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A Lagrangian representation of tangles II

by

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Abstract. The present paper is a continuation of our previous paper [Topology 44 (2005), 747–767], where we extended the Burau representation to oriented tangles. We now study further properties of this construction.

1. Introduction. The Burau representation is a homomorphism from the group of braids on n strands to the group of $(n \times n)$-matrices over the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$. In our work [2], summarized below, we extended this representation to oriented tangles in $\mathbb{R}^3$. Since oriented tangles do not form a group, but a category, the result is a functor $\mathcal{F}$ from the category of oriented tangles to some algebraically defined category: the category of Hermitian modules and Lagrangian relations over Λ (see Section 2). For braids, this functor is equivalent to the Burau representation. For string links, it is equivalent to a construction of Le Dimet [5]. We refer to [6, 7] and references therein for related work on invariants of tangles.

In the present paper, we study further properties of the functor $\mathcal{F}$. The article is organized as follows. In Section 2, we recall the construction of the functor and the main results of [2]. In Section 3, we give a recursive method for the computation of the Lagrangian relation $\mathcal{F}(\tau)$ for any tangle τ with no closed components. In Section 4, we discuss connexions between these Lagrangian relations and the Alexander polynomial of the link obtained as the closure of the tangle. (These connexions are traditionally studied in this context, see e.g. [4, Section 6].) Finally, Section 5 deals with two families of examples: rational tangles and 2-strand tangles.

2. The functor Tangles $\to$ Lagr$_\Lambda$. This section consists of a summary of the main results of [2]. We refer to this article for the proofs and further details.

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2.1. *The category of Lagrangian relations.* Fix throughout this subsection an integral domain Λ (i.e., a commutative ring with unit and without zero-divisors) with ring involution $\Lambda \to \Lambda$, $\lambda \mapsto \widetilde{\lambda}$. A *non-degenerate skew-hermitian form* on a Λ-module H is a form $\omega\colon H \times H \to \Lambda$ such that:

(i) $\omega(\lambda x + \lambda' x', y) = \lambda\omega(x, y) + \lambda'\omega(x', y)$ for all $x, x', y \in H$ and all $\lambda, \lambda' \in \Lambda$;
(ii) $\omega(x, y) = -\widetilde{\omega(y, x)}$ for all $x, y \in H$;
(iii) if $\omega(x, y) = 0$ for all $y \in H$, then $x = 0$.

A *Hermitian Λ-module* is a finitely generated Λ-module H endowed with a non-degenerate skew-Hermitian form ω. The same module H with the opposite form $-\omega$ will be denoted by $-H$.

Hermitian Λ-modules are the objects of our Lagrangian category. To define the morphisms, we need the following preliminary definitions. For a submodule $A \subset H$, denote by $\mathrm{Ann}(A)$ the annihilator of A with respect to ω, that is, the module $\{x \in H \mid \omega(x, a) = 0 \text{ for all } a \in A\}$. We say that A is *Lagrangian* if $A = \mathrm{Ann}(A)$. Given a submodule A of H, set

$$\overline{A} = \{x \in H \mid \lambda x \in A \text{ for a non-zero } \lambda \in \Lambda\}.$$

Note that for any Lagrangian $A \subset H$, we have $A = \overline{A}$.

Let H_1, H_2 be Hermitian Λ-modules. A *Lagrangian relation* between H_1 and H_2 is a Lagrangian submodule of $(-H_1) \oplus H_2$ (the latter is a Hermitian Λ-module in the obvious way). For a Lagrangian relation $N \subset (-H_1) \oplus H_2$, we shall use the notation $N\colon H_1 \Rightarrow H_2$. Given a Hermitian Λ-module H, the submodule

$$\mathrm{diag}_H = \{h \oplus h \in (-H) \oplus H \mid h \in H\}$$

of $H \oplus H$ is clearly a Lagrangian relation $H \Rightarrow H$. It is called the *diagonal Lagrangian relation.* Given two Lagrangian relations $N_1\colon H_1 \Rightarrow H_2$ and $N_2\colon H_2 \Rightarrow H_3$, their composition is defined by $N_2 \circ N_1 = \overline{N_2 N_1}\colon H_1 \Rightarrow H_3$, where $N_2 N_1$ denotes the following submodule of $(-H_1) \oplus H_3$:

$$N_2 N_1 = \{h_1 \oplus h_3 \mid h_1 \oplus h_2 \in N_1 \text{ and } h_2 \oplus h_3 \in N_2 \text{ for a certain } h_2 \in H_2\}.$$

Theorem 2.1. *The Hermitian Λ-modules, as objects, and Lagrangian relations, as morphisms, form a category.*

We shall call this category the *category of Lagrangian relations over Λ.* It will be denoted by $\mathbf{Lagr}_\Lambda$. Lagrangian relations over Λ can be understood as a generalization of unitary Λ-isomorphisms and unitary Q-isomorphisms, where $Q = Q(\Lambda)$ is the field of fractions of Λ. More precisely, let $\mathbf{U}_\Lambda$ be the category of Hermitian Λ-modules and unitary Λ-isomorphisms. Also, let $\mathbf{U}^0_\Lambda$ be the category of Hermitian Λ-modules, where the morphisms between H_1 and H_2 are the unitary Q-isomorphisms between $H_1 \otimes_\Lambda Q$ and $H_2 \otimes_\Lambda Q$.

Finally, given such a unitary Q-isomorphism φ, set $\Gamma^0_\varphi = \{h \oplus \varphi(h) \mid h \in H_1,\, \varphi(h) \in H_2\} \subset H_1 \oplus H_2$.

THEOREM 2.2. *The maps $f \mapsto f \otimes \mathrm{id}_Q$ and $\varphi \mapsto \Gamma^0_\varphi$ define embeddings of categories*

$$\mathbf{U}_\Lambda \hookrightarrow \mathbf{U}^0_\Lambda \hookrightarrow \mathbf{Lagr}_\Lambda.$$

2.2. *The category of oriented tangles.* Let D^2 be the closed unit disk in $\mathbb{R}^2$. Given a positive integer n, denote by x_i the point $((2i-n-1)/n, 0)$ in D^2, for $i = 1, \dots, n$. Let ε and ε' be sequences of ± 1's of respective length n and n'. An $(\varepsilon, \varepsilon')$-*tangle* is the pair consisting of the cylinder $D^2 \times [0,1]$ and its oriented piecewise linear 1-submanifold τ whose oriented boundary $\partial\tau$ is $\sum_{j=1}^{n'} \varepsilon'_j (x'_j, 1) - \sum_{i=1}^{n} \varepsilon_i (x_i, 0)$. Note that for such a tangle to exist, we must have $\sum_i \varepsilon_i = \sum_j \varepsilon'_j$.

Two $(\varepsilon, \varepsilon')$-tangles $(D^2 \times [0,1], \tau_1)$ and $(D^2 \times [0,1], \tau_2)$ are *isotopic* if there exists an auto-homeomorphism h of $D^2 \times [0,1]$, keeping $D^2 \times \{0,1\}$ fixed, such that $h(\tau_1) = \tau_2$ and $h|_{\tau_1} \colon \tau_1 \simeq \tau_2$ is orientation-preserving. We shall denote by $T(\varepsilon, \varepsilon')$ the set of isotopy classes of $(\varepsilon, \varepsilon')$-tangles, and by id_ε the isotopy class of the trivial $(\varepsilon, \varepsilon)$-tangle $(D^2, \{x_1, \dots, x_n\}) \times [0,1]$.

