xi_{\langle i \rangle} = \xi(J_{\langle i \rangle})$ with

$$J_{\langle i+1 \rangle} = f_{[\xi_{\langle i \rangle}]}(J_{\langle i \rangle}). \tag{3.1}$$

We denote by $|\xi_{\langle i \rangle}|$ the length of this sequence.

REMARK 3.3. For future applications we remark that $n(J)$ and $m(J)$ are both bounded from above by some number independent of the interval $J \subset [f_0^{-2}(b_0), b_0]$. Therefore, the definition of the expanded successor of an interval J involves a concatenation of a number of maps f_0 and f_1 that is bounded by some constant independent of J. In particular, there are constants κ_1, $\kappa_2 > 0$ independent of J such that for all $x \in J$ we have

$$\kappa_1 \le |(f_{[0^{n(J)}1\,0^{m(J)}]})'(x)| \le \kappa_2.$$

In what follows we denote by $|I|$ the length of an interval I.

REMARK 3.4. Let J be a closed subinterval in $[f_0^{-2}(b_0), b_0]$. By Lemma 3.1, there is a constant $\kappa > 1$ independent of J such that the expanded successor $J_{\langle 1 \rangle} = f_{[\xi(J)]}(J)$ of J satisfies

$$|J_{\langle 1 \rangle}| = |f_{[\xi(J)]}(J)| \ge \kappa |J|.$$

Moreover, by the definition of $m(J)$, the interval $J_{\langle 1 \rangle}$ intersects $[f_0^{-1}(b_0), b_0]$.

The following lemma is the main result of this subsection.

LEMMA 3.5. *Given a closed interval $J \subset [f_0^{-2}(b_0), b_0]$, there is a number $i(J) \ge 1$ such that the jth expanded successor $J_{\langle j \rangle}$ of J is defined for all $j = 1, \dots, i(J) - 1$ and $J_{\langle i(J) \rangle}$ contains the fundamental domain $[f_0^{-2}(b_0), f_0^{-1}(b_0)]$.*

Proof. Note that the expanded successor is defined for any interval $J \subset (f^{-2}(b_0), b_0]$. Assume, inductively, that for all $j = 0, \dots, i$ the jth expanded successor $J_{\langle j\rangle}$ of $J = J_{\langle 0\rangle}$ is defined and $J_{\langle j\rangle} \subset [f_0^{-2}(b_0), b_0]$. Then the $(i+1)$th expanded successor $J_{\langle i+1\rangle}$ of J is also defined.

Since for every $j = 0, \dots, i$ the interval $J_{\langle j+1\rangle}$ is the successor of $J_{\langle j\rangle}$, by Lemma 3.1 we have

$$|J_{\langle i+1\rangle}| \geq \kappa^{i+1} |J_{\langle 0\rangle}|.$$

Since the size of $[f_0^{-2}(b_0), b_0]$ is bounded there is a first $i(J)$ such $J_{\langle 0\rangle}, J_{\langle 1\rangle}, \dots, J_{\langle i(J)\rangle}$ are defined and $J_{\langle i(J)\rangle}$ is not contained in $[f_0^{-2}(b_0), b_0]$. Since by Remark 3.4 the interval $J_{\langle i(J)\rangle}$ intersects $[f_0^{-1}(b_0), b_0]$, this implies

$$[f_0^{-2}(b_0), f_0^{-1}(b_0)] \subset J_{\langle i(J)\rangle},$$

from which the claimed property follows. ∎

The following result can be of independent interest.

Proposition 3.6 (Sweeping property). *Given a closed interval $H \subset (0,1)$, there is a finite sequence $(\xi_0 \dots \xi_n)$ such that $f_{[\xi_0 \dots \xi_n]}(H)$ contains the fundamental domain $[f_0^{-2}(b_0), f_0^{-1}(b_0)]$.*

Proof. Just note that there exists $m \geq 1$ such that $f_{[0^m 1]}(H) \subset (0, f_0^{-2}(b_0))$. Therefore, there is $k \geq 1$ such that $f_{[0^m 1 0^k]}(H)$ contains an interval $J \subset (f_0^{-2}(b_0), b_0)$ to which we can apply Lemma 3.5. ∎

Definition 3.7 (Expanding sequence). In view of Lemma 3.5, given a closed interval $J \subset [f_0^{-2}(b_0), b_0]$ we consider its (finite) *expanding sequence* $\xi(J)$ obtained by concatenating the finite sequences $\xi_{\langle i\rangle} = \xi(J_{\langle i\rangle})$ corresponding to the expanding successors $\xi_{\langle 1\rangle}, \dots, \xi_{\langle i(J)\rangle}$ of J.

Note that, by the definition of $i(J)$, we have $[f_0^{-2}(b_0), b_0] \subset f_{[\xi(J)]}(J)$ and $|(f_{[\xi(J)]})'(x)| > 1$ for all $x \in J$.

An immediate consequence of Lemma 3.5 and the previous comments is the following lemma.

Lemma 3.8. *Given a closed interval $J \subset [f_0^{-2}(b_0), b_0]$ and its expanding sequence $\xi(J)$, there is a unique expanding fixed point $q_J^* \in J$ of $f_{[\xi(J)]}$. Moreover, $W^u(q_J^*, f_{[\xi(J)]})$ contains $[f_0^{-2}(b_0), f_0^{-1}(b_0)]$.*

Proof. Observe that $J \subset [f_0^{-2}(b_0), b_0] \subset f_{[\xi(J)]}(J)$ and $f_{[\xi(J)]}$ is uniformly expanding in J. ∎

3.3. Lyapunov exponents close to 0. In this section we are going to construct fixed points (of contracting and expanding type) of certain maps $f_{[\xi_0 \dots \xi_{m-1}]}$ whose Lyapunov exponents are arbitrarily close to 0. Here, given $p \in [0,1]$ and an admissible sequence $\xi = (\dots \xi_{-1}.\xi_0\xi_1 \dots) \in \Sigma_2$ of p, the (*forward*) *Lyapunov exponent* of p with respect to ξ is

$$\chi(p,\xi) := \lim_{n\to\infty} \frac{1}{n} \log |(f_{[\xi_0\dots\xi_{n-1}]})'(p)|$$

whenever this limit exists. Otherwise we denote by $\underline{\chi}(p,\xi)$ and $\overline{\chi}(p,\xi)$ the *lower* and the *upper Lyapunov exponent* defined by taking the lower and the upper limit, respectively.

Given a periodic sequence $(\xi_0\dots\xi_{m-1})^{\mathbb{Z}} \in \Sigma_2$ and a point $p_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}} = f_{[\xi_0\dots\xi_{m-1}]}(p_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}})$, we have

$$\chi(p,(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}) = \frac{1}{m} \log |(f_{[\xi_0\dots\xi_{m-1}]})'(p_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}})|. \tag{3.2}$$

We are going to prove the existence of periodic points of contracting and expanding type with Lyapunov exponents arbitrarily close to 0.

PROPOSITION 3.9. *For every $\varepsilon > 0$ there exists a finite sequence $(1^\ell 0^m 1 0^j)$ such that the map $f_{[1^\ell 0^m 1 0^j]}$ is uniformly contracting in $[0,1]$ and its fixed point p is attracting, has a Lyapunov exponent in $(-\varepsilon, 0)$, and has a stable manifold $W^s(p, f_{[1^\ell 0^m 1 0^j]})$ that contains the interval $[0,1]$.*

PROPOSITION 3.10. *For every $\varepsilon > 0$ there exists a finite sequence $(\xi_0\dots\xi_{n-1})$ such that the map $f_{[\xi_0\dots\xi_{n-1}]}$ has an expanding fixed point whose Lyapunov exponent is in $(0,\varepsilon)$.*

Before proving the two propositions, we formulate some preliminary results.

3.3.1. *Tempered distortion.* First, we verify a distortion property. Note that we establish the tempered distortion property that holds true if f_0 is only a C^1 map, instead of focusing on a bounded distortion property that would require the standard $C^{1+\varepsilon}$ assumption.

We will say that an interval $J \subset (0,1)$ *contains at most K consecutive fundamental domains* of f_0 if any orbit of f_0 hits this interval at most $K+1$ times.

LEMMA 3.11 (Tempered distortion). *Given $\widehat{p} \in (0,1)$ and $K \ge 1$, there exists a positive sequence $(\rho_k)_{k\ge 0}$ decreasing to 0 such that for every interval J containing $\widehat{p}$ and containing at most K consecutive fundamental domains of f_0 we have*

$$e^{-k\rho_{|k|}} \frac{|f_0^{\pm k}(J)|}{|J|} \le (f_0^{\pm k})'(x) \le e^{k\rho_{|k|}} \frac{|f_0^{\pm k}(J)|}{|J|}$$

for all $k \in \mathbb{Z}$ and $x \in J$.

Proof. Let $x, y \in J$. As f_0' is bounded away from 0 and the map $y \mapsto \log y$ is Lipschitz if y is bounded away from 0, there exists some positive constant

c and a positive sequence $(\widetilde{\rho}_k)_k$ decreasing to 0 such that for every $k \geq 1$,

$$\left|\log \frac{(f_0^k)'(x)}{(f_0^k)'(y)}\right| \leq c \sum_{n=0}^{k-1} |f_0'(f_0^n(x)) - f_0'(f_0^n(y))| \leq c \sum_{n=0}^{k-1} \widetilde{\rho}_n.$$

Here the latter estimate follows from continuity of f_0' and the fact that $|f_0'(x) - f_0'(y)| \to 0$ as $|x - y| \to 0$ and the observation that $|f_0^n(J)| \to 0$ as $n \to \infty$. Now set

$$\rho_k := \frac{c(\widetilde{\rho}_0 + \cdots + \widetilde{\rho}_{k-1})}{k}$$

and note that $\rho_k \to 0$ as $k \to \infty$. Taking y such that $|(f_0^k)'(y)| = |f^k(J)|/|J|$ we get the claimed property.

Analogously, $|f_0^{-n}(J)| \to 0$ as $n \to \infty$, from which we can deduce the case $k \leq 0$. ∎

3.3.2. *Looping orbits.* We now show that the derivative along a looping orbit starting in and returning to a fixed fundamental domain grows only subexponentially with respect to its length.

LEMMA 3.12. *Given a fundamental domain $J \subset (0,1)$, there exists a number $K \geq 1$ such that for all $m \geq 1$ sufficiently large the interval $f_{[0^m 1]}(J)$ contains at most K consecutive fundamental domains of f_0.*

Proof. For each $m \geq 1$ define $a_m \in (0,1)$ by

$$f_0^m(J) = [1 - a_m, f_0(1 - a_m)].$$

Note that, because $f_0(1) = 1$, $f_0'(1) = \lambda$ and the derivative f_0' is continuous, and $f_0^m(J)$ converges to 1 (and thus $a_m \to 1$) as $m \to \infty$, if m is large enough we have

$$|f_0^m(J)| = f_0(1 - a_m) - (1 - a_m) \approx f_0(1) - a_m f_0'(1) - (1 - a_m) = a_m(1 - \lambda)$$

and hence

$$\frac{1}{2} a_m (1 - \lambda) \leq |f_0^m(J)| \leq 2 a_m (1 - \lambda). \tag{3.3}$$

Recalling the definitions of γ and γ' in (F1.i), one has

$$\gamma' \frac{1}{2} a_m (1 - \lambda) \leq |f_1(f_0^m(J))| \leq \gamma 2 a_m (1 - \lambda). \tag{3.4}$$

Similarly one finds that there is a constant $C > 1$ independent of large m such that $C^{-1} a_m \leq f_1(f_0(1 - a_m)) \leq C a_m$. Hence

$$f_{[0^m 1]}(J) \subset [C^{-1} a_m, C a_m (1 + 2\gamma(1 - \lambda))]. \tag{3.5}$$

Noting that the derivative of f_0 in $f_{[0^m 1]}(J)$ is close to β and larger than some β' close to β, we see that for large m the interval $f_{[0^m 1]}(J)$ contains at most $\ell + 2$ fundamental domains where ℓ is the largest natural number with

$$(\beta')^\ell C^{-1} a_m \leq C a_m (1 + 2\gamma(1 - \lambda)).$$

Notice that a_m cancels and hence ℓ does not depend on m if m is large enough. This finishes the proof of the lemma. ∎

We will now use the above lemma to prove the following.

LEMMA 3.13. *Given a fundamental domain $J \subset (0,1)$ of f_0, there exists $m_0 \geq 1$ and a positive sequence $(\rho_n)_n$ decreasing to 0 such that for every $m \geq m_0$ there exists $j > 0$ such that the interval $f_{[0^m 10^j]}(J)$ intersects J. If $j = j(m)$ is the smallest positive number with this property then*

$$e^{-j\rho_j - m\rho_m} \leq |(f_{[0^m 10^j]})'(x)| \leq e^{j\rho_j + m\rho_m}$$

for every $x \in J$ and every $m \geq m_0$ and $j = j(m)$.

Proof. Let $J = [a, f_0(a)] \subset (0,1)$ be a fundamental domain with respect to f_0. There exists $m_0 \geq 1$ such that for every $m \geq m_0$ and every $x \in J$ we have $f_0^m(x) > f_1^{-1}(a)$, and $f_1(f_0^m(x))$ is in the expanding region of f_0. Moreover, the interval $f_{[0^m 10^j]}(J)$ intersects J for some $j > 0$.

