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A group topology on the free abelian group of cardinality $\mathfrak{c}$ that makes its square countably compact

by

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Abstract. Under $\mathfrak{p} = \mathfrak{c}$, we prove that it is possible to endow the free abelian group of cardinality $\mathfrak{c}$ with a group topology that makes its square countably compact. This answers a question posed by Madariaga-Garcia and Tomita and by Tkachenko. We also prove that there exists a Wallace semigroup (i.e., a countably compact both-sided cancellative topological semigroup which is not a topological group) whose square is countably compact. This answers a question posed by Grant.

1. Introduction

1.1. Some history. It is known that a non-trivial free abelian group does not admit a compact group topology. In 1990, Tkachenko [10] showed that the free abelian group of size $\mathfrak{c}$ can be endowed with a countably compact group topology under CH. In 1998, Tomita [12] obtained such a topology under MA(σ-centered) and, two years later, Koszmider, Tomita and Watson [5] weakened the required form of Martin's axiom to $\mathrm{MA}_{\mathrm{countable}}$. In 2007, Madariaga-Garcia and Tomita [6] established the same result assuming the existence of $\mathfrak{c}$ pairwise incomparable selective ultrafilters (according to the Rudin–Keisler ordering in ω^*); in particular, they showed that the existence of a countably compact group topology on the free abelian group of size $\mathfrak{c}$ is compatible with the total failure of Martin's axiom (in the sense of Baumgartner [1]).

Tomita [12] showed that if a non-trivial free abelian group is endowed with a group topology, then its ωth power cannot be countably compact. Under $\mathfrak{p} = \mathfrak{c}$, we prove that there exists a group topology on the free abelian group of size $\mathfrak{c}$ that makes its square countably compact. This answers a question posed by Madariaga-Garcia and Tomita in [6] and by Tkachenko in [9].

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In 1952, Numakura [7] showed that every compact both-sided cancellative topological semigroup is a topological group. Three years later, Wallace [14] asked whether every countably compact both-sided cancellative topological semigroup is a topological group, and this question remains open in ZFC. Counterexamples to Wallace's question have been called *Wallace semigroups*. In 1996, Robbie and Svetlichny [8] answered Wallace's question in the negative under CH. In the same year, Tomita [11] showed that there exists a Wallace semigroup under $\mathrm{MA}_{\mathrm{countable}}$. It is worth noting that Madariaga-Garcia and Tomita [6] constructed a Wallace semigroup from $\mathfrak{c}$ pairwise incomparable selective ultrafilters.

Tomita [11] showed that the $2^{\mathfrak{c}}$th power of a Wallace semigroup cannot be countably compact. Under $\mathfrak{p} = \mathfrak{c}$, we prove that there exists a both-sided cancellative topological semigroup which is not a topological group and whose square is countably compact. This answers question 4 of [4].

1.2. Basic results, notation and terminology. In what follows, all group topologies are assumed to be Hausdorff. We recall that a topological space X is *countably compact* if every infinite subset of X has an accumulation point.

The following definition was introduced in [2] and is closely related to countable compactness.

DEFINITION 1.1. Let p be a free ultrafilter on ω and let $\{x_n : n \in \omega\}$ be a sequence in a topological space X. We say that $x \in X$ is a *p-limit point* of $\{x_n : n \in \omega\}$ if, for every neighborhood U of x, $\{n \in \omega : x_n \in U\} \in p$. In this case, we write $x = p\text{-}\lim\{x_n : n \in \omega\}$.

The set of all free ultrafilters on ω will be denoted by ω^*.

It is not difficult to prove that a topological space X is countably compact if, and only if, each sequence in X has a p-limit point for some $p \in \omega^*$.

The next two propositions are related to the concept of p-limit and will be used to prove Theorem 2.6.

PROPOSITION 1.2. *If $p \in \omega^*$ and $\{X_i : i \in I\}$ is a family of topological spaces, then $(y_i)_{i\in I} \in \prod_{i\in I} X_i$ is a p-limit point of $\{(x_i^n)_{i\in I} : n \in \omega\} \subset \prod_{i\in I} X_i$ if, and only if, $y_i = p\text{-}\lim\{x_i^n : n \in \omega\}$ for every $i \in I$.*

PROPOSITION 1.3. *Let G be a topological group and $p \in \omega^*$.*

(1) *If $\{x_n : n \in \omega\}$ and $\{y_n : n \in \omega\}$ are sequences in G and $x, y \in G$ are such that $x = p\text{-}\lim\{x_n : n \in \omega\}$ and $y = p\text{-}\lim\{y_n : n \in \omega\}$, then $x + y = p\text{-}\lim\{x_n + y_n : n \in \omega\}$.*
(2) *If $\{x_n : n \in \omega\}$ is a sequence in G and $x \in G$ is such that $x = p\text{-}\lim\{x_n : n \in \omega\}$, then $-x = p\text{-}\lim\{-x_n : n \in \omega\}$.*

If A is a set, then

$$[A]^{\omega} = \{X \subset A : |X| = \omega\}, \qquad [A]^{<\omega} = \{X \subset A : |X| < \omega\}.$$

A *pseudointersection* of a family $\mathcal{G}$ of sets is an infinite set that is $\subset^*$ in every member of $\mathcal{G}$. We say that a family $\mathcal{G}$ of infinite sets has the *strong finite intersection property* (SFIP, for short) if every finite subfamily of $\mathcal{G}$ has infinite intersection. The *pseudointersection number* $\mathfrak{p}$ is the smallest cardinality of any $\mathcal{G} \in [\omega]^{\omega}$ with SFIP but with no pseudointersection.

We denote the set of natural numbers by $\mathbb{N}$, the integers by $\mathbb{Z}$, the rationals by $\mathbb{Q}$ and the reals by $\mathbb{R}$. The unit circle group, which is identified with $\mathbb{R}/\mathbb{Z}$, will be denoted by $\mathbb{T}$ and the set of all non-empty open arcs of $\mathbb{T}$ will be denoted by $\mathcal{B}$.

Let Λ be a set of ordinal numbers and let G be a group. If $f \in G^{\Lambda}$, the *support* of f is the set $\{\lambda \in \Lambda : f(\lambda) \neq 0\}$, which will be indicated by $\operatorname{supp} f$. The direct sum $\bigoplus_{\lambda \in \Lambda} G$ is the set of all elements of G^{Λ} that have finite support and will be denoted by $G^{(\Lambda)}$.

An abelian group F is free abelian if there exist a non-empty set X and a function $\sigma : X \to F$ such that, for every function f from X to an abelian group G, there is a unique group homomorphism $g : F \to G$ satisfying $g \circ \sigma = f$. It is well-known that a free abelian group of size $\mathfrak{c}$ is isomorphic to $\mathbb{Z}^{(\mathfrak{c})}$, and therefore to $\mathbb{Z}^{(\mathfrak{c} \times \omega)}$.

We end this section by presenting some notation that will be used throughout this article.

If $J \in \mathbb{Q}^{(\mathfrak{c} \times \omega)}$, then

$$J = \sum_{(\mu,k) \in \operatorname{supp} J} J(\mu, k) \cdot \chi_{(\mu,k)}$$

where $\chi_{(\mu,k)} : \mathfrak{c} \times \omega \to \mathbb{Q}$ is given by

$$\chi_{(\mu,k)}(\xi, n) = \begin{cases} 1 & \text{if } (\xi, n) = (\mu, k), \\ 0 & \text{if } (\xi, n) \neq (\mu, k). \end{cases}$$

If $(\mu, k) \in \operatorname{supp} J$, we can write

$$J(\mu, k) = \frac{p(J, (\mu, k))}{q(J, (\mu, k))}$$

where $p(J, (\mu, k)), q(J, (\mu, k)) \in \mathbb{Z}$, $\gcd(p(J, (\mu, k)), q(J, (\mu, k))) = 1$ and $q(J, (\mu, k)) > 0$. Define

$$b(J) = \operatorname{lcm}\{q(J, (\mu, k)) : (\mu, k) \in \operatorname{supp} J\}$$

and, for each $(\mu, k) \in \operatorname{supp} J$, set

$$a(J, (\mu, k)) = p(J, (\mu, k)) \cdot \frac{b(J)}{q(J, (\mu, k))}.$$

Finally, define

$$|a(J)| = \max\{|a(J, (\mu, k))| : (\mu, k) \in \operatorname{supp} J\}.$$

2. Countably compact squares of free abelian groups. We will show that a free abelian group of size $\mathfrak{c}$ admits a group topology whose square is countably compact. Since every free abelian group of size $\mathfrak{c}$ is isomorphic to $\mathbb{Z}^{(\mathfrak{c}\times\omega)}$, it suffices to endow any isomorphic copy of $\mathbb{Z}^{(\mathfrak{c}\times\omega)}$ with such a topology.

Our strategy is to construct a group monomorphism $\Phi : \mathbb{Q}^{(\mathfrak{c}\times\omega)} \to \mathbb{T}^{\mathfrak{c}}$ so that $\Phi[\mathbb{Z}^{(\mathfrak{c}\times\omega)}]$ has countably compact square when considered with the subspace topology induced by $\mathbb{T}^{\mathfrak{c}}$. Such an embedding will be obtained "coordinate by coordinate"—that is, we will associate to each $\alpha < \mathfrak{c}$ a group homomorphism $\phi_\alpha : \mathbb{Q}^{(\mathfrak{c}\times\omega)} \to \mathbb{T}$ satisfying two significant conditions and Φ will be the diagonal product of the family $\{\phi_\alpha : \alpha < \mathfrak{c}\}$. One of these conditions will guarantee that Φ is injective and the other will ensure that every component of a pair of sequences in $\Phi[\mathbb{Z}^{(\mathfrak{c})}]$ admits a p-limit point for some $p \in \omega^*$.

Each mapping ϕ_α will be defined in two stages: we will first construct a group homomorphism from a countable subgroup of $\mathbb{Q}^{(\mathfrak{c}\times\omega)}$ to $\mathbb{T}$ by induction, and then we will extend it to the whole group $\mathbb{Q}^{(\mathfrak{c}\times\omega)}$. In every inductive step, we will approximate the values of the group homomorphism by non-empty open arcs of $\mathbb{T}$ with suitable properties. To make this possible, we must deal with appropriate families of sequences in $\mathbb{Q}^{(\mathfrak{c}\times\omega)}$ which we start to sort now.

DEFINITION 2.1. If $f : \omega \to \mathbb{Z}^{(\mathfrak{c}\times\omega)}$, then f is said to be *of type 1* if $|f(n)| > n$ for every $n \in \omega$, where $|f(n)| = \max\{|f(n)(\mu, k)| : (\mu, k) \in \operatorname{supp} f(n)\}$; f is said to be *of type 2* if $\operatorname{supp} f(n) \setminus \bigcup_{m<n} \operatorname{supp} f(m) \neq \emptyset$ for every $n \in \omega$.

The following result can be found in [6].

PROPOSITION 2.2. *Let $g : \omega \to \mathbb{Z}^{(\mathfrak{c}\times\omega)}$. There exists $j : \omega \to \omega$ strictly increasing such that $g \circ j$ is either constant or of type 1 or 2.*

According to Proposition 2.2, every non-trivial sequence in $\mathbb{Z}^{(\mathfrak{c}\times\omega)}$ admits a subsequence of type 1 or 2. Therefore, in order to provide this space with a countably compact topology, it suffices to assign accumulation points to all sequences of type 1 or 2. The advantage of dealing only with those sequences is that there exists enough "freedom" in assigning to them accumulation point—so, approximations by arcs become viable.

The idea of "reducing" the family of sequences to which accumulation points will be assigned will also be used to endow the free abelian group of size $\mathfrak{c}$ with a group topology that makes its square countably compact. In this case, we will consider finite families of sequences in $\mathbb{Q}^{(\mathfrak{c}\times\omega)}$ which will be called *of type A*, *B* and *C*.

A family of type A is composed of sequences that are of type 1 or 2, but whose supports are pairwise disjoint. The calculations in this case are similar to those in [6]. Dealing with families of types B and C requires new ideas, since they contain a pair $\{f_0, f_1\}$ of sequences in $\mathbb{Q}^{(\mathfrak{c}\times\omega)}$ whose supports are not disjoint, and therefore cannot be treated separately. We will consider the ratio $a_n = |f_0(n)(\theta_n, m_n)|/|f_1(n)(\theta_n, m_m)|$ for some $(\theta_n, m_n) \in \operatorname{supp} f_0(n) \cap \operatorname{supp} f_1(n)$.

Families of type C are related to sequences $\{a_n : n \in \omega\}$ converging to irrational numbers. In this case, we will use Kronecker's theorem to work directly with the sequence of pairs. Families of type B are related to sequences $\{a_n : n \in \omega\}$ converging to 0. After dealing with f_0, a smaller arc A will be left and we will need $|f_1(n)(\theta_n, m_n)| \cdot A$ to be large enough in order to deal with f_1.

If $\{a_n : n \in \omega\}$ converges to a non-zero rational number, either we are able to use diophantine equations and obtain a family of type A or we have to write a rational linear combination of sequences in $\mathbb{Z}^{(\mathfrak{c}\times\omega)}$ and consider a family of type B. Thus, we end up working with sequences in $\mathbb{Q}^{(\mathfrak{c}\times\omega)}$ instead of $\mathbb{Z}^{(\mathfrak{c}\times\omega)}$. This forces us to define a topology in $\mathbb{Q}^{(\mathfrak{c}\times\omega)}$ and then take the subspace topology.

Definition 2.3. Let $F = \{f_0, \ldots, f_k\}$ be a finite family of sequences in $\mathbb{Q}^{(\mathfrak{c}\times\omega)}$. We say that F is *of type A* if:

- $f_0, \ldots, f_k : \omega \to \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ are of type 1 or 2;
- $\operatorname{supp} f_i(n) \cap \operatorname{supp} f_j(n) = \emptyset$ for all $n \in \omega$ and $i, j \in \{0, \ldots, k\}$ such that $i \neq j$.