Given an $(\varepsilon, \varepsilon')$-tangle τ_1 and an $(\varepsilon', \varepsilon'')$-tangle τ_2, their *composition* is the $(\varepsilon, \varepsilon'')$-tangle $\tau_2 \circ \tau_1$ obtained by gluing the two cylinders along the disk corresponding to ε' and shrinking the length of the resulting cylinder by a factor 2 (see Figure 1). Clearly, the composition of tangles induces a composition

$$T(\varepsilon, \varepsilon') \times T(\varepsilon', \varepsilon'') \to T(\varepsilon, \varepsilon'')$$

on the isotopy classes of tangles. The *category of oriented tangles* **Tangles** is defined as follows: the objects are the finite sequences ε of ± 1's, and the morphisms are given by $\mathrm{Hom}(\varepsilon, \varepsilon') = T(\varepsilon, \varepsilon')$. The composition is clearly associative, and the trivial tangle id_ε plays the role of the identity endomorphism of ε.

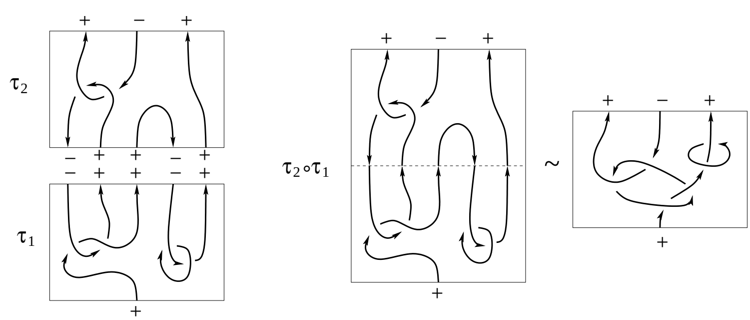

Fig. 1. A tangle composition

An $(\varepsilon, \varepsilon')$-tangle $\tau \subset D^2 \times [0,1]$ is called an *oriented braid* if every component of τ is strictly increasing or strictly decreasing with respect to the projection to $[0,1]$. The finite sequences of ± 1's, as objects, and the isotopy classes of oriented braids, as morphisms, form a subcategory **Braids** of the category of oriented tangles. Finally, an $(\varepsilon, \varepsilon')$-tangle $\tau \subset D^2 \times [0,1]$ is called an *oriented string link* if every component of τ joins $D^2 \times \{0\}$ and $D^2 \times \{1\}$. Oriented string links are the morphisms of a category **Strings** which satisfies

$$\mathbf{Braids} \subset \mathbf{Strings} \subset \mathbf{Tangles},$$

where all the inclusions denote embeddings of categories.

2.3. *The Lagrangian representation.* We denote by $\mathcal{N}(\{x_1, \dots, x_n\})$ an open tubular neighborhood of $\{x_1, \dots, x_n\}$ in $D^2 \subset \mathbb{R}^2$, and by S^2 the 2-sphere $\mathbb{R}^2 \cup \{\infty\}$. Given a sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ of ± 1's, let ℓ_ε be the sum $\sum_{i=1}^n \varepsilon_i$. We denote by D_ε the compact surface

$$D_\varepsilon = \begin{cases} D^2 \setminus \mathcal{N}(\{x_1, \dots, x_n\}) & \text{if } \ell_\varepsilon \neq 0, \\ S^2 \setminus \mathcal{N}(\{x_1, \dots, x_n\}) & \text{if } \ell_\varepsilon = 0, \end{cases}$$

endowed with the counterclockwise orientation, a base point z, and a generating family $\{e_1, \dots, e_n\}$ of $\pi_1(D_\varepsilon, z)$, where e_i is a simple loop turning once around x_i counterclockwise if $\varepsilon_i = +1$, clockwise if $\varepsilon_i = -1$. The same space with the clockwise orientation will be denoted by $-D_\varepsilon$.

The natural epimorphism $\pi_1(D_\varepsilon) \to \mathbb{Z}$, $e_i \mapsto 1$, gives an infinite cyclic covering $\widehat{D}_\varepsilon \to D_\varepsilon$. Choosing a generator t of the group of the covering transformations endows the homology $H_1(\widehat{D}_\varepsilon)$ with a structure of module over $\Lambda = \mathbb{Z}[t, t^{-1}]$. If $\ell_\varepsilon \neq 0$, then D_ε retracts by deformation on the wedge of n circles representing $e_1, \dots, e_n$, and one easily checks that $H_1(\widehat{D}_\varepsilon)$ is a free Λ-module with basis $v_1 = \widehat{e}_1 - \widehat{e}_2, \dots, v_{n-1} = \widehat{e}_{n-1} - \widehat{e}_n$, where $\widehat{e}_i$ is the path in $\widehat{D}_\varepsilon$ lifting e_i starting at some fixed lift $\widehat{z} \in \widehat{D}_\varepsilon$ of z. If $\ell_\varepsilon = 0$, then $H_1(\widehat{D}_\varepsilon) = \bigoplus_i \Lambda v_i / \Lambda \widehat{\gamma}$, where $\widehat{\gamma}$ is a lift of $\gamma = e_1^{\varepsilon_1} \cdots e_n^{\varepsilon_n}$ to $\widehat{D}_\varepsilon$. Note that in any case, $H_1(\widehat{D}_\varepsilon)$ is a free Λ-module.

Let $\langle\ ,\ \rangle \colon H_1(\widehat{D}_\varepsilon) \times H_1(\widehat{D}_\varepsilon) \to \mathbb{Z}$ be the ($\mathbb{Z}$-bilinear, skew-symmetric) intersection form induced by the orientation of D_ε lifted to $\widehat{D}_\varepsilon$. Consider the pairing $\omega_\varepsilon \colon H_1(\widehat{D}_\varepsilon) \times H_1(\widehat{D}_\varepsilon) \to \Lambda$ given by

$$\omega_\varepsilon(x, y) = \sum_k \langle t^k x, y \rangle t^{-k}.$$

It turns out that ω_ε is a non-degenerate skew-Hermitian form with respect to the involution $\Lambda \to \Lambda$ induced by $t \mapsto t^{-1}$. Therefore, $(H_1(\widehat{D}_\varepsilon), \omega_\varepsilon)$ is a Hermitian Λ-module.

Given an $(\varepsilon, \varepsilon')$-tangle $\tau = \tau_1 \cup \cdots \cup \tau_\mu \subset D^2 \times [0,1]$, denote by $\mathcal{N}(\tau)$ an open tubular neighborhood of τ and by X_τ its exterior

$$X_\tau = \begin{cases} (D^2 \times [0,1]) \setminus \mathcal{N}(\tau) & \text{if } \ell_\varepsilon \neq 0, \\ (S^2 \times [0,1]) \setminus \mathcal{N}(\tau) & \text{if } \ell_\varepsilon = 0. \end{cases}$$

Note that $\ell_\varepsilon = \ell_{\varepsilon'}$. We shall orient X_τ so that the induced orientation on ∂X_τ extends the orientation on $(-D_\varepsilon) \sqcup D_{\varepsilon'}$. If $\ell_\varepsilon \neq 0$, we have $H_1(X_\tau) = \bigoplus_{j=1}^{\mu} \mathbb{Z} m_j$, where m_j is a meridian of τ_j oriented so that its linking number with τ_j is 1. If $\ell_\varepsilon = 0$, then $H_1(X_\tau) = \bigoplus_{j=1}^{\mu} \mathbb{Z} m_j / \sum_{i=1}^{n} \varepsilon_i e_i$. The composition of the Hurewicz homomorphism and the homomorphism $H_1(X_\tau) \to \mathbb{Z}$, $m_j \mapsto 1$, gives an epimorphism $\pi_1(X_\tau) \to \mathbb{Z}$ which extends the previously defined homomorphisms $\pi_1(D_\varepsilon) \to \mathbb{Z}$ and $\pi_1(D_{\varepsilon'}) \to \mathbb{Z}$. As before, it determines an infinite cyclic covering $\widehat{X}_\tau \to X_\tau$, so the homology of $\widehat{X}_\tau$ is endowed with a natural structure of module over $\Lambda = \mathbb{Z}[t, t^{-1}]$.