As in the above proof, for each $m \geq m_0$ set

$$f_0^m(J) = [1 - a_m, f_0(1 - a_m)].$$

Just as for (3.3) we obtain

$$\frac{1}{2} a_m (1 - \lambda) \leq |f_0^m(J)| \leq 2 a_m (1 - \lambda).$$

The tempered distortion result in Lemma 3.11 now implies that there is a positive sequence $(\widehat{\rho}_k)_k$ decreasing to 0 such that for all $x \in J$,

$$\frac{1}{2} \frac{a_m(1-\lambda)}{|J|} e^{-m\widehat{\rho}_m} \leq (f_0^m)'(x) \leq 2 \frac{a_m(1-\lambda)}{|J|} e^{m\widehat{\rho}_m}. \tag{3.6}$$

We now consider the fundamental domain of f_0,

$$L := [f_0^{-1}(f_1(1 - a_m)), f_1(1 - a_m)].$$

Note that by Lemma 3.12 the interval $f_{[0^m 1]}(J)$ contains at most K fundamental domains. By our choice the right extreme of L is the right extreme of $f_{[0^m 1]}(J)$ and therefore

$$f_{[0^m 1]}(J) \subset \widetilde{L} := L \cup f_0^{-1}(L) \cup \cdots \cup f_0^{-K}(L).$$

In the next step we compare the lengths of $\widetilde{L}$ and $f_0^m(J)$. Arguing exactly as above, using the fact that $f_0'(0) = \beta$, $f_0'(x) \leq \beta$, $\gamma' \leq |f_1'| \leq \gamma$, and that $f_1(1 - a_m)$ is close to $0 = f_1(1)$ for m large enough, we obtain

$$\frac{1}{2} \gamma a_m (1 - \beta^{-1}) \leq |L| \leq \gamma' a_m (1 - \beta^{-1}). \tag{3.7}$$

Since for $i \geq 0$ we have $\beta^{-i}|L| \leq |f_0^{-i}(L)| \leq (\beta')^{-i}|L|$ for some $1 < \beta' < \beta$, from (3.7) we immediately obtain constants $k_1, k_2 > 0$ such that

$$k_1 a_m \leq |\widetilde{L}| \leq k_2 a_m. \tag{3.8}$$

Fix constants M^- and M^+ such that if J', J'' are any two non-disjoint fundamental domains of f_0 then

$$M^-|J'| \le |J''| \le M^+|J'|. \tag{3.9}$$

For large m the sets $f_{[0^m1]}(J)$ and $\widetilde{L}$ are both to the left of J. Thus there exists a smallest positive integer $j = j(m)$ such that $f_0^j(f_{[0^m1]}(J))$ (and thus $f_0^j(\widetilde{L})$) intersects J for the first time (the same number for both intervals). We now apply the tempered distortion property in Lemma 3.11 to $\widetilde{L}$. Hence, there exists a sequence $(\widetilde{\rho}_k)_k$ decreasing to 0 such that for all $x \in \widetilde{L}$,

$$e^{-j\widetilde{\rho}_j}\frac{|f_0^j(\widetilde{L})|}{|\widetilde{L}|} \le |(f_0^j)'(x)| \le e^{j\widetilde{\rho}_j}\frac{|f_0^j(\widetilde{L})|}{|\widetilde{L}|}.$$

The definition of $\widetilde{L}$ and (3.9) imply that

$$M^-|J| \le |f_0^j(\widetilde{L})| \le \widetilde{M}^+|J|, \quad \text{where} \quad \widetilde{M}^+ = \sum_{j=0}^{K}(M^+)^j.$$

Thus, by the previous two equations, for $x \in \widetilde{L}$ we obtain

$$e^{-j\widetilde{\rho}_j}\frac{M^-|J|}{|\widetilde{L}|} \le |(f_0^j)'(x)| \le e^{j\widetilde{\rho}_j}\frac{\widetilde{M}^+|J|}{|\widetilde{L}|}.$$

This inequality together with (3.8) implies that for all $x \in \widetilde{L}$ (and thus for all $x \in f_{[0^m1]}(J)$) we have

$$\frac{e^{-j\widetilde{\rho}_j}M^-|J|}{k_2a_m} \le |(f_0^j)'(x)| \le \frac{e^{j\widetilde{\rho}_j}\widetilde{M}^+|J|}{k_1a_m}. \tag{3.10}$$

Now putting together (3.6) and (3.10) and recalling that $\gamma' \le |f_1'| \le \gamma$, we see that the factors $|J|$ and a_m cancel. Hence we obtain, for every $x \in J$,

$$e^{-j\widehat{\rho}_j - m\widetilde{\rho}_m}\frac{(1-\lambda)\gamma' M^-}{2k_2} \le |(f_0^j \circ f_1 \circ f_0^m)'(x)| \le e^{j\widehat{\rho}_j + m\widetilde{\rho}_m}\frac{2(1-\lambda)\gamma\widetilde{M}^+}{k_1}.$$

Thus there is some $\widetilde{C} > 1$ independent of m and $j = j(m)$ such that

$$\widetilde{C}^{-1}e^{-j\widehat{\rho}_j - m\widetilde{\rho}_m} \le |(f_0^j \circ f_1 \circ f_0^m)'(x)| \le \widetilde{C}e^{j\widehat{\rho}_j + m\widetilde{\rho}_m}.$$

The claimed property now follows with $\rho_n := \max\{\widehat{\rho}_n, \widetilde{\rho}_n + \frac{1}{2n}\log\widetilde{C}\}$. ■

3.3.3. *Weak contracting and expanding looping orbits.* We are now ready to prove the above propositions.

Proof of Proposition 3.9. Recall that we denoted by $\widehat{p} \in (0,1)$ the attracting fixed point of f_1. Consider a fundamental domain J of f_0 containing $\widehat{p}$ in its interior, and some $\ell_0 \ge 1$ such that for all $\ell \ge \ell_0$ the interval $f_1^\ell([0,1])$ is contained in J. This is possible because $f_1^\ell([0,1])$ converges to $\widehat{p}$.

By Lemma 3.13, there exists $m_0 \geq 1$ and a positive sequence $(\rho_n)_n$ decreasing to zero such that for every $m \geq m_0$ and $j = j(m)$ the intervals $f_{[0^m 10^j]}(J)$ and J intersect and for every $x \in J$ we have

$$e^{-j\rho_j - m\rho_m} \leq |(f_{[0^m 10^j]})'(x)| \leq e^{j\rho_j + m\rho_m}.$$

Recall now the choice of the constants γ and γ' in (F1.i). Therefore, for every $m \geq m_0$ and $j = j(m)$, every $\ell \geq \ell_0$, and every $x \in [0,1]$ we obtain

$$(\gamma')^\ell e^{-j\rho_j - m\rho_m} \leq |(f_{[1^\ell 0^m 10^j]})'(x)| \leq e^{j\rho_j + m\rho_m} \gamma^\ell. \tag{3.11}$$

Let now $\ell = \ell(m,j) \geq \ell_0$ be the smallest such that the right-hand side in (3.11) is < 1, so that

$$\frac{j\rho_j + m\rho_m}{-\log\gamma} < \ell \leq \frac{j\rho_j + m\rho_m}{-\log\gamma} + 1. \tag{3.12}$$

Since $f_1^\ell([0,1]) \subset J$ we can apply the above estimates to any $x \in f_1^\ell([0,1])$. Thus, $f_{[1^\ell 0^m 10^j]}$ is a contraction in $[0,1]$ and hence has a unique fixed point $p_{[1^\ell 0^m 10^j]}$ whose basin of contraction contains $[0,1]$. Moreover, the Lyapunov exponent $\chi(p_{[1^\ell 0^m 10^j]}, \xi)$ with $\xi = (1^\ell 0^m 10^j)^{\mathbb{Z}}$ satisfies

$$M(\ell,m,j) := \frac{\ell\log\gamma' - j\rho_j - m\rho_m}{j+m+\ell+1} \leq \chi(p_{[1^\ell 0^m 10^j]}, \xi) < 0.$$

Using (3.12), we deduce that

$$\frac{1}{j+m+1}\log\gamma'\biggl(\frac{j\rho_j + m\rho_m}{-\log\gamma} + 1\biggr) - \frac{1}{j+m+1}(j\rho_j + m\rho_m) \leq M(\ell,m,j).$$

When we now take m arbitrarily large the index $j = j(m)$ is also large. Hence the lower bound $M(\ell,m,j)$ is arbitrarily close to 0. As the exponent is negative, this finishes the proof of the proposition. ∎

Proof of Proposition 3.10. Take the fundamental domain $I = [f_0^{-1}(b_0), b_0]$. By Lemma 3.13, there exists $m_0 \geq 1$ and a positive sequence $(\rho_k)_k$ decreasing to zero such that for every $m \geq m_0$ and $j = j(m)$ the intervals $f_{[0^m 10^j]}(I)$ and I intersect and for every $x \in I$ we have

$$e^{-j\rho_j - m\rho_m} \leq |(f_{[0^m 10^j]})'(x)| \leq e^{j\rho_j + m\rho_m}. \tag{3.13}$$

This implies, possibly after slightly decreasing $(\rho_k)_k$, that

$$|f_{[0^m 10^j]}(I) \cap (f_0^{-1}(I) \cup I)| \geq e^{-j\rho_j - m\rho_m}. \tag{3.14}$$

We now consider the subinterval

$$J_{m,j} := f_{[0^m 10^j]}(I) \cap (f_0^{-1}(I) \cup I)$$

and its expanded successors as defined in Definition 3.2.

First recall that by Remark 3.3 the number of applications of the maps f_0 and f_1 involved in the definition of the expanded successor of an interval is uniformly bounded from above and below by numbers $N_1 > N_2 \geq 1$ that do not depend on the interval. Moreover, by Remark 3.3 each expanded itinerary has a uniform expansion bounded from below and above by numbers $\kappa_1 > \kappa_2 > 1$ independent of the itinerary.

Therefore, as the length of the interval in (3.14) is bounded from below and each expanded successor involves a uniform expansion bounded from below by κ_2, we need to repeat the expanded successors a finite number $\bar{N} = \bar{N}(m,j)$ of times to obtain the covering of the fundamental domain $[f_0^{-2}(b_0), f_0^{-1}(b_0)]$ as stated in Lemma 3.5. Now we denote by $F_{m,j}$ the resulting concatenated map. Moreover, by construction the interval

$$f_{[\xi_0\dots\xi_n]}(J_{m,j}) := (f_0 \circ F_{m,j})(J_{m,j})$$

covers the original interval I and hence there exists an expanding fixed point $p_{m,j} \in I$ of $f_{[\eta]}$ with $\eta := (0^m 1 0^j \xi_0 \dots \xi_n)$. Moreover, by the comments above, n satisfies $\bar{N}N_2 + 1 \le n < \bar{N}N_1 + 1$.

We finally estimate the Lyapunov exponent of $p_{m,j}$. Using the length estimate of $J_{m,j}$ in (3.14) and the uniform expansion of each expanded successor by a factor of at least κ_2, we can estimate

$$\bar{N} \le \frac{C + j\rho_j + m\rho_m}{\log \kappa_2}, \tag{3.15}$$

where $C > 0$ only depends on the length of $[f_0^{-2}(b_0), f_0^{-1}(b_0)]$. Hence, by (3.13) and since each expanding iterate expands by at most κ_1, the Lyapunov exponent at $p_{m,j}$ satisfies

$$0 < \chi(p_{m,j},\eta) \le \frac{\bar{N}\log\kappa_1 + j\rho_j + m\rho_m}{\bar{N}N_2 + j + m} \le \frac{\bar{N}\log\kappa_1 + j\rho_j + m\rho_m}{j+m},$$

Now (3.15) implies that

$$\chi(p_{m,j},\eta) \le \frac{C + j\rho_j + m\rho_m}{\log\kappa_2}\,\frac{\log\kappa_1}{j+m} + \frac{j\rho_j + m\rho_m}{j+m}.$$

Recall that $j = j(m)$ is large when m is large. Thus, ρ_j, $\rho_m \to 0$ and hence this exponent is arbitrarily close to 0. ∎

3.4. Admissible domains. In this subsection we explore the rich structure of admissible domains.

Notation 3.2. Given $\xi = (\dots\xi_{-1}.\xi_0\xi_1\dots) \in \Sigma_2$ and $m \ge 1$, let

$$I_{[\xi_{-m}\dots\xi_{-1}.]} := f_{\xi_{-1}} \circ \dots \circ f_{\xi_{-m}}([0,1]). \tag{3.16}$$

This is always a non-trivial subinterval of $[0,1]$. Note that $f_{\xi_i}([0,1]) \subset [0,1]$ for every $\xi_i \in \{0,1\}$. Therefore, for each one-sided infinite sequence $\xi = (\dots\xi_{-2}\xi_{-1}.)$ the sets $I_{[\xi_{-m}\dots\xi_{-1}.]}$ form a nested sequence of non-empty compact intervals. Thus, the set

$$I_{[\xi]} := \bigcap_{m\geq 1} I_{[\xi_{-m}\cdots\xi_{-1}.]}$$

is either a singleton or a non-trivial interval. For completeness, for each $n \geq 0$ we write

$$I_{[\xi_{-m}\cdots\xi_{-1}.\xi_0\cdots\xi_n]} := I_{[\xi_{-m}\cdots\xi_{-1}.]}.$$

REMARK 3.14. Note that given a sequence $\xi \in \Sigma_2$, any point $x \in I_{[\xi]}$ is admissible for ξ. Observe that for all $m \geq 1$ the interval $I_{[\xi_{-m}\cdots\xi_{-1}.]}$ is the maximal domain of $f_{[\xi_{-m}\cdots\xi_{-1}.]}$, which justifies our notation.

