FUNDAMENTA
MATHEMATICAE
208 (2010)

C^1 stable maps: examples without saddles

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Abstract. We give here the first examples of C^1 structurally stable maps on manifolds of dimension greater than two that are neither diffeomorphisms nor expanding. It is shown that an Axiom A endomorphism all of whose basic pieces are expanding or attracting is C^1 stable. A necessary condition for the existence of such examples is also given.

1. Introduction. The space of C^1 maps of a compact manifold M will be denoted by $C^1(M)$. An *equivalence* (*conjugacy*) between f and g is a homeomorphism h of M such that $hf = gh$. If such an equivalence exists, the maps f and g are said to be *topologically equivalent*. A map $f \in C^1(M)$ is C^1 *structurally stable*, or simply C^1 *stable*, if there exists a neighborhood $\mathcal{U}$ of f in $C^1(M)$ such that f and g are topologically equivalent for every $g \in \mathcal{U}$.

The characterization of C^1 stable maps is a central problem in dynamics.

For invertible maps of compact manifolds (diffeomorphisms), the characterization of C^1 stability was obtained by Robinson [Rob] and Mañé [Ma2]. A few years earlier, it had been shown by Shub [Sh] that an expanding map is stable. Since then, no new examples of C^1 stable maps on manifolds of dimension greater than one have been discovered.

It is known that if M is a compact manifold then the following conditions are necessary for a map $f \in C^1(M)$ to be C^1 structurally stable:

(1) The set of critical points of f is empty.
(2) The map f is Axiom A without cycles.
(3) If the unstable set of a basic piece Λ intersects another basic piece, then Λ is an expanding basic piece.

Note that this contrasts with the chronology of discovering for diffeomorphisms: sufficient conditions (namely, Axiom A + Strong Transversality)

2010 *Mathematics Subject Classification*: 37C75, 37C20.
Key words and phrases: endomorphisms, stability.

DOI: 10.4064/fm208-1-2

for C^1 structural stability of diffeomorphisms were obtained by C. Robinson in 1976, a long time before R. Mañé obtained a proof of the necessity of these conditions. In the case of noninvertible maps, it is known that hyperbolicity is necessary for stability, but no set of sufficient conditions have been established until now, and no nonexpanding examples have been known. Looking for a characterization of C^1 stable maps, our interest now is to provide sufficient conditions and introduce new examples of C^1 stable maps.

Now we briefly comment on the necessary conditions stated above. The first item is obvious since it concerns C^1 maps. It follows that f is locally invertible and so a covering map. There exist examples of maps (on manifolds of dimension greater than one) having critical points that are C^r structurally stable ($r > 1$) and have nontrivial nonwandering sets (see [IPR1] and [IPR2]).

To explain the second item, it is convenient to briefly recall the history of the definition of Axiom A. The first definition was given by Przytycki ([Prz1]): a map f is Axiom A if its nonwandering set is hyperbolic and the set of periodic points of f is dense in $\Omega(f)$. However, as proved by Przytycki in [Prz2], a map f that has this property and is C^1 Ω-stable, must satisfy the no cycles condition, and also the restriction of f to Λ is injective whenever Λ is a basic piece that is not expanding. Following [Ma1], we include this last condition in the definition of Axiom A.

The proof of the second item is a consequence of the work of several authors. On one hand, Aoki, Moriyasu and Sumi [AMS], adapting the proof of the C^1 stability conjecture given by Mañé [Ma2], proved that C^1 Ω stability implies that the nonwandering set has a hyperbolic structure. The adaptation of Pugh's C^1 closing lemma for critical point free endomorphisms was obtained by Wen [Wen], and trivially implies the density of periodic points in the nonwandering set when the map is stable. As explained in the above paragraph, Przytycki proved that an Ω-stable map cannot have cycles and must satisfy the remaining condition. Actually, the first two items characterize C^1 Ω-stable maps without nonwandering critical points. When the set of critical points intersects the nonwandering set, some conditions were shown to be sufficient for C^1 Ω-stability in [DRRV], but a full characterization was not established, at least in dimensions ≥ 3. The third item was also proved by Przytycki in [Prz2].