Let $i_\tau\colon H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{X}_\tau)$ and $i'_\tau\colon H_1(\widehat{D}_{\varepsilon'}) \to H_1(\widehat{X}_\tau)$ be the homomorphisms induced by the obvious inclusion $\widehat{D}_\varepsilon \sqcup \widehat{D}_{\varepsilon'} \subset \widehat{X}_\tau$. Denote by j_τ the homomorphism $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \to H_1(\widehat{X}_\tau)$ given by $j_\tau(x, x') = i'_\tau(x') - i_\tau(x)$. Finally, set

$$K(\tau) = \ker(j_\tau) \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}).$$

It is proved in [2] that for any tangle τ, the module $\overline{K(\tau)}$ is Lagrangian. It can also be checked that $K(\tau_2 \circ \tau_1) = K(\tau_2) K(\tau_1)$ for any tangles τ_1, τ_2. This leads to the following result.

THEOREM 2.3. *Given a sequence ε of ± 1's, denote by $\mathcal{F}(\varepsilon)$ the Hermitian Λ-module $(H_1(\widehat{D}_\varepsilon), \omega_\varepsilon)$. For $\tau \in T(\varepsilon, \varepsilon')$, let $\mathcal{F}(\tau)$ be the module $\overline{K(\tau)} \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})$. Then $\mathcal{F}$ is a functor* **Tangles** $\to$ $\mathbf{Lagr}_\Lambda$ *which fits in the commutative diagram*

$$\begin{tikzcd} \mathbf{Braids} \arrow[r, hook] \arrow[d] & \mathbf{Strings} \arrow[r, hook] \arrow[d] & \mathbf{Tangles} \arrow[d, "\mathcal{F}"] \\ \mathbf{U}_\Lambda \arrow[r, hook] & \mathbf{U}^0_\Lambda \arrow[r, hook] & \mathbf{Lagr}_\Lambda \end{tikzcd}$$

where the horizontal arrows are the embeddings of categories given in Subections 2.1 *and* 2.2.

COROLLARY 2.4. *Let $\beta \in T(\varepsilon, \varepsilon')$ be an oriented braid. Then there exists a unitary Λ-isomorphism $f_\beta\colon H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{D}_{\varepsilon'})$ such that $\mathcal{F}(\beta) = K(\beta)$ is the graph of f_β.*

3. The module $K(\tau)$. Clearly, any tangle τ can be written as a composition of the *elementary tangles* described in Figure 2, where the orientation of the strands is determined by the signs ε and ε'. We now use this result to study the freeness of $K(\tau)$, and to compute this module recursively.

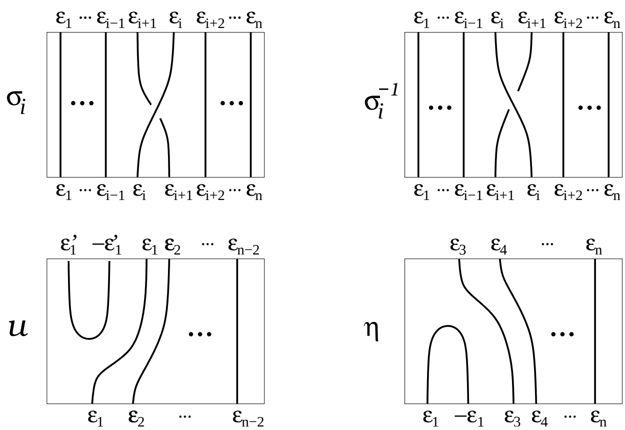

Fig. 2. The elementary tangles

3.1. *Freeness of $K(\tau)$.* In this subsection, we deal with the following question: Given a tangle $\tau \in T(\varepsilon,\varepsilon')$, is the module $K(\tau)$ free? Clearly, this module is contained in the free module $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})$. But since the ring $\Lambda = \mathbb{Z}[t,t^{-1}]$ is not a principal ideal domain, this is not sufficient to conclude that $K(\tau)$ is free. Nevertheless, we have the following result. Let us say that a tangle $\tau \in T(\varepsilon,\varepsilon')$ is *straight* if it has no closed components, and if at least one strand of τ joins D_ε with $D_{\varepsilon'}$.

Proposition 3.1. *If τ is a straight tangle, then the Λ-module $K(\tau)$ is free.*

We shall need the following lemma (see [2] for the proof).

Lemma 3.2. *Consider an exact sequence of Λ-modules $0 \to K \to P \to F$, where P and F are free Λ-modules. Then K is free.*

Lemma 3.3. *Let H, H' and H'' be finitely generated free Λ-modules. Consider free submodules $N_1 \subset H \oplus H'$ and $N_2 \subset H' \oplus H''$ such that $(N_1 \oplus N_2) \cap (0 \oplus \mathrm{diag}_{H'} \oplus 0) = 0$. Then N_2N_1 is a free submodule of $H \oplus H''$.*

Proof. Denote by f_1 (resp. f_1') the homomorphism $N_1 \subset H \oplus H' \xrightarrow{\pi} H$ (resp. $N_1 \subset H \oplus H' \xrightarrow{\pi'} H'$), where π and π' are the canonical projections. Similarly, denote by f_2' and f_2'' the homomorphisms $N_2 \subset H' \oplus H'' \to H'$ and $N_2 \subset H' \oplus H'' \to H''$. Let K be the kernel of $(-f_1') \oplus f_2'\colon N_1 \oplus N_2 \to H'$.

Our assumptions and Lemma 3.2 imply that K is free. We have an exact sequence

$$0 \to (N_1 \oplus N_2) \cap (0 \oplus \mathrm{diag}_{H'} \oplus 0) \to K \xrightarrow{f_1 \oplus f_2''} N_2 N_1 \to 0.$$

Therefore, if $(N_1 \oplus N_2) \cap (0 \oplus \mathrm{diag}_{H'} \oplus 0) = 0$, then $N_2 N_1 = K$ is free. ∎

LEMMA 3.4. *Consider tangles $\tau_1 \in T(\varepsilon, \varepsilon')$ and $\tau_2 \in T(\varepsilon', \varepsilon'')$ such that $\tau_2 \circ \tau_1$ is straight. Then*

$$(K(\tau_1) \oplus K(\tau_2)) \cap (0 \oplus \mathrm{diag}_{H_1(\widehat{D}_{\varepsilon'})} \oplus 0) = 0.$$

Proof. Denote by τ the tangle $\tau_2 \circ \tau_1$. We claim that $H_2(X_\tau) = 0$. Assume first that $\ell_\varepsilon \neq 0$. By excision,

$$H_2(X_\tau) = H_3(D^2 \times [0,1], X_\tau) = H_3(\tau \times D^2, \tau \times S^1) = 0$$

since τ has no closed components. If $\ell_\varepsilon = 0$, consider the Mayer–Vietoris exact sequence associated with the decomposition $X_\tau = ((D^2 \times [0,1]) \setminus \mathcal{N}(\tau)) \cup (D^2 \times [0,1])$:

$$0 \to H_2(X_\tau) \to \mathbb{Z}\gamma \xrightarrow{i} H_1((D^2 \times [0,1]) \setminus \mathcal{N}(\tau)),$$

where γ is a 1-cycle parametrizing ∂D^2. Since one strand of τ joins D_ε to $D_{\varepsilon''}$, we have $i(\gamma) \neq 0 \in H_1((D^2 \times [0,1]) \setminus \mathcal{N}(\tau)) = \mathbb{Z}^\mu$, where μ is the number of components of τ. Therefore, i is injective, so $H_2(X_\tau) = 0$ and the claim is proved.

