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Countable splitting graphs

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Abstract. A graph is called *splitting* if there is a 0-1 labelling of its vertices such that for every infinite set C of natural numbers there is a sequence of labels along a 1-way infinite path in the graph whose restriction to C is not eventually constant. We characterize the countable splitting graphs as those containing a subgraph of one of three simple types.

1. Introduction. We begin with some basic notions from the theory of cardinal characteristics of the continuum (see [1] for a general introduction). A *cardinal characteristic of the continuum* is a term t which denotes the size of an uncountable subset of (some version of) the continuum, i.e., $\aleph_0 < t \leq 2^{\aleph_0}$. (For many such characteristics t and t^* the formula $t = t^*$ is true in one model of set theory and false in another. We therefore have to distinguish between a cardinal characteristic and its denotation in some model. But to avoid notational inconveniences we will leave this distinction implicit.)

An important sort of cardinal characteristic is given by the norms of relations. Following [1], we call a triple (A_-, A_+, A) a *relation* if $A \subseteq A_- \times A_+$, and we call such a relation *sequential* if A_+ is a set of infinite sequences. The *norm* of a relation $\mathbf{A} = (A_-, A_+, A)$ is the smallest cardinality of a set $Y \subseteq A_+$ that contains for every *challenge* $x \in A_-$ a *response* $y \in Y$ such that $(x, y) \in A$:

$$\|\mathbf{A}\| := \min\{|Y|;\, Y \subseteq A_+ \wedge \forall x \in A_- \ \exists y \in Y \ \ (x, y) \in A\}.$$

A well-known sequential relation is $\mathbf{S} := ([\omega]^{\aleph_0}, {}^{\omega}2, sp)$, i.e., challenges are infinite sets of natural numbers, responses are infinite binary sequences, and a challenge C is met by a response g if g *splits* C, i.e., if g restricted to C is not eventually constant (see Section 2 for formal definitions). The norm of this relation is called the *splitting number* $\mathfrak{s}$; it is the smallest size of a set of binary sequences containing for every infinite set C of natural

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numbers an element that splits C, the smallest size of a *splitting family* of binary sequences.

It turns out that one cannot say exactly which size a splitting family has to have; i.e., it is independent of the usual axioms of set theory whether, for example, $\mathfrak{s} = \aleph_1$. And the same is true with respect to the 'response sets' of many other sequential relations. The traditional reaction to such 'individual' independencies has been the study of the relationships between the norms of such relations.

A new approach to dealing with sequential relations, taken in [4] and [5], consists in imposing an additional constraint on a response set restricting the independence of its members: it is required that the sequences making up the response set are sequences of labels that correspond to 1-way infinite paths in a labelled graph. Thus, instead of asking for the smallest cardinality of a response set, one asks for graphs that can be labelled in such a way that the sequences of labels corresponding to 1-way infinite paths constitute a response set.

Let us make these ideas more precise. A *ray* is a 1-way infinite path R in a graph G, i.e., an injective sequence $\omega \to V(G)$ of vertices such that $R(i)R(i+1) \in E(G)$ for all $i \in \omega$. We write $\mathscr{R}_G$ for the set of all rays in G. Given any set M, we say that a function $L\colon V(G) \to M$ is an *M-labelling* of G, and we define the corresponding set of label sequences thus:

$$\mathscr{L}_{G,L} := \{g\colon \omega \to M;\ \exists R \in \mathscr{R}_G\ \ g = L \circ R\}.$$

Then, given a sequential relation $\mathbf{A} := (A_-, A_+, A)$ whose responses are sequences in M, we define a corresponding property of graphs:

$$G \text{ is an } \mathbf{A}\text{-graph} :\Leftrightarrow \exists L\colon V(G) \to M\ \forall x \in A_-\ \exists y \in A_+ \cap \mathscr{L}_{G,L}\ \ (x,y) \in A.$$

So, while the norm of $\mathbf{A}$ is the smallest cardinality of a set Y such that for every challenge there is a response in $A_+ \cap Y$, a graph G is an $\mathbf{A}$-graph if there is a labelling L of its vertices such that for every challenge there is a response in $A_+ \cap \mathscr{L}_{G,L}$. In the first case, one wants to meet all challenges with as few responses as possible, while in the second case, one wants to label a graph so that all challenges can be met by label sequences. The new approach to dealing with sequential relations is motivated by the hypothesis that it might be possible to tell precisely which structure such a 'response graph' has to have, for example, by presenting a few simple structures such that exactly the response graphs contain one of these structures.

In the above example $\mathbf{S} := ([\omega]^{\aleph_0}, {}^{\omega}2, sp)$, we see that a graph G is an $\mathbf{S}$-graph if and only if there is a 0-1 labelling L of its vertices such that for every infinite set C of natural numbers there is a label sequence $g \in \mathscr{L}_{G,L}$ that splits C; these $\mathbf{S}$-graphs are called *splitting graphs*. Put otherwise, a graph G is a splitting graph if and only if there is a 0-1 labelling L of its vertices such that the resulting set $\mathscr{L}_{G,L}$ of label sequences is a splitting family of binary sequences. Note that a graph with a splitting subgraph is itself splitting. It is our aim to present a few simple structures such that exactly the splitting graphs contain one of these structures.

Results of the same type have been presented in [4] for *unbounded graphs* and in [5] for *dominating graphs*, i.e., for **B**-graphs and **D**-graphs where **B** is a sequential relation whose norm is the *(un)bounding number* $\mathfrak{b}$ and where **D** is a sequential relation whose norm is the *dominating number* $\mathfrak{d}$.

After we have defined some basic notions in Section 2, we present, in Section 3, three simple types of splitting graphs. In Sections 4 and 5, we then analyse the graphs that do not have graphs of two of these three types as subgraphs. In the final Section 6, we present our main result, Theorem 12, which states that a countable graph is splitting if and only if it has a subgraph of one of the three simple types.

2. Terminology. Let ${}^{Y}X$ be the set of functions from a set Y to a set X, and let $f{\restriction}X$ be the restriction of a function f to a subset X of its domain. A function s whose domain is an ordinal α and whose range is a subset of X is an *α-sequence in X*; it is a *sequence of length α*. We write the concatenation of sequences s and t as $s^\wedge t$ and sometimes st. We usually denote ω-sequences by $a_0{}^\wedge a_1{}^\wedge\ldots$ or $a_0a_1\ldots$, and finite sequences by $a_0{}^\wedge\ldots{}^\wedge a_n$ or $a_0\ldots a_n$. We write ${}^{<\omega}X$ for the set of finite sequences in X.

A *graph* G is a pair of sets (V,E) where E is a symmetrical relation on V; these sets contain the *vertices* and *edges* of the graph. Given a graph $G=(V,E)$, we write $V(G):=V$ and $E(G):=E$. If xy is an edge, the vertices x and y are called *neighbours*, and the number of neighbours of some vertex x is the *degree* of x.