In Przytycki's last mentioned article, there is an example of an Ω-stable map that satisfies the three items above; it was asked if this example is structurally stable or not. As far as we know, this question has remained unsolved since then. It is our purpose to show that his example is C^1 structurally stable in a forthcoming work. The nonwandering set of Przytycki's map is the union of an attracting fixed point, a saddle type basic piece and an expanding set. The main difficulty in proving the stability comes from the fact that there exist intersections of the unstable manifolds of saddle type basic pieces. Without this type of basic pieces, one has the following simple result.

THEOREM 1. *Let M be a compact manifold. If $f \in C^1(M)$ is an Axiom A map without critical points, and every basic piece is either expanding or attracting, then f is C^1 structurally stable.*

We have known for some time how to prove similar assertions; the ideas are contained in [IP], [IPR1] and [IPR2]. However, we have not known any examples of maps satisfying the hypotheses; the discovery of simple examples in dimensions greater than two motivated us to write the present work.

THEOREM 2. *Let M be a manifold admitting an expanding map, and embed M into some sphere S. Then there exists a noninvertible Axiom A map f in $C^1(M \times S)$ whose nonwandering set is the union of an expanding set and a nonperiodic attractor, and moreover f has no critical points, so Theorem 1 implies that f is C^1 structurally stable.*

A very simple formula is also available in $S^1 \times S^2$: if the extended complex plane is seen as the two-dimensional sphere, then

$$f : S^1 \times S^2 \to S^1 \times S^2, \qquad f(z, w) = (z^2, z/2 + w/3),$$

is a C^1 stable map. The nonwandering set of f is the union of an attracting solenoid A contained in $S^1 \times D(0; 1)$, and an expanding circle $S^1 \times \{\infty\}$. The map f has degree two and is injective in A, so $f^{-1}(A)$ must have a component $A' \neq A$. Behind this simple formula is hidden the fact that A' must be contained in the immediate basin of A. Even when singularities are forbidden, f is not injective in the immediate basin of an injective attractor. It is interesting to realize that this possibility depends on the topology of the attractor. This is the subject of the next result.

A neighborhood U of an attractor Λ of a map f is called *admissible* if it is contained in the basin of attraction of Λ, the closure of $f(U)$ is contained in U, the restriction of f to U is injective and each connected component of U intersects Λ.

DEFINITION 1. The attractor Λ is called *topologically simple* if there exists an admissible neighborhood U of Λ such that for each closed curve γ in U there exists a closed curve γ' in $f(U)$ such that γ and γ' are homotopic and the homotopy is contained in U.

For example, a periodic attractor and a DA attractor are topologically simple, while a solenoid is not.

THEOREM 3. *Let M be a compact connected manifold, and $f \in C^1(M)$ a nonivertible Axiom A map without critical points. If f has a topologically simple attractor Λ, then:*

(1) *The restriction of f to B_0, the immediate basin of attraction of Λ, is injective.*
(2) *If the dimension of M is greater than one, then in the boundary of B_0 there are nonwandering points that do not belong to an expanding basic piece.*

For an Axiom A map, denote by $\Gamma(f)$ the union of the basic pieces that are neither expanding nor attracting (repelling basic pieces that are not expanding are contained in $\Gamma(f)$). The second item of Theorem 3 implies that $\Gamma(f)$ is not empty if there exists a topologically simple attractor. Therefore, under the presence of such an attractor, the hypotheses of Theorem 1 do not hold.

2. Sufficient conditions for stability. In this section we prove Theorem 1. The proof is inspired by that of Theorem C in [IPR2]. In that result, there were critical points in the basin of the attractor but it was assumed that the union of the expanding pieces of the map was completely invariant, which is not the case here.

We begin by recalling the definition of an expanding set: A compact invariant set Λ is *expanding* if there exist constants $C > 0$ and $\lambda > 1$ such that $|Df_x^n(v)| \geq C\lambda^n|v|$ for every $n > 0$ and every vector v. If f is an Axiom A map, then an *expanding basic piece* is an expanding set Λ such that f is transitive on Λ. In this case, there exists an open neighborhood U of Λ such that $f(U) \supset \bar{U}$ and $\bigcap_{n\geq 0} f^{-n}(U) = \Lambda$.

