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Some 2-point sets

by

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Abstract. Chad, Knight & Suabedissen [Fund. Math. 203 (2009)] recently proved, assuming CH, that there is a 2-point set included in the union of countably many concentric circles. This result is obtained here without any additional set-theoretic hypotheses.

A 2-*point set* is a subset of the plane $\mathbb{R}^2$ that meets every line at exactly 2 points. Mazurkiewicz [5] gave the first construction of a 2-point set. There have been other constructions of 2-point sets such as in [3] where it was shown that they exist in arbitrary vector spaces over arbitrary infinite fields. Recently, Chad, Knight & Suabedissen [1] proved that the Continuum Hypothesis implies that there is a 2-point set included in the union of countably many concentric circles. Their conclusion is even stronger, being precisely the statement of the Theorem below. Subsequent to [1], Miller [6] constructed models of ZFC in which the continuum is arbitrarily large and there is a 2-point set that is included in the union of ω_1 circles.

If $0 < r \in \mathbb{R}$, then $C_r = \{x \in \mathbb{R}^2 : \|x\| = r\}$ is the circle of radius r centered at the origin. We prove the following theorem without any additional set-theoretic hypotheses.

THEOREM. *Let $r_0, r_1, r_2, \ldots$ be a strictly increasing, unbounded sequence of positive real numbers. There is a 2-point set $M \subseteq \bigcup\{C_{r_i} : i < \omega\}$.*

Proof. By replacing the sequence $r_0, r_1, r_2, \ldots$ with one of its subsequences, we can assume that one of the following holds:

(A) $\{r_i : i < \omega\}$ is algebraically independent.
(B) Whenever $i, j < \omega$, then r_i is algebraic over r_j.

In the first part of this proof, Part I, we will give a proof of the Theorem assuming (A). In Part II, we modify Part I into a proof of the Theorem assuming (B).

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Part I: *Assume* (A). Let $\mathcal{S} = \{r_i : i < \omega\}$. Extend $\mathcal{S}$ to a transcendence basis $\mathcal{T} \supseteq \mathcal{S}$ for $\mathbb{R}$ over $\mathbb{Q}$. If $X \subseteq \mathbb{R}$, $n < \omega$ and $D \subseteq \mathbb{R}^n$, then we say that D is X-*definable* if it is definable in the ordered field $(\mathbb{R}, +, \cdot, 0, 1, \leq)$ by a first-order formula that involves parameters only from X. By the Tarski–Seidenberg Theorem (see [2]) on the elimination of quantifiers, if $a \in \mathbb{R}$, then $\{a\}$ is X-definable iff a is algebraic over X. If $D \subseteq \mathbb{R}^n$ is $\mathbb{R}$-definable, then $\mathrm{supp}(D)$ (the *support of* D) is the unique smallest subset $X \subseteq \mathcal{T}$ such that D is X-definable.

Let $\mathcal{L}$ be the set of all lines $\ell \subseteq \mathbb{R}^2$. For each $i < \omega$, define $L_i \subseteq \mathcal{L}$ so that $\ell \in L_i$ iff i is the least such that:

(1) $|\ell \cap C_{r_i}| = 2$,
(2) $\mathrm{supp}(\ell) \cap \mathcal{S} \subseteq \{r_0, r_1, \ldots, r_{i-1}\}$.

Clearly, each line is in exactly one L_i. Recursively define $M_{<i}$ and M_i so that $M_{<i} = M_0 \cup M_1 \cup \cdots \cup M_{i-1}$ and M_i is the set of those points $x \in \mathbb{R}^2$ such that there is a line $\ell \in L_i$ such that either:

(3) $\ell \cap M_{<i} = \emptyset$ and $x \in \ell \cap C_{r_i}$; or
(4) $\ell \cap M_{<i}$ is a singleton and x is the first point of $\ell \cap C_{r_i}$ in the lexicographic ordering of $\mathbb{R}^2$.