Let $1\le n\le\omega$. An injective n-sequence P in $V(G)$ is a *path* in G if $P(i)P(i+1)\in E(G)$ for all $i<n-1$. The path P induces a graph $G(P)$: the vertex set of $G(P)$ is simply the range of P, written as $V(P)$, and the edge set of $G(P)$, denoted by $E(P)$, contains all those edges $P(i)P(j)$ where $i,j<n$ and $|i-j|=1$. The graph $G(P)$ is called the *graph of P*. By $\|P\|:=|P|-1=n-1$ we denote the *length* of P, i.e., its number of edges. A path R of length ω is a *ray*. We shall need the following well-known fact (see, for example, [3, p. 200]):

Lemma 1 (König). *Let $V_0,V_1,\ldots$ be an infinite sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that every vertex v in a set V_n with $n\ge 1$ has a neighbour in V_{n-1}. Then G contains a ray $v_0v_1\ldots$ with $v_n\in V_n$ for all n.*

A path P induces an order on its vertices: $P(0)<_P P(1)<_P\cdots$. If x is the last vertex of a path P of length n, the first vertex of a path Q, and the only vertex in $V(P) \cap V(Q)$, we write $P^\wedge Q$ for $(P\restriction n)^\wedge Q$. If $P = Q^\wedge Q'$ is a path, then Q is an *initial segment* of P, and Q' is a *tail* of P. Let $P = x_0 x_1 \ldots$ be a path, and let $n < m$. We denote by $x_n P x_m$ the subpath of P between x_n and x_m, and by $\mathring{x}_n P x_m$ the subpath between x_{n+1} and x_m (analogously for $x_n P \mathring{x}_m$). If $P = x_0 \ldots x_n$ is a path, we call the subpath $\mathring{x}_0 P \mathring{x}_n$ the *interior of* P, written $\mathring{P}$, and we call its vertices the *inner vertices of* P.

A function $L\colon V(G) \to \{0,1\}$ is a 0-1 *labelling* of G. Given a ray R in a graph G labelled by L, the sequence $L \circ R$ is called a *label sequence*, and the set of all label sequences $\{L \circ R;\ R \text{ is a ray in } G\}$ is denoted by $\mathscr{L}_{G,L}$.

A graph G is called *splitting* if there is a 0-1 labelling L of G such that for every infinite set C of natural numbers there is a label sequence $g \in \mathscr{L}_{G,L}$ such that $g\restriction C$ is not eventually constant, i.e., neither $g(n) = 0$ for all but finitely many $n \in C$ nor $g(n) = 1$ for all but finitely many $n \in C$.

3. Examples of splitting graphs. In this section, we introduce three simple types of splitting graphs. It will turn out that every countable splitting graph contains a subgraph of one of these types.

We recall that a graph T is called a *tree* if for any two of its vertices there is exactly one path from one to the other, and that a tree is called a *subdivided binary tree* if each of its vertices has degree 2 or 3, and each of its rays uses a vertex of degree 3.

LEMMA 2. *A subdivided binary tree is splitting.*

Proof. Let T be a subdivided binary tree. We may assume that T has a vertex $r \in V(T)$ of degree 2 which we call the *root* of T. Note that the choice of r imposes a partial order on the vertex set of the tree: $x \le y$ if x is a vertex of the unique path in T from the root to y. The root together with the vertices of degree 3 are called *branching vertices.*

Let $\Sigma := {}^{<\omega}\{r, \ell\}$ be the set of finite sequences in a set $\{r, \ell\}$ with two elements. Now, define inductively a bijection between Σ and the set of branching vertices which respects the order of the branching vertices: Let $x^{\emptyset} := r$, and, if x^σ has been defined, let $x^{\sigma^\wedge r}$ and $x^{\sigma^\wedge \ell}$ be the two branching vertices immediately above the branching vertex x^σ. We call a branching vertex x^σ along a path P which starts in r a *left turn* if P continues with $x^{\sigma^\wedge \ell}$.

To show that T is splitting, we have to define a 0-1 labelling L of T. We let $L(x)$ be the parity of the number of left turns on the path from the root to x (see Figure 1).

To see that L witnesses that T is splitting, let C be any infinite set of natural numbers. Now, build a ray as follows. Start at the root and proceed up the right branch of the tree (i.e., make no left turns) until the number of steps you have taken exceeds the first element of C (this ensures a label 0 for

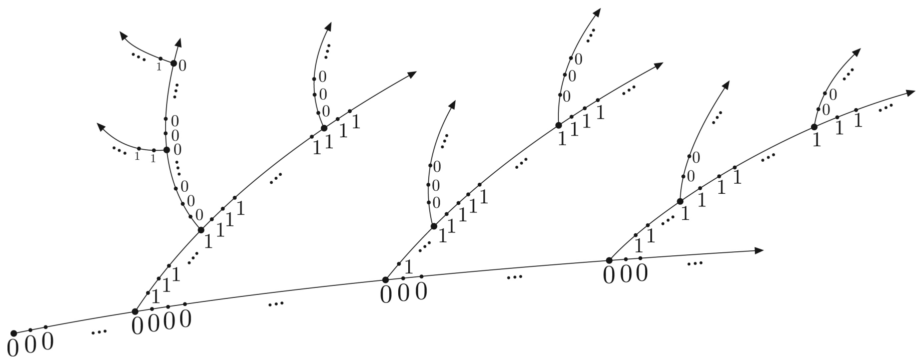

Fig. 1. A labelled subdivided binary tree

this element). At the next branching vertex, make a left turn and continue rightward (i.e., no more left turns) until you have passed another element of C larger than the height of your left turn (this ensures a label 1 for that element of C). At the next branching vertex, again make a left turn and continue rightward until you have ensured a 0 for another element of C. Continue in this fashion. ∎

To give the next example of a splitting graph, we introduce a new concept. Let R be a ray with vertices $R(0) = x_0 \le_R y_0 <_R x_1 \le_R \cdots$. We denote the path $x_n R y_n$ by P_n and the path $y_n R x_{n+1}$ by ${}^nP^a$. Furthermore, for every $n < \omega$, let ${}^nP^b$ be a path from y_n to x_{n+1} with $V(R) \cap V({}^nP^b) = \{y_n, x_{n+1}\}$ and $E(R) \cap E({}^nP^b) = \emptyset$ such that the interiors of the ${}^nP^b$ are pairwise disjoint. The paths ${}^nP^b$ are called *additional paths*. For notational simplicity, we assume that the length of ${}^nP^a$ is not smaller than the length of ${}^nP^b$ for all $n < \omega$, and we let $l(n) := \|{}^nP^a\| - \|{}^nP^b\|$.

We call the union B of the graph of the ray and the graphs of the additional paths a *ray with short cuts* if $l(n) \neq 0$ for all $n < \omega$, and a *ray with equal alternatives* if $l(n) = 0$ for all $n < \omega$. We call R the *basic ray* of B, we call x_0 the *first vertex* of B, and we call y_n a *branching vertex* of B for $n < \omega$. Furthermore, given $z \in V(R)$, we call a subgraph $B' \subseteq B$ the *canonical subgraph in B with first vertex z* if B' is obtained from B by deleting the vertices of the initial segment $R\mathring{z}$ of the basic ray and the inner vertices of those additional paths that have their first vertex in $R\mathring{z}$.