We first claim that the map f is C^1 Ω-stable: as was explained in the introduction, it is sufficient to prove that f is critical points free and Axiom A without cycles. Note that the no cycles condition is a trivial consequence of the nature of the basic pieces and of the absence of critical points; indeed, it may happen that the unstable set of an expanding basic piece contains a different expanding piece (see Fig. 1), but cycles are forbidden since their intersections are nonwandering.

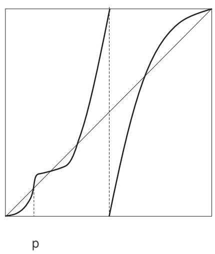

Fig. 1. f is an Axiom A map of S^1. The basic pieces are: two attracting fixed points, a fixed repeller p and an expanding Cantor set K. The preimage $f^{-1}(p)$ is not contained in $\Omega(f)$.

There exists a C^1 neighborhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$ one can define $A(g)$ as the union of the attracting basic pieces of g, $B(g)$ as the union of the basins of the elements of $A(g)$, and $J(g)$ as the union of the preimages of the expanding basic pieces of g. It follows that $J(g) \cup B(g) = M$. Note that for the one-dimensional map f of Fig. 1, the set $J(f)$ is not equal to the union of the expanding basic pieces.

However, $J(f)$ is always an expanding set:

LEMMA 1. *$J(f)$ is an expanding set.*

Proof. We first show that M is the disjoint union of $J(f)$ and $B(f)$. Note that if Λ is an expanding basic piece, then there exists a neighborhood U of Λ such that every point of $U \setminus \Lambda$ eventually escapes from U. If $x \notin B(f)$, then the sequence $\{f^n(x) : n \geq 0\}$ must converge to an expanding basic piece Λ, and this implies that $f^N(x) \in \Lambda$ for some $N > 0$. It follows that $x \in J(f)$. Thus $J(f)$ is compact, invariant and

$$\lim_{n \to +\infty} \|Df^n(v)\| = +\infty \qquad \forall v \in T_{J(f)}M,\ v \neq 0.$$

A standard argument, which we omit here, implies the assertion of the lemma. ■

It is a well known fact that hyperbolic attractors are stable. As the restriction of f to a neighborhood of $A(f)$ is a diffeomorphism onto its image, there exists an equivalence h between the restrictions of f and G to neighborhoods $U(f)$ of $A(f)$ and $U(g)$ of $A(g)$. Moreover, the equivalence can be taken as close to the identity as wished by shrinking the neighborhood $\mathcal{U}$ of f.

Denote by h the equivalence between f and g referred to above. The first step of the proof consists in extending h to $B(f)$. This would be trivial if f were a diffeomorphism. Indeed, let x be a point in $f^{-k}(U(f))$; to define $h(x)$ one has to choose a g^k-preimage of $h(f(x))$, and there are a lot of them. However, our arguments will imply that there exists one of these preimages that is closest to x. This is easy to prove for a finite number of preimages, but at each step, one is forced to shrink the neighborhood of f. A different argument will be used when the preimages taken are sufficiently close to an expanding set. Lemma 2 will provide precise estimates, and Lemma 3 will explain the order of choices of neighborhoods and constants. The second step of the proof consists in extending h to the complement $J(f)$ of $B(f)$.

Some definitions and notations are in order before proceeding to the statement of the first lemma. Define $U_k(f) = f^{-k}(U(f))$. Let d denote the distance in M, and $B(x; r)$ be the ball of center x and radius r. As the maps have no critical points and the manifold M is compact, there exists $\epsilon_0 > 0$ such that $g(x) = g(y)$ implies $x = y$ or $d(x, y) > \epsilon_0$ whenever $g \in \mathcal{U}$. If W is a subset of M and $\delta > 0$, denote by $\mathcal{N}_\delta(W)$ the set of homeomorphisms $h : W \to h(W) \subset M$ that are δ-close to the identity of W. If, moreover, h conjugates the corresponding restrictions of f and g, then we write $h \in \mathcal{N}_\delta(W; g)$.