Since X_τ has the homotopy type of a 2-dimensional CW-complex and $H_2(X_\tau) = 0$, we have $H_2(\widehat{X}_\tau) = 0$. The decomposition $X_\tau = X_{\tau_1} \cup X_{\tau_2}$ gives the Mayer–Vietoris exact sequence

$$H_2(\widehat{X}_\tau) = 0 \to H_1(\widehat{D}_{\varepsilon'}) \xrightarrow{j} H_1(\widehat{X}_{\tau_1}) \oplus H_1(\widehat{X}_{\tau_2}).$$

Therefore,

$$\begin{aligned} 0 = \ker(j) &= \{x \in H_1(\widehat{D}_{\varepsilon'}) \mid j_{\tau_1}(0 \oplus x) = j_{\tau_2}(x \oplus 0) = 0\} \\ &\cong (\ker(j_{\tau_1}) \oplus \ker(j_{\tau_2})) \cap (0 \oplus \mathrm{diag}_{H_1(\widehat{D}_{\varepsilon'})} \oplus 0) \end{aligned}$$

and the lemma is proved. ∎

LEMMA 3.5. *Let τ be an elementary tangle, as described in Figure 2. Then $K(\tau)$ is a free Λ-module.*

Proof. Note that X_τ has the homotopy type of a 1-dimensional connected CW-complex Y_τ (unless τ is one of the 1-strand tangles u and η, in which case $K(\tau) = 0$). Therefore, $H_1(\widehat{X}_\tau)$ is the kernel of $\partial\colon C_1(\widehat{Y}_\tau) \to C_0(\widehat{Y}_\tau)$. Since the latter two modules are free, Lemma 3.2 implies that $H_1(\widehat{X}_\tau)$ is free. Now, consider the exact sequence

$$0 \to K(\tau) \hookrightarrow H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \xrightarrow{j_\tau} H_1(\widehat{X}_\tau).$$

Since $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})$ and $H_1(\widehat{X}_\tau)$ are free, the conclusion follows from Lemma 3.2. ■

Proof of Proposition 3.1. Any tangle τ can be written as a composition of the elementary tangles given in Figure 2. Since $K(\tau_2 \circ \tau_1) = K(\tau_2)K(\tau_1)$, the result follows from Lemmas 3.3–3.5. ■

Recall that for a Λ-module K, its *rank* $\mathrm{rk}_\Lambda K$ is defined by $\mathrm{rk}_\Lambda K = \dim_Q(K \otimes_\Lambda Q)$, where $Q = Q(\Lambda)$ is the field of fractions of Λ.

Proposition 3.6. *Consider $\tau \in T(\varepsilon, \varepsilon')$ with ε of length n and ε' of length n'. Then the rank of $K(\tau)$ is given by*

$$\mathrm{rk}_\Lambda K(\tau) = \begin{cases} 0 & \text{if } n = n' = 0, \\ (n+n')/2 - 1 & \text{if } \ell_\varepsilon \neq 0 \text{ or } nn' = 0 \text{ and } (n, n') \neq (0,0), \\ (n+n')/2 - 2 & \text{if } \ell_\varepsilon = 0 \text{ and } nn' > 0. \end{cases}$$

Proof. Since $K(\tau)$ is a Lagrangian submodule of $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})$, we have $\mathrm{rk}_\Lambda K(\tau) = \frac{1}{2}\mathrm{rk}_\Lambda(H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}))$. If ε has length n, we know that

$$\mathrm{rk}_\Lambda H_1(\widehat{D}_\varepsilon) = \begin{cases} 0 & \text{if } n = 0, \\ n - 1 & \text{if } \ell_\varepsilon \neq 0, \\ n - 2 & \text{if } \ell_\varepsilon = 0 \text{ and } n > 0. \end{cases}$$

The result follows. ■

3.2. *Recursive computation of $K(\tau)$.* Throughout this subsection, let I_k denote the identity $(k \times k)$-matrix. Consider two finitely generated free Λ-modules H and H' with fixed bases. A homomorphism of Λ-modules $f\colon H \to H'$ is canonically described by its matrix M_f, and the composition of homomorphisms corresponds to the product of matrices. What about morphisms in the Lagrangian category? A free submodule N of $H \oplus H'$ is determined by a matrix of the inclusion $N \subset H \oplus H'$ with respect to a basis of N. We will say that $N \subset H \oplus H'$ is *encoded* by this matrix. For example, the graph of an isomorphism $f\colon H \to H'$ is encoded by the matrix $\binom{I}{M_f}$.

Let H, H', H'' be finitely generated free Λ-modules with fixed basis. Consider free submodules $N_1 \subset H \oplus H'$ and $N_2 \subset H' \oplus H''$. A choice of a basis for N_1 and N_2 determines matrices $\binom{M_1}{M_1'}$ and $\binom{M_2'}{M_2''}$ of the inclusions $N_1 \subset H \oplus H'$ and $N_2 \subset H' \oplus H''$. By Lemma 3.3, if $(N_1 \oplus N_2) \cap (0 \oplus \mathrm{diag}_{H'} \oplus 0) = 0$, then $N_2 N_1$ is free. A natural question is: how can we compute a matrix of the inclusion $N_2 N_1 \subset H \oplus H'$ from the matrices $\binom{M_1}{M_1'}$ and $\binom{M_2'}{M_2''}$?

Lemma 3.7. *If $(N_1 \oplus N_2) \cap (0 \oplus \mathrm{diag}_{H'} \oplus 0) = 0$, then the inclusion of $N_2 N_1$ in $H \oplus H''$ is encoded by the matrix $\binom{M_1 W_1}{M_2'' W_2}$, where $\binom{W_1}{W_2}$ is a matrix of the inclusion of $K = \{x \in N_1 \oplus N_2 \mid (-M_1', M_2') \cdot x = 0\}$ in $N_1 \oplus N_2$.*

Proof. We will assume the notation of the proof of Lemma 3.3. By definition, M_1, M_1', M_2' and M_2'' are the matrices of f_1, f_1', f_2' and f_2'' with respect to the bases of N_1, N_2, H, H' and H''. Furthermore, we saw in the proof of Lemma 3.3 that $K = \ker((-f_1') \oplus f_2')$ is free. Let $\binom{W_1}{W_2}$ be a matrix of the inclusion $K \subset N_1 \oplus N_2$ with respect to a basis of K and the fixed basis of $N_1 \oplus N_2$. By definition, $N_2N_1 = (f_1 \oplus f_2'')(K)$. Clearly, $\ker(f_1 \oplus f_2'') \cap K = (N_1 \oplus N_2) \cap (0 \oplus \mathrm{diag}_{H'} \oplus 0)$. Since the latter module is assumed to be trivial, $f_1 \oplus f_2''$ restricted to K gives an isomorphism onto N_2N_1. The lemma follows. ∎

Lemma 3.7 gives the following recursive method for the computation of $K(\tau)$, where τ is a straight tangle.

Proposition 3.8. *Let $\tau_1 \in T(\varepsilon, \varepsilon')$ and $\tau_2 \in T(\varepsilon', \varepsilon'')$ be tangles such that $\tau_2 \circ \tau_1$ is straight. Then $K(\tau_1)$, $K(\tau_2)$ and $K(\tau_2 \circ \tau_1)$ are free. Furthermore, if the inclusions $K(\tau_1) \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})$ and $K(\tau_2) \subset H_1(\widehat{D}_{\varepsilon'}) \oplus H_1(\widehat{D}_{\varepsilon''})$ are encoded by matrices $\binom{M_1}{M_1'}$ and $\binom{M_2'}{M_2''}$, then $K(\tau_2 \circ \tau_1) \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon''})$ is encoded by the matrix $\binom{M_1W_1}{M_2''W_2}$, where $\binom{W_1}{W_2}$ is a matrix of the inclusion $\{x \in K(\tau_1) \oplus K(\tau_2) \mid (-M_1', M_2') \cdot x = 0\} \subset K(\tau_1) \oplus K(\tau_2)$.*

Therefore, the computation of $K(\tau)$ for any straight tangle τ boils down to the computation of this module for the elementary tangles u, η, σ_i and σ_i^{-1}. Let us state the result and refer to [2] for the easy proof.