Lemma 3. *A ray with short cuts is splitting.*

Proof. Let B be a ray with short cuts as above. By deleting the inner vertices of some additional paths, we may assume that $l(n) + l(n+1) \le \|x_{n+2} R x_{n+3}\|$ for all $n < \omega$.

To show that B is splitting, we now define a 0-1 labelling L of B. Beginning with the vertex x_1 and the label 0, we label on $x_1 R x_2$ alternately

$l(0)$ vertices with 0 and $l(0)$ vertices with 1. For the induction step, assume that the vertices of the paths $x_1Rx_2, \mathring{x}_2Rx_3, \ldots, \mathring{x}_{n-1}Rx_n$ are labelled, let $c := L(x_n)$, and let c' be the other label. Beginning with the larger neighbour of x_n (larger with respect to $<_R$) and the label c, we label on $\mathring{x}_nRx_{n+1}$ alternately $l(n-1)$ vertices with c and $l(n-1)$ vertices with c'. (The remaining vertices may be labelled arbitrarily.)

To see that this is splitting, let C be any infinite set of natural numbers. By taking appropriate short cuts, we now build a ray which induces a sequence that splits C. Suppose we have defined an initial segment P of this ray, and suppose that k is the largest element in C for which $P(k)$ is defined. Our aim is to extend P to some branching vertex y_n so that for some new element i from C the following holds: no matter whether we continue by following ${}^nP^a$ on the basic ray or by taking the short cut ${}^nP^b$, the vertex corresponding to i is one of $V(x_{n+1}Rx_{n+2})$. The periodic labelling of these vertices then implies that the vertex corresponding to i is labelled differently depending on which path we choose. We can therefore ensure a label different from $L(P(k))$.

It is helpful to introduce the following notions. Let P be a finite path from x_0 to y_n. We call the vertices $V(x_{n+1}Rx_{n+2})$ the *target vertices* of P, and we define two extensions of P:

$$a(P) := P^\wedge\, {}^nP^{a\wedge}x_{n+1}Rx_{n+2}, \quad b(P) := P^\wedge\, {}^nP^{b\wedge}x_{n+1}Rx_{n+2}.$$

As indicated above, suppose that an initial segment P of the desired ray has been defined, and suppose that k is the largest element in C for which $P(k)$ is defined. We then let $i \in C - \{0, \ldots, k\}$ be minimal with the property that there is a branching vertex y_m such that a path $P^?$ extending P to y_m has the property that $\|a(P^?)x_{m+1}\| \leq i < \|a(P^?)\|$.

If $i \leq \|b(P^?)\|$, then we are done. The expressions $a(P^?)(i)$ and $b(P^?)(i)$ are defined since $\|b(P^?)\| \geq i$, and these expressions stand for target vertices of $P^?$ since $\|a(P^?)x_{m+1}\| \leq i$. The definition of L then implies that the vertices $a(P^?)(i)$ and $b(P^?)(i)$ are labelled differently.

If $i > \|b(P^?)\|$, we let $P^* := b(P^?)y_{m+1}$ (see Figure 2). The expressions $a(P^*)(i)$ and $b(P^*)(i)$ are defined since

$$\|b(P^*\| \geq \|b(P^?)y_{m+1}{}^\wedge\, {}^{m+1}P^b\| + l(m) + l(m+1) = \|a(P^?)\| > i,$$

which is a consequence of the definitions of $b(P^*)$ and $a(P^?)$ and the assumption that $l(m) + l(m+1)$ is not larger than $\|x_{m+2}Rx_{m+3}\|$. The expressions $a(P^*)(i)$ and $b(P^*)(i)$ stand for target vertices of P^* since

$$\|a(P^*)x_{m+2}\| = \|b(P^?)y_{m+1}{}^\wedge\, {}^{m+1}P^a\| = \|b(P^?)\| < i.$$

Again the definition of L then implies that the vertices $a(P^*)(i)$ and $b(P^*)(i)$ are labelled differently. ■

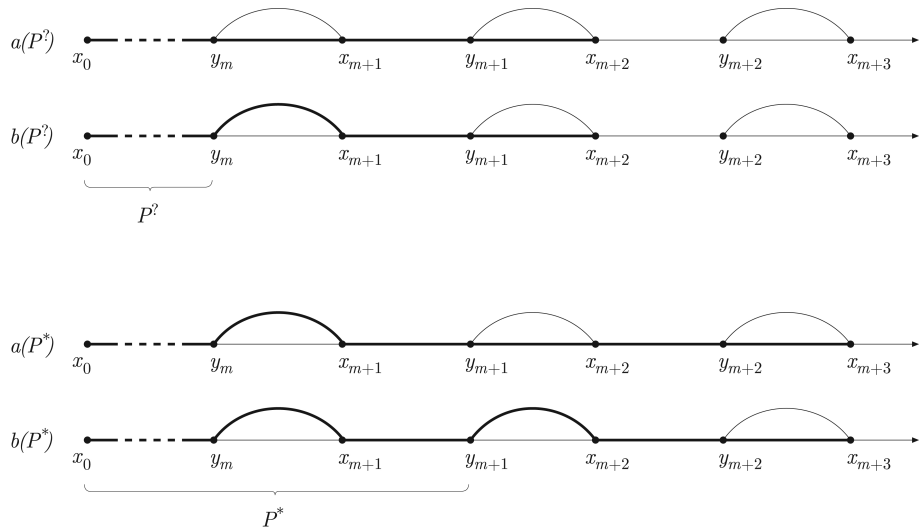

Fig. 2. The extensions of the path $P^?$ and those of the path P^*

Let B be a ray with equal alternatives as in the above definition. To be splitting, B has to have an additional property. We call B *bounded* if there is a number $m < \omega$ such that $\|P_n\| \leq m$ for all $n < \omega$; such a number m is called a *bound* for B.

It turns out to be helpful to reformulate this concept. Let us call a sequence $\langle n_l;\ l < m \rangle$ $(m \leq \omega)$ of natural numbers *connected* if $n_{l+1} = n_l + 1$ for all $l < m$. If I is a set of natural numbers, ordered in the natural way, we let $I(n)$ be the length of the nth maximal connected sequence in I (for $n \geq 1$). Let I, for example, be the set $\{5, 8, 9, 10, 16\}$; then $I(1) = 1$, i.e., the length of the first maximal connected sequence in I equals 1, $I(2) = 3$, and $I(3) = 1$. Given B, a ray with equal alternatives as above, we let

$$\mathrm{Const}(B) := \Big\{ i \in \omega;\ R(i) \in \bigcup_{n<\omega} V(P_n) \Big\} \quad \text{and} \quad \mathrm{Var}(B) := \omega - \mathrm{Const}(B).$$

Note that a ray B with equal alternatives is bounded if and only if there is a number $m < \omega$ such that $\mathrm{Const}(B)(n) \leq m$ for all $n \geq 1$.