If F is of type A, put $d(F) = 1$. We say that F if *of type B* if:

- $f_2, \ldots, f_k : \omega \to \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ are of type 1 or 2;
- $f_0(n) = (1/d(F))\tilde{f}_0(n)$ and $f_1(n) = (1/d(F))\tilde{f}_1(n)$ for every $n \in \omega$, where $\tilde{f}_0, \tilde{f}_1 : \omega \to \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ are of type 1 or 2 and $d(F)$ is a positive integer;
- $\operatorname{supp} f_0(n) \subset \operatorname{supp} f_1(n)$ for every $n \in \omega$;
- $\operatorname{supp} f_i(n) \cap \operatorname{supp} f_j(n) = \emptyset$ for all $n \in \omega$ and $i, j \in \{2, \ldots, k\}$ such that $i \neq j$;
- $\operatorname{supp} f_i(n) \cap \operatorname{supp} f_j(n) = \emptyset$ for all $n \in \omega$, $i \in \{0, 1\}$ and $j \in \{2, \ldots, k\}$;
- for each $n \in \omega$, there exists $(\theta_n, m_n) \in \operatorname{supp} f_0(n)$ such that the sequence
$$\left\{ \frac{f_0(n)(\theta_n, m_n)}{f_1(n)(\theta_n, m_n)} : n \in \omega \right\}$$
is strictly monotonic and converges to 0.

We say that F is *of type C* if:

- $f_0, \dots, f_k : \omega \to \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ are of type 1 or 2;
- $\operatorname{supp} f_0(n) = \operatorname{supp} f_1(n)$ for every $n \in \omega$;
- $\operatorname{supp} f_i(n) \cap \operatorname{supp} f_j(n) = \emptyset$ for all $n \in \omega$ and $i, j \in \{2, \dots, k\}$ such that $i \neq j$;
- $\operatorname{supp} f_i(n) \cap \operatorname{supp} f_j(n) = \emptyset$ for all $n \in \omega$, $i \in \{0, 1\}$ and $j \in \{2, \dots, k\}$;
- for each $n \in \omega$, there exists $(\theta_n, m_n) \in \operatorname{supp} f_0(n)$ such that the sequence
$$\left\{ \frac{f_0(n)(\theta_n, m_n)}{f_1(n)(\theta_n, m_n)} : n \in \omega \right\}$$
is strictly monotonic and converges to $\xi \in \mathbb{R} \setminus \mathbb{Q}$.

If F is of type C, put $d(F) = 1$.

The set $\mathcal{F}$ of all families of type A, B or C enables us not only to recover a subsequence of any pair of sequences in $\mathbb{Z}^{(\mathfrak{c}\times\omega)}$, but also to construct the coordinates ϕ_α of the embedding Φ. The following two propositions support these statements. Their proofs will be presented in Sections 3 and 4.

PROPOSITION 2.4. *Let $g, h : \omega \to \mathbb{Z}^{(\mathfrak{c}\times\omega)}$. There exist $F \in \mathcal{F}$, $j : \omega \to \omega$ strictly increasing and $\tilde{g}, \tilde{h} \in \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ such that $(g \circ j)(n) = \tilde{g} + \sum_{f\in F} a_f f(n)$ and $(h \circ j)(n) = \tilde{h} + \sum_{f\in F} b_f f(n)$ for every $n \in \omega$, where $a_f, b_f \in \mathbb{Z}$ for every $f \in F$.*

Before stating the next proposition, we fix an enumeration $\{J_\alpha : \alpha < \mathfrak{c}\}$ of $\mathbb{Q}^{(\mathfrak{c}\times\omega)} \setminus \{0\}$ and an enumeration $\{F_\xi : 0 < \xi < \mathfrak{c}\}$ of $\mathcal{F}$ such that

$$(*) \qquad \bigcup_{n\in\omega} \operatorname{supp} f(n) \subset \xi \times \omega \qquad \text{for every } f \in F_\xi \text{ and every } \xi \in]0, \mathfrak{c}[.$$

The cardinality of F_ξ will be denoted by $n(F_\xi)$ and we will write $F_\xi = \{f_{\xi,0}, \dots, f_{\xi,n(F_\xi)-1}\}$.

PROPOSITION 2.5. ($\mathfrak{p} = \mathfrak{c}$) *For each $\alpha < \mathfrak{c}$ and each $\xi \in]0, \mathfrak{c}[$ there exists $S_{\xi,\alpha} \in [\omega]^\omega$ such that if $\alpha < \beta < \mathfrak{c}$, then $S_{\xi,\beta} \subset^* S_{\xi,\alpha}$. There also exists a group homomorphism $\phi_\alpha : \mathbb{Q}^{(\mathfrak{c}\times\omega)} \to \mathbb{T}$ such that:*

(i) *$\phi_\alpha(J_\alpha) \neq 0 + \mathbb{Z}$;*
(ii) *The sequence $\{\phi_\alpha(f_{\xi,i}(n)) : n \in S_{\xi,\alpha}\}$ converges to $\phi_\alpha(\chi_{(\xi,i)})$ for every $i \in \{0, \dots, n(F_\xi) - 1\}$.*

We end this section by showing how Propositions 2.4 and 2.5 can be used to endow the free abelian group of size $\mathfrak{c}$ with a group topology that makes its square countably compact.

THEOREM 2.6. ($\mathfrak{p} = \mathfrak{c}$) *There exists a group topology on the free abelian group of cardinality $\mathfrak{c}$ that makes its square countably compact.*

Proof. It follows from Proposition 2.5(i) that

$$\Phi : \mathbb{Q}^{(\mathfrak{c}\times\omega)} \to \mathbb{T}^{\mathfrak{c}}, \qquad J \mapsto \Phi(J),$$

given by

$$\Phi(J)(\alpha) = \phi_\alpha(J) \quad \text{for every } \alpha < \mathfrak{c}$$

is a group monomorphism. Thus, $\Phi[\mathbb{Z}^{(\mathfrak{c}\times\omega)}]$ is isomorphic to $\mathbb{Z}^{(\mathfrak{c}\times\omega)}$, and since $\mathbb{T}^{\mathfrak{c}}$ is a topological group, the subspace topology induced by $\mathbb{T}^{\mathfrak{c}}$ in $\Phi[\mathbb{Z}^{(\mathfrak{c}\times\omega)}]$ turns $\Phi[\mathbb{Z}^{(\mathfrak{c}\times\omega)}]$ into a topological group.

Let $g, h : \omega \to \Phi[\mathbb{Z}^{(\mathfrak{c}\times\omega)}]$. It follows from Proposition 2.4 that there exist $\xi \in \,]0, \mathfrak{c}[$, $j : \omega \to \omega$ strictly increasing and $\tilde{g}, \tilde{h} \in \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ such that

$$(\Phi^{-1} \circ g \circ j)(n) = \tilde{g} + \sum_{i<n(F_\xi)} a_{\xi,i} \cdot f_{\xi,i}(n)$$

and

$$(\Phi^{-1} \circ h \circ j)(n) = \tilde{h} + \sum_{i<n(F_\xi)} b_{\xi,i} \cdot f_{\xi,i}(n)$$

for every $n \in \omega$, where $a_{\xi,i}, b_{\xi,i} \in \mathbb{Z}$ for every $i < n(F_\xi)$.

Fix $p_\xi \in \omega^*$ containing $\{S_{\xi,\alpha} : \alpha < \mathfrak{c}\}$. According to Proposition 2.5(ii), the sequence $\{\phi_\alpha(f_{\xi,i}(n)) : n \in S_{\xi,\alpha}\}$ converges to $\phi_\alpha(\chi_{(\xi,i)})$ for all $i \in \{0, \ldots, n(F_\xi) - 1\}$ and $\alpha < \mathfrak{c}$. Thus,

$$\phi_\alpha(\chi_{(\xi,i)}) = p_\xi\text{-}\lim\{\phi_\alpha(f_{\xi,i}(n)) : n \in \omega\}$$

for all $i \in \{0, \ldots, n(F_\xi) - 1\}$ and $\alpha < \mathfrak{c}$.

It follows from Proposition 1.2 that

$$\Phi(\chi_{(\xi,i)}) = p_\xi\text{-}\lim\{\Phi(f_{\xi,i}(n)) : n \in \omega\}$$

and Proposition 1.3 implies that

$$\Phi\Big(\tilde{g} + \sum_{i<n(F_\xi)} a_{\xi,i} \cdot \chi_{(\xi,i)}\Big) = p_\xi\text{-}\lim\Big\{\Phi\Big(\tilde{g} + \sum_{i<n(F_\xi)} a_{\xi,i} \cdot f_{\xi,i}(n)\Big) : n \in \omega\Big\},$$

$$\Phi\Big(\tilde{h} + \sum_{i<n(F_\xi)} b_{\xi,i} \cdot \chi_{(\xi,i)}\Big) = p_\xi\text{-}\lim\Big\{\Phi\Big(\tilde{h} + \sum_{i<n(F_\xi)} b_{\xi,i} \cdot f_{\xi,i}(n)\Big) : n \in \omega\Big\}.$$

Consequently, $(\Phi(\tilde{g} + \sum_{i<n(F_\xi)} a_{\xi,i} \cdot \chi_{(\xi,i)}), \Phi(\tilde{h} + \sum_{i<n(F_\xi)} b_{\xi,i} \cdot \chi_{(\xi,i)})) = p_\xi\text{-}\lim\{(g(n), h(n)) : n \in \omega\}$. Therefore, $\Phi[\mathbb{Z}^{(\mathfrak{c}\times\omega)}] \times \Phi[\mathbb{Z}^{(\mathfrak{c}\times\omega)}]$ is a countably compact group. ∎

3. Proof of Proposition 2.4. In this section, we restate and prove Proposition 2.4, which shows that it is possible to recover a subsequence of any pair of sequences in $\mathbb{Z}^{(\mathfrak{c}\times\omega)}$ from an element of $\mathcal{F}$ and from translations in $\mathbb{Z}^{(\mathfrak{c}\times\omega)}$.

PROPOSITION 2.4. *Let $g, h : \omega \to \mathbb{Z}^{(\mathfrak{c}\times\omega)}$. There exist $F \in \mathcal{F}$, $j : \omega \to \omega$ strictly increasing and $\tilde{g}, \tilde{h} \in \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ such that $(g \circ j)(n) = \tilde{g} + \sum_{f\in F} a_f f(n)$*

and $(h\circ j)(n)=\tilde{h}+\sum_{f\in F}b_f f(n)$ *for every* $n\in\omega$*, where* $a_f,b_f\in\mathbb{Z}$ *for every* $f\in F$*.*

Proof. Let $g_0,g_1,h_0,h_1:\omega\to\mathbb{Z}^{(\mathfrak{c}\times\omega)}$ be given by

$$g_0(n)=\sum_{(\mu,k)\in\operatorname{supp}g(n)\setminus\operatorname{supp}h(n)}g(n)(\mu,k)\cdot\chi_{(\mu,k)},$$

$$g_1(n)=\sum_{(\mu,k)\in\operatorname{supp}g(n)\cap\operatorname{supp}h(n)}g(n)(\mu,k)\cdot\chi_{(\mu,k)},$$

$$h_0(n)=\sum_{(\mu,k)\in\operatorname{supp}h(n)\setminus\operatorname{supp}g(n)}h(n)(\mu,k)\cdot\chi_{(\mu,k)},$$

$$h_1(n)=\sum_{(\mu,k)\in\operatorname{supp}h(n)\cap\operatorname{supp}g(n)}h(n)(\mu,k)\cdot\chi_{(\mu,k)}.$$

Note that $g(n)=g_0(n)+g_1(n)$ and $h(n)=h_0(n)+h_1(n)$ for every $n\in\omega$.

It follows from Proposition 2.2 that there exists $j_1:\omega\to\omega$ strictly increasing such that $g_0\circ j_1$, $g_1\circ j_1$, $h_0\circ j_1$ and $h_1\circ j_1$ are of type 1, 2 or constant. If $g_1\circ j_1$ or $h_1\circ j_1$ are constant, it is not difficult to realize that there exist $F\in[{}^{\omega}\mathbb{Q}^{(\mathfrak{c}\times\omega)}]^{<\omega}$ of type A and $\tilde{g},\tilde{h}\in\mathbb{Z}^{(\mathfrak{c}\times\omega)}$ such that $(g\circ j_1)(n)=\tilde{g}+\sum_{f\in F}a_f f(n)$ and $(h\circ j_1)(n)=\tilde{h}+\sum_{f\in F}b_f f(n)$ for every $n\in\omega$, where $a_f,b_f\in\mathbb{Z}$ for every $f\in F$. Hence, we can suppose that $g_1\circ j_1$ and $h_1\circ j_1$ are of type 1 or 2.