Clearly, $M_i \subseteq C_{r_i}$ for each $i < \omega$. Let $M = \bigcup\{M_i : i < \omega\}$. Obviously, $M \subseteq \bigcup\{C_{r_i} : i < \omega\}$. We will show that M is a 2-point set.

Consider an arbitrary line $\ell \in \mathcal{L}$, and let $i < \omega$ be such that $\ell \in L_i$. We want to show that $|\ell \cap M| = 2$. Clearly, (1), (3) and (4) imply that $|\ell \cap M_{<i}| + |\ell \cap M_i| \geq 2$, so $|\ell \cap M| \geq 2$.

It remains to show that $|\ell \cap M| \leq 2$. Suppose the contrary, and let $j < \omega$ be the least such that $|\ell \cap M_{\leq j}| \geq 3$. [*Notation*: $M_{\leq j} = M_{<j} \cup M_j$.]

Suppose $j = i$. It follows from (3) and (4) that in order for $|\ell \cap M_{\leq i}| \geq 3$, it must be that there are $w \in C_{r_i}$ and $\ell' \in L_i$ such that $\ell \cap \ell' = \{w\}$. But then r_i is $(\mathrm{supp}(\ell) \cup \mathrm{supp}(\ell'))$-definable, so $r_i \in \mathrm{supp}(\ell) \cup \mathrm{supp}(\ell')$, contradicting (2).

Thus, $j \neq i$ and, as in the previous paragraph, there are $w \in C_{r_j}$ and $\ell' \in L_j$ such that $\ell \cap \ell' = \{w\}$. Therefore, $r_j \in \mathrm{supp}(\ell) \cup \mathrm{supp}(\ell')$. Clearly, (2) implies that $j < i$ and $r_j \notin \mathrm{supp}(\ell')$, so $r_j \in \mathrm{supp}(\ell)$. Since $|\ell \cap M_{\leq j}| \geq 3$ and $|\ell \cap M_j| \leq 2$, it must be that $|\ell \cap M_{<j}| \in \{1, 2\}$.

First, suppose that $|\ell \cap M_{<j}| = 2$, and let $y, z \in \ell \cap M_{<j}$ be distinct. Then, since $r_j \in \mathrm{supp}(\ell)$, it follows that $r_j \in \mathrm{supp}(\{y, z\})$, so assume $r_j \in \mathrm{supp}(y)$. Let $k < j$ be such that $y \in M_k$. Then there is $\ell'' \in L_k$ such that $y \in \ell'' \cap C_{r_k}$, so $r_j \in \mathrm{supp}(\ell'') \cup \{r_k\}$, contradicting (2).

Second, suppose that $|\ell \cap M_{<j}| = 1$, and let $\ell \cap M_{<j} = \{x\}$. Then there is $k < j$ such that $x \in M_k$. Let $\ell \cap M_j = \{y, z\}$ and let $\ell', \ell'' \in L_j$ be such that $y \in \ell' \cap C_{r_j}$ and $z \in \ell'' \cap C_{r_j}$. It then follows from the following lemma that r_j is $\big(\mathrm{supp}(\ell') \cup \mathrm{supp}(\ell'') \cup \mathrm{supp}(x)\big)$-definable, contradicting (2) and thereby completing the proof assuming (A).

LEMMA 1. *Suppose that $\ell', \ell'' \in \mathcal{L}$ are distinct and $x \in \mathbb{R}^2 \backslash (\ell' \cup \ell'')$. Then there are at most finitely many $r > 0$ such that there are $y \in \ell' \cap C_r$ and $z \in \ell'' \cap C_r$ with x, y, z being collinear.*

Lemma 1 is Lemma 4.1 of [1]. As stated in [1], it says that there are at most 23 possible r; this does not seem to be the optimal number.

Before starting Part II of this proof, we prove a simple lemma.