Lemma 4. *A bounded ray with equal alternatives is splitting.*

Proof. Let B be a bounded ray with equal alternatives, and let $m < \omega$ be a bound for B. Given $z \in V(B)$ and a ray R' with $R'(0) = x_0$ and $z \in V(R')$, we say that the vertex *z has a partner* if ${R'}^{-1}(z) \in \mathrm{Var}(B)$.

To show that B is splitting, we now define a 0-1 labelling L of B. We let all vertices on the basic ray R be labelled with 0, and we let all the other vertices be labelled with 1.

To see that this is splitting, let C be any infinite set of natural numbers. The idea of the proof is fairly simple. Since the intervals where we have no choice are bounded in length by m, some translate of C (by an amount $\leq m$) contains infinitely many places where a choice can be made. We only have to start our ray a little late to handle the translation and then make the required choices.

To be precise, we will show that there are finitely many rays with equal alternatives contained in B, say ${}^0B,\dots,{}^kB$, such that $C-\{0,\dots,m\}\subseteq\bigcup_{j\leq k}\mathrm{Var}({}^jB)$. This has the consequence that there is one of them, say ${}^jB$, that has the property that $\mathrm{Var}({}^jB)\cap C$ is infinite; then the label sequence induced by some ray R^* whose first vertex is the first vertex of ${}^jB$ splits C.

To this end, we choose, for all $j<\omega$, a vertex $z_j\in V(R)$ with minimal distance to the vertex x_0 such that there is a vertex $z_j^*\geq_R z_j$ with a distance of n_j steps from z_j that has a partner. Either $z_j=x_0$, or there is some branching vertex y_i that is the smaller neighbour of z_j^* (smaller with respect to $<_R$); this is a consequence of the minimality assumption. Therefore, the distance between x_0 and z_j is not larger than $m+1$. Now, the claim follows for those canonical subgraphs in B whose first vertices are $R(0),\dots,R(m+1)$ respectively. ■

The idea of boundedness can be generalized. A finite set $\mathscr{B}=\{{}^0B,\dots,{}^tB\}$ of pairwise disjoint rays with equal alternatives is called *bounded* if there is a number $m<\omega$ such that $(\bigcap_{s\leq t}\mathrm{Const}({}^sB))(n)\leq m$ for all $n\geq 1$ (see Figure 3).

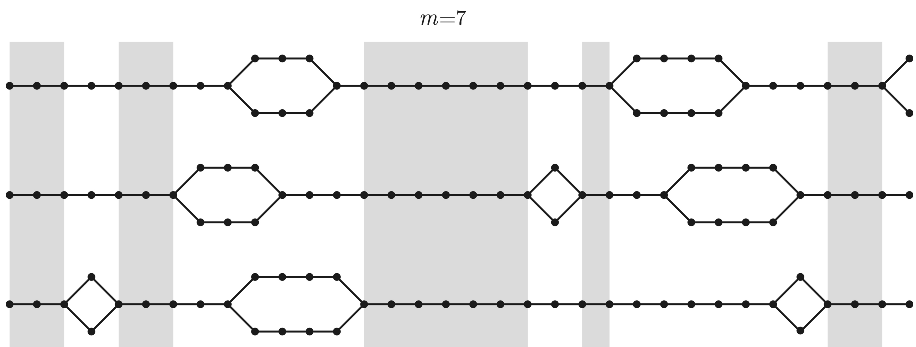

Fig. 3. A bounded system of rays with equal alternatives with bound 7

Lemma 5. *A bounded system of rays with equal alternatives is splitting.*

Proof. The proof is similar to the previous one. Let $\mathscr{B}=\{{}^0B,\dots,{}^tB\}$ be a bounded system of rays with equal alternatives, let $m<\omega$ be a bound for $\mathscr{B}$, and let ${}^sR$ be the basic ray of ${}^sB$ with first vertex ${}^sx_0$ (for all $s\leq t$). As above, we label all the vertices on the basic rays with 0, and all the other vertices with 1.

Let C be any infinite set of natural numbers. As in the proof of the previous lemma, it suffices to show that there are finitely many rays with equal alternatives contained in $\bigcup_{s\le t} {}^sB$, say ${}^0B^*, \ldots, {}^kB^*$, such that $C - \{0, \ldots, m\}$ is contained in $\bigcup_{j\le k} \mathrm{Var}({}^jB^*)$.

To this end, we choose, for all $j < \omega$ and $s \le t$, a vertex $z^s_j \in V({}^sR)$ with minimal distance to ${}^sx$ such that there is a vertex $z^*_j \ge_{^sR} z_j$ with a distance of n_j steps from z_j that has a partner. For all $j < \omega$, we then choose $s \le t$ such that the distance between ${}^sx$ and z^s_j is minimal, and we let $z_j := z^s_j$. As in the proof of the previous lemma, it follows that the distance between z_j and 'its' ${}^sx$ is not larger than $m + 1$. Here, the claim follows for those $(t+1)\cdot(m+1)$ canonical subgraphs whose first vertices are ${}^sR(0), \ldots, {}^sR(m + 1)$ $(s \le t)$ respectively. ∎

4. Small and tiny graphs. In the remaining sections, we will demonstrate that a countable graph is not splitting if it does not contain a subdivided binary tree, a ray with short cuts, or a bounded system of rays with equal alternatives (together with the results of the previous section, this will yield our characterization of the countable splitting graphs). In this section, we analyse the graphs that do not contain a ray with short cuts.

To begin, we have to introduce some further notions. Recall that a path with first vertex x and last vertex y is called an *x-y path*. Let $X, Y \subseteq V(G)$ be sets of vertices; then an x-y path P with $V(P) \cap X = \{x\}$ and $V(P) \cap Y = \{y\}$ is an *X-Y path*. We call a family of $\{x\}$-Y paths $\mathscr{F}$ an *x-Y fan* if $V(P) \cap V(P') = \{x\}$ for all distinct $P, P' \in \mathscr{F}$. Given a subgraph $H \subseteq G$, we call an x-y path P an *H-path* if $V(H) \cap V(P) = \{x, y\}$ and $E(P) \cap E(H) = \emptyset$. (Note that the set of inner vertices of an H-path is *disjoint* from the vertex set of H.) A vertex z is an *articulation* of a graph G if there are vertices x and y, distinct from z and from each other, such that z is a vertex of every x-y path.

Furthermore, recall the notion of an *end* of a graph. If the rays R and R' in some graph G have infinitely many vertices in common or if there is a disjoint infinite set of $V(R)$-$V(R')$ paths in G, then R and R' are called *equivalent*; the corresponding equivalence classes of rays are the *ends* of G; the *degree* of an end Ω of G is the supremum of the cardinalities of sets of disjoint rays in Ω. We shall need the following simple lemma.