Let

$$A=\left\{\frac{(g_1\circ j_1)(n)(\mu,k)}{(h_1\circ j_1)(n)(\mu,k)}:(\mu,k)\in\operatorname{supp}(g_1\circ j_1)(n)=\operatorname{supp}(h_1\circ j_1)(n),\ n\in\omega\right\}.$$

If A is a finite set—say, $A=\{p_0/q_0,\dots,p_k/q_k\}$ where $p_i,q_i\in\mathbb{Z}\setminus\{0\}$, $q_i>0$ and $\gcd(p_i,q_i)=1$ for every $i\in\{0,\dots,k\}$—consider

$$g_{1,i}(n)=\sum_{\substack{(\mu,k)\in\operatorname{supp}(g_1\circ j_1)(n),\\ \frac{(g_1\circ j_1)(n)(\mu,k)}{(h_1\circ j_1)(n)(\mu,k)}=\frac{p_i}{q_i}}}(g_1\circ j_1)(n)(\mu,k)\cdot\chi_{(\mu,k)},$$

$$h_{1,i}(n)=\sum_{\substack{(\mu,k)\in\operatorname{supp}(h_1\circ j_1)(n),\\ \frac{(g_1\circ j_1)(n)(\mu,k)}{(h_1\circ j_1)(n)(\mu,k)}=\frac{p_i}{q_i}}}(h_1\circ j_1)(n)(\mu,k)\cdot\chi_{(\mu,k)},$$

for all $n\in\omega$ and $i\in\{0,\dots,k\}$. Note that

$$(g_1\circ j_1)(n)=\sum_{i=0}^{k}g_{1,i}(n),\qquad(h_1\circ j_1)(n)=\sum_{i=0}^{k}h_{1,i}(n).$$

If $i\in\{0,\dots,k\}$ and $n\in\omega$, then $\operatorname{supp}g_{1,i}(n)=\operatorname{supp}h_{1,i}(n)$. Moreover,

$$\frac{g_{1,i}(n)(\mu,k)}{h_{1,i}(n)(\mu,k)}=\frac{p_i}{q_i}$$

for every $(\mu, k) \in \operatorname{supp} g_{1,i}(n) = \operatorname{supp} h_{1,i}(n)$. Since $\gcd(p_i, q_i) = 1$, it follows that $q_i \mid h_{1,i}(n)(\mu, k)$ for every $(\mu, k) \in \operatorname{supp} g_{1,i}(n) = \operatorname{supp} h_{1,i}(n)$. Let $f_i : \omega \to \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ be given by

$$f_i(n)(\mu, k) = \frac{h_{1,i}(n)(\mu, k)}{q_i}$$

for every $(\mu, k) \in \mathfrak{c} \times \omega$. We have

$$(g_1 \circ j_1)(n) = \sum_{i=0}^{k} p_i \cdot f_i(n), \qquad (h_1 \circ j_1)(n) = \sum_{i=0}^{k} q_i \cdot f_i(n).$$

Choose $j_2 : \omega \to \omega$ strictly increasing and such that $f_i \circ j_2$ is constant or of type 1 or 2, and define $s_i = f_i \circ j_2$ for every $i \in \{0, \dots, k\}$. Put also $s_{k+1} = g_0 \circ j_1 \circ j_2$ and $s_{k+2} = h_0 \circ j_1 \circ j_2$. Let $\{n_0, \dots, n_l\}$ be a strictly increasing enumeration of the set $I = \{i \in \{0, \dots, k\} : s_i \text{ is not constant}\}$. If s_{k+1} and s_{k+2} are constant, put $F = \{s_{n_i} : 0 \le i \le l\}$; if s_{k+1} is not constant and s_{k+2} is constant, put $F = \{s_{n_i} : 0 \le i \le l\} \cup \{s_{k+1}\}$; if s_{k+1} is constant and s_{k+2} is not, put $F = \{s_{n_i} : 0 \le i \le l\} \cup \{s_{k+2}\}$; if s_{k+1} and s_{k+2} are not constant, put $F = \{s_{n_i} : 0 \le i \le l\} \cup \{s_{k+1}\} \cup \{s_{k+2}\}$. We see that F is of type A, and therefore belongs to $\mathcal{F}$.

Note that

$$(g \circ j_1 \circ j_2)(n) = \sum_{i\in\{0,\dots,k\}\setminus I} s_i(0) + s_{k+1}(n) + \sum_{i=0}^{l} p_{n_i} \cdot s_{n_i},$$

$$(h \circ j_1 \circ j_2)(n) = \sum_{i\in\{0,\dots,k\}\setminus I} s_i(0) + s_{k+2}(n) + \sum_{i=0}^{l} q_{n_i} \cdot s_{n_i},$$

for every $n \in \omega$.

If A is infinite, there exist $j_2 : \omega \to \omega$ strictly increasing and $\{(\theta_n, m_n) : n \in \omega\} \subset \mathfrak{c} \times \omega$ such that $g_1 \circ j_1 \circ j_2$ and $h_1 \circ j_1 \circ j_2$ are of type 1 or 2, $(\theta_n, m_n) \in \operatorname{supp}(g_1 \circ j_1 \circ j_2)(n) \cap \operatorname{supp}(h_1 \circ j_1 \circ j_2)(n)$ for every $n \in \omega$, and

$$\frac{(g_1 \circ j_1 \circ j_2)(n)(\theta_n, m_n)}{(h_1 \circ j_1 \circ j_2)(n)(\theta_n, m_n)} \to \xi$$

strictly monotonically for some $\xi \in [-\infty, +\infty]$.

If $\xi = 0$, put $s_0 = g_1 \circ j_1 \circ j_2$, $s_1 = h_1 \circ j_1 \circ j_2$, $s_2 = g_0 \circ j_1 \circ j_2$ and $s_3 = h_0 \circ j_1 \circ j_2$. Consider $I = \{i \in \{0, 1, 2, 3\} : s_i \text{ is not constant}\}$. Note that $0, 1 \in I$. If $2, 3 \in I$, then $F = \{s_0, s_1, s_2, s_3\}$ is of type B, and therefore belongs to $\mathcal{F}$. Define $j = j_1 \circ j_2$ and $\tilde{g} = \tilde{h} = 0$. Then $(g \circ j)(n) = s_0(n) + s_2(n)$ and $(h \circ j)(n) = s_1(n) + s_3(n)$ for every $n \in \omega$. If $2 \in I$ and $3 \notin I$, put $F = \{s_0, s_1, s_2\}$, $j = j_1 \circ j_2$, $\tilde{g} = 0$ and $\tilde{h} = s_3(0)$. If $2 \notin I$ and $3 \in I$, put $F = \{s_0, s_1, s_3\}$, $j = j_1 \circ j_2$, $\tilde{g} = s_2(0)$ and $\tilde{h} = 0$. Finally, if $2, 3 \notin I$, put $F = \{s_0, s_1\}$, $j = j_1 \circ j_2$, $\tilde{g} = s_2(0)$ and $\tilde{h} = s_3(0)$.

If $\xi = -\infty$ or $\xi = +\infty$, then $\frac{(h_1 \circ j_1 \circ j_2)(n)(\theta_n, m_n)}{(g_1 \circ j_1 \circ j_2)(n)(\theta_n, m_n)} \to 0$; now proceed as in case $\xi = 0$.

If $\xi \in \mathbb{R} \setminus \mathbb{Q}$, put $s_0 = g_1 \circ j_1 \circ j_2$, $s_1 = h_1 \circ j_1 \circ j_2$, $s_2 = g_0 \circ j_1 \circ j_2$ and $s_3 = h_0 \circ j_1 \circ j_2$. Consider $I = \{i \in \{0,1,2,3\} : s_i$ is not constant$\}$. Note that $0, 1 \in I$. If $2, 3 \in I$, then $F = \{s_0, s_1, s_2, s_3\}$ is of type C, and therefore belongs to $\mathcal{F}$. Define $j = j_1 \circ j_2$ and $\tilde{g} = \tilde{h} = 0$. Then $(g \circ j)(n) = s_0(n) + s_2(n)$ and $(h \circ j)(n) = s_1(n) + s_3(n)$ for every $n \in \omega$. The other cases ($2 \in I$ and $3 \notin I$; $2 \notin I$ and $3 \in I$; $2, 3 \notin I$) are treated in an analogous way.

If $\xi \in \mathbb{Q} \setminus \{0\}$, then $\xi = p/q$, where $p, q \in \mathbb{Z} \setminus \{0\}$, $q > 0$ and $\gcd(p, q) = 1$. Set

$$r_0(n) = \frac{q \cdot (g_1 \circ j_1 \circ j_2)(n) - p \cdot (h_1 \circ j_1 \circ j_2)(n)}{q},$$

$$r_1(n) = \frac{(h_1 \circ j_1 \circ j_2)(n)}{q},$$

for every $n \in \omega$. Put also $r_2 = g_0 \circ j_1 \circ j_2$ and $r_3 = h_0 \circ j_1 \circ j_2$. There exists $j_3 : \omega \to \omega$ strictly increasing such that

$$n \mapsto q \cdot (g_1 \circ j_1 \circ j_2 \circ j_3)(n) - p \cdot (h_1 \circ j_1 \circ j_2 \circ j_3)(n)$$

is of type 1, 2 or constant. If the sequence is of type 1 or 2, put $\tilde{s}_0 = q \cdot (g_1 \circ j_1 \circ j_2 \circ j_3) - p \cdot (h_1 \circ j_1 \circ j_2 \circ j_3)$, $\tilde{s}_1 = h_1 \circ j_1 \circ j_2 \circ j_3$, $s_2 = r_2 \circ j_3$ and $s_3 = r_3 \circ j_3$. Set $I = \{i \in \{2, 3\} : s_i$ is not constant$\}$. If $2, 3 \in I$, then $F = \{s_0, s_1, s_2, s_3\}$ is of type B, where $s_0(n) = (1/q)\tilde{s}_0(n)$ and $s_1(n) = (1/q)\tilde{s}_1(n)$ for every $n \in \omega$. Put $j = j_1 \circ j_2 \circ j_3$, $\tilde{g} = 0$ and $\tilde{h} = 0$. It follows that $g \circ j(n) = s_0(n) + p \cdot s_1(n) + s_2(n)$ and $h \circ j(n) = q \cdot s_1(n) + s_3(n)$ for every $n \in \omega$. The other cases ($2 \in I$ and $3 \notin I$; $2 \notin I$ and $3 \in I$; $2, 3 \notin I$) are treated in an analogous way.

If the sequence $n \mapsto q \cdot (g_1 \circ j_1 \circ j_2 \circ j_3)(n) - p \cdot (h_1 \circ j_1 \circ j_2 \circ j_3)(n)$ is constant, there exists $J \in \mathbb{Z}^{(\mathfrak{c} \times \omega)}$ such that

$$q \cdot (g_1 \circ j_1 \circ j_2 \circ j_3)(n)(\mu, k) - p \cdot (h_1 \circ j_1 \circ j_2 \circ j_3)(n)(\mu, k) = J(\mu, k)$$

for all $(\mu, k) \in \mathfrak{c} \times \omega$ and $n \in \omega$.

Fix $(\mu, k) \in \mathfrak{c} \times \omega$. Since $\gcd(p, q) = 1$, the diophantine equation $qx - py = J(\mu, k)$ has infinitely many solutions. If $x = x_{(\mu,k)}$, $y = y_{(\mu,k)}$ is a particular solution of $qx - py = J(\mu, k)$, then all of its solutions are given by $x = x_{(\mu,k)} - pt$, $y = y_{(\mu,k)} - qt$ for $t \in \mathbb{Z}$. Hence, for every $n \in \omega$ and $(\mu, k) \in \operatorname{supp}(g_1 \circ j_1 \circ j_2 \circ j_3)(n) = \operatorname{supp}(h_1 \circ j_1 \circ j_2 \circ j_3)(n)$, there exists $t_{n,(\mu,k)} \in \mathbb{Z}$ such that

$$(g_1 \circ j_1 \circ j_2 \circ j_3)(n)(\mu, k) = x_{(\mu,k)} - pt_{n,(\mu,k)},$$

$$(h_1 \circ j_1 \circ j_2 \circ j_3)(n)(\mu, k) = y_{(\mu,k)} - qt_{n,(\mu,k)}.$$

Note that if $(\mu, k) \notin \operatorname{supp} J$, then one can put $x_{(\mu,k)} = 0$, $y_{(\mu,k)} = 0$. We also remark that if $p, q > 0$, one can choose $x_{(\mu,k)}, y_{(\mu,k)} \geq 0$. Hence, for every $n \in \omega$,

$$(g_1 \circ j_1 \circ j_2 \circ j_3)(n) = \sum_{(\mu,k)\in \operatorname{supp}(g_1\circ j_1\circ j_2\circ j_3)(n)} g_1 \circ j_1 \circ j_2 \circ j_3)(n)(\mu,k) \cdot \chi_{(\mu,k)}$$
$$= \sum_{(\mu,k)\in \operatorname{supp} J} x_{(\mu,k)} \cdot \chi_{(\mu,k)} + p \cdot \sum_{(\mu,k)\in \operatorname{supp}(g_1\circ j_1\circ j_2\circ j_3)(n)} -t_{n,(\mu,k)} \cdot \chi_{(\mu,k)}$$

and, analogously,

$$(h_1 \circ j_1 \circ j_2 \circ j_3)(n) = \sum_{(\mu,k)\in \operatorname{supp}(h_1\circ j_1\circ j_2\circ j_3)(n)} (h_1 \circ j_1 \circ j_2 \circ j_3)(n)(\mu,k) \cdot \chi_{(\mu,k)}$$
$$= \sum_{(\mu,k)\in \operatorname{supp} J} y_{(\mu,k)} \cdot \chi_{(\mu,k)} + q \cdot \sum_{(\mu,k)\in \operatorname{supp}(h_1\circ j_1\circ j_2\circ j_3)(n)} -t_{n,(\mu,k)} \cdot \chi_{(\mu,k)}.$$

Moreover,

$$n \mapsto \sum_{(\mu,k)\in \operatorname{supp} J} x_{(\mu,k)} \cdot \chi_{(\mu,k)}$$

and

$$n \mapsto \sum_{(\mu,k)\in \operatorname{supp} J} y_{(\mu,k)} \cdot \chi_{(\mu,k)}$$

are constant.