Proposition 3.9. *Let $\tau \in T(\varepsilon, \varepsilon')$ be an elementary tangle with $\ell_\varepsilon \neq 0$. The inclusion $K(\tau) \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})$ with respect to the canonical basis of $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})$ is encoded by the matrix $\binom{M}{M'}$, where*

- *$M = (0 \;\; I_{n-3})$ and $M' = \binom{1}{0} \oplus I_{n-3}$ if $\tau = u$;*
- *$M = \binom{1}{0} \oplus I_{n-3}$ and $M' = (0 \;\; I_{n-3})$ if $\tau = \eta$;*
- *$M = I_{n-1}$ and $M' = M_{f_i}^\varepsilon$ if $\tau = \sigma_i^\varepsilon$, for $\varepsilon = \pm 1$ and $1 \leq i \leq n-1$, with*

$$M_{f_1} = \begin{pmatrix} -t^{\varepsilon_2} & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3}, \qquad M_{f_{n-1}} = I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ t^{\varepsilon_n} & -t^{\varepsilon_n} \end{pmatrix},$$

$$M_{f_i} = I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t^{\varepsilon_{i+1}} & -t^{\varepsilon_{i+1}} & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2} \quad \text{for } 2 \leq i \leq n-2.$$

4. The Alexander polynomial. Let $\tau \subset D^2 \times [0,1]$ be an $(\varepsilon, \varepsilon)$-tangle, with ε of length n. The *closure of* τ is the oriented link $\widehat{\tau} \subset S^3$ obtained from τ by adding n oriented parallel strands in $S^3 \setminus (D^2 \times [0,1])$ as indicated in Figure 3. The orientation of these strands is determined by ε in order to obtain a well defined oriented link $\widehat{\tau}$. In this section, we show

how the Alexander polynomial $\Delta_{\widehat{\tau}}$ of $\widehat{\tau}$ is related to the module $K(\tau) \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_\varepsilon)$.

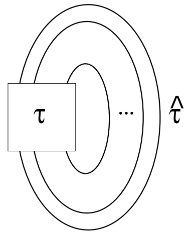

Fig. 3. The closure $\widehat{\tau}$ of an oriented tangle τ

4.1. *Basics.* Let Λ be a unique factorization domain. Consider a finite presentation $\Lambda^r \xrightarrow{f} \Lambda^g \to M \to 0$ of a Λ-module M. We will denote by $\Delta(M)$ the greatest common divisor of the $(g \times g)$-minors of the matrix of f. It is well known that, up to multiplication by units of Λ, the element $\Delta(M)$ of Λ only depends on the isomorphism class of M. Furthermore, if $0 \to A \to B \to C \to 0$ is an exact sequence of Λ-modules, then $\Delta(B) \doteq \Delta(A)\Delta(C)$, where $\doteq$ denotes equality up to multiplication by units of Λ.

We briefly recall the definition of the 1-variable Alexander polynomial of an oriented link $L \subset S^3$. Denote by X_L the exterior of L in S^3, and consider the epimorphism $\pi_1(X_L) \to \mathbb{Z}$ given by the total linking number with L. It induces an infinite cyclic covering $\widehat{X}_L \to X_L$. The $\mathbb{Z}[t, t^{-1}]$-module $H_1(\widehat{X}_L)$ is called the *Alexander module* of L and the Laurent polynomial $\Delta_L(t) = \Delta(H_1(\widehat{X}_L))$ is the *Alexander polynomial* of L. It is defined up to multiplication by $\pm t^\nu$, with $\nu \in \mathbb{Z}$.

4.2. *A factorization of the Alexander polynomial.* Throughout this subsection we use the notation of Section 2.

Lemma 4.1. *For $\tau \in T(\varepsilon, \varepsilon)$ with $\ell_\varepsilon \neq 0$,*

$$(t^{\ell_\varepsilon} - 1)\Delta_{\widehat{\tau}}(t) \doteq (t-1)\Delta(A),$$

where A is the cokernel of $i'_\tau - i_\tau\colon H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{X}_\tau)$.

Proof. Consider the compact manifold Y_τ obtained by pasting X_τ and $X_{\mathrm{id}_\varepsilon}$ along $D_\varepsilon \sqcup D_\varepsilon$. The epimorphisms $\pi_1(X_\tau) \to \mathbb{Z}$ and $\pi_1(X_{\mathrm{id}_\varepsilon}) \to \mathbb{Z}$ extend to an epimorphism $\pi_1(Y_\tau) \to \mathbb{Z}$, which defines a $\mathbb{Z}$-covering $\widehat{Y}_\tau \to Y_\tau$. Hence, we have the Mayer–Vietoris exact sequence

$$H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_\varepsilon) \xrightarrow{\alpha_1} H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{X}_\tau) \xrightarrow{\beta} H_1(\widehat{Y}_\tau) \xrightarrow{\partial} H_0(\widehat{D}_\varepsilon) \oplus H_0(\widehat{D}_\varepsilon)$$
$$\xrightarrow{\alpha_0} H_0(\widehat{D}_\varepsilon) \oplus H_0(\widehat{X}_\tau),$$

where $\alpha_1(x,y) = (x+y, i_\tau(x)+i'_\tau(y))$. Since $H_0(\widehat{D}_\varepsilon) = H_0(\widehat{X}_\tau) = \Lambda/(t-1)$, the module $\mathrm{Im}(\partial) = \ker(\alpha_0)$ is equal to $\Lambda/(t-1)$. This and the equality $A = \mathrm{Im}(\beta)$ lead to the exact sequence

$$0 \to A \hookrightarrow H_1(\widehat{Y}_\tau) \to \Lambda/(t-1) \to 0.$$

Hence, $\Delta(H_1(\widehat{Y}_\tau)) \doteq (t-1)\Delta(A)$.

Clearly, $X_{\widehat{\tau}}$ is the union of Y_τ and $D^2 \times S^1$ along a torus $T \subset \partial Y_\tau$. The epimorphism $\pi_1(X_{\widehat{\tau}}) \to \mathbb{Z}$ given by the total linking number with $\widehat{\tau}$ extends the previously defined epimorphism $\pi_1(Y_\tau) \to \mathbb{Z}$. Therefore, the exact sequence of the pair $(\widehat{X}_{\widehat{\tau}}, \widehat{Y}_\tau)$ gives

$$0 \to H_2(\widehat{Y}_\tau) \to H_2(\widehat{X}_{\widehat{\tau}}) \to \Lambda/(t^{\ell_\varepsilon} - 1) \to H_1(\widehat{Y}_\tau) \to H_1(\widehat{X}_{\widehat{\tau}}) \to 0.$$

Note that both $H_2(\widehat{Y}_\tau)$ and $H_2(\widehat{X}_{\widehat{\tau}})$ are free Λ-modules. (This follows from the fact that $X_{\widehat{\tau}}$ and Y_τ have the homotopy type of a 2-dimensional *CW*-complex, and from Lemma 3.2.) If $H_2(\widehat{X}_{\widehat{\tau}}) = 0$, then we have $\Delta(H_1(\widehat{Y}_\tau)) \doteq (t^{\ell_\varepsilon} - 1)\Delta(H_1(\widehat{X}_{\widehat{\tau}})) = (t^{\ell_\varepsilon} - 1)\Delta_{\widehat{\tau}}(t)$ and the lemma holds. If $H_2(\widehat{X}_{\widehat{\tau}}) \neq 0$, then $H_2(\widehat{Y}_\tau) \neq 0$ so both modules have positive rank. By an Euler characteristic argument, the rank of $H_1(\widehat{X}_{\widehat{\tau}})$ and $H_1(\widehat{Y}_\tau)$ is also positive. Therefore, $\Delta(H_1(\widehat{X}_{\widehat{\tau}})) = \Delta(H_1(\widehat{Y}_\tau)) = 0$, and the lemma is proved. ∎

THEOREM 4.2. *Let $\tau \in T(\varepsilon, \varepsilon)$ be a tangle with $\ell_\varepsilon \neq 0$, such that $K(\tau)$ is free. Then*

$$\frac{t^{\ell_\varepsilon} - 1}{t-1}\,\Delta_{\widehat{\tau}}(t) \doteq \det(M' - M)\,\Delta(\mathrm{coker}(j_\tau)),$$

where $\binom{M}{M'}$ is a matrix of the inclusion $K(\tau) \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_\varepsilon)$.