Lemma 6. *A countable graph that does not contain a subdivided binary tree has only countably many ends.*

Proof. Let G be a countable graph, and let $C_0, C_1, \ldots$ be the components of G. As a countable connected graph, every component C_i has a normal spanning tree T_i (cf. [3, pp. 205–206]). Every end of C_i contains exactly one normal ray of T_i (cf. [3, p. 205]). If G has uncountably many ends, some

C_i has uncountably many ends, and therefore some T_i contains uncountably many normal rays. Inductively, it is now possible to construct in T_i a subdivided binary tree (cf. [3, p. 238]). ∎

It is easy to see that a graph with an end of degree at least 2 contains a ray with short cuts so that we can restrict our attention to graphs which have only ends of degree 1: Let G be a graph that contains two disjoint rays R and R' which are linked by a disjoint infinite set of $V(R)$-$V(R')$ paths in G. Suppose that we have defined an 'initial part' of a ray with short cuts by following R, R', and some of the $V(R)$-$V(R')$ paths. We then take a look at three $V(R)$-$V(R')$ paths lying ahead, say $P_1 = x_1 \ldots y_1$, $P_2 = x_2 \ldots y_2$, $P_3 = x_3 \ldots y_3$ with $x_1 <_R x_2 <_R x_3$ and $y_1 <_{R'} y_2 <_{R'} y_3$. Now, either $\|x_2 R x_3\| < \|x_2 P_2 y_2{}^\wedge y_2 R' y_3{}^\wedge y_3 P_3 x_3\|$, or $\|y_2 R' y_3\| < \|y_2 P_2 x_2{}^\wedge x_2 R x_3{}^\wedge x_3 P_3 y_3\|$. Let us assume the former case obtains. Possibly by using P_1, we extend the initial segment of our basic ray to the vertex x_2. Then we use $x_2 P_2 y_2{}^\wedge y_2 R' y_3{}^\wedge y_3 P_3 x_3$ as a continuation of the basic ray and $x_2 R x_3$ as the next short cut. By continuing in this manner we obtain a ray with short cuts.

We will now introduce the basic notions of the following lemma. Let R be a ray, and let $H := G(R)$ be the graph of R. We call the pair (H, R) a *fragment in the first sense*, and we say that R is the *basic ray* of this fragment.

Let R be a ray with vertices $R(0) = x_0 \le_R y_0 <_R x_1 \le_R \cdots$. The path $x_n R y_n$ is denoted by P_n, and the path $y_n R x_{n+1}$ by P_n^*. Furthermore, let $X_n \supsetneq G(P_n^*)$ be graphs such that the following conditions are satisfied for all $n < \omega$:

- every $v \in V(X_n) - V(P_n^*)$ is a vertex of some $G(P_n^*)$-path,
- $V(X_n) \cap \bigcup_{m<\omega} V(P_m) = \{y_n, x_{n+1}\}$,
- $V(X_n) \cap \bigcup_{m>n} V(X_m) = \{x_{n+1}\} \cap \{y_{n+1}\}$,
- there is no inner vertex from P_n^* that is an articulation of X_n.

We then let H be the union of the graph of R and the graphs X_n for all $n < \omega$. We call the pair (H, R) a *fragment in the second sense*, or, more informatively, a *fragment in the second sense with respect to the infinite sequence* (P_n, P_n^*, X_n). We say that R is the *basic ray* of this fragment, and we call the graphs X_n the *additional graphs* of this fragment. If (H, R) is a fragment in the second sense with respect to (P_n, P_n^*, X_n), we let $x_n := P_n(0)$ and $y_n := P_n^*(0)$ for all $n < \omega$.

A *small* graph is composed of fragments in a way that reflects the types of rays that exist in the graph, i.e., we call a graph G *small* if there is a set U of fragments (H, R) such that the following two conditions are satisfied:

- for all ends Ω of G there is exactly one $(H, R) \in U$ such that $R \in \Omega$,
- for all rays R' of G there is exactly one $(H, R) \in U$ such that

$$|V(R') \cap V(R)| = \aleph_0$$

and $R'(n) \in V(H)$ for all but finitely many $n < \omega$.

In this situation, we also say, more informatively, that G is *small with respect to U*.

Let us now present the announced characterization of graphs that have only ends of degree 1.

LEMMA 7. *All ends of a graph G have degree* 1 *if and only if G is small.*

Proof. $\Leftarrow$: Let G be a graph that is small with respect to U, and suppose there is an end Ω and two disjoint rays R_1 and R_2 from Ω. It follows that R_1 and R_2 belong to the same fragment $(H, R) \in U$. Therefore, these rays are equivalent to the ray R and have tails in the graph H. But this contradicts the fact that (H, R) is a fragment.

$\Rightarrow$: Let G be a graph that has only ends of degree 1, let $\{\Omega_\alpha;\ \alpha < \kappa\}$ be the set of ends of G for some cardinal κ, and let R'_α be any element of Ω_α. We say that a vertex v *dominates* a ray R if there is an infinite v-$V(R)$ fan in G. Given $\alpha < \kappa$, we let A_α be the set of those vertices that dominate R'_α. It is easy to see that A_α is finite since Ω_α has degree 1: If a ray R is dominated by infinitely many vertices v_n, $n < \omega$, you can build two disjoint rays R_1 and R_2 which are equivalent to R. Begin with R_1: Let v_0 be the first vertex. Take some v_0-$V(R)$ path P^0; then follow R and take some v_2-$V(R)$ path P^1 disjoint from P^0. Continue with R_2: Let v_1 be the first vertex. Take some v_1-$V(R)$ path P^2 disjoint from P^0 and P^1 which ends at some vertex larger than the vertices of R already used (in the sense of $>_R$); then follow R and take some v_3-$V(R)$ path P^3 disjoint from P^0, P^1, and P^2. Go back to R_1 and continue in the same fashion.

Given $\alpha < \kappa$, let us now define subgraphs H_α. To this end, we let R_α be some tail of R'_α that has no vertices in A_α. We let $H_\alpha^0 := G(R_\alpha)$. For a limit ordinal γ, we assume that H_α^β has been defined for all $\beta < \gamma$, and we let $H_\alpha^\gamma := \bigcup_{\beta<\gamma} H_\alpha^\beta$. For the induction step, we assume that H_α^γ has been defined for some ordinal γ, and we check whether there is an H_α^γ-path which does not use a vertex from A_α; if there is one, we let P_α^γ be such a path, and we let $H_\alpha^{\gamma+1} := H_\alpha^\gamma \cup G(P_\alpha^\gamma)$; otherwise, we let $H_\alpha^{\gamma+1} := H_\alpha^\gamma$. Finally, we let $H_\alpha := \bigcup_\gamma H_\alpha^\gamma$.