Fix $j_4 : \omega \to \omega$ strictly increasing such that the sequence $s_0(n) = \sum_{(\mu,k)\in \operatorname{supp}(g_1\circ j_1\circ j_2\circ j_3\circ j_4)(n)} -t_{n,(\mu,k)} \cdot \chi_{(\mu,k)}$ is of type 1 or 2. Define $j = j_1 \circ j_2 \circ j_3 \circ j_4$, $\tilde{g} = \sum_{(\mu,k)\in \operatorname{supp} J} x_{(\mu,k)} \cdot \chi_{(\mu,k)}$, $\tilde{h} = \sum_{(\mu,k)\in \operatorname{supp} J} y_{(\mu,k)} \cdot \chi_{(\mu,k)}$, $s_1 = g_0 \circ j$ and $s_2 = h_0 \circ j$. Set $I = \{i \in \{1,2\} : s_i \text{ is not constant}\}$. If $1, 2 \in I$, then $F = \{s_0, s_1, s_2\}$ is of type A, and therefore belongs to $\mathcal{F}$. Moreover, $(g \circ j)(n) = \tilde{g} + p \cdot s_0(n) + s_1(n)$ and $(h \circ j)(n) = \tilde{h} + q \cdot s_0(n) + s_2(n)$, for every $n \in \omega$. The other cases ($1 \in I$ and $2 \notin I$; $1 \notin I$ and $2 \in I$; $1, 2 \notin I$) are treated in an analogous way. ∎

4. Proof of Proposition 2.5. This section is devoted to proving Proposition 2.5. As mentioned, we will concern ourselves primarily with the construction of group homomorphisms from countable subgroups of $\mathbb{Q}^{(\mathfrak{c}\times\omega)}$ to $\mathbb{T}$. These countable subgroups will be of the form $\mathbb{Q}^{(E\times\omega)}$ for some $E \in [\mathfrak{c}]^\omega$ and the following proposition shows how to construct inductively a suitable E. We remark that property $(*)$ of $\{F_\xi : 0 < \xi < \mathfrak{c}\}$ will be used to carry out the induction.

PROPOSITION 4.1. *If $J \in \mathbb{Q}^{(\mathfrak{c}\times\omega)} \setminus \{0\}$, there exists $E \in [\mathfrak{c}]^\omega$ such that* $\operatorname{supp} J \subset E \times \omega$ *and such that if $\xi \in E \setminus \{0\}$, then* $\bigcup_{n\in\omega} \operatorname{supp} f(n) \subset E \times \omega$ *for every $f \in F_\xi$.*

Proof. Define $E(0) = \omega$. If $\xi \in \,]0, \mathfrak{c}[$, define by induction

$$E(\xi) = \{\xi\} \cup \bigcup_{\mu \in X_\xi} E(\mu)$$

where

$$X_\xi = \Big\{\theta < \mathfrak{c} : \exists m \in \omega \text{ such that } (\theta, m) \in \bigcup_{i<n(F_\xi)} \bigcup_{n\in\omega} \operatorname{supp} f_{\xi,i}(n)\Big\}$$

and let

$$E = \bigcup_{\zeta \in X_J} E(\zeta)$$

where

$$X_J = \{\theta < \mathfrak{c} : \exists m \in \omega \text{ such that } (\theta, m) \in \operatorname{supp} J\}.$$

It is clear that $\operatorname{supp} J \subset E \times \omega$. An inductive argument shows that $E(\xi) \in [\mathfrak{c}]^\omega$ for every $\xi < \mathfrak{c}$, and therefore $E \in [\mathfrak{c}]^\omega$. Another inductive argument shows that if $\alpha \in E(\beta)$, then $E(\alpha) \subset E(\beta)$. Thus, if $\xi \in E \setminus \{0\}$, then $\bigcup_{n\in\omega} \operatorname{supp} f_{\xi,i}(n) \subset E \times \omega$ for every $i < n(F_\xi)$. ■

The next three lemmas are the technical part relative to the types A, B and C respectively and will be used to prove Lemma 4.5, which will be necessary in the successor step of the induction in Proposition 4.6. Their proofs can be skipped on a first reading, without affecting the understanding of what follows.

We recall that $\mathcal{B}$ denotes the set of all non-empty open arcs of $\mathbb{T}$.

LEMMA 4.2. *Let $\epsilon > 0$, $A_0, \ldots, A_k \in \mathcal{B}$, $G \in [\mathfrak{c} \times \omega]^{<\omega}$, $\psi : G \to \mathcal{B}$ and $\{H_0, \ldots, H_k\} \subset \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ such that $\operatorname{supp} H_i \cap \operatorname{supp} H_j = \emptyset$ for all $i, j \le k$ with $i \ne j$. For each $i \le k$, let $(\mu_i, k_i) \in \operatorname{supp} H_i$ be such that:*

(1) *$|H_i(\mu_i, k_i)|\epsilon > 4$ and $\delta(\psi(\mu_i, k_i)) \ge \epsilon$, or*
(2) *$(\mu_i, k_i) \notin G$.*

Denote $G \cup \operatorname{supp} H_0 \cup \cdots \cup \operatorname{supp} H_k$ by $\tilde{G}$. There exists $\tilde{\epsilon} \le \epsilon/2$ such that for every $\tilde{\tilde{\epsilon}} \le \tilde{\epsilon}$ there exists $\tilde{\psi} : \tilde{G} \to \mathcal{B}$ satisfying the following conditions:

(i) *$\overline{\tilde{\psi}(\xi, n)} \subset \psi(\xi, n)$ for every $(\xi, n) \in G$;*
(ii) *$\delta(\tilde{\psi}(\xi, n)) = \tilde{\tilde{\epsilon}}$ for every $(\xi, n) \in \tilde{G}$;*
(iii) *$\delta(\sum_{(\xi,n)\in\operatorname{supp} H_i} H_i(\xi, n) \cdot \tilde{\psi}(\xi, n)) < \epsilon$ for every $i \le k$;*
(iv) *$A_i \cap \sum_{(\xi,n)\in\operatorname{supp} H_i} H_i(\xi, n) \cdot \tilde{\psi}(\xi, n) \ne \emptyset$ for every $i \le k$.*

Proof. Define

$$\tilde{\epsilon} = \min\bigg\{\bigg\{\frac{\delta(\psi(\xi, n))}{2} : (\xi, n) \in G\bigg\} \cup \bigg\{\frac{\epsilon}{2 \cdot \sum_{\substack{(\xi,n)\in\operatorname{supp} H_i \\ i\le k}} |H_i(\xi, n)|}\bigg\}\bigg\}$$

and choose $\tilde{\tilde{\epsilon}} \le \tilde{\epsilon}$.

If $(\xi, n) \in \tilde{G} \setminus \bigcup_{i\le k} \operatorname{supp} H_i$, define $\tilde{\psi}(\xi, n)$ as the element of $\mathcal{B}$ centered at the middle point of $\psi(\xi, n)$ with diameter $\tilde{\tilde{\epsilon}}$.

Fix $i \in \{0, \ldots, k\}$. If $(\xi, n) \in \operatorname{supp} H_i \setminus \{(\mu_i, k_i)\}$, define $z_{(\xi,n)} \in \mathbb{T}$ in the following way: if $(\xi, n) \in G$, then $z_{(\xi,n)}$ is the middle point of $\psi(\xi, n)$; if $(\xi, n) \notin G$, then $z_{(\xi,n)}$ is chosen arbitrarily. Set

$$x_i = \sum_{(\xi,n)\in \operatorname{supp} H_i \setminus \{(\mu_i,k_i)\}} H_i(\xi, n) \cdot z_{(\xi,n)}.$$

If $(\mu_i, k_i) \notin G$, choose $z_{(\mu_i,k_i)} \in \mathbb{T}$ such that

$$\text{(a)} \qquad x_i + H_i(\mu_i, k_i) \cdot z_{(\mu_i,k_i)} \in A_i.$$

If $(\mu_i, k_i) \in G$, let $\tilde{z}_{(\mu_i,k_i)}$ be the middle point of $\psi(\mu_i, k_i)$ and A be the open arc of $\mathbb{T}$ centered at $\tilde{z}_{(\mu_i,k_i)}$ with diameter $\epsilon/4$. Note that $H_i(\mu_i, k_i) \cdot A = \mathbb{T}$, and therefore there exists $z_{(\mu_i,k_i)} \in A$ such that

$$\text{(b)} \qquad x_i + H_i(\mu_i, k_i) \cdot z_{(\mu_i,k_i)} \in A_i.$$

For each $(\xi, n) \in \bigcup_{i\le k} \operatorname{supp} H_i$, let $\tilde{\psi}(\xi, n)$ be the open arc of $\mathbb{T}$ centered at $z_{(\xi,n)}$ with diameter $\tilde{\tilde{\epsilon}}$.

Conditions (i) and (ii) are clearly satisfied, (iii) follows from the choice of $\tilde{\epsilon}$, and (iv) follows from (a) and (b). ∎

LEMMA 4.3. *Let $d \in \mathbb{N} \setminus \{0\}$, $\epsilon > 0$, $A_0, A_1 \in \mathcal{B}$ with $\delta(A_0) \ge \epsilon$ and $\delta(A_1) \ge \epsilon$, $G \in [\mathfrak{c} \times \omega]^{<\omega}$, $\psi : G \to \mathcal{B}$ and $\{H_0, H_1\} \subset \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ where* $\operatorname{supp} H_0 \subset \operatorname{supp} H_1$. *Let $(\mu, k), (\nu, l) \in \operatorname{supp} H_0$ (not necessarily distinct) be such that:*

1. $\delta(\psi(\mu, k)) \ge \epsilon$ *if* $(\mu, k) \in G$;
2. $|H_0(\nu, l)|\epsilon > 4d$ *and* $\delta(\psi(\nu, l)) \ge \epsilon$ *if* $(\nu, l) \in G$;
3. $|H_1(\mu, k)|\epsilon \ge 4d|H_0(\mu, k)|$.

Denote $G \cup \operatorname{supp} H_0 \cup \operatorname{supp} H_1$ by $\tilde{G}$. There exists $\tilde{\epsilon} \le \epsilon/2$ such that for every $\tilde{\tilde{\epsilon}} \le \tilde{\epsilon}$ there exists $\tilde{\psi} : \tilde{G} \to \mathcal{B}$ satisfying the following conditions:

(i) $d \cdot \overline{\tilde{\psi}(\xi, n)} \subset \psi(\xi, n)$ *for every* $(\xi, n) \in G$;
(ii) $\delta(\tilde{\psi}(\xi, n)) = \tilde{\tilde{\epsilon}}$ *for every* $(\xi, n) \in \tilde{G}$;
(iii) $\delta(\sum_{(\xi,n)\in \operatorname{supp} H_i} H_i(\xi, n)) \cdot \tilde{\psi}(\xi, n)) < \epsilon$ *for every* $i < 2$;
(iv) $A_i \cap \sum_{(\xi,n)\in \operatorname{supp} H_i} H_i(\xi, n) \cdot \tilde{\psi}(\xi, n) \ne \emptyset$ *for every* $i < 2$.

Proof. Define

$$\tilde{\epsilon} = \min\left\{\left\{\frac{\delta(\psi(\xi, n))}{2d} : (\xi, n) \in G\right\} \cup \left\{\frac{\epsilon}{2d \cdot \sum_{\substack{(\xi,n)\in \operatorname{supp} H_i \\ i<2}} |H_i(\xi, n)|}\right\}\right\}$$

and choose $\tilde{\tilde{\epsilon}} < \tilde{\epsilon}$.

If $(\xi, n) \in \tilde{G} \setminus \bigcup_{i<2} \operatorname{supp} H_i$ define $\tilde{\psi}(\xi, n)$ as the element of $\mathcal{B}$ centered at the dth root of the middle point of $\psi(\xi, n)$ with diameter $\tilde{\tilde{\epsilon}}$.

If $(\nu,l)=(\mu,k)$, then for each $(\xi,n)\in \operatorname{supp} H_1\setminus\{(\mu,k)\}$, define $z_{(\xi,n)}\in\mathbb{T}$ in the following way: if $(\xi,n)\in G$, then $d\cdot z_{(\xi,n)}$ is the middle point of $\psi(\xi,n)$; if $(\xi,n)\notin G$, then $z_{(\xi,n)}$ is chosen arbitrarily. Set also

$$x_0=\sum_{(\xi,n)\in \operatorname{supp} H_0\setminus\{(\mu,k)\}} H_0(\xi,n)\cdot z_{(\xi,n)},$$
$$x_1=\sum_{(\xi,n)\in \operatorname{supp} H_1\setminus\{(\mu,k)\}} H_1(\xi,n)\cdot z_{(\xi,n)}.$$

Fix $\tilde{z}_{(\mu,k)}\in\mathbb{T}$ such that $H_0(\mu,k)\cdot\tilde{z}_{(\mu,k)}$ is the middle point of A_0-x_0; if $(\mu,k)\in G$, we also require that $d\cdot\tilde{z}_{(\mu,k)}$ is contained in the open arc of $\mathbb{T}$ centered at the middle point of $\psi(\mu,k)$ with diameter $\epsilon/4$. Let A be the arc centered at $\tilde{z}_{(\mu,k)}$ with diameter $\epsilon/(4d|H_0(\mu,k)|)$. From (3), it follows that there exists $z_{(\mu,k)}\in A$ such that

$$H_1(\mu,k)\cdot z_{(\mu,k)}\in A_1-x_1. \tag{c}$$

We also have

$$H_0(\mu,k)\cdot z_{(\mu,k)}\in A_0-x_0. \tag{d}$$

If $(\nu,l)\neq(\mu,k)$, then for each $(\xi,n)\in \operatorname{supp} H_1\setminus\{(\mu,k),(\nu,l)\}$, define $z_{(\xi,n)}\in\mathbb{T}$ in the following way: if $(\xi,n)\in G$, then $d\cdot z_{(\xi,n)}$ is the middle point of $\psi(\xi,n)$; if $(\xi,n)\notin G$, then $z_{(\xi,n)}$ is chosen arbitrarily.

If $(\mu,k)\in G$, let $d\cdot\tilde{z}_{(\mu,k)}$ be the middle point of $\psi(\mu,k)$. If $(\mu,k)\notin G$, let $\tilde{z}_{(\mu,k)}$ be an arbitrary element of $\mathbb{T}$. Choose $z_{(\nu,l)}$ such that $H_0(\nu,l)\cdot z_{(\nu,l)}$ is the middle point of $A_0-\tilde{x}_0-x$; if $(\nu,l)\in G$, we also require that $d\cdot z_{(\nu,l)}$ is contained in the open arc of $\mathbb{T}$ centered at the middle point of $\psi(\nu,l)$ with diameter $\epsilon/4$, where

$$\tilde{x}_0=\sum_{(\xi,n)\in \operatorname{supp} H_0\setminus\{(\mu,k),(\nu,l)\}} H_0(\xi,n)\cdot z_{(\xi,n)},\qquad x=H_0(\mu,k)\cdot\tilde{z}_{(\mu,k)}.$$

Define

$$x_0=\tilde{x}_0+H_0(\nu,l)\cdot z_{(\nu,l)}.$$

It follows that x is the middle point of A_0-x_0.