LEMMA 2.

(1) *Let ρ be a positive constant less than or equal to $\epsilon_0/2$. Then there exist $\delta = \delta(\rho) > 0$ and a neighborhood $\mathcal{U} = \mathcal{U}(\rho)$ of f such that, if W is any subset of M and h belongs to $\mathcal{N}_\delta(W; g)$ for some $g \in \mathcal{U}$, then there exists a unique extension h' of h in $\mathcal{N}_\rho(W \cup f^{-1}(W); g)$.*

(2) *There exist a positive number δ_0 and a neighborhood V of $J(f)$ such that the following property holds. Given any $\delta < \delta_0$ there exists a neighborhood $\mathcal{U}$ of f such that, given any $W \subset V$, any $g \in \mathcal{U}$ and any $h \in \mathcal{N}_\delta(W; g)$, there exists a unique extension h' of h in $\mathcal{N}_\delta(W \cup f^{-1}(W); g)$.*

Proof. (1) Let $x \in f^{-1}(W)$; one has to prove that there exists a unique $x' \in B(x; \rho)$ such that $g(x') = h(f(x))$.

Note that given any $\rho > 0$ there exists a neighborhood $\mathcal{U}$ of f and a positive number δ such that, for every g in $\mathcal{U}$ and $x \in M$,

$$g(B(x; \rho)) \supset B(g(x); 2\delta).$$

Note also that if $\rho < \epsilon_0/2$, then $g_{|B(x;\rho)}$ is a homeomorphism onto its image.

To prove part (1) it suffices to show that $h(f(x)) \in B(g(x); 2\delta)$. But

$$d(h(f(x)), g(x)) \le d(h(f(x)), f(x)) + d(f(x), g(x)) \le 2\delta$$

if the C^0 distance between f and g is less than δ. This defines h in $f^{-1}(W)$; it is a homeomorphism since it is open by definition (locally $h = g^{-1}hf$). Moreover $h(x) = h(y)$ implies $h(f(x)) = h(f(y))$, hence $f(x) = f(y)$ and so $x = y$ because h is close to the identity.

(2) If $\mathcal{U}$ is a small neighborhood of f, and V is a small neighborhood of $J(f)$, then Lemma 1 implies that there exists a number $\lambda > 1$ and an adapted metric in M such that Dg_x λ-expands any direction, for any $x \in V$ and $g \in \mathcal{U}$. Using part (1), let $\delta_0 = \delta(\epsilon_0/2)$. Now, if $\delta < \delta_0$, and if for some $g \in \mathcal{U}(\epsilon_0/2)$ one has an $h \in \mathcal{N}_\delta(W; g)$, then there is an extension h' of h to $f^{-1}(W)$ that still conjugates f and g. It must be shown that the extension remains in the δ-neighborhood of the identity. Indeed, if $f(x) \in W$ and $h(x) = x'$, then $d(f(x), f(x')) \ge \lambda d(x, x')$. Moreover,

$$d(f(x), f(x')) \le d(f(x), g(x')) + d(g(x'), f(x')) \le \delta + d_0,$$

where d_0 is the C^0 distance between f and g. Taking $\mathcal{U}$ so small that $d_0 < (\lambda - 1)\delta$, it follows that $d(x, x') < \delta$. ∎

By Lemma 2(1), one can extend h to $U_1(f)$ if the neighborhood $\mathcal{U}$ is small enough. This can be repeated a finite number of times, but it is not enough to cover $B(f)$. We will show how the second part of the lemma implies that close to the boundary of the basin, the distance from h to the identity will not increase. We are using here the fact that the complement of the basin is an expanding set. It follows that there exist neighborhoods V of $J(f)$ and $\mathcal{U}$ of f, and a positive constant $\lambda > 1$, such that for an adapted metric, Dg_x expands vectors at a rate at least λ for any $x \in V$ and $g \in \mathcal{U}$. Note that the neighborhood V of $J(f)$ can be taken backward invariant for every $g \in \mathcal{U}$.