We now let $U := \{(H_\alpha, R_\alpha);\ \alpha < \kappa\}$, and we claim that G is small with respect to U. As every end has degree 1, for all rays R' there is exactly one $\alpha < \kappa$ such that $|V(R') \cap V(R_\alpha)| = \aleph_0$ and $R'(n) \in V(H_\alpha)$ for all but finitely many $n < \omega$. If $H_\alpha = G(R_\alpha)$, the pair (H_α, R_α) is a fragment in the first sense. If not, we need to find infinitely many articulations $z \in V(R_\alpha)$ of H_α;

it is then possible to define paths $P_{\alpha,n}$, paths $P^*_{\alpha,n}$, and graphs $X_{\alpha,n}$ so that (H_α, R_α) is a fragment in the second sense with respect to $(P_{\alpha,n}, P^*_{\alpha,n}, X_{\alpha,n})$. That it is possible to find these articulations is an easy and known consequence of the theorem of Menger ([1]). ∎

Our next aim is to analyse the graphs that do not contain a ray with short cuts. A fragment (H, R) in the first sense is said to be *rigid*. A fragment (H, R) in the second sense with respect to (P_n, P^*_n, X_n) is said to be *rigid* if there are no two y_n-x_{n+1} paths of different length in an additional graph X_n. We call a graph G that is small with respect to U *tiny with respect to* U if every fragment in U is rigid. Now, we are in a position to characterize the graphs that do not contain a ray with short cuts.

THEOREM 8. *A graph G does not contain a ray with short cuts if and only if G is tiny.*

Proof. Since the implication from right to left is easy, we only show the other one. Let G be a graph not containing a ray with short cuts. As a consequence of Lemma 7, there is a set of fragments U such that G is small with respect to U. Our aim is to replace each fragment $(H, R) \in U$ by a rigid fragment (H', R) so that G is small (and therefore tiny) with respect to the resulting set $\{(H', R);\ (H, R) \in U\}$.

To this end, let (H, R) be an arbitrary fragment in U. If (H, R) is a fragment in the first sense, we let $H' := H$. Let us therefore assume that (H, R) is a fragment in the second sense with respect to (P_n, P^*_n, X_n). First, suppose there are only finitely many $n < \omega$ such that there are two y_n-x_{n+1} paths in X_n of different length; we then let H' be the result of deleting from H the vertices in $V(X_n) - V(P^*_n)$ for those n. Now, suppose there are infinitely many n such that there are y_n-x_{n+1} paths Q^1_n and Q^2_n in X_n of different length. We may assume that the paths Q^1_n and y_nRx_{n+1} have different length and that Q^1_n is shorter than y_nRx_{n+1}. There has to be a subpath $Q_n = y_n \ldots z$ of Q^1_n that is a $G(R)$-path shorter than y_nRz. But now the graphs of these infinitely many paths Q_n together with the graph of R form a ray with short cuts contained in G (contradiction). ∎

5. Minute graphs. In this section, we characterize the graphs that contain neither a ray with short cuts nor a bounded system of rays with equal alternatives. Let μ and κ be ordinals with $\mu \le \kappa$, and let G be tiny with respect to $U := \{(H_\alpha, R_\alpha);\ \alpha < \kappa\}$ where (H_α, R_α) is a fragment in the first sense for all $\alpha < \mu$, and a fragment in the second sense with respect to

([1]) In [2], this is a special case of Lemma 9 (let $m(\Omega_\alpha) := 1$). Bruhn and Stein remark that this lemma can be obtained from results of Polat [6].

$(P_{\alpha,n}, P^*_{\alpha,n}, X_{\alpha,n})$ for all $\alpha \geq \mu$ $(\alpha < \kappa)$. Given $\alpha \geq \mu$, we let $x_{\alpha,n} := P_{\alpha,n}(0)$ and $y_{\alpha,n} := P^*_{\alpha,n}(0)$ for all $n < \omega$.

We begin by defining a system of rays containing the rays R_α, $\alpha < \kappa$. Given $\alpha < \kappa$ and $a \in \mathbb{Z}$, we let R^a_α be the tail of R_α whose first vertex is $R_\alpha(a)$ if $a \geq 0$, and a ray such that R_α is a tail of it and $R^a_\alpha(a) = R_\alpha(0)$ if $a < 0$. (Note that R^a_α does not have to be a ray in G if $a < 0$.) We let

$$\mathrm{Const}(\alpha, a) := \begin{cases} \{i \in \omega;\ R^a_\alpha(i) \in V(R_\alpha)\} & \text{for } \alpha < \mu, \\ \left\{i \in \omega;\ R^a_\alpha(i) \in \bigcup\limits_{n<\omega} V(P_{\alpha,n})\right\} & \text{for } \alpha \geq \mu. \end{cases}$$

Given $\alpha \geq \mu$ and $a \in \mathbb{Z}$, we now define a ray with equal alternatives corresponding to the pair (α, a). Let $n^* < \omega$ be minimal such that $y_{\alpha,n^*} \in V(R^a_\alpha)$. We then call a graph B a *ray with equal alternatives corresponding to* (α, a) if B is a ray with equal alternatives containing R^a_α as its basic ray whose additional paths are $y_{\alpha,n}$-$x_{\alpha,n+1}$ paths in $X_{\alpha,n}$ for all $n \geq n^*$.

There is a ray with equal alternatives corresponding to (α, a) for every $\alpha \geq \mu$ and $a \in \mathbb{Z}$: Since $X_{\alpha,n} \neq G(P^*_{\alpha,n})$, there is a nontrivial $G(R_\alpha)$-path $Q_{\alpha,n} = y_{\alpha,n} \ldots z$ in $X_{\alpha,n}$; let $Q_{\alpha,n}$ be such a path with z as large as possible (with respect to $<_{R_\alpha}$). Since all $y_{\alpha,n}$-$x_{\alpha,n+1}$ paths in $X_{\alpha,n}$ have the same length, it follows that $z = x_{\alpha,n+1}$.

For $\alpha \geq \mu$ and $a \in \mathbb{Z}$, we let B^a_α be such a ray with equal alternatives corresponding to (α, a). Note that $\mathrm{Const}(\alpha, a) = \mathrm{Const}(B^a_\alpha)$ if B^a_α is defined. Note furthermore that $B^a_\alpha \subseteq H_\alpha$ if $a \geq 0$.

Our next aim is to define a function that maps the rays of G to pairs in $\kappa \times \mathbb{Z}$. To this end, let R be any ray in G. There is exactly one $\alpha < \kappa$ such that $|V(R) \cap V(R_\alpha)| = \aleph_0$ and $R(i) \in V(H_\alpha)$ for all but finitely many $i < \omega$. Furthermore, if $\alpha < \mu$, there is exactly one $a \in \mathbb{Z}$ such that, for all but finitely many $i < \omega$, $R(i) = R^a_\alpha(i)$, and if $\alpha \geq \mu$, there is exactly one $a \in \mathbb{Z}$ such that the following two conditions are satisfied for all but finitely many $i < \omega$:

$$R(i) \in \bigcup_{n<\omega} V(P_{\alpha,n}) \ \Leftrightarrow\ R^a_\alpha(i) \in \bigcup_{n<\omega} V(P_{\alpha,n}),$$
$$R^a_\alpha(i) \in \bigcup_{n<\omega} V(P_{\alpha,n}) \ \Rightarrow\ R(i) = R^a_\alpha(i).$$

In each case, we call (α, a) the *pair corresponding to* R.