Let A be the arc centered at $\tilde{z}_{(\mu,k)}$ with diameter $\epsilon/(4d|H_0(\mu,k)|)$. From (3), it follows that there exists $z_{(\mu,k)}\in A$ such that

$$H_1(\mu,k)\cdot z_{(\mu,k)}\in A_1-x_1. \tag{c$'$}$$

We also have

$$H_0(\mu,k)\cdot z_{(\mu,k)}\in A_0-x_0. \tag{d$'$}$$

For each $(\xi,n)\in\bigcup_{i<2}\operatorname{supp} H_i$, let $\tilde{\psi}(\xi,n)$ be the open arc of $\mathbb{T}$ centered at $z_{(\xi,n)}$ with diameter $\tilde{\tilde{\epsilon}}$.

Conditions (i) and (ii) are clearly satisfied, (iii) follows from the choice of $\tilde{\epsilon}$, and (iv) follows from (c), (d), (c′) and (d′). ■

If $\xi \in \mathbb{R} \setminus \mathbb{Q}$, it follows from Kronecker's theorem that $\{(x+\mathbb{Z}, \xi x+\mathbb{Z}) : x \in \mathbb{R}\}$ is a dense subset of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Thus, for each $\epsilon > 0$, there exists $l > 0$ such that if $I \subset \mathbb{R}$ is an interval of length greater than l, then $\{(x+\mathbb{Z}, \xi x+\mathbb{Z}) : x \in I\}$ is ϵ-dense in $\mathbb{T}^2$. Fix such an $l = l(\epsilon, \xi)$.

LEMMA 4.4. *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$, $\epsilon > 0$, $A_0, A_1 \in \mathcal{B}$ with $\delta(A_0) \geq \epsilon$ and $\delta(A_1) \geq \epsilon$, $G \in [\mathfrak{c} \times \omega]^{<\omega}$, $\psi : G \to \mathcal{B}$ and $\{H_0, H_1\} \subset \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ where $\operatorname{supp} H_0 = \operatorname{supp} H_1$. Let $(\mu, k) \in \operatorname{supp} H_0$ and $a > l(\epsilon/8, \xi)$ be such that:*

(1) $\delta(\psi(\mu, k)) \geq \epsilon$ *if* $(\mu, k) \in G$;
(2) $|H_1(\mu, k)|\epsilon \geq 4a$;
(3) $\left|\dfrac{|H_0(\mu, k)|}{|H_1(\mu, k)|} \cdot a - \xi \cdot a\right| < \dfrac{\epsilon}{8}$.

Denote $G \cup \operatorname{supp} H_0 \cup \operatorname{supp} H_1$ by $\tilde{G}$. There exists $\tilde{\epsilon} \leq \epsilon/2$ such that for every $\tilde{\tilde{\epsilon}} \leq \tilde{\epsilon}$ there exists $\tilde{\psi} : \tilde{G} \to \mathcal{B}$ satisfying the following conditions:

(i) $\overline{\tilde{\psi}(\xi, n)} \subset \psi(\xi, n)$ *for every* $(\xi, n) \in G$;
(ii) $\delta(\tilde{\psi}(\xi, n)) = \tilde{\tilde{\epsilon}}$ *for every* $(\xi, n) \in \tilde{G}$;
(iii) $\delta(\sum_{(\xi,n)\in \operatorname{supp} H_i} H_i(\xi, n) \cdot \tilde{\psi}(\xi, n)) < \epsilon$ *for every* $i < 2$;
(iv) $A_i \cap \sum_{(\xi,n)\in \operatorname{supp} H_i} H_i(\xi, n) \cdot \tilde{\psi}(\xi, n) \neq \emptyset$ *for every* $i < 2$.

Proof. Define

$$\tilde{\epsilon} = \min\left\{\left\{\frac{\delta(\psi(\xi, n))}{2} : (\xi, n) \in G\right\} \cup \left\{\frac{\epsilon}{2 \cdot \sum_{\substack{(\xi,n)\in \operatorname{supp} H_i \\ i<2}} |H_i(\xi, n)|}\right\}\right\}$$

and choose $\tilde{\tilde{\epsilon}} \leq \tilde{\epsilon}$.

If $(\xi, n) \in \tilde{G} \setminus \bigcup_{i<2} \operatorname{supp} H_i$, define $\tilde{\psi}(\xi, n)$ as the element of $\mathcal{B}$ centered at the middle point of $\psi(\xi, n)$ with diameter $\tilde{\tilde{\epsilon}}$.

For each $(\xi, n) \in \operatorname{supp} H_1 \setminus \{(\mu, k)\}$, define $z_{(\xi,n)} \in \mathbb{T}$ in the following way: if $(\xi, n) \in G$, then $z_{(\xi,n)}$ is the middle point of $\psi(\xi, n)$; if $(\xi, n) \notin G$, then $z_{(\xi,n)}$ is chosen arbitrarily. Set also

$$x_0 = \sum_{(\xi,n)\in \operatorname{supp} H_0 \setminus \{(\mu,k)\}} H_0(\xi, n) \cdot z_{(\xi,n)},$$
$$x_1 = \sum_{(\xi,n)\in \operatorname{supp} H_1 \setminus \{(\mu,k)\}} H_1(\xi, n) \cdot z_{(\xi,n)}.$$

If $(\mu, k) \in G$, choose $\tilde{z}_{(\mu,k)} \in \mathbb{T}$ such that $\tilde{z}_{(\mu,k)}$ is the middle point of $\psi(\mu, k)$. If $(\mu, k) \notin G$, choose $\tilde{z}_{(\mu,k)}$ arbitrarily.

Let A be the arc centered at $\tilde{z}_{(\mu,k)}$ with diameter $\epsilon/4$. In order to show that there exists $z_{(\mu,k)} \in A$ such that

$$H_0(\mu,k)\cdot z_{(\mu,k)} \in A_0 - x_0, \qquad H_1(\mu,k)\cdot z_{(\mu,k)} \in A_1 - x_1,$$

it suffices to prove that $\{(H_1(\mu,k)\cdot x, H_0(\mu,k)\cdot x) : x \in A\}$ is $\epsilon/4$-dense in $\mathbb{T}^2$. This occurs if, and only if,

$$X = \left\{\left(x+\mathbb{Z}, \frac{H_0(\mu,k)}{H_1(\mu,k)}\cdot x+\mathbb{Z}\right) : x \in H_1(\mu,k)\cdot\tilde{A}\right\}$$

is $\epsilon/4$-dense in $\mathbb{T}^2$, where $\tilde{A}$ is an interval of $\mathbb{R}$ such that $\tilde{A}+\mathbb{Z} = A$.

From the choice of a and from (3), it follows that

$$\left\{\left(x+\mathbb{Z}, \frac{H_0(\mu,k)}{H_1(\mu,k)}\cdot x+\mathbb{Z}\right) : x \in]0,a[\right\}$$

is $\epsilon/4$-dense in $\mathbb{T}^2$. Thus, from (2),

$$Y = \left\{\left(x+\mathbb{Z}, \frac{H_0(\mu,k)}{H_1(\mu,k)}\cdot x+\mathbb{Z}\right) : x \in]0,|H_1(\mu,k)|\cdot\epsilon/4[\right\}$$

is also $\epsilon/4$-dense in $\mathbb{T}^2$. Since $]0,|H_1(\mu,k)|\cdot\epsilon/4[= H_1(\mu,k)\cdot\tilde{A}+r$ for some $r\in\mathbb{R}$, we have

$$Y = X + \left(r+\mathbb{Z}, \frac{H_0(\mu,k)}{H_1(\mu,k)}\cdot r+\mathbb{Z}\right),$$

and since translations in $\mathbb{T}^2$ are isometries, it follows that X is $\epsilon/4$-dense in $\mathbb{T}^2$.

Fix $z_{(\mu,k)} \in A$ such that

$$\text{(e)} \qquad H_0(\mu,k)\cdot z_{(\mu,k)} \in A_0 - x_0,$$

$$\text{(f)} \qquad H_1(\mu,k)\cdot z_{(\mu,k)} \in A_1 - x_1.$$

For each $(\xi,n) \in \bigcup_{i\leq 2}\operatorname{supp} H_i$, let $\tilde{\psi}(\xi,n)$ be the open arc of $\mathbb{T}$ centered at $z_{(\xi,n)}$ with diameter $\tilde{\tilde{\epsilon}}$.

Conditions (i) and (ii) are clearly satisfied, (iii) follows from the choice of $\tilde{\epsilon}$, and (iv) follows from (4) and (4). ∎

Lemma 4.5. *Let $d\in\mathbb{N}\setminus\{0\}$, $\epsilon>0$, $A_0,\ldots,A_k\in\mathcal{B}$ with $\delta(A_i)\geq\epsilon$ for every $i\leq k$, $G\in[\mathfrak{c}\times\omega]^{<\omega}$ and $\psi : G\to\mathcal{B}$ such that $\delta(\psi(\theta,m))\geq\epsilon$ for every $(\theta,m)\in G$. Let $F=\{f_0,\ldots,f_k\}\in[{}^{\omega}\mathbb{Q}^{(\mathfrak{c}\times\omega)}]^{<\omega}$ be of type A, B or C. For every sufficiently large n in ω, there exists $\tilde{\epsilon}\leq\epsilon/2$ such that for every $\tilde{\tilde{\epsilon}}\leq\tilde{\epsilon}$ there exists $\tilde{\psi} : \tilde{G} = G\cup\operatorname{supp} f_0(n)\cup\cdots\cup\operatorname{supp} f_k(n)\to\mathcal{B}$ satisfying the following conditions:*

(i) *$d\cdot\overline{\tilde{\psi}(\theta,m)}\subset\psi(\theta,m)$ for every $(\theta,m)\in G$;*
(ii) *$\delta(\tilde{\psi}(\theta,m)) = \tilde{\tilde{\epsilon}}$ for every $(\theta,m)\in\tilde{G}$;*
(iii) *$\delta(\sum_{(\theta,m)\in\operatorname{supp} f_i(n)} a(f_i(n),(\theta,m))\cdot\tilde{\psi}(\theta,m))<\epsilon$ for every $i\leq k$;*
(iv) *$A_i\cap\sum_{(\theta,m)\in\operatorname{supp} f_i(n)} a(f_i(n),(\theta,m))\cdot\tilde{\psi}(\theta,m)\neq\emptyset$ for every $i\leq k$.*

Proof. Let $\phi : G \to \mathcal{B}$ be such that $\delta(\phi(\theta, m)) = \epsilon/d$ and $d \cdot \phi(\theta, m) = \psi(\theta, m)$ for every $(\theta, m) \in G$. We will consider each type separately.

CASE 1: F is of type A. In this case, $a(f_i(n), (\theta, m)) = f_i(n)(\theta, m)$ for all $n \in \omega$, $(\theta, m) \in \operatorname{supp} f_i(n)$ and $i \le k$. Fix $i \in \{0, \dots, k\}$. If f_i is of type 1, there are only finitely many n's such that $|f_i(n)|\epsilon \le 4d$; if f_i is of type 2, there are only finitely many n's such that $\operatorname{supp} f_i(n) \subset G$. Therefore, for all but finitely many n's we have $|f_i(n)|\epsilon > 4d$ or $\operatorname{supp} f_i(n) \setminus G \neq \emptyset$, for every $i \le k$. Choose such an n. Applying Lemma 4.2 for $\epsilon/d > 0$, $A_0, \dots, A_k \in \mathcal{B}$, $G \in [\mathfrak{c} \times \omega]^{<\omega}$, $\phi : G \to \mathcal{B}$ and $\{f_0(n), \dots, f_k(n)\} \subset \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ we obtain $\tilde{\epsilon} \le \epsilon/(2d)$ such that, for every $\tilde{\tilde{\epsilon}} \le \tilde{\epsilon}$, there exists $\tilde{\psi} : \tilde{G} \to \mathcal{B}$ satisfying the following conditions:

- $\overline{d \cdot \tilde{\psi}(\theta, m)} \subset d \cdot \phi(\theta, m) = \psi(\theta, m)$ for every $(\theta, m) \in G$;
- $\delta(\tilde{\psi}(\theta, m)) = \tilde{\tilde{\epsilon}}$ for every $(\theta, m) \in \tilde{G}$;
- $\delta(\sum_{(\theta,m)\in \operatorname{supp} f_i(n)} a(f_i(n), (\theta, m)) \cdot \tilde{\psi}(\theta, m)) < \epsilon/d \le \epsilon$ for every $i \le k$;
- $A_i \cap \sum_{(\theta,m)\in \operatorname{supp} f_i(n)} a(f_i(n), (\theta, m)) \cdot \tilde{\psi}(\theta, m) \neq \emptyset$ for every $i \le k$.

CASE 2: F is of type B. In this case, $f_0 = (1/\tilde{d})\tilde{f}_0$ and $f_1 = (1/\tilde{d})\tilde{f}_1$, where $\tilde{f}_0, \tilde{f}_1 : \omega \to \mathbb{Z}^{(\mathfrak{c}\times\omega)}$ and $\tilde{d}$ is a positive integer. We have $a(f_i(n), (\theta, m)) = \tilde{f}_i(n)(\theta, m)$ for all $n \in \omega$, $(\theta, m) \in \operatorname{supp} f_i(n)$ and $i < 2$, and we also have $a(f_i(n), (\theta, m)) = f_i(n)(\theta, m)$ for all $n \in \omega$, $(\theta, m) \in \operatorname{supp} f_i(n)$ and $i \in \{2, \dots, k\}$. For all but finitely many n's we have $|f_i(n)|\epsilon > 4d$ or $\operatorname{supp} f_i(n) \setminus G \neq \emptyset$, for every $i \le k$. Also, for all but finitely many n's there exists $(\theta_n, m_n) \in \operatorname{supp} f_0(n)$ such that $|\tilde{f}_1(n)(\theta_n, m_n)|\epsilon > 4d|\tilde{f}_0(n)(\theta_n, m_n)|$. Choose such an n.