Proof. Since $K(\tau) = \ker(j_\tau)$, we have the exact sequence

$$0 \to K(\tau) \hookrightarrow H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_\varepsilon) \xrightarrow{j_\tau} H_1(\widehat{X}_\tau) \xrightarrow{\pi} \mathrm{coker}(j_\tau) \to 0.$$

The module A defined by the exact sequence $H_1(\widehat{D}_\varepsilon) \xrightarrow{i'_\tau - i_\tau} H_1(\widehat{X}_\tau) \xrightarrow{p} A \to 0$ fits in the sequence

$$K(\tau) \xrightarrow{\alpha} H_1(\widehat{D}_\varepsilon) \xrightarrow{\beta} A \xrightarrow{\gamma} \mathrm{coker}(j_\tau) \to 0,$$

where $\alpha(x,y) = y - x$ for $x, y \in H_1(\widehat{D}_\varepsilon)$, $\beta = p \circ i_\tau = p \circ i'_\tau$, and $\gamma(\zeta) = \pi(z)$ for $\zeta = p(z) \in A$, $z \in H_1(\widehat{X}_\tau)$. We leave to the reader the proof that this sequence is exact. It then splits into two exact sequences

$$K(\tau) \xrightarrow{\alpha} H_1(\widehat{D}_\varepsilon) \xrightarrow{\beta} \mathrm{Im}(\beta) \to 0, \qquad 0 \to \mathrm{Im}(\beta) \hookrightarrow A \to \mathrm{coker}(j_\tau) \to 0.$$

The latter sequence implies that $\Delta(A) \doteq \Delta(\mathrm{coker}(j_\tau))\Delta(\mathrm{Im}(\beta))$. By Lemma 4.1, we get $(t^{\ell_\varepsilon} - 1)\Delta_{\widehat{\tau}}(t) \doteq (t-1)\Delta(\mathrm{Im}(\beta))\Delta(\mathrm{coker}(j_\tau))$. The former sequence is just a finite presentation of the module $\mathrm{Im}(\beta)$. Furthermore, if a matrix of the inclusion $K(\tau) \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_\varepsilon)$ is given by $\binom{M}{M'}$, then a matrix of α is given by $M' - M$. Since $K(\tau)$ is a Lagrangian submodule of $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_\varepsilon)$, its rank is equal to the rank of $H_1(\widehat{D}_\varepsilon)$. Therefore, M and M' are square matrices and $\Delta(\mathrm{Im}(\beta)) \doteq \det(M' - M)$. ∎

We have the following generalization of [1, Theorem 3.11]. (There, all the strands of the braid must be oriented in the same direction.)

Corollary 4.3. *If $\beta \in T(\varepsilon, \varepsilon)$ is an oriented braid with $\ell_\varepsilon \neq 0$, then*

$$\frac{t^{\ell_\varepsilon} - 1}{t - 1}\, \Delta_{\widehat{\beta}}(t) \doteq \det(M_{f_\beta} - I),$$

where M_{f_β} is a matrix of $f_\beta \colon H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{D}_\varepsilon)$ (cf. Corollary 2.4) and I is the identity matrix.

Proof. By Corollary 2.4, $K(\beta)$ is the graph of f_β. Therefore, its inclusion in $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_\varepsilon)$ is given by the matrix $\binom{I}{M_{f_\beta}}$. Furthermore, $\widehat{D}_\varepsilon$ is a deformation retract of $\widehat{X}_\beta$, so the homomorphism j_β is onto. The equality then follows from Theorem 4.2. ∎

A tangle $\tau \in T(\varepsilon, \varepsilon')$ is said to be *topologically trivial* if the oriented pair $(D^2 \times [0,1], \tau)$ is homeomorphic to the oriented pair $(D^2 \times [0,1], \mathrm{id}_{\varepsilon''})$ for some ε''. For instance, the oriented braids are topologically trivial, as are the elementary tangles described in Figure 2. Note that a topologically trivial tangle with $\ell_\varepsilon \neq 0$ is always straight. Therefore, $K(\tau)$ is a free module if $\ell_\varepsilon \neq 0$.

Corollary 4.4. *Consider a topologically trivial tangle $\tau \in T(\varepsilon, \varepsilon)$ with $\ell_\varepsilon \neq 0$, and let $\binom{M}{M'}$ be a matrix of the inclusion $K(\tau) \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_\varepsilon)$. Then there is a divisor $\delta \in \Lambda$ of $(t^{\ell_\varepsilon} - 1)/(t - 1)$ such that*

$$\delta \Delta_{\widehat{\tau}}(t) \doteq \det(M' - M).$$

Proof. Let h be the homeomorphism between $(D^2 \times [0,1], \tau)$ and $(D^2 \times [0,1], \mathrm{id}_{\varepsilon''})$. The induced isomorphism $h_\sharp \colon \pi_1(X_\tau) \to \pi_1(X_{\mathrm{id}_{\varepsilon''}})$ is compatible with the epimorphisms $\pi_1(X_\tau) \to \mathbb{Z}$ and $\pi_1(X_{\mathrm{id}_{\varepsilon''}}) \to \mathbb{Z}$. Therefore, h lifts to a homeomorphism $\widehat{h} \colon \widehat{X}_\tau \to \widehat{X}_{\mathrm{id}_{\varepsilon''}}$.

Denote by B_ε the compact surface $(\partial D^2 \times [0,1]) \cup (D_\varepsilon \times \{0,1\})$. Since $\widehat{X}_{\mathrm{id}_{\varepsilon''}}$ retracts by deformation on $\widehat{D}_{\varepsilon''} \subset \widehat{B}_{\varepsilon''}$, the manifold $\widehat{X}_\tau$ retracts by deformation on $\widehat{C} = \widehat{h}^{-1}(\widehat{D}_{\varepsilon''}) \subset \widehat{B}_\varepsilon$. This leads to the following commutative diagram of inclusion homomorphisms:

$$\begin{tikzcd}
H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_\varepsilon) \arrow[rr, "j\circ i"] \arrow[d, "i"'] & & H_1(\widehat{X}_\tau) \\
H_1(\widehat{B}_\varepsilon) \arrow[urr, "j"] & & H_1(\widehat{C}) \arrow[ll, "k"'] \arrow[u, "j\circ k"']
\end{tikzcd}$$

where $j \circ k$ is an isomorphism. Denote by $\pi\colon H_1(\widehat{X}_\tau) \to \operatorname{coker}(j \circ i)$ and $\pi'\colon H_1(\widehat{B}_\varepsilon) \to \operatorname{coker}(i)$ the canonical projections. Consider the homomorphism $\varphi\colon \operatorname{coker}(j \circ i) \to \operatorname{coker}(i)$ given by

$$\varphi(\pi(x)) = \pi' \circ k \circ (j \circ k)^{-1}(x)$$

for $x \in H_1(\widehat{X}_\tau)$. We easily check that φ is a well defined injective homomorphism. Therefore, $\Delta(\operatorname{coker}(j_\tau)) = \Delta(\operatorname{coker}(j \circ i))$ divides $\Delta(\operatorname{coker}(i))$. The exact sequence of the pair $(\widehat{B}_\varepsilon, \widehat{D}_\varepsilon \sqcup \widehat{D}_\varepsilon)$ gives

$$H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_\varepsilon) \xrightarrow{i} H_1(\widehat{B}_\varepsilon) \to \Lambda/(t^{\ell_\varepsilon} - 1) \to \Lambda/(t-1) \to 0.$$

Therefore, $\Delta(\operatorname{coker}(i)) \doteq (t^{\ell_\varepsilon} - 1)/(t-1)$. The result now follows from Theorem 4.2. ■

5. Examples. Given a topologically trivial tangle $\tau \in T(\varepsilon, \varepsilon)$ with $\ell_\varepsilon \neq 0$, Propositions 3.8, 3.9 and Corollary 4.4 provide a method for the computation of the Alexander polynomial $\Delta_{\widehat{\tau}}$. We now give several examples of such computations.