We now return to our analysis of the graphs that contain neither a ray with short cuts nor a bounded system of rays with equal alternatives. Let G be tiny with respect to some set U as in the preceding discussion. We call G *minute with respect to* U if for all finite $M \subseteq \kappa \times \mathbb{Z}$,

$$\bigcap_{(\alpha,a) \in M} \mathrm{Const}(\alpha, a) = \aleph_0,$$

and we use this concept to present a characterization of graphs that contain neither a ray with short cuts nor a bounded system of rays with equal alternatives.

THEOREM 9. *A graph G is minute if and only if G contains neither a ray with short cuts nor a bounded system of rays with equal alternatives.*

Proof. $\Rightarrow$: Let G be a minute graph. Theorem 8 implies that G does not contain a ray with short cuts.

Suppose G contains a bounded system of rays with equal alternatives, i.e., suppose ${}^0B, \ldots, {}^tB$ are disjoint rays with equal alternatives contained in G with basic rays ${}^0R, \ldots, {}^tR$ such that there is a natural number r with $(\bigcap_{s\le t} \mathrm{Const}({}^sB))(n) \le r$ for all $n \ge 1$ (recall that $I(n)$ is the length of the nth maximal connected sequence in I, for any $I \subseteq \omega$ ordered in the natural way).

For $s \le t$ and $q \le r$, we let ${}^{s,q}B$ be the canonical subgraph in ${}^sB$ with first vertex ${}^sR(q)$, and we let $(\alpha_{s,q}, a_{s,q})$ be the pair corresponding to the basic ray of ${}^{s,q}B$. It follows that $\bigcap_{s\le t,\, q\le r} \mathrm{Const}(\alpha_{s,q}, a_{s,q}) < \aleph_0$ (contradiction).

$\Leftarrow$: Let G be a graph that contains neither a ray with short cuts nor a bounded system of rays with equal alternatives. Theorem 8 implies that G is tiny with respect to some set $U := \{(H_\alpha, R_\alpha);\ \alpha < \kappa\}$ for some ordinal κ. As in the preceding discussion, we assume that there is an ordinal $\mu \le \kappa$ such that (H_α, R_α) is a fragment in the first sense if $\alpha < \mu$, and a fragment in the second sense with respect to $(P_{\alpha,n}, P^*_{\alpha,n}, X_{\alpha,n})$ if $\alpha \ge \mu$.

Suppose G is not minute, i.e., there are finitely many $\alpha_0, \ldots, \alpha_m$ in κ and $a_0, \ldots, a_m$ in $\mathbb{Z}$ such that $\bigcap_{\ell\le m} \mathrm{Const}(\alpha_\ell, a_\ell)$ is finite; we may assume that $\alpha_\ell \ge \mu$ for all $\ell \le m$, i.e., each H_{α_ℓ} is a fragment in the second sense.

Our first aim is to replace each a_ℓ by some b_ℓ so that no b_ℓ is negative while $\bigcap_{\ell\le m} \mathrm{Const}(\alpha_\ell, b_\ell)$ is still finite. We achieve this by letting $b_\ell := a_\ell + k$ for all $\ell \le m$ where $k := \max_{\ell\le m}\{-a_\ell;\ a_\ell < 0\}$. Note that

$$\mathrm{Const}(B^{b_\ell}_{\alpha_\ell}) = \mathrm{Const}(\alpha_\ell, b_\ell) = \{n - k;\ n \in \mathrm{Const}(\alpha_\ell, a_\ell) \wedge n \ge k\}$$

for all $\ell \le m$; therefore, $\bigcap_{\ell\le m} \mathrm{Const}(B^{b_\ell}_{\alpha_\ell})$ is indeed still finite.

Our next aim is to replace the $B^{b_\ell}_{\alpha_\ell}$ involved by $B^{c_\ell}_{\alpha_\ell}$ in such a way that the resulting $B^{c_\ell}_{\alpha_\ell}$ are 'as often as possible' pairwise disjoint, i.e., we would like to get rid of pairs $(B^{b_\ell}_{\alpha_\ell}, B^{b_k}_{\alpha_k})$ if $V(B^{b_\ell}_{\alpha_\ell}) \cap V(B^{b_k}_{\alpha_k}) \ne \emptyset$ although $\alpha_\ell \ne \alpha_k$. We achieve this by letting $c_\ell := b_\ell + r$ for all $\ell \le m$ where r is defined as follows: Let $r_{\ell,k} := 0$ if $\alpha_\ell = \alpha_k$; otherwise, let $r_{\ell,k}$ be the smallest number such that $B^{b_\ell + r_{\ell,k}}_{\alpha_\ell}$ and $B^{b_k + r_{\ell,k}}_{\alpha_\ell}$ are disjoint (the existence of such a number is guaranteed by the definition of 'small'); then let $r := \max_{\ell,k\le m} r_{\ell,k}$. Note that, for all $\ell \le m$,

$$\mathrm{Const}(B^{c_\ell}_{\alpha_\ell}) = \{n - r;\ n \in \mathrm{Const}(B^{b_\ell}_{\alpha_\ell}) \wedge n \ge r\};$$

therefore, $\bigcap_{\ell\le m} \mathrm{Const}(B^{c_\ell}_{\alpha_\ell})$ is still finite. We let ${}^\ell B := B^{c_\ell}_{\alpha_\ell}$.

Now, let $\mathscr{B}$ be a maximal subset of $\{{}^\ell B;\ \ell \le m\}$ whose elements are pairwise disjoint, and let ${}^{i_0}B, {}^{i_1}B, \ldots$ be an enumeration of $\{{}^\ell B;\ \ell \le m\} - \bigcup \mathscr{B}$. The assumption that G does not contain a bounded system of rays with equal alternatives implies that for every $m^* < \omega$, there is $n \ge 1$ such that $(\bigcap_{B \in \mathscr{B}} \mathrm{Const}(B))(n) > m^*$. Let q be the smallest number such that

$$(\sharp)\qquad \exists m^* < \omega\ \forall n \ge 1 \quad \Big(\bigcap_{B \in \mathscr{B}} \mathrm{Const}(B) \cap \bigcap_{s \le q} \mathrm{Const}({}^{i_s}B)\Big)(n) \le m^*;$$

such a number q exists since $\bigcap_{\ell \le m} \mathrm{Const}({}^\ell B)$ is finite. The minimality of q implies that

$$(+)\qquad \forall m^* < \omega\ \exists n \ge 1 \quad \Big(\bigcap_{B \in \mathscr{B}} \mathrm{Const}(B) \cap \bigcap_{s < q} \mathrm{Const}({}^{i_s}B)\Big)(n) > m^*.$$

Let ${}^tB$ be an element of $\mathscr{B}$ such that ${}^tB$ and ${}^{i_q}B$ are not disjoint. Since ${}^tB$ and ${}^{i_q}B$ are not disjoint, we know that $\alpha_t = \alpha_{i_q}$. Let us assume, without loss of generality, that $c_t = c_{i_q} + s$ for some $s < \omega$. We infer that $\mathrm{Const}({}^tB) \cap \mathrm{Const}({}^{i_q}B))(n) \ge \mathrm{Const}({}^{i_q}B)(n) - s$ for all $n \ge 1$. But this contradicts the formulas $(\sharp)$ and $(+)$. ∎

6. The characterization of splitting graphs. In this final section, we characterize the countable splitting graphs. We begin with a lemma concerning countable graphs without a subdivided binary tree.