Note that any $\xi \in \Sigma_2$ is given by $\xi = \xi^-.\xi^+$, where $\xi^+ \in \Sigma_2^+ := \{0,1\}^{\mathbb{N}}$ and $\xi^- \in \Sigma_2^- := \{0,1\}^{-\mathbb{N}}$.

PROPOSITION 3.15. *We have the following properties:*

(1) *The set $\{\xi \in \Sigma_2 \colon I_{[\xi]}$ is a single point$\}$ is residual in Σ_2.*
(2) *Given $\xi^+ \in \Sigma_2^+$, the set $\{\xi^- \in \Sigma_2^- \colon I_{[\xi^-.\xi^+]}$ is non-trivial$\}$ is uncountable and dense in Σ_2^-.*

Moreover, for every closed interval $J \subset [0,1)$ and every $\xi^+ \in \Sigma_2^+$ the set of $\xi^- \in \Sigma_2^-$ with $I_{[\xi^-.\xi^+]} \supset J$ is uncountable.

We postpone the proof of this proposition until the end of this section. Let us first collect some basic properties of the admissible domains $I_{[\xi]}$.

LEMMA 3.16. *Given a finite sequence $(\xi_{-m}\ldots\xi_{-1})$, if $\xi_{-k} = 1$ for at least two indices $k \in \{1,\ldots,m\}$ then $I_{[\xi_{-m}\cdots\xi_{-1}.]} \subset (0,1)$.*

Proof. Let $m \geq 1$ be the smallest such that $\xi_{-m} = 1$. By property (F01) we have $I_{[\xi_{-m}\cdots\xi_{-1}.]}([0,1]) = [0,c_1] \subset [0,1)$. Recall that $f_0([0,1]) = [0,1]$ and $f_0(0) = 0$. If $k \geq m$ is the smallest such that $\xi_{-k} = 1$ then

$$I_{[\xi_{-k}\cdots\xi_{-m}\cdots\xi_{-1}.]} = [f_1 \circ f_0^{k-m-1}(c_1), f_1(0)],$$

where $f_1 \circ f_0^{k-m-1}(c_1) > 0$ and $f_1(0) < 1$. This implies that for all $\ell \geq k$ we have $I_{[\xi_{-\ell}\cdots\xi_{-1}.]} \subset (0,1)$. This proves the lemma. ∎

Note that $f_0^k([0,1]) = [0,1]$ implies that

$$I_{[0^k\xi_{-m}\cdots\xi_{-1}.]} = (f_{\xi_{-1}} \circ \ldots \circ f_{\xi_{-m}})(f_0^k([0,1])) = I_{[\xi_{-m}\cdots\xi_{-1}.]}.$$

This yields the following result.

LEMMA 3.17. *For all $k \geq 1$ we have $I_{[0^k\xi_{-m}\cdots\xi_{-1}.]} = I_{[\xi_{-m}\cdots\xi_{-1}.]}$.*

LEMMA 3.18. *Given $\xi = (\ldots\xi_{-1}.\xi_0\xi_1\ldots) \in \Sigma_2$ satisfying $\xi_{-m} = 0$ for all $m \geq m_0$ for some $m_0 \geq 1$, the domain $I_{[\xi]} = I_{[\xi_{-m_0}\cdots\xi_{-1}.]}$ is a non-trivial interval.*

Proof. Recall that $I_{[\xi_{-m}\cdots\xi_{-1}.]}$ is a non-trivial interval for any $m \geq 1$. Hence, the claim follows from Lemma 3.17. ■

We now start investigating the structure of admissible domains. Recall the choice of the constants γ and β in (F1.i) and (F0.i).

DEFINITION 3.19. We call a sequence $\xi = (\dots\xi_{-1}.\xi_0\xi_1\dots) \in \Sigma_2$ *asymptotically contracting* if for every $m \geq 1$ we have

$$\limsup_{m\to\infty} \gamma^{k_m}\beta^{m-k_m} = 0, \quad \text{where} \quad k_m = \sum_{i=1}^{m} \xi_{-i}. \tag{3.17}$$

With (3.16) and properties (F0.i) and (F1.i), if ξ is asymptotically contracting the length of the interval $I_{[\xi]}$ satisfies

$$|I_{[\xi]}| \leq \lim_{m\to\infty} \gamma^{k_m}\beta^{m-k_m} = 0.$$

The following lemma is hence an immediate consequence.

LEMMA 3.20. *For every asymptotically contracting sequence $\xi \in \Sigma_2$ the set $I_{[\xi]}$ consists of a single point. Moreover, if $\xi = (\xi_0 \dots \xi_{m-1})^{\mathbb{Z}}$ is periodic, then this point is an attracting fixed point of the map $f_{[\xi_0\cdots\xi_{m-1}]}$.*

REMARK 3.21. Given any asymptotically contracting sequence $\eta = \eta^-.\eta^+ \in \Sigma_2$, the set $\{\eta^-.\xi^+ \colon \xi^+ \in \Sigma_2^+\}$ consists of only asymptotically contracting sequences. Observe that this set is clearly uncountable.

LEMMA 3.22. *The function $\xi \mapsto |I_{[\xi]}|$ is upper semicontinuous but not continuous. However, it is continuous at every $\eta \in \Sigma_2$ for which $I_{[\eta]}$ is a single point.*

Proof. Let $\eta \in \Sigma_2$. Since the $I_{[\eta_{-n}\cdots\eta_{-1}.]}$ form a nested sequence of compact intervals, their length is non-increasing and for any $\varepsilon > 0$ there exists $N \geq 1$ such that $|I_{[\eta]}| \leq |I_{[\eta_{-n}\cdots\eta_{-1}.]}| < |I_{[\eta]}| + \varepsilon$ for every $n \geq N$. For every $\xi \in \Sigma_2$ with $d(\xi,\eta) \leq \delta := \sum_{|i|>N} 2^{-|i|}$ we have $\xi_i = \eta_i$ for every $|i| \leq n$ and thus $|I_{[\xi_{-n}\cdots\xi_{-1}.]}| = |I_{[\eta_{-n}\cdots\eta_{-1}.]}|$. Since $I_{[\xi_{-n}\cdots\xi_{-1}.]}$ is also nested, we obtain

$$|I_{[\xi]}| \leq |I_{[\xi_{-n}\cdots\xi_{-1}.]}| = |I_{[\eta_{-n}\cdots\eta_{-1}.]}| < |I_{[\eta]}| + \varepsilon,$$

which implies upper semicontinuity.

If $I_{[\eta]}$ is a single point only then $|I_{[\eta_{-n}\cdots\eta_{-1}.]}| = |I_{[\xi_{-n}\cdots\xi_{-1}.]}| < \varepsilon$ and

$$\big||I_{[\eta]}| - |I_{[\xi]}|\big| \leq \big||I_{[\eta]}| - |I_{[\eta_{-n}\cdots\eta_{-1}.]}|\big| + |I_{[\xi_{-n}\cdots\xi_{-1}.]}| - |I_{[\xi]}| \leq 2\varepsilon.$$

This implies continuity at η.

Finally, observe that the function $\xi \mapsto |I_{[\xi]}|$ is not continuous in general. Indeed, recall that $I_{[\eta]} = [0,1]$ for $\eta = (0^{\mathbb{Z}})$. However, as every sequence $\xi \in \Sigma_2$ with $\xi_{-i} = 1$ for all i large enough is asymptotically contracting, Lemma 3.20 shows that $I_{[\xi]}$ then contains only one point. Clearly, such a ξ can be chosen arbitrarily close to η. ■

To complete our analysis of admissible domains let us consider the sets $I_{[\xi]}$ that contain a repelling point.

LEMMA 3.23. *For every periodic sequence $\xi = (\xi_0\xi_1\ldots\xi_{m-1})^{\mathbb{Z}} \in \Sigma_2 \setminus \{0^{\mathbb{Z}}\}$ for which $f_{[\xi_0\ldots\xi_{m-1}]}$ has a repelling fixed point q, the set $I_{[\xi]}$ is a non-trivial interval. Moreover, this interval contains points p_∞ and $\widetilde{p}_\infty$ with $p_\infty < q < \widetilde{p}_\infty$ that are fixed for $f^2_{[\xi_0\ldots\xi_{m-1}]}$.*

Proof. Assume that q is a repelling fixed point of $f_{[\xi_0\ldots\xi_{m-1}]}$. Consider a point $p \in W^u_{\mathrm{loc}}(q, f_{[\xi_0\ldots\xi_{m-1}]})$ and note that the sequence $p_k := f^{2k}_{[\xi_0\ldots\xi_{m-1}]}(p)$, $k \geq 1$, is well-defined. Assume that $p < q$. Note that $f^{2k}_{[\xi_0\ldots\xi_{m-1}]}$ preserves orientation and hence we can set $p_\infty := \lim_{k\to\infty} p_k \in [0,1]$. Observe that $f^2_{[\xi_0\ldots\xi_{m-1}]}(p_\infty) = p_\infty$. Since $p \in W^u_{\mathrm{loc}}(q, f_{[\xi_0\ldots\xi_{m-1}]})$ and $p \neq q$, we conclude $p_\infty \neq q$. Then, since $f^2_{[\xi_0\ldots\xi_{m-1}]}$ preserves orientation, we conclude that $[p_\infty, q] \subset I_{[\xi]}$.

Completely analogously, we can show that $I_{[\xi]}$ contains an interval $[q, \widetilde{p}_\infty]$ where $\widetilde{p}_\infty = f^2_{[\xi_0\ldots\xi_{m-1}]}(\widetilde{p}_\infty) \neq q$. This completes the proof. ∎

Observe that, as a consequence of Proposition 3.10, there are infinitely many periodic sequences ξ such that $I_{[\xi]}$ contains a repelling periodic point and therefore is non-trivial, by Lemma 3.23.

We now start by analyzing further properties of admissible domains. Continuing Remark 3.21, we show that the set of asymptotically contracting sequences is in fact much richer.

LEMMA 3.24. *Given any $\xi^+ \in \Sigma_2^+$, the set*

$$\mathcal{A}_{\xi^+} := \{\eta = \eta^-.\xi^+ : \eta^- \in \Sigma_2^- \text{ and } \eta \text{ is asymptotically contracting}\}$$

is uncountable.

Proof. Fix any $\xi^+ \in \Sigma_2^+$. To prove that $\mathcal{A}_{\xi^+} \subset \Sigma_2$ is uncountable we use the standard Cantor diagonal argument.

Assume for contradiction that $\mathcal{A}_{\xi^+}$ is countable. Notice that for every asymptotically contracting sequence $\xi = (\ldots\xi_{-1}.\xi_0\xi_1\ldots) \in \Sigma_2$, we cannot have $\xi_{-i} = 0$ for all large enough i. Let $\widetilde{\mathcal{A}}_{\xi^+} \subset \mathcal{A}_{\xi^+}$ consist of the sequences $\xi = (\ldots\xi_{-1}.\xi^+)$ for which $\xi_{-i} = 0$ for infinitely many i. Clearly, this set is also countable. Consider some enumeration of it,

$$\widetilde{\mathcal{A}}_{\xi^+} = \{\xi^1 = (\ldots 1^{L^1_2}0^{K^1_2}1^{L^1_1}0^{K^1_1}.\xi^+), \xi^2 = (\ldots 1^{L^2_2}0^{K^2_2}1^{L^2_1}0^{K^2_1}.\xi^+), \ldots\}.$$

Here we allow also $K^k_1 = 0$, in which case the symbol 0 is neglected. We now construct a "new" sequence that is in $\widetilde{\mathcal{A}}_{\xi^+}$ but not in that enumeration. Let $\bar{L}_1 = L^1_1 + 1$ and for $k \geq 2$ choose

$$\bar{L}_{k+1} > \max\{\bar{L}_k, L^{k+1}_{k+1}\} \quad \text{such that} \quad \beta\cdot\gamma^{\sum_{i=1}^{k+1}\bar{L}_i}\cdot\beta^k < \frac{1}{2^k}.$$

Observe that the sequence $(\dots 01^{\bar L_2}01^{\bar L_1}0.\xi^+)$ is asymptotically contracting. By construction this sequence is not in the enumeration of $\widetilde{\mathcal{A}}_{\xi^+}$ above, which is a contradiction. Hence, $\widetilde{\mathcal{A}}_{\xi^+}$ (and thus $\mathcal{A}_{\xi^+}$) is uncountable. ∎

The following lemma shows that the set of sequences ξ having a non-trivial admissible domain $I_{[\xi]}$ is also very rich.