Applying Lemma 4.3 for $d \in \mathbb{N} \setminus \{0\}$, $\epsilon/d > 0$, $A_0, A_1 \in \mathcal{B}$, $\bar{G} = G \cap (\operatorname{supp} f_0(n) \cup \operatorname{supp} f_1(n))$, $\bar{\psi} = \psi\restriction_{\bar{G}} : \bar{G} \to \mathcal{B}$ and $\{\tilde{f}_0(n), \tilde{f}_1(n)\} \subset [\mathbb{Z}^{(\mathfrak{c}\times\omega)}]^{<\omega}$, we obtain $\tilde{\bar{G}} = \bar{G} \cup \operatorname{supp} f_0(n) \cup \operatorname{supp} f_1(n)$ and $\tilde{\bar{\epsilon}} \le \epsilon/(2d)$ such that, for every $\tilde{\tilde{\bar{\epsilon}}} \le \tilde{\bar{\epsilon}}$, there exists $\tilde{\bar{\psi}} : \tilde{\bar{G}} \to \mathcal{B}$ satisfying the following conditions:

- $d \cdot \tilde{\bar{\psi}}(\theta, m) \subset \bar{\psi}(\theta, m) = \psi(\theta, m)$ for every $(\theta, m) \in \bar{G}$;
- $\delta(\tilde{\bar{\psi}}(\theta, m)) = \tilde{\tilde{\bar{\epsilon}}}$ for every $(\theta, m) \in \tilde{\bar{G}}$;
- $\delta(\sum_{(\theta,m)\in \operatorname{supp} f_i(n)} \tilde{f}_i(n)(\theta, m) \cdot \tilde{\bar{\psi}}(\theta, m)) < \epsilon/d \le \epsilon$ for every $i < 2$;
- $A_i \cap \sum_{(\theta,m)\in \operatorname{supp} f_i(n)} \tilde{f}_i(n)(\theta, m) \cdot \tilde{\bar{\psi}}(\theta, m) \neq \emptyset$ for every $i < 2$.

Applying Lemma 4.2 for $\epsilon/d > 0$, $A_2, \dots, A_k \in \mathcal{B}$, $\hat{G} = G \setminus (\operatorname{supp} f_0(n) \cup \operatorname{supp} f_1(n))$, $\hat{\phi} = \phi\restriction_{\hat{G}} : \hat{G} \to \mathcal{B}$ and $\{f_2(n), \dots, f_k(n)\} \subset [\mathbb{Z}^{(\mathfrak{c}\times\omega)}]^{<\omega}$, we obtain $\tilde{\hat{G}} = \hat{G} \cup \operatorname{supp} f_2(n) \cup \dots \cup \operatorname{supp} f_k(n)$ and $\tilde{\hat{\epsilon}} \le \epsilon/(2d)$ such that, for every $\tilde{\tilde{\hat{\epsilon}}} \le \tilde{\hat{\epsilon}}$, there exists $\tilde{\hat{\phi}} : \tilde{\hat{G}} \to \mathcal{B}$ satisfying the following conditions:

- $d \cdot \tilde{\hat{\phi}}(\theta, m) \subset d \cdot \hat{\phi}(\theta, m) = \psi(\theta, m)$ for every $(\theta, m) \in \hat{G}$;

- $\delta(\tilde{\hat{\phi}}(\theta,m)) = \tilde{\tilde{\hat{\epsilon}}}$ for every $(\theta,m) \in \tilde{\hat{G}}$;
- $\delta(\sum_{(\theta,m)\in \operatorname{supp} f_i(n)} f_i(n)(\theta,m) \cdot \tilde{\hat{\phi}}(\theta,m)) < \epsilon/d \leq \epsilon$ for every $i \in \{2,\ldots,k\}$;
- $A_i \cap \sum_{(\theta,m)\in \operatorname{supp} f_i(n)} f_i(n)(\theta,m) \cdot \tilde{\hat{\phi}}(\theta,m) \neq \emptyset$ for every $i \in \{2,\ldots,k\}$.

Put $\tilde{\epsilon} = \min\{\tilde{\bar{\epsilon}}, \tilde{\hat{\epsilon}}\}$. If $(\theta,m) \in \tilde{\bar{G}}$, define $\tilde{\psi}(\theta,m) = \tilde{\bar{\psi}}(\theta,m)$, and if $(\theta,m) \in \tilde{\hat{G}}$, define $\tilde{\psi}(\theta,m) = \tilde{\hat{\phi}}(\theta,m)$, where $\tilde{\bar{\psi}}$ and $\tilde{\hat{\phi}}$ are related to $\tilde{\epsilon}$. Note that $\tilde{\psi}$ is well-defined since $\tilde{\bar{G}} \cap \tilde{\hat{G}} = \emptyset$.

Case 3: F is of type C. In this case, $a(f_i(n),(\theta,m)) = f_i(n)(\theta,m)$ for all $n \in \omega$, $(\theta,m) \in \operatorname{supp} f_i(n)$ and $i \leq k$. For all but finitely many n's we have $|f_i(n)|\epsilon > 4d$ or $\operatorname{supp} f_i(n) \setminus G \neq \emptyset$, for every $i \leq k$. Fix $a > l(\epsilon/8,\xi)$, where $\xi \in \mathbb{R}\setminus\mathbb{Q}$ is the limit of the sequence $\left\{\frac{f_0(n)(\theta_n,m_n)}{f_1(n)(\theta_n,m_n)} : n \in \omega\right\}$. There are only finitely many n's such that

$$\left|\frac{|f_0(n)(\theta_n,m_n)|}{|f_1(n)(\theta_n,m_n)|}\cdot a - \xi\cdot a\right| \geq \frac{\epsilon}{8d}.$$

Also, there are only finitely many n's such that $|f_1(n)(\theta_n,m_n)|\epsilon < 4a$.

Therefore for all but finitely many n's we have $|f_i(n)|\epsilon > 4d$ or $\operatorname{supp} f_i(n)\setminus G \neq \emptyset$, for every $i \leq k$. Also, there exists $(\theta_n,m_n) \in \operatorname{supp} f_1(n)$ such that

$$\left|\frac{|f_0(n)(\theta_n,m_n)|}{|f_1(n)(\theta_n,m_n)|}\cdot a - \xi\cdot a\right| < \frac{\epsilon}{8d}$$

and $|f_1(n)(\theta_n,m_n)|\epsilon \geq 4a$. Choose such an n.

Applying Lemma 4.4 for $\xi \in \mathbb{R}\setminus\mathbb{Q}$, $\epsilon/d > 0$, $A_0, A_1 \in \mathcal{B}$, $\bar{G} = G \cap (\operatorname{supp} f_0(n) \cup \operatorname{supp} f_1(n))$, $\bar{\phi} = \phi\restriction_{\bar{G}} : \bar{G} \to \mathcal{B}$ and $\{\tilde{f}_0(n), \tilde{f}_1(n)\} \subset [\mathbb{Z}^{(\mathfrak{c}\times\omega)}]^{<\omega}$, we obtain $\tilde{\bar{G}} = \bar{G} \cup \operatorname{supp} f_0(n) \cup \operatorname{supp} f_1(n)$ and $\tilde{\bar{\epsilon}} \leq \epsilon/(2d)$ such that, for every $\tilde{\tilde{\bar{\epsilon}}} \leq \tilde{\bar{\epsilon}}$, there exists $\tilde{\bar{\phi}} : \tilde{\bar{G}} \to \mathcal{B}$ satisfying the following conditions:

- $d \cdot \tilde{\bar{\phi}}(\theta,m) \subset d \cdot \bar{\phi}(\theta,m) = \psi(\theta,m)$ for every $(\theta,m) \in G$;
- $\delta(\tilde{\bar{\phi}}(\theta,m)) = \tilde{\tilde{\bar{\epsilon}}}$ for every $(\theta,m) \in \tilde{G}$;
- $\delta(\sum_{(\theta,m)\in \operatorname{supp} f_i(n)} f_i(n)(\theta,m) \cdot \tilde{\bar{\phi}}(\theta,m)) < \epsilon/d \leq \epsilon$ for every $i < 2$;
- $A_i \cap \sum_{(\theta,m)\in \operatorname{supp} f_i(n)} f_i(n)(\theta,m) \cdot \tilde{\bar{\phi}}(\theta,m) \neq \emptyset$ for every $i < 2$.

Applying Lemma 4.2 for $\epsilon/d > 0$, $A_2,\ldots,A_k \in \mathcal{B}$, $\hat{G} = G \setminus (\operatorname{supp} f_0(n) \cup \operatorname{supp} f_1(n))$, $\hat{\phi} = \phi\restriction_{\hat{G}} : \hat{G} \to \mathcal{B}$ and $\{f_2(n),\ldots,f_k(n)\} \subset [\mathbb{Z}^{(\mathfrak{c}\times\omega)}]^{<\omega}$, we obtain $\tilde{\hat{G}} = \hat{G} \cup \operatorname{supp} f_2(n) \cup \cdots \cup \operatorname{supp} f_k(n)$ and $\tilde{\hat{\epsilon}} \leq \epsilon/(2d)$ such that, for every $\tilde{\tilde{\hat{\epsilon}}} \leq \tilde{\hat{\epsilon}}$, there exists $\tilde{\hat{\phi}} : \tilde{\hat{G}} \to \mathcal{B}$ satisfying the following conditions:

- $d \cdot \tilde{\hat{\phi}}(\theta,m) \subset d \cdot \hat{\phi}(\theta,m) = \psi(\theta,m)$ for every $(\theta,m) \in \hat{G}$;
- $\delta(\tilde{\hat{\phi}}(\theta,m)) = \tilde{\tilde{\hat{\epsilon}}}$ for every $(\theta,m) \in \tilde{\hat{G}}$;

- $\delta(\sum_{(\theta,m)\in\operatorname{supp} f_i(n)} f_i(n)(\theta,m)\cdot\tilde{\hat{\phi}}(\theta,m)) < \epsilon/d \le \epsilon$ for every $i \in \{2,\dots,k\}$;
- $A_i \cap \sum_{(\theta,m)\in\operatorname{supp} f_i(n)} f_i(n)(\theta,m)\cdot\tilde{\hat{\phi}}(\theta,m) \neq \emptyset$ for every $i\in\{2,\dots,k\}$.

Put $\tilde{\epsilon} = \min\{\tilde{\bar{\epsilon}}, \tilde{\hat{\epsilon}}\}$. If $(\theta,m)\in\tilde{\bar{G}}$, define $\tilde{\psi}(\theta,m) = \tilde{\bar{\phi}}(\theta,m)$, and if $(\theta,m)\in\tilde{\hat{G}}$, define $\tilde{\psi}(\theta,m) = \tilde{\hat{\phi}}(\theta,m)$, where $\tilde{\bar{\phi}}$ and $\tilde{\hat{\phi}}$ are related to $\tilde{\epsilon}$. ∎

Proposition 4.6. *Let $J\in\mathbb{Q}^{(\mathfrak{c}\times\omega)}\setminus\{0\}$ and $E\in[\mathfrak{c}]^\omega$ be as in Proposition 4.1. For each $\xi\in E\setminus\{0\}$, let $R_\xi\in[\omega]^\omega$. There exists a group homomorphism $\phi\restriction_{\mathbb{Q}^{(E\times\omega)}} : \mathbb{Q}^{(E\times\omega)}\to\mathbb{T}$ such that:*

(i) *$\phi\restriction_{\mathbb{Q}^{(E\times\omega)}}(J)\neq 0+\mathbb{Z}$;*
(ii) *for each $\xi\in E\setminus\{0\}$, there exists $S_\xi\in[R_\xi]^\omega$ such that the sequence $\{\phi\restriction_{\mathbb{Q}^{(E\times\omega)}}(f_{\xi,i}(n)) : n\in S_\xi\}$ converges to $\phi\restriction_{\mathbb{Q}^{(E\times\omega)}}(\chi_{(\xi,i)})$ for every $i\in\{0,\dots,n(F_\xi)-1\}$.*

Proof. Let $\{\theta_n : n\in\omega\}$ be an enumeration of $E\setminus\{0\}$ such that

$$|\{n\in\omega : \theta=\theta_n\}| = \omega$$

for every $\theta\in E\setminus\{0\}$. Let also $\{e_n : n\in\omega\}$ be an enumeration of $E\times\omega$. We will make an inductive construction in order to obtain a group homomorphism $\phi\restriction_{\mathbb{Q}^{(E\times\omega)}} : \mathbb{Q}^{(E\times\omega)}\to\mathbb{T}$ satisfying (i) and (ii).