Lemma 10. *A countable, minute graph G that does not contain a subdivided binary tree is minute with respect to some countable U.*

Proof. Let G be a countable graph not containing a subdivided binary tree such that G is minute with respect to some U. From Lemma 6 we know that G has only countably many ends. The definition of 'small' then implies that U is countable. ∎

The final gap in the proof of the characterization of the countable splitting graphs is closed by the following lemma.

Lemma 11. *A graph that is minute with respect to some countable U is not splitting.*

Proof. Let G be minute with respect to $U = \{(H_j, R_j);\ j < \omega\}$. We may assume that each (H_j, R_j) is a fragment in the second sense with respect to $(P_{j,n}, P^*_{j,n}, X_{j,n})$. Recall from the discussion in Section 5 the definitions of R^a_j, $\mathrm{Const}(j,a)$, and 'pair corresponding to R'. Recall furthermore that for every ray R in G, the following two conditions are satisfied for all but finitely many $i < \omega$ with respect to the pair (j,a) corresponding to R:

$$(\sharp_i)\qquad \begin{aligned} R(i) \in \bigcup_{n<\omega} V(P_{j,n}) \;\Leftrightarrow\; R^a_j(i) \in \bigcup_{n<\omega} V(P_{j,n}),\\ R^a_j(i) \in \bigcup_{n<\omega} V(P_{j,n}) \;\Rightarrow\; R(i) = R^a_j(i). \end{aligned}$$

We let $f = (f_1, f_2)$ be a bijection between ω and $\omega \times \mathbb{Z}$, and we let $C_m :=$ Const$(f_1(m), f_2(m))$ for all $m < \omega$. (A number i is in C_m if and only if $R^{f_2(m)}_{f_1(m)}(i) \in \bigcup_{n<\omega} V(P_{f_1(m),n})$.) Since G is minute with respect to U, we infer that $\bigcap_{\ell\leq m} C_\ell$ is infinite for all $m < \omega$.

To see that G is not splitting, let any 0-1 labelling L of G be given. For $m < \omega$ and $c \in \{0, 1\}$, we let ${}^cC_m$ be the set of those $i \in C_m$ with $L(R^{f_2(m)}_{f_1(m)}(i)) = c$, i.e., we let ${}^cC_m$ be the set $C_m \cap (L \circ R^{f_2(m)}_{f_1(m)})^{-1}(c)$. Given a binary sequence $\sigma\colon m + 1 \to \{0, 1\}$ of length $m < \omega$, we let ${}^\sigma C_m :=$ $\bigcap_{l\leq m} {}^{\sigma(l)}C_l$. Our aim is to find an infinite binary sequence g such that ${}^{g\restriction 1}C_0 \supseteq {}^{g\restriction 2}C_1 \supseteq \cdots$ and $|{}^{g\restriction(m+1)}C_m| = \aleph_0$ for all $m < \omega$.

Suppose there is such a sequence g. We then let $C^*_m := {}^{g\restriction(m+1)}C_m$. Now, we choose an increasing sequence $n_0 < n_1 < \cdots$ such that $n_m \in C^*_m$ for all $m < \omega$, and we let $C := \{n_m;\, m < \omega\}$. To complete the proof, we show that the sequence $(L \circ R)\restriction C$ converges for every ray R in G.

To this end, let R be any ray in G, and let $(j, a) := (f_1(m), f_2(m))$ be the pair corresponding to R (for some natural number m). We then let i^* be the smallest number such that the conditions $(\sharp_i)$ are satisfied for all $i \geq i^*$. For all $i \in C^*_m$, we know that $L(R^a_j(i)) = L(R^{f_2(m)}_{f_1(m)}(i)) = \sigma(m)$. We infer that $R(i) = \sigma(m)$ for all $i \in C^*_m$ with $i \geq i^*$. If we then let $m^* := \max\{m, i^*\}$, we know that $L(R(n_k)) = \sigma(m)$ for all $k \geq m^*$.

To define the sequence g, we use Lemma 1. Let

$$V_m := \{{}^\sigma C_m;\, \sigma\colon m + 1 \to 2 \wedge |{}^\sigma C'_m| = \aleph_0\}.$$

It is not difficult to see that each V_m is non-empty. Given $k \leq m$ and $c_m, c_{m-1}, \ldots, c_k \in \{0, 1\}$, we let

$$M^k := \bigcap_{\ell\leq m} C_\ell \cap \bigcap_{k\leq\ell\leq m} (L \circ R^{f_2(\ell)}_{f_1(\ell)})^{-1}\,(c_\ell).$$

By induction, we will define $c_m, c_{m-1}, \ldots, c_0 \in \{0, 1\}$ such that M^k is infinite for all $k \leq m$. (We then let $\sigma := c_0 \ldots c_m$ so that ${}^\sigma C_m = M^0$ is an element of V_m.) We already remarked that the set $\bigcap_{\ell\leq m} C_\ell$ is infinite. There has to be a $c_m \in \{0, 1\}$ such that M^m is infinite, and given $c_m, c_{m-1}, \ldots, c_k$ in $\{0, 1\}$ with $k > 0$ such that M^k is infinite, there has to be a $c_{k-1} \in \{0, 1\}$ such that M^{k-1} is infinite.

To apply Lemma 1, we have to define a graph on $\bigcup_{m<\omega} V_m$. To this end, given an element ${}^{\sigma^\wedge c}C_m$ in V_m for $m > 0$ and $c \in \{0, 1\}$, we let the element

${}^{\sigma}C_{m-1}$ in V_{m-1} be a neighbour of it. Since we may assume that distinct V_m are disjoint, Lemma 1 gives a ray ${}^{g\restriction 1}C_0\,{}^{g\restriction 2}C_1 \ldots$ where g is an infinite binary sequence. ∎

Now, we are in a position to characterize the countable splitting graphs.

THEOREM 12. *A countable graph G is splitting if and only if a subdivided binary tree, a ray with short cuts, or a bounded system of rays with equal alternatives is contained in G.*

Proof. The implication from right to left is given by Lemmas 2, 3, and 5.

For the other direction, suppose G is a countable graph that contains neither a subdivided binary tree, nor a ray with short cuts, nor a bounded system of rays with equal alternatives. Theorem 9 then implies that G is minute; Lemma 10 shows that G is minute with respect to some countable U; and Lemma 11 tells us that G is not splitting. ∎

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