LEMMA 2. *Let $\varphi(y_0, y_1, \ldots, y_{m-1}, u_0, u_1, \ldots, u_{n-1}, x)$ be a formula in the language of ordered fields, and let $a_0, a_1, \ldots, a_{m-1} \in \mathbb{R}$. Then there are only finitely many $b \in \mathbb{R}$ that are algebraic over $\{a_0, a_1, \ldots, a_{m-1}\}$ for which there are $t_0, t_1, \ldots, t_{n-1} \in \mathbb{R}$ that are algebraically independent over $\{a_0, a_1, \ldots, a_{m-1}\}$ such that $\varphi(\overline{a}, \overline{t}, x)$ defines b in $\mathbb{R}$.*

Proof. This lemma is a consequence of the o-minimality of $\mathbb{R}$ considered as an ordered field. (See [2, Chapter 2].) Let $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, be the $\{a_0, a_1, \ldots, a_{n-1}\}$-definable function such that whenever $d_0, d_1, \ldots, d_{m-1}, b \in \mathbb{R}$ are such that $\varphi(\overline{a}, \overline{d}, x)$ defines b, then $\overline{d} \in D$ and $f(\overline{d}) = b$. If $t_0, t_1, \ldots, t_{n-1} \in \mathbb{R}$ are algebraically independent over $\{a_0, a_1, \ldots, a_{m-1}\}$, $\overline{t} \in D$ and $f(\overline{t}) = b$ is algebraic over $\{a_0, a_1, \ldots, a_{m-1}\}$, then f is constantly b on some neighborhood U of $\overline{t}$. If $b_0, b_1, b_2, \ldots$ are infinitely many distinct such possibilities for b, then there are corresponding neighborhoods $U_0, U_1, U_2, \ldots \subseteq D$ such that f is constantly b_i on U_i. But this is impossible by o-minimality. ∎

PART II: *Assume* (B). If r_0 is algebraic, let $\mathcal{S} = \emptyset$, and if r_0 is transcendental, let $\mathcal{S} = \{r_0\}$. As in Part I, extend $\mathcal{S}$ to a transcendence basis $\mathcal{T}$. With Lemma 2 in mind, we make the following ad hoc definition. If $A \subseteq \{r_0, r_1, r_2, \ldots\}$ and $i < \omega$, then $\mathrm{Def}(A, i)$ is the set of all $\mathbb{R}$-definable $D \subseteq \mathbb{R}^n$, for some $n < \omega$, such that D is $(\mathcal{T} \cup A)$-definable by a formula having length at most i.

We now recursively get a subsequence $r_{k_0}, r_{k_1}, r_{k_2}, \ldots$ that is sufficiently fast growing. To be definitive, let $k_0 = 0$, and then let k_{i+1} be the least $k > k_i$ such that the following hold, where $A = \{r_j : j \leq k_i\}$:

(5) If $\ell, \ell' \in \mathcal{L} \cap \mathrm{Def}(A, i+1)$, $\ell \cap \ell' = \{w\}$, $r = \|w\|$ and r is algebraic over $\{r_0\}$, then $r_k > r$.

(6) [cf. Lemma 1] If $\ell', \ell_1, \ell_2 \in \mathcal{L} \cap \mathrm{Def}(A, i+1)$, $w \in \ell'$, $\|w\| \in A$, $w \notin \ell_1 \cup \ell_2$, $y \in \ell_1$, $z \in \ell_2$, w, y, z are collinear, $\|y\| = \|z\| = r$ and r is algebraic over r_0, then $r_k > r$.

Lemma 2 guarantees that k_{i+1} is well defined. For notational convenience and without loss of generality, we will assume that $r_{k_i} = r_i$ for all $i < \omega$.

For each $i < \omega$, define $L_i \subseteq \mathcal{L}$ so that $\ell \in L_i$ iff i is the least such that (1) and the following hold:

(7) $\ell \in \mathrm{Def}(\{r_0, r_1, \ldots, r_{i-1}\}, i)$.

Clearly, each line is in exactly one L_i. Notice that the set $\mathcal{L} \cap \mathrm{Def}(A, i+1)$ occurring in (5) and (6) is $L_0 \cup L_1 \cup \cdots \cup L_{i+1}$. Define $M_{<i}$ and M_i just as in (3) and (4) in Part I, and then define M the same way.