LEMMA 3. *Given any $\delta > 0$ there exists a neighborhood $\mathcal{U}$ of f such that the set $\mathcal{N}_\delta(B(f); g)$ is empty for no $g \in \mathcal{U}$.*

Proof. Fix an admissible neighborhood $U(f)$ of $A(f)$. Then choose neighborhoods V of $J(f)$ and $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is λ-expanding in V. By Lemma 1 there exists a positive integer k such that $V \cup U_k(f) = M$.

If $\mathcal{U}$ is sufficiently small and ρ is a small positive constant one can obtain, for each fixed $g \in \mathcal{U}$, an $h \in \mathcal{N}_\rho(U(f), g)$; then, repeatedly applying Lemma 2(1), there exists an extension of h in $\mathcal{N}_\delta(U_k(f); g)$, again denoted by h.

Now, since V is backward invariant, the fundamental neighborhood $U_{k+1}(f) \setminus U_k(f)$ is contained in V, so that Lemma 2(2) gives an extension of h to $U_{k+1}(f)$, and this extension remains in the δ-neighborhood of the identity. By induction the homeomorphism h is extended to $\bigcup_{n>0} U_n(f) = B(f)$ to a conjugacy between $f_{|B(f)}$ and $g_{|B(g)}$ that is δ-close to the identity in $B(f)$. ∎

It remains to prove the second part of the theorem, which consists in extending h to the whole manifold.

Let ϵ be a constant of expansivity of the restriction of f to $J(f)$, that is, for every $z \neq w$ in $J(f)$, there exists $N \geq 0$ such that $d(f^N(z), f^N(w)) > \epsilon$. For every g in a neighborhood of f the same ϵ is a constant of expansivity for the restriction of g to $J(g)$.

By Lemma 3, one can choose $\mathcal{U}$ such that the distance between the identity and h is less than $\epsilon/2$, where $h : B(f) \to B(g)$ is a conjugacy between f and some fixed $g \in \mathcal{U}$. Let $x \in \partial B(f)$ and $\{x_n\}$ a sequence in $B(f)$ that converges to x. We claim that the sequence $\{h(x_n)\}$ converges. Otherwise, one can choose accumulation points $z \neq y$ of the set $\{h(x_n)\}$. By the choice of ϵ there exists $N \geq 0$ such that $d(g^N(y), g^N(z)) > \epsilon$. Then the sequence $\{hf^N(x_n) : n > 0\}$ accumulates at $g^N(y)$ and $g^N(z)$, but as $\{f^N(x_n)\}$ converges to $f^N(x)$, a contradiction appears because h is $\epsilon/2$-close to the identity. This proves the claim. Define h on the boundary of $B(f)$ as the limit of $\{h(x_n)\}$. The claim implies that h is continuous and surjective. Finally, h is injective because two points z and w with the same image would have $d(f^n(z), f^n(w))$ eventually greater than ϵ, while $h(f^n(z)) = h(f^n(w))$ for every $n > 0$. This extends h to the closure of $B(f)$, which equals the whole manifold unless $J(f)$ has nonempty interior, in which case the map is expanding and the stability already established by Shub.

3. Existence of examples. This section is devoted to the proof of Theorem 2. Let T be an expanding map of degree greater than one on a manifold M and assume that there exists an embedding J from M into S^n. Consider S^n as the one-point compactification of $\mathbb{R}^n$ and assume that JM is contained in the ball $B(0;1)$. To simplify notation we will also assume that J is the inclusion. Let $\alpha > 0$ be such that $T(x) = T(y)$ implies $x = y$ or $|Jx - Jy| > \alpha$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. For each $z \in M$ let $p_z : \mathbb{R}^n \to \mathbb{R}^n$ be given by $p_z(w) = aw + (1-a)z$, where $0 < a < 1$ is to be chosen. Note that p_z can be extended to a diffeomorphism of S^n having an attracting fixed point at z and a repelling fixed point at ∞. Define $f : M \times S^n \to M \times S^n$ by $f(z, w) = (T(z), p_z(w))$. Note that f is a locally invertible map with the same degree and class of differentiability as T.