Lemma 3.25. *For every closed interval $J \subset (0,1)$ and every $\xi^+ \in \Sigma_2^+$ the set*

$$\mathcal{A}_{\xi^+,J} := \{\xi = (\xi^-.\xi^+)\colon \xi^- \in \Sigma_2^- \text{ and } I_{[\xi]} \supset J\}$$

is uncountable.

Proof. First, let us show that $\mathcal{A}_{\xi^+,J}$ is non-empty. Note that given any $J \subset (0,1)$, there exists $k = k(J) \ge 1$ such that $f_0^{-k}(J) \subset (0,c_1)$, where $c_1 = f_1(0)$. Let now $J_0 := J$ and $K_1 := k(J_0) = k(J)$ and note that by the definition in (3.16) we have

$$J_0 \subset f_0^{K_1}([0,c_1]) = (f_0^{K_1} \circ f_1)([0,1]) = I_{[10^{K_1}.]}.$$

For $\ell \ge 0$ define recursively

$$K_{\ell+1} \ge k(J_\ell) \quad \text{such that} \quad f_0^{-K_{\ell+1}}(J_\ell) \subset (0,c_1)$$

and

$$J_{\ell+1} := (f_1^{-1} \circ f_0^{-K_{\ell+1}})(J_\ell).$$

Then

$$J_{\ell+1} = (f_1^{-1} \circ f_0^{-K_{\ell+1}} \circ \dots \circ f_1^{-1} \circ f_0^{-K_1})(J) = f_{[10^{K_{\ell+1}}\dots 10^{K_1}.]}(J) \subset [0,1]$$

and hence $J \subset I_{[10^{K_{\ell+1}}\dots 10^{K_1}.]}$ for every $\ell \ge 1$ (recall Remark 3.14). This implies $J \subset I_{[\xi]}$ for $\xi = (\dots 10^{K_\ell}\dots 10^{K_1}.\xi^+)$, proving that $\mathcal{A}_{\xi^+,J}$ is non-empty.

We point out that, since $f_0^{-m}(J_\ell) \subset (0,c_1)$ for every $m > K_{\ell+1}$, we can repeat the construction above replacing at each step K_ℓ by any $\overline{K}_\ell > K_\ell$. In this way, we get a new sequence $\overline{\xi}$ such that $J \subset I_{[\overline{\xi}]}$. In particular, the set of sequences ξ such that $J \subset I_{[\xi]}$ is infinite.

The above remark guarantees that the set $\mathcal{A}_{\xi^+,J}$ is infinite. To prove that it is uncountable we use again the Cantor diagonal argument. Arguing towards a contradiction, assume that $\mathcal{A}_{\xi^+,J}$ is countable and consider its subset defined by

$$\begin{aligned}\widetilde{\mathcal{A}}_{\xi^+,J} := \{\xi = (\dots 0^{K_3}10^{K_2}10^{K_1}.\xi^+) \in \Sigma_2\colon & \\ & \text{there are infinitely many blocks of 0s of length } K_\ell \\ & \text{satisfying } K_{\ell+1} > K_\ell \text{ for all } \ell \ge 1 \\ & \text{and } I_{[\xi_{-m}\dots\xi_{-1}.]} \supset J \text{ for all } m \ge 1\}\end{aligned}$$

and its enumeration

$$\widetilde{\mathcal{A}}_{\xi^+,J} = \{\xi^1 = (\dots 10^{K^1_\ell} \dots 10^{K^1_1}.\xi^+), \xi^2 = (\dots 10^{K^2_\ell} \dots 10^{K^2_1}.\xi^+), \dots\}.$$

Let now $J'_0 = J$ and $\bar{K}_1 = \max\{K^1_1 + 1, k(J'_0) + 1\}$ with $J'_0 \subset f_{[0^{\bar{K}_1}]}([0, c_1])$ and write $J'_1 := (f_1^{-1} \circ f_0^{-\bar{K}_1})([0, 1])$. Note that $J'_1 \subset I_{[10^{\bar{K}_1}]}$. Arguing inductively, for $\ell \geq 2$ choose

$$\bar{K}_{\ell+1} > \max\{\bar{K}_\ell, K^{\ell+1}_{\ell+1}, k(J'_\ell) + 1\} \quad \text{and} \quad J'_{\ell+1} := (f_1^{-1} \circ f_0^{-\bar{K}_{\ell+1}})(J'_\ell).$$

If we bear in mind the above remark and argue as above, these choices give

$$I_{[10^{\bar{K}_{\ell+1}}10^{\bar{K}_\ell}1 \dots 10^{\bar{K}_1}.]} \supset J'_{\ell+1}.$$

Clearly, none of the sequences $(\dots 10^{\bar{K}_\ell} \dots 10^{\bar{K}_1}.\xi^+)$ is in $\widetilde{\mathcal{A}}_{\xi^+,J}$, contradicting that $\widetilde{\mathcal{A}}_{\xi^+,J}$ is countable. Hence $\mathcal{A}_{\xi^+,J}$ is uncountable. ∎

We finally provide the proof of our proposition.

Proof of Proposition 3.15. We first prove that the set of sequences with trivial spines is residual. As an immediate consequence of Definition 3.19, given any sequence $\xi = (\dots \xi_{-1}.\xi_0\xi_1 \dots) \in \Sigma_2$, for any $m \geq 1$ the sequence $\xi' = (\dots 11\xi_{-m} \dots \xi_{-1}.\xi_0\xi_1\xi_2 \dots)$ is asymptotically contracting. Moreover, by Lemma 3.20 the domain $I_{[\xi']}$ is a single point only. Clearly, the distance between ξ and ξ' can be made arbitrarily small by increasing m. This proves that the sequences ξ such that $I_{[\xi]}$ is trivial are dense in Σ_2. Given $\varepsilon > 0$, set

$$\mathcal{A}_\varepsilon := \{\xi \in \Sigma_2 \colon |I_{[\xi]}| \leq \varepsilon\}.$$

The second statement in Lemma 3.22 in particular implies that $\mathcal{A}_\varepsilon$ contains an open and dense subset of Σ_2. Thus, $\bigcap_{n\geq 1} \mathcal{A}_{1/n}$ contains a residual subset that consists of sequences for which $I_{[\xi]}$ is a single point. This proves the first part of the proposition.

We now look at the set of sequences with non-trivial spines. Given any sequence $\xi = (\dots \xi_{-1}.\xi_0\xi_1 \dots) \in \Sigma_2$, recall that $I_{[\xi_{-m}\dots\xi_{-1}.]}$ for any $m \geq 1$ is a non-trivial interval. By Lemma 3.17 we have $I_{[0^k\xi_{-m}\dots\xi_{-1}.]} = I_{[\xi_{-m}\dots\xi_{-1}.]}$ for any $k \geq 1$. Further, recall that $I_{[\xi_{-m}\dots\xi_{-1}.]} = I_{[\xi_{-m}\dots\xi_{-1}.\xi_0\dots\xi_m]}$. Thus, the sequence $\xi' = (\dots 00\xi_{-m} \dots \xi_{-1}.\xi_0\xi_1 \dots)$ satisfies $I_{[\xi']} = I_{[\xi_{-m}\dots\xi_{-1}.]}$ and hence contains an interval. Clearly, the distance between ξ and ξ' can be made arbitrarily small by increasing m. Together with Lemma 3.25, this proves the second part of the proposition. ∎

3.5. Gap in the Lyapunov spectrum. We finally establish some gap in the spectrum of Lyapunov exponents.

Proposition 3.26 (Spectral gap). *Let*

$$\widetilde{\chi} := \sup\{\chi(p, \xi) \colon p \in [0, 1],\ \xi \in \Sigma_2 \setminus E\},$$

where

$$E := \{\xi = (\xi^-.\xi^+) \colon \xi^+ = (\xi_0 \dots \xi_k 0^{\mathbb{N}}),\, k \ge 0\}.$$

Then $\widetilde{\beta} := \exp \widetilde{\chi} < \beta$.

Proof. The idea of the proof is quite simple although the proof itself is a bit technical. Note that the exponent $\chi(p,\xi)$ could be close to β only if the orbit $\{f_{[\xi_0\dots\xi_m]}(p)\}_{m\ge1}$ stays arbitrarily close to 0 infinitely often. Note also that each visit close to 0 is preceded by a visit close to 1. This implies that the effect of expansion (iterates of f_0 close to 0) will be compensated by a (previous) contraction (iterates of f_0 close to 1 and some iterates of f_1) that will force the exponent to decrease. Now we will provide the details.

For simplicity of exposition we assume that f_0 is non-linear in a neighborhood of 0. A similar proof can be done in the general case. Let $\delta > 0$ be close to 0 and define

$$H_0 := [0,\delta], \qquad H_0' := f_1^{-1}(H_0) = [h_0', 1].$$

Recalling condition (F01), we see that if δ is small enough then

$$f_1(H_0) \cap H_0 = \emptyset, \quad f_1(H_0') \cap H_0' = \emptyset, \quad f_1([0,1]) \cap H_0' = \emptyset. \tag{3.18}$$

We first introduce some constants that will be used throughout the proof:

$$\widetilde{\alpha} := \min_{H_0'} |f_1'|, \quad \widehat{\alpha} := \max_{H_0'} |f_1'|. \tag{3.19}$$

Note that $\widetilde{\alpha}/\widehat{\alpha} \sim 1$ if δ is small enough. Further define

$$\beta_0 := \sup\{f_0'(x) \colon x \notin H_0\} = \sup\{f_0'(x), f_1'(x) \colon x \notin H_0\} < \beta.$$

Observe that $\beta_0 < \beta$ follows from our simplifying assumption that f_0 is non-linear close to 0. Let

$$\beta_0' := \inf\{f_0'(x) \colon x \in H_0\} < \beta, \quad \lambda_0' := \sup\{f_0'(x) \colon x \in H_0'\}.$$

Note that if δ is small then λ_0' and β_0' are close to λ and β and thus

$$|\log \lambda| \frac{\log \beta}{\log \beta_0'} - |\log \lambda_0'| + \frac{\log \beta}{2} < \log \beta_0'. \tag{3.20}$$

To prove the proposition, note that it is enough to consider the case that $\xi_i = 0$ for infinitely many $i \ge 1$. Indeed, otherwise, because f_1 is a contraction, we have $\chi(p,\xi) < 0$. Moreover, by replacing p by some iterate, we can assume that $p \ne 0$. Further, we can assume that the orbit $\{f_{[\xi_0\dots\xi_m]}(p)\}_{m\ge0}$ hits the interval H_0 infinitely many times. Indeed, otherwise this orbit is contained in the interval $(\delta,1]$ in which the derivatives f_0' and f_1' are upper bounded by β_0' and thus the Lyapunov exponent of $\chi(p,\xi)$ is upper bounded by $\log \beta_0'$. Hence, without loss of generality, possibly replacing p by some positive iterate, we can assume that $p \in H_0$ and $f_{\xi_0}(p) \notin H_0$.

For every $m \ge 0$ write

$$p_{m+1} := f_{[\xi_0\dots\xi_m]}(p).$$

We define three increasing sequences $(r_k)_k$, $(e_k)_k$, $(i_k)_k$ of positive integers as follows (compare Fig. 4): $i_k < r_k \le e_k < i_{k+1}$,

(3.21) $$p_j \in H_0 \quad \text{if and only if} \quad r_k \le j \le e_k \text{ for some } k,$$

(3.22) $$p_j \in H_0' \quad \text{if and only if} \quad i_k \le j \le r_k - 1 \text{ for some } k.$$

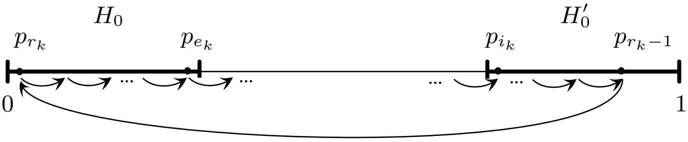

Fig. 4. Definition of the sequences $(r_k)_k$, $(e_k)_k$, $(i_k)_k$

Note that indeed by our choices the only way of entering H_0 is by coming from H_0' after applying f_1, and the only way of entering H_0' is after applying f_0. More precisely: Since $f_1(H_0) \cap H_0 = \emptyset$ (recall (3.18)), we have $\xi_j = 0$ for every $j \in \{r_k, \dots, e_k - 1\}$ whenever $r_k < e_k$. By definition of r_k we have $p_{r_k} \in H_0$ and $p_{r_k-1} \notin H_0$ and thus $p_{r_k-1} > p_{r_k}$. Since f_0 is an increasing function, we have $p_{r_k} \ne f_0(p_{r_k-1})$. Thus,

$$p_{r_k} = f_1(p_{r_k-1}), \qquad p_{r_k-1} \in H_0', \quad \text{and} \quad \xi_{r_k-1} = 1.$$

Since $f_1([0,1]) \cap H_0' = \emptyset$ (recall (3.18)), we have $\xi_j = 0$ for every index $j \in \{i_k - 1, \dots, r_k - 2\}$. In particular this implies that

(3.23) $$p_{i_k} \in [h_0', f_0(h_0')).$$

By the definitions of H_0 and of the sequences above we have $p_j \notin H_0$ for all $j \in \{e_k + 1, i_{k+1} - 1\}$, and therefore

(3.24) $$\log |(f_{[\xi_{e_k+1} \dots \xi_{i_{k+1}-1}]})'(p_{e_k+1})| < (\beta_0)^{i_{k+1}-e_k-1}.$$

Denote by N_k the number of iterates of the point p_{i_k} in H_0', that is,

$$N_k := r_k - i_k - 1.$$

CLAIM 3.27. *We have* $p_{r_k} \ge \widetilde{\alpha}\widehat{\alpha}^{-1}\lambda^{N_k+1}\delta$.