Put $G_0 = \operatorname{supp} J\cup\{(\theta_0,i) : i<n(F_{\theta_0})\}\cup\{e_0\}$. For each $(\xi,n)\in G_0$, choose $y_{(\xi,n)}\in\mathbb{R}$ such that

$$\sum_{(\xi,n)\in\operatorname{supp} J} J(\xi,n)\cdot y_{(\xi,n)} = \frac{1}{2}$$

and define

$$x_{(\xi,n)} = \frac{1}{b(J)}\cdot y_{(\xi,n)} + \mathbb{Z}.$$

We have

$$\sum_{(\xi,n)\in\operatorname{supp} J} a(J,(\xi,n))\cdot x_{(\xi,n)} = \frac{1}{2}+\mathbb{Z}.$$

Let $\psi_0(\xi,n)$ be the open arc of $\mathbb{T}$ centered at $x_{(\xi,n)}$ with diameter $r_0/b(J)$ where

$$r_0 = \frac{1}{4\cdot\sum_{(\xi,n)\in\operatorname{supp} J}|a(J,(\xi,n))|}.$$

Since

$$\frac{1}{2}+\mathbb{Z}\in\sum_{(\xi,n)\in\operatorname{supp} J} a(J,(\xi,n))\cdot\psi_0(\xi,n)$$

and

$$\delta\Big(\sum_{(\xi,n)\in\operatorname{supp} J} a(J,(\xi,n))\cdot\psi_0(\xi,n)\Big) \le \sum_{(\xi,n)\in\operatorname{supp} J} |a(J,(\xi,n))|\delta(\psi_0(\xi,n)) \le \frac{1}{4}$$

it follows that

$$0+\mathbb{Z} \notin \sum_{(\xi,n)\in \operatorname{supp} J} a(J,(\xi,n)) \cdot \psi_0(\xi,n).$$

This concludes the first step of the induction. We remark that ψ_0 will be used to show that condition (i) of this proposition is satisfied.

Now, we start the successor stage. Fix $m \in \omega$ and suppose we have defined $r_m > 0$, $b_{-1} = 0$, $b_{m-1} \in R_{\theta_{m-1}}$ (if $m \geq 1$), $G_m \in [E\times\omega]^{<\omega}$ and $\psi_m : G_m \to \mathcal{B}$ such that $\delta(\psi_m(\xi,n)) = r_m/(b(J)\cdot\prod_{j<m} d(F_{\theta_j}))$ for every $(\xi,n) \in G_m$. Before stating the Claim that takes care of step $m+1$, we briefly comment on its statement.

Condition (1) of the Claim is used to make the sequence $\{r_k : k \in \omega\}$ of positive real numbers converge to 0. This is important to define the required homomorphism, since the lengths of the arcs of the function ψ_k are related to r_k. Conditions (1), (4) and (5) are used to define $\phi{\restriction}_{\mathbb{Q}^{(E\times\omega)}}(\chi_{(\xi,n)})$ as an intersection of decreasing arcs.

Roughly speaking, $\psi_{m+1}(\xi,n)$ is associated to a root of a point of $\psi_m(\xi,n)$ and the size of the arcs $\psi_{m+1}(\xi,n)$ guarantees that such a root is uniquely defined. This is necessary, since we want to embed a vector space over $\mathbb{Q}$ into $\mathbb{T}$.

Finally, conditions (1), (6) and (7) are used to produce a triangular inequality which shows that the image of each element of a family of type A, B or C is sent near to the image of their pre-assigned accumulation points. This last fact, together with condition (2) and the definition of b_m, is used to show that the pre-assigned accumulation points are preserved. Condition (3) keeps track of the domain of the arc functions, which needs to be finite at each stage, but increasing to $E\times\omega$.

Claim. *There exist $r_{m+1} > 0$, $b_m \in R_{\theta_m}$, $G_{m+1} \in [E\times\omega]^{<\omega}$ and $\psi_{m+1} : G_{m+1} \to \mathcal{B}$ satisfying the following conditions:*

(1) $2r_{m+1} \leq r_m$;
(2) $b_m > b_{m-1}$;
(3) $G_{m+1} = G_m \cup \operatorname{supp} f_{\theta_m,0}(b_m) \cup\cdots\cup \operatorname{supp} f_{\theta_m,n(F_{\theta_m})-1}(b_m) \cup \{e_{m+1}\} \cup \{(\theta_{m+1},i) : i < n(F_{\theta_{m+1}})\}$;
(4) $d(F_{\theta_m})\cdot\overline{\psi_{m+1}(\xi,n)} \subset \psi_m(\xi,n)$ *for every* $(\xi,n)\in G_m$;
(5) $\delta(b(J)\cdot\prod_{j<m+1} d(F_{\theta_j})\cdot\psi_{m+1}(\xi,n)) = r_{m+1}$ *for every* $(\xi,n)\in G_{m+1}$;
(6) $\delta(\sum_{(\xi,n)\in\operatorname{supp} f_{\theta_m,i}(b_m)} a(f_{\theta_m,i}(b_m),(\xi,n))\cdot\psi_{m+1}(\xi,n)) < r_m/(b(J)\cdot\prod_{j<m} d(F_{\theta_j}))$ *for every* $i < n(F_{\theta_m})$;
(7) $\psi_m(\theta_m,i)\cap\sum_{(\xi,n)\in\operatorname{supp} f_{\theta_m,i}(b_m)} a(f_{\theta_m,i}(b_m),(\xi,n))\cdot\psi_{m+1}(\xi,n) \neq \emptyset$ *for every* $i < n(F_{\theta_m})$.

Proof of the claim. Since F_{θ_m} is of type A, B or C and R_{θ_m} is infinite, one can choose $b_m \in R_{\theta_m}$ so that $b_m > b_{m-1}$ and Lemma 4.5 can be applied for $d(F_{\theta_m})$, $r_m/(b(J) \cdot \prod_{j<m+1} d(F_{\theta_j}))$, $\psi_m(\theta_m, 0), \ldots, \psi_m(\theta_m, n(F_{\theta_m}) - 1)$, G_m, ψ_m, F_{θ_m} and b_m. We obtain $\tilde{G} = G_m \cup \operatorname{supp} f_{\theta_m,0}(b_m) \cup \cdots \cup \operatorname{supp} f_{\theta_m,n(F_{\theta_m})-1}(b_m)$, $\tilde{\epsilon} \le r_m/(2b(J) \cdot \prod_{j<m+1} d(F_{\theta_j}))$ and, for $r_{m+1} = \tilde{\epsilon}/(b(J) \cdot \prod_{j<m+1} d(F_{\theta_j}))$, there exists $\tilde{\psi} : \tilde{G} \to \mathcal{B}$ satisfying (i)–(iv) of Lemma 4.5. Define $G_{m+1} = \tilde{G} \cup \{e_{m+1}\} \cup \{(\theta_{m+1}, i) : i < n(F_{\theta_{m+1}})\}$. If $(\xi, n) \in \tilde{G}$, define $\psi_{m+1}(\xi, n) = \tilde{\psi}(\xi, n)$. If $(\xi, n) \in G_{m+1} \setminus \tilde{G}$, let $\psi_{m+1}(\xi, n)$ be an element of $\mathcal{B}$ with diameter $r_{m+1}/(b(J) \cdot \prod_{j<m+1} d(F_{\theta_j}))$. ■

By finite induction, we have $r_m > 0$, $b_m \in R_{\theta_m}$, $G_m \in [E \times \omega]^{<\omega}$ and $\psi_m : G_m \to \mathcal{B}$ satisfying (1)–(7) for every $m \in \omega$. Note that $\bigcup_{m\in\omega} G_m = E \times \omega$.

Since $\mathbb{T}$ is a complete metric space and $(r_m)_{m\in\omega}$ is a sequence of positive real numbers that converges to 0, and since (4) and (5) hold, we conclude that if $(\xi, n) \in E \times \omega$, then

$$\bigcap_{m \ge N_{(\xi,n)}} b(J) \cdot \prod_{j<m} d(F_{\theta_j}) \cdot \psi_m(\xi, n) = \bigcap_{m \ge N_{(\xi,n)}} b(J) \cdot \prod_{j<m} d(F_{\theta_j}) \cdot \overline{\psi_m(\xi, n)}$$

is a one-element set, where $N_{(\xi,n)} = \min\{m \in \omega : (\xi, n) \in G_m\}$. Denote by $\phi(\chi_{(\xi,n)})$ the unique element of this set.

If $m \ge N_{(\xi,n)}$, there exists a unique element of $\psi_m(\xi, n)$ whose multiplication by $b(J) \cdot \prod_{j<m} d(F_{\theta_j})$ is equal to $\phi(\chi_{(\xi,n)})$. We shall denote this element by

$$\phi\bigg(\frac{1}{b(J) \cdot \prod_{j<m} d(F_{\theta_j})} \cdot \chi_{(\xi,n)}\bigg).$$

For each $(\xi, n) \in E \times \omega$, consider

$$G_{(\xi,n)} = \bigg\{\frac{1}{b(J) \cdot \prod_{j<m} d(F_{\theta_j})} \cdot \chi_{(\xi,n)} \in \mathbb{Q}^{(E\times\omega)} : m \ge N_{(\xi,n)}\bigg\}$$

and extend ϕ to a group homomorphism $\phi{\restriction}_G : G \to \mathbb{T}$, where G is the group generated by $\bigcup_{(\xi,n)\in G\times\omega} G_{(\xi,n)}$.

Since $\mathbb{T}$ is a divisible group, one can extend $\phi{\restriction}_G$ to a group homomorphism $\phi{\restriction}_{\mathbb{Q}^{(E\times\mathfrak{c})}} : \mathbb{Q}^{(E\times\omega)} \to \mathbb{T}$. It remains to show that conditions (i) and (ii) are satisfied.

We have

$$\begin{aligned} \phi{\restriction}_{\mathbb{Q}^{(E\times\mathfrak{c})}}(J) &= \sum_{(\xi,n)\in\operatorname{supp} J} a(J, (\xi, n)) \cdot \phi{\restriction}_{\mathbb{Q}^{(E\times\mathfrak{c})}}\bigg(\frac{1}{b(J)} \cdot \chi_{(\xi,n)}\bigg) \\ &\in \sum_{(\xi,n)\in\operatorname{supp} J} a(J, (\xi, n)) \cdot \psi_0(\xi, n). \end{aligned}$$

Since

$$0+\mathbb{Z}\notin\sum_{(\xi,n)\in\operatorname{supp} J} a(J,(\xi,n))\cdot\psi_0(\xi,n)$$

we conclude that $\phi\restriction_{\mathbb{Q}^{(E\times\mathfrak{c})}}(J)\neq 0+\mathbb{Z}$. Therefore, (i) is satisfied.

Fix $\xi\in E\setminus\{0\}$ and $i\in\{0,\dots,n(F_\xi)-1\}$. Set $I=\{m\in\omega:\xi=\theta_m\}$ and $S_\xi=\{b_m: m\in I\}$. It is clear that $S_\xi\in[R_\xi]^\omega$. We will show that $\{\phi\restriction_{\mathbb{Q}^{(E\times\mathfrak{c})}}(f_{\xi,i}(n)):n\in S_\xi\}$ converges to $\phi\restriction_{\mathbb{Q}^{(E\times\mathfrak{c})}}(\chi_{(\xi,i)})$. Since

$$\begin{aligned}&\phi\restriction_{\mathbb{Q}^{(E\times\mathfrak{c})}}(f_{\theta_m,i}(b_m))\\ &\qquad\in\sum_{(\xi,n)\in\operatorname{supp} f_{\theta_m,i}(b_m)} a(f_{\theta_m,i}(b_m),(\xi,n))\cdot b(J)\cdot\prod_{j<m} d(F_{\theta_j})\cdot\psi_{m+1}(\xi,n)\end{aligned}$$

and

$$\phi\restriction_{\mathbb{Q}^{(E\times\mathfrak{c})}}(\chi_{(\theta_m,i)})\in b(J)\cdot\prod_{j<m} d(F_{\theta_j})\cdot\psi_m(\theta_m,i)$$

it follows from (6) and (7) that $\delta(\phi\restriction_{\mathbb{Q}^{(E\times\mathfrak{c})}}(f_{\theta_m,i}(b_m)),\phi\restriction_{\mathbb{Q}^{(E\times\mathfrak{c})}}(\chi_{(\theta_m,i)}))\leq 2r_m$, and therefore condition (ii) is also satisfied. ∎

We are ready to extend each group homomorphism obtained from Proposition 4.6 to the whole group $\mathbb{Q}^{(\mathfrak{c}\times\omega)}$.

Proposition 4.7. *Let $J\in\mathbb{Q}^{(\mathfrak{c}\times\omega)}\setminus\{0\}$. For each $\xi\in\,]0,\mathfrak{c}[$, let $R_\xi\in[\omega]^\omega$. There exists a group homomorphism $\phi:\mathbb{Q}^{(\mathfrak{c}\times\omega)}\to\mathbb{T}$ such that:*

- (i) $\phi(J)\neq 0+\mathbb{Z}$;
- (ii) *for each $\xi\in\,]0,\mathfrak{c}[$, there exists $S_\xi\in[R_\xi]^\omega$ such that the sequence $\{\phi(f_{\xi,i}(n)):n\in S_\xi\}$ converges to $\phi(\chi_{(\xi,i)})$ for every $i\in\{0,\dots,n(F_\xi)-1\}$.*

Proof. According to Proposition 4.1, there exists $E\in[\mathfrak{c}]^\omega$ such that $\operatorname{supp} J\subset E\times\omega$ and such that if $\xi\in E\setminus\{0\}$, then $\bigcup_{n\in\omega}\operatorname{supp} f(n)\subset E\times\omega$, for every $f\in F_\xi$.

It follows from Proposition 4.6 that there exists a group homomorphism $\phi\restriction_{\mathbb{Q}^{(E\times\omega)}}:\mathbb{Q}^{(E\times\omega)}\to\mathbb{T}$ such that:

- (1) $\phi\restriction_{\mathbb{Q}^{(E\times\omega)}}(J)\neq 0+\mathbb{Z}$;
- (2) for each $\xi\in E\setminus\{0\}$, there exists $S_\xi\in[R_\xi]^\omega$ such that the sequence $\{\phi\restriction_{\mathbb{Q}^{(E\times\omega)}}(f_{\xi,i}(n)):n\in S_\xi\}$ converges to $\phi\restriction_{\mathbb{Q}^{(E\times\omega)}}(\chi_{(\xi,i)})$ for every $i\in\{0,\dots,n(F_\xi)-1\}$.