Again, $M_i \subseteq C_{r_i}$, so $M \subseteq \bigcup\{C_{r_i} : i < \omega\}$. We will show that M is a 2-point set by an argument that parallels the one in Part I.

Consider some $i < \omega$ and an arbitrary $\ell \in L_i$. As in Part I, $|\ell \cap M| \geq 2$, so it remains to show that $|\ell \cap M| \leq 2$. Suppose the contrary, and let $j < \omega$ be the least such that $|\ell \cap M_{\leq j}| \geq 3$.

Suppose $j = i$. It follows from (3) and (4) that in order for $|\ell \cap M_{\leq i}| \geq 3$, it must be that there are $w \in C_{r_i}$ and $\ell' \in L_i$ such that $\ell \cap \ell' = \{w\}$. But this contradicts (5).

Thus, $j \neq i$ and there are $w \in C_{r_j}$ and $\ell' \in L_j$ such that $\ell \cap \ell' = \{w\}$. So again by (5), it cannot be that $j > i$. Thus $j < i$, $w \in C_{r_j}$ and $|\ell \cap M_j| \leq 2$, so it must be that $|\ell \cap M_{<j}| \in \{1, 2\}$.

First, suppose that $|\ell \cap M_{<j}| = 2$, and let $\{y, z\} = \ell \cap M_{<j}$. Let $k_1, k_2 < j$, $\ell_1 \in L_{k_1}$ and $\ell_2 \in L_{k_2}$ be such that $y \in \ell_1 \cap C_{r_{k_1}}$ and $z \in \ell_2 \cap C_{r_{k_2}}$. Clearly, this contradicts (6).

Second, suppose that $|\ell \cap M_{<j}| = 1$, and let $\ell \cap M_{<j} = \{x\}$. Then, there is $k < j$ such that $x \in M_k$. Let $\ell \cap M_j = \{y, z\}$ and let $\ell_1, \ell_2 \in L_j$ be such that $\{y\} = \ell \cap \ell_1$ and $\{z\} = \ell \cap \ell_2$. Clearly, this contradicts (6), completing Part II and the proof of the Theorem. ∎

A long-standing open problem (see [4]) is whether there is a Borel 2-point set. Very closely related to this is the question: Can the existence of a 2-point set be proved in ZF (that is, ZFC with the Axiom of Choice deleted). In the absence of a positive answer to this last question, one can ask for weak consequences of AC that imply the existence of a 2-point set. The following is an example of such a consequence, although I am unable to say what its strength is relative to other consequences:

(⁎) *There is a real-closed subfield $\mathbb{F} \subseteq \mathbb{R}$ such that the transcendence degree of $\mathbb{R}$ over $\mathbb{F}$ is $\aleph_0$.*

The following is a consequence of Part I of the proof of the Theorem.

Corollary. (ZF) *If (⁎), then there is a 2-point set.*

Proof. Let $\mathbb{F}$ be as in (⁎). Let $\mathcal{S} = \{r_0, r_1, r_2, \ldots\}$ be a transcendence basis for $\mathbb{R}$ over $\mathbb{F}$ such that $r_0, r_1, r_2, \ldots$ is a strictly increasing, unbounded sequence of positive real numbers. In Part I of the proof of the Theorem, we would extend $\mathcal{S}$ to a transcendence basis $\mathcal{T} \supseteq \mathcal{S}$ for $\mathbb{R}$ over $\mathbb{Q}$. We see that the construction in the proof is independent of the actual choice of $\mathcal{T}$. In the absence of AC, it may be impossible to get any such $\mathcal{T}$; however, if we modify the definition of support so that $\mathrm{supp}(D)$ is $X \cup \mathbb{F}$, where X is the smallest subset $X \subseteq \mathcal{S}$ such that D is $(X \cup \mathbb{F})$-definable, then the proof still works. ∎

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