Proof. Recall that by (3.23) we have $p_{i_k} < f_0(h_0')$ and hence

$$f_0^{N_k}(p_{i_k}) = p_{r_k-1} < f_0^{N_k}(f_0(h_0')) = f_0^{N_k+1}(h_0').$$

Since $h_0' = f_1^{-1}(\delta)$, with (3.19) we can estimate $1 - h_0' \ge \widehat{\alpha}^{-1}\delta$. Hence, by (F0.i) we can estimate

$$1 - p_{r_k-1} > 1 - f_0^{N_k+1}(h_0') \ge \lambda^{N_k+1}\widehat{\alpha}^{-1}\delta.$$

Finally, since $f_1(0) = 1$, we have

$$p_{r_k} = f_1(p_{r_k-1}) \ge \widetilde{\alpha}\lambda^{N_k+1}\widehat{\alpha}^{-1}\delta. \ \blacksquare$$

By Claim 3.27, $e_k - r_k \le \widetilde{M}_k + 1$, where $\widetilde{M}_k$ is defined by

$$(\beta_0')^{\widetilde{M}_k}\widetilde{\alpha}\widehat{\alpha}^{-1}\lambda^{N_k+1}\delta = \delta,$$

that is,

$$\widetilde{M}_k \le M_k := \left\lfloor \frac{(N_k+1)|\log\lambda| - \log\widetilde{\alpha}/\widehat{\alpha}}{\log\beta_0'} \right\rfloor + 1. \tag{3.25}$$

Let us now estimate the finite-time Lyapunov exponents associated to each of the finite sequences $(\xi_{i_k}\dots\xi_{e_k})$.

Claim 3.28. *There exists $\beta_0'' < \beta$ such that*

$$\frac{\log|(f_{[\xi_{i_k}\dots\xi_{e_k}]})'(p_{i_k})|}{e_k - i_k + 1} \le \log\beta_0''.$$

Proof. We can freely assume that the number of iterations in the interval H_0 is the maximum possible (clearly this is the case that maximizes the derivative $f_{[\xi_{i_k}\dots\xi_{e_k}]}$). That is, suppose that $e_k - r_k = M_k$. Recalling the definition of $\widehat{\alpha}$ in (3.19), with the above we obtain

$$\frac{\log|(f_{[\xi_{i_k}\dots\xi_{e_k}]})'(p_{i_k})|}{e_k - i_k + 1} \le \frac{M_k\log\beta + \log\widehat{\alpha} - N_k|\log\lambda_0'|}{M_k + N_k + 1}.$$

From (3.25) we conclude that

$$\begin{aligned}
&\frac{\log|(f_{[\xi_{i_k}\dots\xi_{e_k}]})'(p_{i_k})|}{e_k - i_k + 1}\\
&\quad\le \frac{1}{M_k+N_k+1}\left[\left((N_k+1)|\log\lambda| - \log\frac{\widetilde{\alpha}}{\widehat{\alpha}}\right)\frac{\log\beta}{\log\beta_0'} + \log\beta - N_k|\log\lambda_0'|\right]\\
&\quad\le \frac{N_k}{M_k+N_k+1}\left(|\log\lambda|\frac{\log\beta}{\log\beta_0'} - |\log\lambda_0'|\right) + \frac{\log\beta}{M_k+N_k+1}\\
&\qquad+ \frac{1}{M_k+N_k+1}\left(|\log\lambda| - \log\frac{\widetilde{\alpha}}{\widehat{\alpha}}\right)\frac{\log\beta}{\log\beta_0'}\\
&\quad\le \left(|\log\lambda|\frac{\log\beta}{\log\beta_0'} - |\log\lambda_0'| + \frac{\log\beta}{2}\right)\\
&\qquad+ \frac{1}{M_k+N_k+1}\left[|\log\lambda| - \log\frac{\widetilde{\alpha}}{\widehat{\alpha}}\right]\frac{\log\beta}{\log\beta_0'},
\end{aligned}$$

where in the last line we also used $M_k + N_k + 1 \ge 2$. By (3.20),

$$\frac{\log|(f_{[\xi_{i_k}\dots\xi_{e_k}]})'(p_{i_k})|}{e_k - i_k + 1} < \log\beta_0' + \frac{1}{M_k+N_k+1}\left[|\log\lambda| - \log\frac{\widetilde{\alpha}}{\widehat{\alpha}}\right]\frac{\log\beta}{\log\beta_0'}.$$

Let $L \ge 1$ be large enough such that the rightmost term in the last estimate is less than $\varepsilon := (\log\beta - \log\beta_0')/2$ whenever $M_k + N_k + 1 \ge L$, and thus the claim is proved if $M_k + N_k + 1 \geq L$. In the finitely many remaining possible cases with $|f_1'| \leq \gamma$ (recall (F1.i)) we can estimate

$$\frac{\log|(f_{[\xi_{i_k}\dots\xi_{e_k}]})'(p_{i_k})|}{e_k - i_k + 1} \leq \max_{\ell=1,\dots,L} \frac{\log(\beta^\ell\gamma)}{\ell+1} < \max_{\ell=1,\dots,L} \frac{\ell}{\ell+1}\log\beta < \log\beta.$$

Hence, with

$$\beta_0'' := \exp\max\left\{\log\beta_0' + \varepsilon, \max_\ell \frac{\ell}{\ell+1}\log\beta\right\}$$

we have $\beta_0'' < \beta$. This proves the claim. ∎

We are now ready to get an upper bound for $\chi(p,\xi)$. It is enough to consider segments of orbits corresponding to exit times e_k and starting at the point p_1:

$$\frac{\log|(f_{[\xi_1\dots\xi_{e_k}]})'(p_1)|}{e_k} \leq \sum_{j=0}^{k} \frac{e_j - i_j + 1}{e_k} \frac{\log|(f_{[\xi_{i_j}\dots\xi_{e_j}]})'(p_{i_j})|}{e_j - i_j + 1} + \frac{i_{j+1} - e_j - 1}{e_k} \frac{\log|(f_{[\xi_{e_j+1}\dots\xi_{i_{j+1}-1}]})'(p_{e_{j+1}})|}{i_{j+1} - e_j - 1}.$$

By (3.24) and Claim 3.28 this derivative is bounded from above by $\max\{\log\beta_0, \log\beta_0''\} < \log\beta$. This completes the proof of the proposition. ∎

4. Transverse homoclinic intersections

4.1. The maximal invariant set. In this section we are going to prove that the maximal invariant set of F in the cube $\mathbf{C}$ (recall the definition in (2.1)) is the homoclinic class of a saddle of index $u+1$.

Theorem 4.1. *Given the periodic point $q^* = q^*_{I_0} \in [0,1]$ and the expanding sequence $\xi = \xi(I_0)$ provided by Lemma 3.8 applied to the interval $I_0 = [f_0^{-1}(b_0), b_0]$, set $\widehat{q} = \varpi^{-1}(\xi)$. Then the periodic point $Q^* = (\widehat{q}, q^*)$ has index $u+1$ and $\Lambda_F = H(Q^*, F)$.*

Note that this result implies in particular that Λ_F is transitive and contains both saddles of index $u+1$ and of index u and thus is not hyperbolic. To prove Theorem 4.1, we will use the properties of the iterated function system in Section 3. This translation is possible by the skew structure of F. In fact, the following remark follows immediately from this structure.

Remark 4.2. Given a periodic sequence $\xi = (\xi_0\dots\xi_{m-1})^{\mathbb{Z}} \in \Sigma_2$ and a fixed point $r \in [0,1]$ of the map $f_{[\xi_0\dots\xi_{m-1}]}$, there is a canonically associated saddle point $R = (r^s, r^u, r)$ where $(r^s, r^u) = \varpi^{-1}((\xi_0\dots\xi_{m-1})^{\mathbb{Z}})$. If

$|f'_{[\xi_0\dots\xi_{m-1}]}(r)| \neq 1$ then the saddle R is hyperbolic and

$$\begin{aligned}\{r^s\} \times [0,1]^u \times W^u(r, f_{[\xi_0\dots\xi_{m-1}]}) \subset W^u(R,F),\\ [0,1]^s \times \{r^u\} \times W^s(r, f_{[\xi_0\dots\xi_{m-1}]}) \subset W^s(R,F).\end{aligned}$$

Note that if $|f'_{[\xi]}(r)| > 1$ (respectively, $|f'_{[\xi]}(r)| < 1$) then the saddle R has index $u+1$ (respectively, u).

We introduce some notation. Given a periodic sequence $(\xi_0\dots\xi_{m-1})^{\mathbb{Z}} \in \Sigma_2$ and a fixed point $r_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}} = f_{[\xi_0\dots\xi_{m-1}]}(r_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}})$, we will consider the point

$$R_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}} := (\varpi^{-1}((\xi_0\dots\xi_{m-1})^{\mathbb{Z}}), r_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}}),$$

which is periodic under F. Notice that there can exist fibers that contain more than one periodic point, so in general the points $r_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}}$ and hence $R_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}}$ are not unique.

Remark 4.3. Note that Remark 4.2 implies, in particular, that for every periodic point $R \in \Lambda_F \setminus \{Q,P\}$ the manifold $W^u(R,F)$ transversely intersects $[0,1]^s \times \{0^u\} \times (0,1) \subset W^s(P,F)$, and $W^s(R,F)$ transversely intersects $\{0^s\} \times [0,1]^u \times (0,1) \subset W^u(Q,F)$.

Remark 4.4. If $R = R_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}} = (r^s, r^u, r)$ is a periodic point such that $W^s(r, f_{[\xi_0\dots\xi_{m-1}]})$ contains the forward orbit of either 0 or 1, then $W^u(P,F)$ intersects $W^s(R,F)$ transversely.

We have the following relation for periodic points with index u.

Lemma 4.5. *Consider a periodic sequence* $\xi = (\xi_0\dots\xi_{m-1})^{\mathbb{Z}} \in \Sigma_2$ *with* $\xi \neq 0^{\mathbb{Z}}$ *and an associated periodic point of the map* F,

$$R = R_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}} = (r^s, r^u, r_{(\xi_0\dots\xi_{m-1})^{\mathbb{Z}}}) = (r^s, r^u, r).$$

(1) *If R has index u and if the stable manifold $W^s(r, f_{[\xi_0\dots\xi_{m-1}]})$ contains $[0,1]$ then R is homoclinically related to P.*
(2) *If R has index $u+1$ and if the unstable manifold $W^u(r, f_{[\xi_0\dots\xi_{m-1}]})$ contains a fundamental domain of f_0 in $(0,1)$ then for every saddle $\widetilde{R} \in \Lambda_F$ with $\widetilde{R} \neq Q$ the manifolds $W^s(\widetilde{R},F)$ and $W^u(R,F)$ intersect transversely.*

Proof. Note that as $\xi \neq 0^{\mathbb{Z}}$, we have $r \in (0,1)$ and thus $R \neq Q$, P.

Suppose that R has index u. Remark 4.3 implies that $W^u(R,F)$ transversely intersects $W^s(P,F)$. To see that $W^u(P,F)$ transversely intersects $W^s(R,F)$ (and thus R and P are homoclinically related) it suffices to note that $\{y^s\} \times [0,1]^u \times \{f_1(0)\} \subset W^u(P,F)$ for some $y^s \in (0,1)^s$, and that $[0,1] \subset W^s(r, f_{[\xi_0\dots,\xi_{m-1}]})$. Remark 4.2 then implies (1).

To prove (2) note that our assumptions imply that for some fundamental domain $D = [d, f_0(d)] \subset (0,1)$ of f_0 and some $z^s \in (0,1)^s$ we have

$$\varDelta := \{z^s\} \times [0,1]^u \times D \subset W^u(R,F).$$

Remark 4.3 applied to $\widetilde{R}$ implies that $W^s(\widetilde{R},F)$ accumulates at $W^s(Q,F)$ from the right and therefore (from the definition of f_1) $W^s(\widetilde{R},F)$ accumulates to $\{0^s\} \times [0,1]^u \times \{1\}$ from the left. In particular, for every $\delta > 0$ there are $x \in (1-\delta, 1) \subset (f_0(d), 1)$ and $z^u \in (0,1)$ such that

$$\varUpsilon := [0,1]^s \times \{(z^u, x)\} \subset W^s(\widetilde{R},F). \tag{4.1}$$

The choice of x implies that some negative iterate of $\varUpsilon$ by F transversely meets $\varDelta \subset W^u(R,F)$. Thus $W^s(\widetilde{R},F)$ transversely intersects $W^u(R,F)$, ending the proof of the lemma. ∎

Remark 4.6. Note that (4.1) and the fact that x can be taken arbitrarily close to 1 implies that for every saddle $R \in \varLambda_F$, $R \neq Q$, every fundamental domain D of f_0 in $(0,1)$, and every $x^s \in [0,1]$ we have $\{x^s\} \times [0,1]^u \times D \pitchfork W^s(R,F) \neq \emptyset$.