We make the following choices: if $a < \alpha/(\alpha + 2)$, then $(1-a)\alpha/2a > 1$; take a number $r \in (1, (1-a)\alpha/2a)$, and define $U = M \times B(0; r)$.

Claim 1. *The closure of $f(U)$ is contained in U.*

Note that $p_z(B(0;r))$ is equal to the ball $B((1-a)z; ar)$, whose closure is contained in $B(0;r)$, because $r > 1$ implies that $(1-a)|z| + ar \le (1-a) + ar < r$. This implies the claim.

Claim 2. *The restriction of f to U is injective.*

Assume that $f(z, w) = f(z_1, w_1)$ with $z \ne z_1$, $|w| < r$ and $|w_1| < r$. This implies that $Tz = Tz_1$, so $|z - z_1| > \alpha$. Moreover, $p_z(w) = p_{z_1}(w_1)$ implies that $aw + (1-a)z = aw_1 + (1-a)z_1$. But this is impossible because

$$|a(w - w_1) + (1-a)(z - z_1)| \ge (1-a)\alpha - 2ar > 0$$

by the choice of r.

Claim 3. *The intersection Λ of the future iterates of U is a transitive hyperbolic attractor.*

This part of the construction is a trivial generalization of the solenoid attractor: the solenoid is obtained when M is the circle S^1, $T(z) = z^2$ and $n = 2$. Consider the inverse limit of T, that is, the set Σ of sequences $\underline{z} = \{z(m) : m \ge 0\}$ such that $T(z(m)) = z(m-1)$ for every $m > 1$, and endow it with the product topology. Given $z \in M$ let $U_z = \{z\} \times B(0;r)$. If $\underline{z} = \{z_m\} \in \Sigma$, note that $f^n(U_{z(m)})$ is a decreasing sequence of relatively compact sets whose diameters converge to 0, which implies that their closures intersect in a unique point, denoted $i(z(0))$. It is then easily seen that $i : \Sigma \to \Lambda$ is a homeomorphism realizing a conjugacy between the restriction of f to Λ and the shift σ given by $\sigma(\underline{z})(m) = T(z(m))$.

Claim 4. *The basin of attraction of Λ is equal to $M \times S^n \setminus M \times \{\infty\}$.*

Note that $|p_z(w)| \le a|w|+(1-a)$, but the function $x \in \mathbb{R} \mapsto ax+(1-a) \in \mathbb{R}$ has a fixed attractor at $x = 1$ that attracts every $x > 1$. It follows that for any $w \in S^n \setminus \{\infty\}$ such that $|w| > 1$, there exists a positive k such that $f^k(z, w) \in U$.

Claim 5. *f is Axiom A with $\Gamma(f) = \emptyset$.*

Note that $M \times \{\infty\}$ is an expanding basic piece. It follows that the nonwandering set of f is the union of Λ with this expanding set. By Claim 2, the restriction of f to Λ is injective. The claim and the theorem are proved.

4. Proof of Theorem 3. We first give a short description of the proof. The hypothesis on the attractor Λ implies that the restriction of f to B_0 (the immediate basin of Λ) is injective. Next it is assumed that $\Gamma(f)$ does not intersect the boundary of B_0. Applying Lemma 1 it follows that ∂B_0 is contained in an expanding set. We will show that this contradicts the hypotheses of the theorem.

Let Λ be a topologically simple attractor of a noninvertible Axiom A map f in a manifold of dimension at least two. The immediate basin of Λ is the union of the connected components of the basin that intersect Λ. Taking an iterate of f one can assume that the attractor Λ is connected; hence B_0 is connected and $f(B_0) = B_0$.

Let U be an admissible neighborhood of Λ such that every closed curve in U is homotopic to a closed curve in $f(U)$, with the homotopy contained in U. Define by induction an increasing sequence of open sets as follows: let $U_0 = U$ and U_n be the connected component of $f^{-1}(U_{n-1})$ that contains U_{n-1}. The first four claims give the proof that f is injective in B_0.