5.1. *Rational links.* For integers $a_1, \dots, a_n$, denote by $\sigma(a_1, \dots, a_n)$ the following unoriented 3-strand braid:

$$\sigma(a_1, \dots, a_n) = \begin{cases} \sigma_2^{a_1}\sigma_1^{-a_2}\sigma_2^{a_3}\cdots\sigma_2^{a_n} & \text{if } n \text{ is odd,} \\ \sigma_2^{a_1}\sigma_1^{-a_2}\sigma_2^{a_3}\cdots\sigma_1^{-a_n} & \text{if } n \text{ is even.} \end{cases}$$

Consider the unoriented 3-strand tangle $\tau(a_1, \dots, a_n) = \tau_n \circ \sigma(a_1, \dots, a_n)$, where

$$\tau_n = \begin{cases} u \circ \eta & \text{if } n \text{ is odd,} \\ u \circ \eta \circ \sigma_2 \circ \sigma_1 & \text{if } n \text{ is even.} \end{cases}$$

(Recall Figure 2 for the definition of the tangles u, η and σ_i.) Finally, denote by $C(a_1, \dots, a_n)$ the unoriented link given by the closure of $\tau(a_1, \dots, a_n)$. Such a link is called a *rational link* or a 2-*bridge link* (see [3] and Figure 4 for examples).

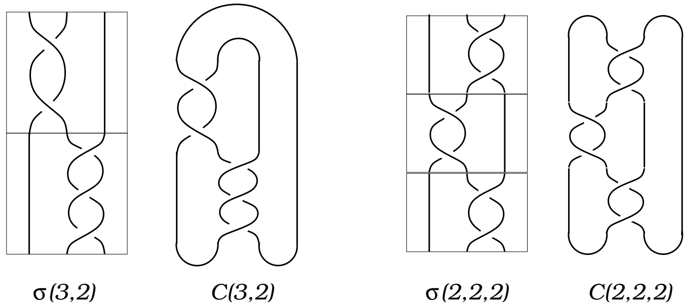

Fig. 4. Rational tangles and rational links

Consider the oriented link L obtained by endowing $C(a_1,\dots,a_n)$ with an orientation. (Note that there is no canonical way to do so: L is not uniquely determined by the integers $(a_1,\dots,a_n)$.) This turns $\sigma(a_1,\dots,a_n)$ into an oriented braid β. Using Proposition 3.9, one easily computes the associated matrix $M_{f_\beta} = \left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{smallmatrix}\right)$, where $m_{ij} \in \Lambda$ for $i,j = 1,2$.

PROPOSITION 5.1. *The Alexander polynomial of L is given by*

$$\Delta_L(t) \doteq \begin{cases} m_{21} & \text{if } n \text{ is odd,} \\ m_{11} & \text{if } n \text{ is even.}\end{cases}$$

Proof. Assume first that n is odd. Consider the decomposition $\tau = \tau_n \circ \beta$. In the canonical bases v_1, v_2 of $H_1(\widehat{D}_\varepsilon)$ and v'_1, v'_2 of $H_1(\widehat{D}_{\varepsilon'})$, the inclusion $K(\tau_n) \subset H_1(\widehat{D}_{\varepsilon'}) \oplus H_1(\widehat{D}_\varepsilon)$ is encoded by the matrix $\left(\begin{smallmatrix} M' \\ M\end{smallmatrix}\right)$ with $M' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)$. Furthermore, the inclusion $K(\beta) \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})$ is encoded by the matrix $\left(\begin{smallmatrix} I \\ M_{f_\beta}\end{smallmatrix}\right)$. Since M_{f_β} is invertible, the solutions of the system $(-M_{f_\beta}, M') \cdot x = 0$ are given by $\left(\begin{smallmatrix} W_1 \\ W_2\end{smallmatrix}\right) = \left(\begin{smallmatrix} M_{f_\beta}^{-1}M' \\ I\end{smallmatrix}\right)$. By Proposition 3.8, $K(\tau)$ is encoded by $\left(\begin{smallmatrix} M_{f_\beta}^{-1}M' \\ M\end{smallmatrix}\right)$. By Corollary 4.4,

$$\begin{aligned}\Delta_L(t) &\doteq \det(M - M_{f_\beta}^{-1}M') \doteq \det(M_{f_\beta}M - M') \\ &= \det\left(\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\right) \doteq m_{21}.\end{aligned}$$

If n is even, we have $M' = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0\end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)$. This leads to $\Delta_L(t) \doteq m_{11}$. ∎

For example, consider an oriented knot K obtained by orienting the knot $C(3,2)$ described in Figure 4. The corresponding oriented braid β is the composition of five elementary braids, leading to

$$\begin{aligned} M_{f_\beta} &= \begin{pmatrix} -t^\varepsilon & 1 \\ 0 & 1\end{pmatrix}^{-2}\begin{pmatrix} 1 & 0 \\ t^\varepsilon & -t^\varepsilon\end{pmatrix}\begin{pmatrix} 1 & 0 \\ t^{-\varepsilon} & -t^{-\varepsilon}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ t^\varepsilon & -t^\varepsilon\end{pmatrix} \\ &= \begin{pmatrix} 2t^{-2\varepsilon} - 3t^{-\varepsilon} + 2 & t^{-\varepsilon} - 1 \\ 2t^\varepsilon - 1 & -t^\varepsilon\end{pmatrix},\end{aligned}$$

where ε is ± 1 according to the orientation of K. By Proposition 5.1, we have $\Delta_K(t) \doteq 2t - 3 + 2t^{-1}$.

Let L be an oriented link obtained by orienting $C(2,2,2)$ so that the linking number of the components is $+2$. Here, we get

$$M_{f_\beta} = \begin{pmatrix} 1 & 0 \\ t^\varepsilon & -t^\varepsilon \end{pmatrix}^2 \begin{pmatrix} -t^\varepsilon & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -t^{-\varepsilon} & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ t^\varepsilon & -t^\varepsilon \end{pmatrix}^2$$
$$= \begin{pmatrix} t^{2\varepsilon} - 2t^\varepsilon + 2 & t^\varepsilon - t^{2\varepsilon} \\ 2(t^\varepsilon - t^{2\varepsilon})(t^{2\varepsilon} - t^\varepsilon + 1) & 2t^{4\varepsilon} - 2t^{3\varepsilon} + t^{2\varepsilon} \end{pmatrix},$$

where the sign $\varepsilon = \pm 1$ is given by the global orientation of L. Therefore, $\Delta_L(t) \doteq 2(t-1)(t-1+t^{-1})$. Finally, if we orient $C(2,2,2)$ so that the linking number of the components is -2, the resulting oriented link L' has Alexander polynomial $\Delta_{L'} \doteq (t-1)(t-4+t^{-1})$.

5.2. 2-*strand tangles.* In this subsection, we use the techniques introduced above to define an invariant of $(2,2)$-tangles formed by two arcs and having no closed components. This invariant is a pair of elements of Λ defined up to simultaneous multiplication by a unit of Λ. We study the behavior of this invariant under the basic transformations of $(2,2)$-tangles introduced by Conway [3].

Consider a tangle $\tau \in T(\varepsilon, \varepsilon')$ with no closed components, where ε and ε' are sequences of ± 1's of length 2. By bending τ, we get a tangle $\tau^b \in T(\emptyset, \mu)$ where $\emptyset$ is the empty sequence and $\mu = (\varepsilon'_1, \varepsilon'_2, -\varepsilon_2, -\varepsilon_1)$. This is illustrated in Figure 5.