We continue exploring the skew-product structure and the strong un/stable directions of the global transformation F.

Remark 4.7. Consider an interval $I \subset [0,1]$, a point $y^s \in [0,1]^s$, and the disk $\varDelta = \{y^s\} \times [0,1]^u \times I$. Given a finite sequence $\xi = (\xi_0 \dots \xi_m)$ with Notation 3.1 there is some $\overline{y}^s \in [0,1]^s$ such that

$$F^{m+1}(\varDelta_{[\xi]}) = \{\overline{y}^s\} \times [0,1]^u \times f_{[\xi]}(I).$$

Lemma 4.8. *Given the periodic point $Q^* = (q^s, q^u, q^*) = (\widehat{q}, q^*)$ in Theorem 4.1, the unstable manifold $W^u(Q^*,F)$ transversely intersects the s-disk $[0,1]^s \times \{(x^u, x)\}$ for any $(x^u, x) \in [0,1]^u \times (0,1)$.*

Proof. Consider the finite sequence $\xi = \xi(I_0)$ associated to q^* as provided by Lemma 3.8. Recall that by Lemma 3.8 the fundamental domain $D = [f_0^{-2}(b_0), f_0^{-1}(b_0)]$ is contained in $W^u(q^*, f_{[\xi]})$. This implies that

$$\varDelta_0 := \{q^s\} \times [0,1]^u \times D \subset W^u(Q^*,F).$$

Let us consider the following forward iterations of $\varDelta_0$ under F. For $i \geq 0$ define recursively

$$\varDelta_{i+1} := F^{i+1}(\varDelta_0 \cap \mathbf{C}_{[0^{i+1}]}) = F(\varDelta_i \cap \mathbf{C}_0) = \{q^s_{i+1}\} \times [0,1]^u \times f_0^{i+1}(D)$$

for some point $q^s_{i+1} \in [0,1]^s$. Observe that

$$\bigcup_{i \geq 0} f_0^i(D) \supset [f_0^{-2}(b_0), 1).$$

Thus for every $x \in [f_0^{-2}(b_0), 1)$ there is some point $y^s(x) \in [0,1]^s$ such that

$$\Upsilon_x = \{y^s(x)\} \times [0,1]^u \times \{x\} \subset W^u(Q^*, F).$$

This implies that the lemma holds when $x \in [f_0^{-2}(b_0), 1)$.

To complete the proof, first observe that, by (F01)(iii) for any $x \in (0, f_0^{-2}(b_0))$ one has

$$x' = f_1^{-1}(x) \in [f_0^{-2}(b_0), 1).$$

Thus, we can consider the disk $\Upsilon_{x'}$ and the point $z^s(x') \in [0,1]^s$ given by

$$F(\Upsilon_{x'} \cap \mathbf{C}_1) = \{z^s(x')\} \times [0,1]^u \times \{f_1(x')\} = \{z^s(x')\} \times [0,1]^u \times \{x\}.$$

By construction, the u-disk $F(\Upsilon_{x'} \cap \mathbf{C}_1)$ is contained in $W^u(Q^*, F)$ and intersects the s-disk $[0,1]^s \times \{(x^u, x)\}$. This ends the proof of the lemma. ■

Remark 4.9. Given $X = (x^s, x^u, x) \in \Lambda_F$, we denote by $W^{ss}(X, F)$ (resp. $W^{uu}(X, F)$) the strong stable manifold of X (resp. the strong unstable manifold of X) defined as the unique invariant manifold tangent to E^s (resp. E^u) at X and of dimension s (resp. dimension u). Note that

$$[0,1]^s \times \{(x^u, x)\} \subset W^{ss}(X, F), \qquad \{0^s\} \times [0,1]^u \times \{x\} \subset W^{uu}(X, F).$$

Remark 4.9 and Lemma 4.8 immediately imply the following.

Corollary 4.10. *For every $X = (x^s, x^u, x) \in \Lambda_F$ with $x \in (0,1)$ we have $W^u(Q^*, F) \pitchfork W^{ss}(X, F) \neq \emptyset$. In particular, $W^u(Q^*, F) \pitchfork W^s(R, F)$ for every saddle $R \in \Lambda_F \setminus \{P, Q\}$.*

Notation 4.1. For a point $X \in \mathbf{C}$ and a number $i \in \mathbb{Z}$ such that $F^i(X) \in \mathbf{C}$ let us write

$$X_i = F^i(X) = (x_i^s, x_i^u, x_i).$$

Given $x^s \in \mathbb{R}^s$ we denote $B^s_\delta(x^s) := \{x \in \mathbb{R}^s : d(x, x^s) < \delta\}$. We will also use the analogous notation $B^u_\delta(x^u)$.

The following proposition is the main step in the proof of Theorem 4.1.

Proposition 4.11. *Let $X = (x^s, x^u, x) \in \Lambda_F$ be such that $x_i \in (0,1)$ for infinitely many $i \leq 0$. For $\delta > 0$, the disk*

$$\Delta^s_\delta(X) := B^s_\delta(x^s) \times \{(x^u, x)\}$$

transversely intersects $W^u(Q^*, F)$. Set

$$X(\delta) := (x^s(\delta), x^u, x) \in \Delta^s_\delta(X) \pitchfork W^u(Q^*, F).$$

Then for every $\varepsilon > 0$ the disk

$$\Delta^{cu}_\varepsilon(X(\delta)) := \{x^s(\delta)\} \times B^u_\varepsilon(x^u) \times [x - \varepsilon, x + \varepsilon] \subset W^u(Q^*, F)$$

intersects $W^s(Q^*, F)$ transversely.

Proof. Since $x_i \in (0,1)$ for infinitely many $i \le 0$, the uniform expansion in the s-direction with respect to F^{-1} implies that there is some iterate $i \le 0$ such that $x_i \in (0,1)$ and

$$[0,1]^s \times \{(x_i^u, x_i)\} \subset F^i(\varDelta_\delta^s(X)).$$

Thus, by Lemma 4.8, $F^i(\varDelta_\delta^s(X))$ intersects $W^u(Q^*,F)$ transversely and hence $\varDelta_\delta^s(X)$ intersects $W^u(Q^*,F)$ transversely.

Note that, since $X \in \varLambda_F$, the definition of $X(\delta)$ implies that $X(\delta) \in \varLambda_F^+$. Consider now the forward orbit of $X(\delta)$ and let $\xi = (\xi_0\xi_1\ldots) \in \varSigma_2^+$ be the one-sided sequence determined by

$$F^i(X(\delta)) \in \mathbf{C}_{\xi_i}, \quad i \ge 0.$$

We let $\varDelta_0 := \varDelta_\varepsilon^{cu}(X(\delta))$ and (using Notation 3.1) recursively define, for $i \ge 0$,

$$\varDelta_{i+1} := F(\varDelta_i \cap \mathbf{C}_{\xi_i}) = F^{i+1}(\varDelta_0 \cap \mathbf{C}_{[\xi_0\ldots\xi_i]}).$$

The uniform expansion in the u-direction implies that there is a least iterate i_0 such that we cover the unstable vertical direction, that is,

$$\varDelta_{i_0} = \{y^s(\delta)\} \times [0,1]^u \times L \tag{4.2}$$

for some $y^s(\delta) \in [0,1]^s$ and some interval $L \subset (0,1)$. Clearly, this covering property is also true for any $i \ge i_0$.

Notice that, in general, we have no information about the location of the interval L. Thus, in principle, we cannot apply our preliminary results about expanding itineraries in Section 3.2 and we need to consider some additional iterates of L. More precisely, first consider some image H of L under the iterated function system such that we can apply these arguments to H. Recall that, in particular, such an interval H must be contained in $[f_0^{-2}(b_0), b_0]$. Take j_0 large enough such that $f_0^{j_0}(L)$ is close enough to 1 and

$$f_{[0^{j_0}1]}(L) \subset (0, f_0^{-1}(a_0)) = (0, f_0^{-2}(b_0)).$$

Consider now the smallest $\ell_0 \ge 0$ such that

$$f_{[0^{j_0}10^{\ell_0}]}(L) \cap (f_0^{-2}(b_0), b_0] \ne \emptyset$$

and consider the finite sequence $\eta := (0^{j_0}10^{\ell_0})$. Let

$$H := f_{[\eta]}(L) \cap [f_0^{-2}(b_0), b_0]$$

and consider the disk

$$\widetilde{\varDelta} := F^{j_0+1+\ell_0}(\varDelta_{i_0} \cap \mathbf{C}_{[\eta]}) = \{\widetilde{y}^s\} \times [0,1]^u \times H, \tag{4.3}$$

where $\widetilde{y}^s$ is some point in $[0,1]^s$. In comparison to (4.2), this disk is now appropriate to apply our arguments on expanding itineraries.

By Remark 4.6, if H contains a fundamental domain of f_0 then $\widetilde{\varDelta}$ meets $W^s(Q^*,F)$ transversely and, since $\widetilde{\varDelta}$ is a positive iterate of $\varDelta_\varepsilon^{cu}(X(\delta))$, we are already done in this case.

In the general case we will see that some forward iterate of H will contain the fundamental domain $[f_0^{-2}(b_0), f_0^{-1}(b_0)]$. To prove that, we apply our results about expanded successors in Section 3.2. By Lemma 3.5, there exist expanded successors $H = H_{\langle 0\rangle}, H_{\langle 1\rangle}, \dots, H_{\langle i(H)\rangle}$ of H such that $H_{\langle i(H)\rangle}$ contains $[f_0^{-2}(b_0), f_0^{-1}(b_0)]$. Together with the expanded successor, for $j = 0, \dots, i = i(H)$, we obtain an expanded finite sequence $\xi_{\langle j\rangle} = \xi(H_{\langle j(H)\rangle})$ of length $|\xi_{\langle j\rangle}|$ (recall (3.1)).

We now define recursively a sequence of disks as follows. Let $\widehat{\Delta}_0 := \widetilde{\Delta}$ with $\widetilde{\Delta}$ defined in (4.3) and for $j = 0, \dots, i(H) - 1$ let

$$\widehat{\Delta}_{j+1} := F^{|\xi_{\langle j\rangle}|}(\widehat{\Delta}_j \cap \mathbf{C}_{[\xi_{\langle j\rangle}]}).$$

Notice that

$$\widehat{\Delta}_{j+1} = \{y^s_{j+1}\} \times [0,1]^u \times f_{[\xi_{\langle j\rangle}]}(H_{\langle j\rangle}) = \{y^s_{j+1}\} \times [0,1]^u \times H_{\langle j+1\rangle}$$

for some point $y^s_{j+1} \in [0,1]^s$. As $H_{\langle i(H)\rangle}$ contains the fundamental domain $[f_0^{-2}(b_0), f_0^{-1}(b_0)]$ of f_0, by Remark 4.6 the disk $\widehat{\Delta}_{i(H)}$ meets $W^s(Q^*, F)$ transversely. Hence, the proposition is proved in that case also. ∎

As a consequence of the proof of Proposition 4.11 we obtain the following.

Remark 4.12. Observe that $\{(0^s, 0^u)\} \times (0,1) \subset H(Q^*, F)$. As the homoclinic class is a closed set, we can conclude that $\{(0^s, 0^u)\} \times [0,1] \subset H(Q^*, F)$. In particular, $P, Q \in H(Q^*, F)$.

Remark 4.13. The proof of the proposition implies that for any hyperbolic periodic point $R \neq Q$ of index $u+1$ the manifolds $W^u(R, F)$ and $W^s(Q^*, F)$ intersect transversely.

This remark, Corollary 4.10, and the fact that being homoclinic is an equivalence relation, together imply the following result.

Corollary 4.14. *Every pair of saddles of index $u+1$ in Λ_F that are different from Q are homoclinically related.*

We finally formulate a simple fact.

Lemma 4.15. *Given any sequence $\xi \in \Sigma_2$, the point $(x^s, x^u) = \varpi^{-1}(\xi)$, and some point $x \in I_{[\xi]}$, we have $X = (x^s, x^u, x) \in \Lambda_F$.*

Proof. Recall Notation 4.1. As an immediate consequence of the skew-product structure of F, by the definition of $I_{[\xi]}$ we have $x_i \in [0,1]$ for all $i \in \mathbb{Z}$. Since F is topologically conjugate to the shift map one finds that $(x^s_i, x^u_i) \in \widehat{\mathbf{C}}$ for all $i \in \mathbb{Z}$. Hence $F^i(X) \in \mathbf{C}$ for all $i \in \mathbb{Z}$ and thus $X \in \Lambda_F$. ∎

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Clearly, given any $X = (x^s, x^u, x) \in H(Q^*, F)$, we have $(x^s, x^u) \in \widehat{\mathbf{C}}$. It hence remains to prove $\Lambda_F \subset H(Q^*, F)$. We consider two cases.