Let $\{\alpha_\xi:\xi<\mathfrak{c}\}$ be a strictly increasing enumeration of $\mathfrak{c}\setminus E$. Choose $S_{\alpha_0}\in[R_{\alpha_0}]^\omega$ such that $\{\phi\restriction_{\mathbb{Q}^{(E\times\omega)}}(f_{\alpha_0,i}(n)):n\in S_{\alpha_0}\}$ is convergent for every $i < n(F_{\alpha_0})$. Note that this is possible, since $\alpha_0 = \min(\mathfrak{c} \setminus E)$, $\bigcup_{i<n(F_{\alpha_0})} \bigcup_{n\in\omega} \operatorname{supp} f_{\alpha_0,i}(n) \subset \alpha_0 \times \omega$ and $\mathbb{T}$ is sequentially compact.

Denote by $\tilde{\phi}\restriction_{\mathbb{Q}^{((E\cup\{\alpha_0\})\times\omega)}}(\chi_{(\alpha_0,i)})$ the limit point of $\{\phi\restriction_{\mathbb{Q}^{(E\times\omega)}}(f_{\alpha_0,i}(n)) : n \in S_{\alpha_0}\}$ for every $i < n(F_{\alpha_0})$. If $i \geq n(F_{\alpha_0})$, define $\tilde{\phi}\restriction_{\mathbb{Q}^{((E\cup\{\alpha_0\})\times\omega)}}(\chi_{(\alpha_0,i)})$ arbitrarily. Finally, if $H \in \mathbb{Q}^{(E\times\omega)}$ put $\tilde{\phi}\restriction_{\mathbb{Q}^{((E\cup\{\alpha_0\})\times\omega)}}(H) = \phi\restriction_{\mathbb{Q}^{(E\times\omega)}}(H)$.

Let H_{α_0} be the subgroup of $\mathbb{Q}^{(\mathfrak{c}\times\omega)}$ generated by $\mathbb{Q}^{(E\times\omega)} \cup \{\chi_{(\alpha_0,n)} : n \in \omega\}$. It is possible to extend $\tilde{\phi}\restriction_{\mathbb{Q}^{((E\cup\{\alpha_0\})\times\omega)}}$ to a group homomorphism from H_{α_0} to $\mathbb{T}$, and since $\mathbb{T}$ is divisible, it is possible to extend $\tilde{\phi}\restriction_{\mathbb{Q}^{((E\cup\{\alpha_0\})\times\omega)}}$ to a group homomorphism $\phi\restriction_{\mathbb{Q}^{((E\cup\{\alpha_0\})\times\omega)}} : \mathbb{Q}^{((E\cup\{\alpha_0\})\times\omega)} \to \mathbb{T}$ such that $\phi\restriction_{\mathbb{Q}^{((E\cup\{\alpha_0\})\times\omega)}}(J) \neq 0 + \mathbb{Z}$ and so that, for each $\xi \in (E \cup \{\alpha_0\}) \setminus \{0\}$, the sequence $\{\phi\restriction_{\mathbb{Q}^{((E\cup\{\alpha_0\})\times\omega)}}(f_{\xi,i}(n)) : n \in S_\xi\}$ converges to $\phi\restriction_{\mathbb{Q}^{((E\cup\{\alpha_0\})\times\omega)}}(\chi_{(\xi,i)})$ for every $i \in \{0, \ldots, n(F_\xi) - 1\}$.

By induction, we obtain $S_\xi \in [R_\xi]^\omega$ for every $\xi \in \,]0, \mathfrak{c}[$ and a group homomorphism $\phi : \mathbb{Q}^{(\mathfrak{c}\times\omega)} \to \mathbb{T}$ satisfying (i) and (ii). ∎

The assumption $\mathfrak{p} = \mathfrak{c}$ together with Proposition 4.7 implies Proposition 2.5, which will be restated and proved below.

PROPOSITION 2.5. ($\mathfrak{p} = \mathfrak{c}$) *For each $\alpha < \mathfrak{c}$ and $\xi \in \,]0, \mathfrak{c}[$ there exists $S_{\xi,\alpha} \in [\omega]^\omega$ such that if $\alpha < \beta < \mathfrak{c}$, then $S_{\xi,\beta} \subset^* S_{\xi,\alpha}$. There also exists a group homomorphism $\phi_\alpha : \mathbb{Q}^{(\mathfrak{c}\times\omega)} \to \mathbb{T}$ such that:*

(i) *$\phi_\alpha(J_\alpha) \neq 0 + \mathbb{Z}$;*
(ii) *the sequence $\{\phi_\alpha(f_{\xi,i}(n)) : n \in S_{\xi,\alpha}\}$ converges to $\phi_\alpha(\chi_{(\xi,i)})$ for every $i \in \{0, \ldots, n(F_\xi) - 1\}$.*

Proof. For each $\xi \in \,]0, \mathfrak{c}[$, put $R_{\xi,0} = \omega$. Applying Proposition 4.7 to $J = J_0$ and $R_\xi = R_{\xi,0}$, we obtain a group homomorphism $\phi_0 : \mathbb{Q}^{(\mathfrak{c}\times\omega)} \to \mathbb{T}$ such that $\phi_0(J_0) \neq 0 + \mathbb{Z}$ and $S_{\xi,0} \in [R_{\xi,0}]^\omega$ such that the sequence $\{\phi_0(f_{\xi,i}(n)) : n \in S_{\xi,0}\}$ converges to $\phi_0(\chi_{(\xi,i)})$ for every $i \in \{0, \ldots, n(F_\xi) - 1\}$.

Fix $\beta < \mathfrak{c}$ and suppose that $S_{\xi,\alpha} \in [\omega]^\omega$ is defined for every $\alpha < \beta$ so that $S_{\xi,\delta} \subset^* S_{\xi,\gamma}$ for all $\gamma < \delta < \beta$ and $\xi \in \,]0, \mathfrak{c}[$. Suppose also that we have constructed a group homomorphism $\phi_\alpha : \mathbb{Q}^{(\mathfrak{c}\times\omega)} \to \mathbb{T}$ such that $\phi_\alpha(J_\alpha) \neq 0 + \mathbb{Z}$ and the sequence $\{\phi_\alpha(f_{\xi,i}(n)) : n \in S_{\xi,\alpha}\}$ converges to $\phi_\alpha(\chi_{\xi,i})$ for all $i \in \{0, \ldots, n(F_\xi) - 1\}$ and $\alpha < \beta$. We shall show that it is possible to choose $S_{\xi,\beta} \in [\omega]^\omega$ so that $S_{\xi,\beta} \subset^* S_{\xi,\alpha}$ for all $\alpha < \beta$ and $\xi \in \,]0, \mathfrak{c}[$ and that it is also possible to construct a group homomorphism $\phi_\beta : \mathbb{Q}^{(\mathfrak{c}\times\omega)} \to \mathbb{T}$ such that $\phi_\beta(J_\beta) \neq 0 + \mathbb{Z}$ and the sequence $\{\phi_\beta(f_{\xi,i}(n)) : n \in S_{\xi,\beta}\}$ converges to $\phi_\beta(\chi_{\xi,i})$ for every $i \in \{0, \ldots, n(F_\xi) - 1\}$.

If β is a successor ordinal—say, $\beta = \alpha + 1$—put $R_{\xi,\beta} = S_{\xi,\alpha}$ for every $\xi \in \,]0, \mathfrak{c}[$ and apply Proposition 4.7 to $J = J_\beta$ and $R_\xi = R_{\xi,\beta}$. If β is a limit ordinal, consider, for each $\xi \in \,]0, \mathfrak{c}[$, the family $\{S_{\xi,\alpha} : \alpha < \beta\}$. By inductive hypothesis, this family has the SFIP, and since we are assuming $\mathfrak{p} = \mathfrak{c}$, it has a pseudointersection $R_{\xi,\beta}$. Then, apply Proposition 4.7 to $J = J_\beta$ and $R_\xi = R_{\xi,\beta}$. ∎

5. Concerning Wallace's problem. It is consistent (with ZFC) that Wallace semigroups can have the square countably compact.

THEOREM 5.1. ($\mathfrak{p} = \mathfrak{c}$) *There exists a both-sided cancellative topological semigroup which is not a topological group and whose square is countably compact.*

Proof. Consider $S = \{\Phi(J) \in \mathbb{T}^{\mathfrak{c}} : J \in \mathbb{N}^{(\mathfrak{c}\times\omega)}\}$. Clearly, S is a both-sided cancellative topological semigroup which is not a topological group. Let $\{(g(n), h(n)) : n \in \omega\}$ be a sequence in $S\times S$, where $g, h : \omega \to \Phi[\mathbb{N}^{(\mathfrak{c}\times\omega)}]$. According to the proof of Proposition 2.4, there exist $\xi \in]0, \mathfrak{c}[$, $j : \omega \to \omega$ strictly increasing and $\tilde{g}, \tilde{h} \in \mathbb{N}^{(\mathfrak{c}\times\omega)}$ such that

$$(\Phi^{-1} \circ g \circ j)(n) = \tilde{g} + \sum_{i<n(F_\xi)} a_{\xi,i} \cdot f_{\xi,i}(n),$$
$$(\Phi^{-1} \circ h \circ j)(n) = \tilde{h} + \sum_{i<n(F_\xi)} b_{\xi,i} \cdot f_{\xi,i}(n),$$

for every $n \in \omega$, where $a_{\xi,i}, b_{\xi,i} \in \mathbb{N}$ for every $i < n(F_\xi)$.

It was shown in Theorem 2.6 that

$$\Big(\Phi\Big(\tilde{g} + \sum_{i<n(F_\xi)} a_{\xi,i} \cdot \chi_{(\xi,i)}\Big), \Phi\Big(\tilde{h} + \sum_{i<n(F_\xi)} b_{\xi,i} \cdot \chi_{(\xi,i)}\Big)\Big) = p_\xi\text{-}\lim\{(g(n), h(n)) : n \in \omega\}$$

for some $p_\xi \in \omega^*$. As $\Phi(\tilde{g} + \sum_{i<n(F_\xi)} a_{\xi,i} \cdot \chi_{(\xi,i)}), \Phi(\tilde{h} + \sum_{i<n(F_\xi)} b_{\xi,i} \cdot \chi_{(\xi,i)}) \in \Phi[\mathbb{N}^{(\mathfrak{c}\times\omega)}]$, it follows that $S \times S$ is countably compact. ∎

6. Final remarks. In 1990, Comfort [3] asked for which cardinals $\kappa \le 2^{\mathfrak{c}}$ there exists a topological group G such that G^α is countably compact for every $\alpha < \kappa$ and G^κ is not countably compact. It was shown by Tomita [13] that, assuming a cardinal arithmetic and the existence of $2^{\mathfrak{c}}$ selective ultrafilters, every $\kappa \le 2^{\mathfrak{c}}$ admits such a topological group. However, these groups have finite order 2.

Tomita [12] showed that the ωth power of a non-trivial topological free abelian group cannot be countably compact. Tomita [11] also showed that the $2^{\mathfrak{c}}$th power of a Wallace semigroup cannot be countably compact.

These results motivate the following questions:

PROBLEM 6.1. *For which cardinals $\kappa \in]3, \omega]$ does there exist a group topology on the free abelian group of cardinality $\mathfrak{c}$ whose powers smaller than κ are countably compact?*

Problem 6.2. *Is it true that every group of cardinality $\mathfrak{c}$ that admits a countably compact group topology admits one whose square is countably compact?*

Problem 6.3. *For which cardinals $\kappa \leq 2^{\mathfrak{c}}$ does there exist a group topology on a non-torsion abelian group G such that G^{α} is countably compact for every $\alpha < \kappa$ and G^{κ} is not countably compact?*

Problem 6.4. *For which cardinals $\kappa \in]3, 2^{\mathfrak{c}}]$ does there exist a Wallace semigroup whose powers smaller than κ are countably compact?*

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References

[1] J. E. Baumgartner, *Sacks forcing and the total failure of Martin's axiom*, Topology Appl. 19 (1985), 211–225.

[2] A. R. Bernstein, *A new kind of compactness for topological spaces*, Fund. Math. 66 (1970), 185–193.

[3] W. W. Comfort, *Problems on topological groups and other homogeneous spaces*, in: Open Problems in Topology, J. van Mill and G. M. Reed (eds.), North-Holland, 1990, 313–347.

[4] D. L. Grant, *Sequentially compact cancellative topological semigroups: some progress on the Wallace problem*, Ann. New York Acad. Sci. 704 (1993), 150–154.

[5] P. B. Koszmider, A. H. Tomita and S. Watson, *Forcing countably compact group topologies on a larger free Abelian group*, Topology Proc. 25 (2000), 563–574.

[6] R. E. Madariaga-Garcia and A. H. Tomita, *Countably compact topological group topologies on free Abelian groups from selective ultrafilters*, Topology Appl. 154 (2007), 1470–1480.

[7] K. Numakura, *On bicompact semigroups*, Math. J. Okayama Univ. 1 (1952), 99–108.

[8] D. Robbie and S. Svetlichny, *An answer to A. D. Wallace's question about countably compact cancellative semigroups*, Proc. Amer. Math. Soc. 124 (1996), 325–330.

[9] M. G. Tkachenko, *Topological groups: between compactness and $\aleph_0$-boundedness*, in: Recent Progress in General Topology II, M. Hušek and J. van Mill (eds.), North-Holland, 2002, 515–543.

[10] —, *Countably compact and pseudocompact topologies on free Abelian groups*, Soviet Math. (Izv. VUZ) 34 (1990), 79–86.

[11] A. H. Tomita, *The Wallace problem: a counterexample from $MA_{countable}$ and p-compactness*, Canad. Math. Bull. 39 (1996), 486–498.
[12] —, *The existence of initially ω_1-compact group topologies on free Abelian groups is independent of ZFC*, Comment. Math. Univ. Carolin. 39 (1998), 401–413.
[13] —, *A solution to Comfort's question on the countable compactness of powers of a topological group*, Fund. Math. 186 (2005), 1–24.
[14] A. D. Wallace, *The structure of topological semigroups*, Bull. Amer. Math. Soc. 61 (1955), 95–112.

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