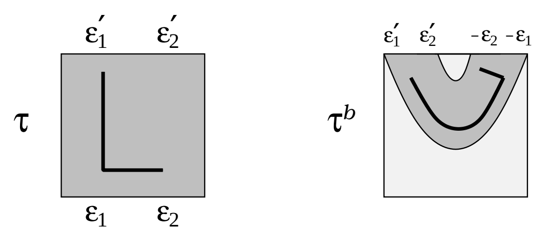

Fig. 5. The tangle τ^b obtained by bending τ

LEMMA 5.2. *The submodule $K(\tau^b)$ of $H_1(\widehat{D}_\mu)$ is free of rank one.*

Proof. One can write τ^b as a composition $\tau^b = \tau' \circ u$, where $u \in T(\emptyset, \widetilde{\varepsilon})$ is the elementary 1-strand "cup" tangle and $\tau' \in T(\widetilde{\varepsilon}, \varepsilon)$ is a straight tangle. Since $H_1(\widehat{D}_\emptyset) = H_1(\widehat{D}_{\widetilde{\varepsilon}}) = 0$, we have $K(u) = 0$. Now, $K(\tau^b) = K(\tau')$, which is free by Proposition 3.1. Its rank is one by Proposition 3.6. ∎

Recall from Subsection 2.3 that $H_1(\widehat{D}_\mu) = (\Lambda v_1 \oplus \Lambda v_2 \oplus \Lambda v_3)/\Lambda\widehat{\gamma}$, where $v_i = \widehat{e}_i - \widehat{e}_{i+1}$ and $\gamma = e_1^{\varepsilon_1} \cdots e_4^{\varepsilon_4}$. Therefore, $H_1(\widehat{D}_\mu)$ is free with basis v_1, v_2. Using this fact and Lemma 5.2, the inclusion $K(\tau^b) \subset H_1(\widehat{D}_\mu)$ is given by a matrix $\binom{m_1}{m_2}$ with $m_1, m_2 \in \Lambda$, unique up to multiplication by $\pm t^\nu$ with $\nu \in \mathbb{Z}$. We denote this fact by $\tau \sim (m_1, m_2)$.

For concreteness, we shall assume throughout the rest of the discussion that $\varepsilon = \varepsilon' = (-1, +1)$ as for the tangle τ in Figure 6. (The other five cases can be treated similarly.) Consider the tangles τ_1, τ_2, τ_3 and τ_4 shown in Figure 6: τ_1 is obtained from τ by a horizontal reflexion, τ_2 by a rotation to the angle $\pi/2$, τ_3 by addition of a twist to the right, and τ_4 by addition of a twist to the top.

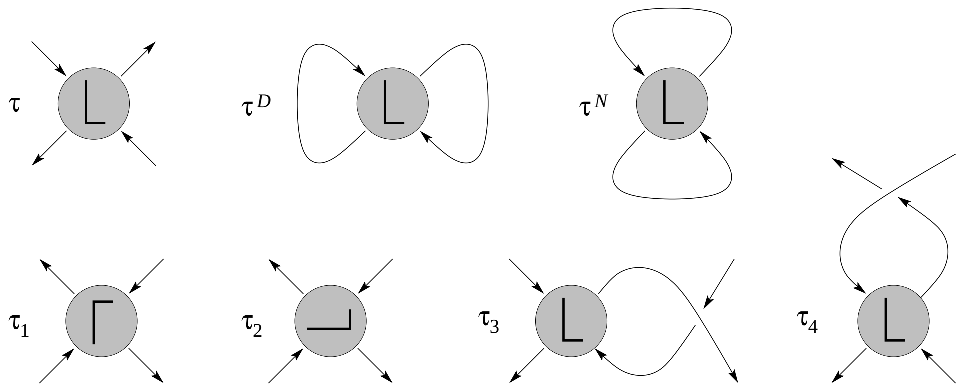

Fig. 6. Tangles with two strands

PROPOSITION 5.3. *If $\tau \sim (m_1, m_2)$, then $\tau_1 \sim (m_1, -m_2)$, $\tau_2 \sim (m_2, -m_1)$, $\tau_3 \sim (tm_1, m_1 - m_2)$ and $\tau_4 \sim (m_2 - tm_1, m_2)$.*

Proof. We have $\tau^b \in T(\emptyset, \mu)$ with $\mu = (-1, +1, -1, +1)$, while $\tau_1^b, \tau_2^b \in T(\emptyset, \mu')$ where $\mu' = (+1, -1, +1, -1)$. Hence, $H_1(\widehat{D}_{\mu'}) = (\Lambda v_1' \oplus \Lambda v_2' \oplus \Lambda v_3')/\Lambda(v_1' + v_3')$. The horizontal reflexion induces an isomorphism $H_1(\widehat{D}_\mu) \to H_1(\widehat{D}_{\mu'})$ given by $v_1 \mapsto -v_3' = v_1'$ and $v_2 \mapsto -v_2'$. Hence, $\tau_1 \sim (m_1', m_2')$ with $\binom{m_1'}{m_2'} = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\binom{m_1}{m_2} = \binom{m_1}{-m_2}$. Similarly, the rotation to the angle $\pi/2$ induces an isomorphism $H_1(\widehat{D}_\mu) \to H_1(\widehat{D}_{\mu'})$ given by the matrix $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Thus, $\tau_2 \sim (m_2, -m_1)$. Note that $\tau_3^b \in T(\emptyset, \mu'')$, where $\mu'' = (-1, -1, +1, +1)$. The transformation from τ^b to τ_3^b can be understood as a composition $\tau_3^b = \sigma \circ \tau^b$, where σ is a spherical braid. By the results of Subsection 3.2, the isomorphism $H_1(\widehat{D}_\mu) \to H_1(\widehat{D}_{\mu''})$ corresponding to σ is given by $v_1 \mapsto v_1'' + t^{-1}v_2''$ and $v_2 \mapsto -t^{-1}v_2''$. Therefore, $\tau_3 \sim (m_1, t^{-1}(m_1 - m_2))$, which is equivalent to $(tm_1, m_1 - m_2)$. The case of τ_4 is similar. ∎

PROPOSITION 5.4. *If τ is topologically trivial and $\tau \sim (m_1, m_2)$, then the oriented links τ^D and τ^N described in Figure 6 have the Alexander module*

$$H_1(\widehat{X}_{\tau^D}) = \Lambda/(m_1) \quad \text{and} \quad H_1(\widehat{X}_{\tau^N}) = \Lambda/(m_2).$$

In particular, $\Delta_{\tau^D}(t) \doteq m_1$ and $\Delta_{\tau^N}(t) \doteq m_2$.

Proof. Since τ is topologically trivial, $H_1(\widehat{X}_{\tau^b}) = H_1(\widehat{X}_\tau) = \Lambda$ and the inclusion homomorphism $j\colon H_1(\widehat{D}_\mu) = \Lambda v_1 \oplus \Lambda v_2 \to H_1(\widehat{X}_\tau)$ is onto (cf. the proof of Corollary 4.4). Therefore, the greatest common divisor of $j(v_1)$ and $j(v_2)$ is 1. Hence, the kernel $K(\tau^b)$ of j is generated by $j(v_2)v_1 - j(v_1)v_2$, so $m_1 = j(v_2)$ and $m_2 = -j(v_1)$. Since the exterior of τ^D in S^3 can be written $X_{\tau^D} = X_\tau \cup X_{\mathrm{id}}$, we have the Mayer–Vietoris exact sequence

$$H_1(\widehat{D}_\mu) \xrightarrow{\varphi} H_1(\widehat{X}_\tau) \oplus H_1(\widehat{X}_{\mathrm{id}}) \to H_1(\widehat{X}_{\tau^D}) \to 0.$$

Clearly, $H_1(\widehat{X}_{\mathrm{id}}) = \Lambda v_1$ and a matrix of φ is given by $\left(\begin{smallmatrix} j(v_1) & j(v_2) \\ 1 & 0 \end{smallmatrix}\right)$. It is equivalent to $(j(v_2)) = (m_1)$, so $H_1(\widehat{X}_{\tau^D}) = \Lambda/(j(v_2)) = \Lambda/(m_1)$. With the notation of Figure 6, we have $\tau^N = (\tau_2)^D$. Hence, the formula for τ^N follows from the formula for τ^D and from Proposition 5.3. ∎

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