Case 1: *$X = (x^s, x^u, x) \in \Lambda_F$ and $x_{-i} \in (0,1)$ for infinitely many $i > 0$.* By Proposition 4.11, and using the notation there, there exists a point

$$X(\delta) = (x^s(\delta), x^u, x) \in \Delta^s_\delta(X) \cap W^u(Q^*, F).$$

Note that it belongs to the forward invariant set Λ_F^+ and the disk $\Delta^{cu}_\varepsilon(X(\delta)) \subset W^u(Q^*, F)$ transversely intersects $W^s(Q^*, F)$ and hence contains a transverse homoclinic point of Q^*. Thus

$$\Delta^{cu}_\varepsilon(X(\delta)) \cap H(Q^*, F) \neq \emptyset.$$

By Proposition 4.11 we have $X(\delta) \in H(Q^*, F)$. As δ can be taken arbitrarily small, the point $X(\delta)$ can be taken arbitrarily close to X and thus $X \in H(Q^*, F)$. This implies the theorem in Case 1.

Case 2: *There is i_0 such that $X = (x^s, x^u, x) \in \Lambda_F$ and $x_{-i} \in \{0,1\}$ for all $i \geq i_0$.* Replacing X by its iterate $F^{-i_0}(X)$, we can assume that $x_{-i} \in \{0,1\}$ for all $i \geq 0$. We now distinguish two more cases.

Case 2.1: $x_0 = 1$. Since $f_\ell(x) = 1$ if and only if $\ell = 0$ and $x = 1$, as the only possibility for the backward branch of X we must have $x_{-i} = 1$ for all $i \geq 0$. Moreover, the sequence $\xi = \varpi^{-1}(x_0)$ must satisfy $\xi_{-i} = 0$ for all $i \geq 0$. Hence $X_{-i} \in \mathbf{C}_0$ for all $i \geq 0$ and therefore X is of the form $(0^s, x^u, 1)$.

Note that $I_{[\xi]} = [0,1]$. Thus, by Lemma 4.15, given any $\tau > 0$ for every $y \in (1-\tau, 1)$ there is a point $Y = (0^s, x^u, y)$ in Λ_F. Note that these points form an uncountable set. Since the set of all preimages

$$\{f_{[\xi_{-m}\dots\xi_{-1}.]}(\{0,1\}) \colon (\xi_{-m}\dots\xi_{-1}) \in \{0,1\}^{-m},\ m \geq 1\}$$

is countable, without loss of generality we can assume that the point Y and its preimages with central coordinates y_{-i} additionally satisfy $y_{-i} \in (0,1)$ for all $i \geq 0$. Now applying Case 1 to Y we conclude that $Y \in H(Q^*, F)$. Since a homoclinic class is a closed set and Y can be chosen arbitrarily close to X, we obtain $X \in H(Q^*, F)$.

Case 2.2: $x_0 = 0$. To distinguish the two possible types of backward branches of X in this case, observe that $f_\ell(x) = 0$ if either $x = 0$ and $\ell = 0$ or $x = 1$ and $\ell = 1$.

Case 2.2a: *$x_{-i} = 0$ for all $i \geq 0$.* Hence in this case $X_{-i} \in \mathbf{C}_0$ for all $i \geq 0$, and we can conclude as in Case 2.1.

Case 2.2b: *There exists a first index i such that $x_{-i} = 1$.* Replacing X by the iterate $f^{-i}(X)$, we can now conclude as in Case 2.1.

This proves that $\Lambda_F \subset H(Q^*, F)$ and hence proves the theorem. ∎

4.2. Particular cases. Supplementing the results in the previous section we obtain, under additional mild hypotheses on the maps f_0, f_1, further properties of the homoclinic class.

First, we assume that the following Kupka–Smale-like condition is satisfied:

($\mathrm{F_{KS}}$) Every periodic point of any composition $f_{[\xi_0\dots\xi_m]}$ is hyperbolic.

Note that this condition is generic among pairs of maps f_0, f_1 satisfying conditions (F0), (F1), and (F01).

THEOREM 4.16. *Under the additional hypothesis* ($\mathrm{F_{KS}}$)*, for every periodic sequence* $\xi = (\xi_0 \dots \xi_{m-1})^{\mathbb{Z}}$ *there is a periodic point* R_ξ *of* F *of index* u *in the fiber of* ξ *(that is,* $\pi(R_\xi) = \xi$*) that is homoclinically related to* P*.*

Proof. The arguments in the proof of Lemma 3.23 and the hypothesis ($\mathrm{F_{KS}}$) together imply that the set

$$\bigcap_{k\in\mathbb{N}} (f_{[\xi_0\dots\xi_m]})^{2k}([0,1])$$

is either an attracting fixed point of $f_{[\xi_0\dots\xi_m]}$ or an interval whose extremes are hyperbolic attracting periodic points of $f_{[\xi_0\dots\xi_m]}$. In either case, consider one such attracting point and denote it by r_ξ. By construction $W^s(r, f_{[\xi_0\dots\xi_m]})$ contains either 0 or 1. By Remark 4.4, this implies that $W^s(R_\xi, F)$ transversely intersects $W^u(P,F)$. On the other hand, by Remark 4.3 we know that $W^u(R_\xi, F)$ and $W^s(P,F)$ intersect transversely. This implies that the saddles R_ξ and P are homoclinically related, proving the proposition. ∎

Recall that in the previous case under conditions (F0), (F1), (F01) we have $H(P,F) \subset \Lambda_F$. We now consider another particular case. Assume that:

($\mathrm{F_B}$) If $f_1([0,1]) = [0,c]$ then $f_0'(x) \in (0,1)$ for all $x \in [c,1]$.

THEOREM 4.17. *Under the additional hypothesis* ($\mathrm{F_B}$) *we have* $H(P,F) = \Lambda_F = H(Q^*,F)$*.*

The argument is somewhat analogous to the one for Theorem 4.1. Moreover, it follows very closely the exposition in [7, Section 6.2] using a construction of so-called *blenders*.

First, we have a completely analogous version of Proposition 4.11.

PROPOSITION 4.18. *Let* $Y = (y^s, y^u, y) \in \Lambda_F$ *be such that* $y_i \in (0,1)$ *for infinitely many* $i \geq 0$*. For* $\delta > 0$*, the disk*

$$\Delta^u_\delta(Y) := \{y^s\} \times B^u_\delta(y^u) \times \{y\}$$

transversely intersects $W^s(P,F)$*. Let*

$$Y(\delta) := (y^s, y^u(\delta), y) \in \Delta^u_\delta(Y) \pitchfork W^s(P,F).$$

Then for every small $\varepsilon > 0$ *the disk*

$$\Delta^{cs}_\varepsilon(Y(\delta)) := B^s_\varepsilon(y^s) \times \{y^u(\delta)\} \times [y-\varepsilon, y+\varepsilon] \subset W^s(P,F)$$

intersects $W^u(P,F)$ *transversely.*

After proving this proposition the proof of Theorem 4.17 is identical to the one of Theorem 4.1, so we refrain from giving the details.

Proof of Proposition 4.18. The first steps are identical to the ones of the proof of Proposition 4.11.

Further, to show that $\Delta_\varepsilon^{cs}(Y(\delta)) \pitchfork W^s(P,F) \neq \emptyset$, we consider the iterate of an interval $J := [y-\varepsilon, y+\varepsilon]$ under the maps $f_{[\xi_{-m}\cdots\xi_{-1}.]}$. First note that, under the hypothesis ($\mathrm{F_B}$), the maps f_0^{-1} and f_1^{-1} are uniformly expanding in $[c,1]$ and $[0,c]$, respectively, with derivatives having moduli $\geq \kappa > 1$. An immediate consequence (see also the Lemma in [6]) is that $f_{[\xi_{-m}\cdots\xi_{-1}.]}(J)$ for large m contains the point c. Just observe that if $c \notin f_{[\xi_{-m}\cdots\xi_{-1}.]}(J)$ then either $f_{[\xi_{-m}\cdots\xi_{-1}.]}(J) \in [0,c)$ or $f_{[\xi_{-m}\cdots\xi_{-1}.]}(J) \in (c,1]$. In the first case let $\xi_{-m-1} = 0$ while in the second let $\xi_{-m-1} = 1$. Consequently, $|f_{[\xi_{-m-1}\cdots\xi_{-1}.]}(J)| \geq \kappa^{m+1}|J|$. Thus, there is a first m with the desired property.

Now, to finish the proof, note that the skew-product structure implies that there exist $\widetilde{y}^u \in [0,1]^u$ such that

$$[0,1]^s \times \{\widetilde{y}^u\} \times f_{[\xi_{-m}\cdots\xi_{-1}.]}(J) \subset W^s(P,F)$$

and, recalling that $c = f_1(0)$, this implies that there is $\widetilde{y}^s \in [0,1]^s$ such that

$$\{\widetilde{y}^s\} \times [0,1]^u \times \{f_1(0)\} \subset W^u(P,F).$$

This means that $\Delta_\varepsilon^{cs}(Y(\delta)) \pitchfork W^u(P,F) \neq \emptyset$. ∎

5. Lyapunov exponents in the central direction. We now continue our discussion of Lyapunov exponents started in Section 3.3. Recall that, due to the skew-product structure and our hypotheses, the splitting in (2.2) is dominated and for every Lyapunov regular point coincides with the Oseledec splitting provided by the multiplicative ergodic theorem. Here, in particular, a point S is *Lyapunov regular* if for $i = uu$, c, and ss for every $v \in E_S^i$ the limit

$$\chi_i(S) := \lim_{n\to\pm\infty} \frac{1}{n} \log \|dF_S^n(v)\| \tag{5.1}$$

exists. In the following we will focus only on the *Lyapunov exponent* $\chi_{\mathrm{c}}(S)$ *associated to the central direction* E^c. Observe that given a Lyapunov regular point $S = (s^s, s^u, s) \in \Lambda_F$ and $\xi = (\dots\xi_{-1}.\xi_0\xi_1\dots) \in \Sigma_2$ given by $\xi = \varpi((s^s, s^u))$, we have

$$\chi_c(S) = \lim_{n\to\infty} \frac{1}{n} \log |(f_{[\xi_0\dots\xi_{n-1}]})'(s)|. \tag{5.2}$$

Clearly, $\chi_c(S)$ is well-defined for every periodic point.

5.1. Spectra of Lyapunov exponents. Let us consider spectra of central exponents from various points of view.

5.1.1. *Spectrum related to periodic points.* Given a saddle S, we define the *spectrum of saddles homoclinically related to* S by

$$\mathcal{I}_{\text{homrel}}(S) := \{\chi_c(R)\colon R \text{ hyperbolic periodic homoclinically related to } S\},$$

and the *periodic point spectrum of the homoclinic class of* S by

$$\mathcal{I}_{\text{per}}(H(S,F)) := \{\chi_c(R)\colon R \in H(S,F) \text{ periodic}\}.$$

Clearly, $\mathcal{I}_{\text{homrel}}(S) \subset \mathcal{I}_{\text{per}}(H(S,F))$. Let Q^* be the saddle provided by Theorem 4.1. Since the homoclinic class $H(Q^*,F)$ coincides with the maximal invariant set Λ_F, we have

$$\mathcal{I}_{\text{homrel}}(Q) \cup \mathcal{I}_{\text{homrel}}(Q^*) \cup \mathcal{I}_{\text{homrel}}(P) \subset \mathcal{I}_{\text{per}}(H(Q^*,F)). \tag{5.3}$$

Moreover, $\mathcal{I}_{\text{homrel}}(Q) = \{\log\beta\}$ by Lemma 2.9.

Let us recall the following standard fact (see also [2, Corollary 2]).

Lemma 5.1. *Given two saddles S and S' that are homoclinically related and satisfy $\chi_c(S) \le \chi_c(S')$, we have*

$$[\chi_c(S),\chi_c(S')] \subset \overline{\mathcal{I}_{\text{homrel}}(S)} = \overline{\mathcal{I}_{\text{homrel}}(S')}.$$

Proof. Recall that if S and S' are homoclinically related then there exists a horseshoe $\Lambda_{S,S'} \subset \Lambda_F$ that contains both saddles. In particular $\Lambda_{S,S'}$ is a uniformly hyperbolic locally maximal and transitive set with respect to F. The existence of a Markov partition implies that we can construct orbits in the hyperbolic set that spend a fixed proportion of time close to S and S', respectively. This is enough to obtain periodic points in $\Lambda_{S,S'}$ with Lyapunov exponents dense in $[\chi_c(S),\chi_c(S')]$. Finally, any such periodic orbit is homoclinically related to S and S'. ∎

5.1.2. *Spectrum of Lyapunov regular points.* Let

$$\mathcal{I}_{\text{reg}}(H(S,F)) := \{\chi_c(R)\colon R \in H(S,F) \text{ Lyapunov regular}\}.$$

We finally obtain the possible spectrum of central Lyapunov exponents. Recall the definition of $\widetilde{\beta}$ in Proposition 3.26 that is the largest Lyapunov exponent as in (5.2) that is different from β.

Proposition 5.2. *Let*

$$\beta^* := \exp\sup\{\chi\colon \chi \in \mathcal{I}_{\text{reg}}(H(Q^*,F)),\ \chi \ne \log\beta\}.$$

We have

$$\begin{aligned}\overline{\mathcal{I}_{\text{per}}(H(Q^*,F))} &= [\log\lambda,\log\beta^*] \cup \{\log\beta\}\\ &\subset \overline{\mathcal{I}_{\text{reg}}(H(Q^*,F))} \subset [\log\lambda,\log\widetilde{\beta}] \cup \{\log\beta\}.\end{aligned}$$

Remark 5.3. Using different methods involving shadowing-like properties, one can in fact show that $[\log\lambda,\log\widetilde{\beta}] \cup \{\log\beta\}$ is the set of all possible upper/lower central Lyapunov exponents, that is, all exponents that are obtained when we replace lim by lim sup/lim inf in (5.1). Hence, in particular, we have equalities in Proposition 5.2. For the details we refer to [12].