Claim 1. *The restriction of f to U_n is a covering map.*

The restriction is locally injective because f has no critical points. To prove that it is a covering map it suffices to show that it is proper. Let $\{x_k\}$ be a sequence in U_n converging to a point $x \notin U_n$. The sequence $\{f(x_k)\}$ converges to a point y in the closure of U_{n-1}. We have to prove that $y \notin U_{n-1}$. If $y \in U_{n-1}$, then there exists a ball B centered at x such that $f(B) \subset U_{n-1}$, which is absurd since $U_n \cup B$ is a connected set whose image is contained in U_{n-1} and strictly contains U_n.

Claim 2. *Every closed curve in U_n is homotopic to a closed curve contained in U_{n-1}.*

Indeed, given a closed curve γ contained in U_n let γ' be a closed curve in $f(U_0)$ that is homotopic to $f^n(\gamma)$. This implies that every f^n-lift of γ' is a closed curve contained in U_{n-1}, and one of them is homotopic to γ.

Claim 3. *There exists a map g defined in $\tilde{U} = \bigcup U_n$ such that $g(f(x)) = x$.*

Define $g : f(U_0) \to U_0$ as the inverse of $f_{|U_0}$. Assume g was extended until U_{n-1} and take any $x \in U_n$. If γ_i, $i = 1, 2$, are curves in U_n joining a point in Λ to x, then $\gamma_1\gamma_2^{-1}$ is a closed curve in U_n that has a homotopic curve γ' in U_{n-1}. As γ' has a closed lift under f, namely $g(\gamma')$, it follows that any f-lift of $\gamma_1\gamma_2^{-1}$ is closed, and one of them is homotopic to $g(\gamma')$. Therefore, the corresponding f-lifts of γ_1 and γ_2 have the same end point x', which must be sent to x by f. This allows us to define $g(x) = x'$, thus extending g to a diffeomorphism from U_n with the property $f(g(x)) = x$. Note also that $g(U_n) = U_{n+1}$.

Claim 4. *The restriction of f to $B_0(\Lambda)$ is injective.*

The above claim implies that f is injective on $\tilde{U}$. It remains to show that $\tilde{U} = B_0(\Lambda)$. Indeed, let $x \in B_0(\Lambda)$ and let α be a curve in $B_0(\Lambda)$ joining x to a point in Λ. There exists $K > 0$ such that $f^K(\alpha) \subset U_0$, but as U_K is the connected component of $f^{-K}(U_0)$ that contains U_0, we conclude that $\alpha \subset U_K$, whence $x \in U_K$.

This proves assertion (1) of the theorem. To prove (2) assume, by contradiction, that $\Gamma(f) \cap \partial B_0 = \emptyset$. This implies by Lemma 1 that the boundary of B_0 is an expanding set.

As ∂U_0 is compact, there exist open balls $C_1, \ldots, C_N$ covering ∂U_0 such that the closure of each C_j is contained in $B_0 \setminus \Lambda$.

Claim 5. *For each $1 \leq j \leq N$, the diameter of $g^n(C_j)$ converges to* 0 *as n tends to ∞.*

Take a neighborhood V of the boundary of B_0 such that f is expanding in V. Then the inverse g of f is a contraction defined in $V \cap B_0$. There exists k_0 such that $g^k(C_j)$ is contained in $V \cap B_0$ for every $k \geq k_0$ and every $1 \leq j \leq N$. This implies the claim.

Moreover, the boundary of U_k is contained in $\bigcup_{j=1}^N g^k(C_j)$. It follows that the boundary of U_k converges as $k \to \infty$ to a finite union of single points, which constitute the boundary of B_0. As M is a connected manifold of dimension greater than one, the closure of B_0 is the whole manifold, and f is a diffeomorphism, contradicting the assumptions.

Acknowledgments. Many colleagues, visiting our seminar, participated in different parts of this work. We want to specially thank Pablo Lessa, Rafael Potrie, Alfonso Artigue and Peter Haissinski.

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Received 20 June 2009;
in revised form 20 November 2009