Proof of Proposition 5.2. We first prove that $(0, \log \beta^*) \subset \mathcal{I}_{\rm per}(H(Q^*, F))$. Note that by Proposition 3.10 for every $\varepsilon > 0$ there exist a finite sequence $(\xi_0 \dots \xi_{m-1})$ and a fixed point $q_{(\xi_0 \dots \xi_{m-1})^{\mathbb{Z}}} \in (0,1)$ of $f_{[\xi_0 \dots \xi_{m-1}]}$ that has Lyapunov exponent in $(0,\varepsilon)$. Therefore, the corresponding periodic point $Q_{(\xi_0 \dots \xi_{m-1})^{\mathbb{Z}}}$ has central Lyapunov exponent in $(0,\varepsilon)$. By Corollary 4.14 this point is homoclinically related to Q^*. By Lemma 5.1 and (5.3) we hence obtain

$$[0, \chi_c(Q^*)] \subset \overline{\mathcal{I}_{\rm per}(H(Q^*, F))}.$$

Similarly,

$$[\chi_c(Q^*), \log \beta^*] \subset \overline{\mathcal{I}_{\rm per}(H(Q^*, F))},$$

proving that

$$[0, \log \beta^*] \subset \overline{\mathcal{I}_{\rm per}(H(Q^*, F))}.$$

Now we consider the negative part of the spectrum. Note that by Proposition 3.9 for every $\varepsilon > 0$ there exist a finite sequence $(\xi_0 \dots \xi_{m-1})$ and a fixed point $p_{(\xi_0 \dots \xi_{m-1})^{\mathbb{Z}}}$ of $f_{[\xi_0 \dots \xi_{m-1}]}$ that has Lyapunov exponent in $(-\varepsilon, 0)$ and whose stable manifold contains $[0,1]$. By Lemma 4.5(1) the corresponding hyperbolic periodic point $P_{(\xi_0 \dots \xi_{m-1})^{\mathbb{Z}}}$ is homoclinically related to the fixed point P. Exactly as above, we obtain

$$[\log \lambda, 0] \subset \overline{\mathcal{I}_{\rm per}(H(P, F))}.$$

Since $H(P,F) \subset H(Q^*, F)$, this proves

$$[\log \lambda, \log \beta^*] \cup \{\log \beta\} \subset \overline{\mathcal{I}_{\rm per}(H(Q^*, F))}.$$

By the definition of β^* we have

$$\mathcal{I}_{\rm per}(H(Q^*, F)) \subset [\log \lambda, \log \beta^*] \cup \{\log \beta\}.$$

Clearly, $\mathcal{I}_{\rm per}(H(Q^*, F)) \subset \mathcal{I}_{\rm reg}(H(Q^*, F))$. Finally, note that by Proposition 3.26, any Lyapunov regular point is either contained in the stable manifold of Q and hence has exponent $\log \beta$, or has exponent less than or equal to $\log \widetilde{\beta}$, proving

$$\mathcal{I}_{\rm reg}(H(Q^*, F)) \subset [\log \lambda, \log \widetilde{\beta}] \cup \{\log \beta\}.$$

This finishes the proof of the proposition. ∎

5.1.3. *Spectrum of ergodic measures.* The following results will be needed in the following section. We denote by $\mathcal{M}(\Lambda)$ the set of F-invariant Borel probability measures supported on a set Λ, and by $\mathcal{M}_{\rm erg}(\Lambda)$ the subset of ergodic measures. For $\mu \in \mathcal{M}(\Lambda)$ let

$$\chi_c(\mu) := \int \log \|dF|_{E^c}\| \, d\mu.$$

Denote by δ_Q the Dirac measure at Q and consider

$$\mathcal{I}_{\rm erg}(H(Q^*,F)) := \{\chi_c(\mu)\colon \mu\in\mathcal{M}_{\rm erg}(H(Q^*,F))\setminus\{\delta_Q\}\}.$$

The following is an immediate consequence of Proposition 3.26.

Proposition 5.4. *We have* $(\log\widetilde\beta,\log\beta)\cap\mathcal{I}_{\rm erg}(H(Q^*,F))=\emptyset$.

5.2. Phase transitions. In this section we continue our analysis of spectral properties and study equilibrium states. Recall that, given a continuous potential $\varphi\colon \Lambda_F\to\mathbb{R}$, an F-invariant Borel probability measure ν is called an *equilibrium state* of φ with respect to $F|_{\Lambda_F}$ if

$$h_\nu(F)+\int\varphi\,d\nu=\max_{\mu\in\mathcal{M}(F|\Lambda_F)}\Big(h_\mu(F)+\int\varphi\,d\mu\Big),$$

where $h_\mu(F)$ denotes the measure-theoretic entropy of μ. Notice that as the central direction has dimension one, such a maximizing measure indeed exists by [11, Corollary 1.5] (see also [9]). Without loss of generality, we can always assume that this measure is ergodic. Indeed, given an equilibrium state for φ that is non-ergodic, any ergodic measure in its ergodic decomposition is also an equilibrium state for φ. Note that

$$P(\varphi)=\max_{\mu\in\mathcal{M}(F|\Lambda_F)}\Big(h_\mu(F)+\int\varphi\,d\mu\Big) \tag{5.4}$$

is the *topological pressure* of φ with respect to $F|_{\Lambda_F}$ (see [24]). An equilibrium state for the zero potential $\varphi=0$ is simply a *measure of maximal entropy* $h(F)=h(F|_{\Lambda_F})$.

We will study the family of continuous potentials

$$\varphi_t := -t\log\|dF|_{E^c}\|,\quad t\in\mathbb{R},$$

and will continue denoting $P(t)=P(\varphi_t)$. Note that $t\mapsto P(t)$ is convex (and hence continuous and differentiable on a residual set; see also [24] for further details). Recall that a number $\alpha\in\mathbb{R}$ is said to be a *subgradient* at t if $P(t+s)\ge P(t)+s\alpha$ for all $s\in\mathbb{R}$.

Lemma 5.5. *For any $t\in\mathbb{R}$ and any equilibrium state μ_t of the potential φ_t the number $-\chi_c(\mu_t)$ is a subgradient of $s\mapsto P(s)$ at $s=t$. If moreover $s\mapsto P(s)$ is differentiable at $s=t$ then $\chi_c(\mu_t)=-P'(t)$.*

Proof. Given $t\in\mathbb{R}$ and an equilibrium state μ_t, it follows from the variational principle (5.4) that for all $s\in\mathbb{R}$ we have

$$P(t+s)\ge h_{\mu_t}(F)-(t+s)\chi_c(\mu_t)=P(t)-s\chi_c(\mu_t),$$

that is, $-\chi_c(\mu_t)$ is a subgradient at t.

If the pressure is differentiable at t then this subgradient is unique and thus all equilibrium states of φ_t have the same exponent given by $-P'(t)$. ∎

We now derive the existence of a *first order phase transition*, that is, a parameter t at which the pressure function is not differentiable. Note that in our case, by Lemma 5.5, this is equivalent to the existence of a parameter t and (at least) two equilibrium states for φ_t with different central exponents.

PROPOSITION 5.6. *There exists a parameter* $t_Q \in [-h(F)/(\log\beta - \log\widetilde{\beta}), 0)$ *such that for every* $t \le t_Q$ *the measure* δ_Q *is an equilibrium state for* φ_t *and* $P(t) = -t\log\beta$. *Moreover,* $s \mapsto P(s)$ *is not differentiable at* $s = t_Q$ *and*

$$D^-P(t_Q) = -\log\beta \quad \textit{and} \quad D^+P(t_Q) \ge -\log\widetilde{\beta}.$$

Proof. We first show that δ_Q is an equilibrium state for some t. Recall that (F0.i) implies that $\chi_c(\delta_Q) = \log\beta$. Note that the variational principle (5.4) implies that for every $t \in \mathbb{R}$ we have

$$P(t) \ge h_{\delta_Q}(F) - t\chi_c(\delta_Q) = -t\log\beta. \tag{5.5}$$

Aiming for a contradiction, assume that there exists no $t \in \mathbb{R}$ such that δ_Q is an equilibrium state for φ_t. Then, by (5.4), $P(t) > -t\log\beta$ for all t. By Proposition 5.4 there is no ergodic F-invariant measure different from δ_Q with central Lyapunov exponent within the interval $(\log\widetilde{\beta}, \infty)$. Hence, for any $t \in \mathbb{R}$ any ergodic equilibrium state of φ_t different from δ_Q has exponent $\le \log\widetilde{\beta}$. In particular, for every $t < 0$ we have

$$P(t) = h_{\mu_t}(F) + |t|\chi_c(\mu_t) \le h(F) + |t|\log\widetilde{\beta} \tag{5.6}$$

for some ergodic equilibrium state μ_t of φ_t. Summarizing, for $t < 0$ we have

$$|t|\log\beta < P(t) \le h(F) + |t|\log\widetilde{\beta}. \tag{5.7}$$

But this is a contradiction if $t < 0$ and $|t| < h(F)/(\log\beta - \log\widetilde{\beta})$. Therefore, there exists $t \in \mathbb{R}$ such that δ_Q is an equilibrium state for φ_t.

Since $P(0) = h(F) \ge \log 2 > 0$, by continuity of $t \mapsto P(t)$ we have

$$t_Q := \max\{t \in \mathbb{R} \colon P(t) = -t\log\beta\} < 0.$$

Consider $\tau \in (t_Q, 0)$ and an ergodic equilibrium state μ_τ for φ_τ. Since $P(\tau) \ne -\tau\log\beta$ we have $\mu_\tau \ne \delta_Q$. Hence, by Proposition 5.4, we have $\chi(\mu_\tau) \le \log\widetilde{\beta}$. Note again that $P(\tau) > -\tau\log\beta$. Thus, arguing as in (5.6) and (5.7), we get $|\tau| < h(F)/(\log\beta - \log\widetilde{\beta})$. In particular, this implies

$$|t_Q| \le \frac{h(F)}{\log\beta - \log\widetilde{\beta}}.$$

This completes the first part of the lemma.

It remains to estimate the left and right derivatives at t_Q. Consider again $\tau \in (t_Q, 0)$ and an ergodic equilibrium state μ_τ for φ_τ. As above, $\mu_\tau \neq \delta_Q$ and $\chi_c(\mu_\tau) \le \log\widetilde{\beta}$. By Lemma 5.5, $-\chi_c(\mu_\tau)$ is a subgradient at τ. Hence, $D^+P(t_Q) \ge -\log\widetilde{\beta}$. On the other hand, by the definition of t_Q, we have $D^-P(t_Q) = -\log\beta$. Hence $t \mapsto P(t)$ is not differentiable at t_Q. ∎

6. Proofs of the main results

Proof of Theorem 1. Items (A) and (C) follow immediately from Propositions 3.15 and 3.26, respectively. Finally, (B) is a one-dimensional version of Theorem 2(B) using also transitivity in item (D), so we will omit its proof. ∎

Proof of Theorem 2. Item (A.a) follows from Theorem 1(A) together with the skew-product structure of F.

By Theorem 4.1 there is a saddle Q^* of index $u+1$ such that the homoclinic class $H = H(Q^*, F)$ coincides with the locally maximal invariant set Λ_F in $\mathbf{C}$. This homoclinic class coincides with the closure of all saddle points homoclinically related to Q^*. By Corollary 4.14 every saddle of index $u+1$ in Λ_F is homoclinically related to Q^*. By Lemma 3.23 the spine of every periodic point of index $u+1$ is non-trivial. Hence, the set of all points with non-trivial spines is dense in Λ_F. This shows (A.b).

The first part of item (B) follows from the previous arguments. Recall that P has index u and that its homoclinic class of P is non-trivial (Lemma 2.6) and therefore contains infinitely many saddles of index u. Since this class is contained in Λ_F we are done.

The first part of item (C) follows from Theorem 1(C) together with the skew-product structure of F. The fact that the spectrum contains an interval and the existence of a phase transition follow immediately from Propositions 5.2 and 5.6.

Again, by Theorem 4.1 for the saddle Q^* of index $u+1$ the homoclinic class $H = H(Q^*, F)$ is the locally maximal set in $\mathbf{C}$ and hence contains the non-trivial class $H(P, F)$. Finally, Lemmas 2.9 implies that $H(Q, F) = \{Q\} \subset H$. This proves item (D) and hence the theorem. ∎

Acknowledgments. This paper was partially supported by CNPq, Faperj, Pronex (Brazil) and the Alexander von Humboldt Foundation (Germany). The authors thank the Departamento de Matemática Pura, Universidade de Porto, and IMPAN Warsaw for support and hospitality, as well as the participants of the DynNonHyp ANR-meeting Dijon, June 2010, M. Rams for useful comments, and an anonymous referee for careful reading and relevant suggestions.

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Received 17 January 2011;
in revised form 9 September 2011