Brunnian local moves of knots and Vassiliev invariants

by

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Dedicated to Professor Takao Matumoto for his 60th birthday

Abstract. K. Habiro gave a necessary and sufficient condition for knots to have the same Vassiliev invariants in terms of $C_k$-moves. In this paper we give another geometric condition in terms of Brunnian local moves. The proof is simple and self-contained.

1. Introduction. We will define local moves via tangles. Our definition follows [11], [12]. A tangle $T$ is a disjoint union of properly embedded arcs in the unit 3-ball $B^3$. Here $T$ contains no closed arcs. A tangle $T$ is trivial if there exists a properly embedded disk in $B^3$ that contains $T$. A local move is a pair of trivial tangles $(T_1, T_2)$ with $\partial T_1 = \partial T_2$ such that for each component $t$ of $T_1$ there exists a component $u$ of $T_2$ with $\partial t = \partial u$. Two local moves $(T_1, T_2)$ and $(U_1, U_2)$ are equivalent, denoted by $(T_1, T_2) \cong (U_1, U_2)$, if there is an orientation preserving self-homeomorphism $\psi : B^3 \to B^3$ such that $\psi(t_i)$ and $U_i$ are ambient isotopic in $B^3$ relative to $\partial B^3$ for $i = 1, 2$. A local move $(T_1, T_2)$ is trivial if $(T_1, T_2)$ is equivalent to the local move $(T_1, T_1)$. Note that $(T_1, T_2)$ is trivial if and only if $T_1$ and $T_2$ are ambient isotopic in $B^3$ relative to $\partial B^3$.

Let $(T_1, T_2)$ be a local move, and let $t_1, \ldots, t_k$ and $u_1, \ldots, u_k$ be the components of $T_1$ and $T_2$ respectively with $\partial t_i = \partial u_i$ ($i = 1, \ldots, k$). We call $(T_1, T_2)$ a $k$-component Brunnian local move ($k \geq 2$), or $B_k$-move, if each local move $(T_1 - t_i, T_2 - u_i)$ is trivial ($i = 1, \ldots, k$) [10]. If $(T_1, T_2)$ is Brunnian, then $(T_2, T_1)$ is also Brunnian. For example, a crossing change is a $B_2$-move, the delta-move defined in [7] is a $B_3$-move, and a $C_k$-move defined in [3], [4] is a $B_{k+1}$-move.
Let $K_1$ and $K_2$ be oriented knots in the three-sphere $S^3$ with a fixed orientation. We say that $K_2$ is obtained from $K_1$ by a local move $(T_1, T_2)$ if there is an orientation preserving embedding $h : B^3 \to S^3$ such that $(h^{-1}(K_1), h^{-1}(K_2)) \cong (T_1, T_2)$ and $K_1 - h(B^3) = K_2 - h(B^3)$ as oriented tangles. Two oriented knots $K_1$ and $K_2$ are $B_k$-equivalent if $K_2$ is obtained from $K_1$ by a finite sequence of $B_k$-moves and ambient isotopies. This relation is an equivalence relation on knots.

We have the following geometric condition for knots to have the same value of Vassiliev invariant.

**Theorem 1** (cf. Goussarov–Habiro Theorem [4, 2]). Two knots $K_1$ and $K_2$ are $B_{l+1}$-equivalent if and only if their values of any Vassiliev invariant of order $\leq l - 1$ are equal.

**Remark.** The authors of [5] and [6] showed independently that $B_{l+1}$- and $C_l$-equivalence classes coincide. Therefore, the theorem above and the Goussarov–Habiro Theorem are consequences of each other. Although Theorem 1 can be obtained as a corollary of the Goussarov–Habiro Theorem the author believes that a new and self-contained proof is worth presenting. Moreover, the arguments used in our proof are shorter and simpler compared to those given in [4, 2] and [12] for the proof of the Goussarov–Habiro Theorem.

Let $l$ be a positive integer and let $k_1, \ldots, k_l \geq 2$ be integers. Suppose that for each $P \subset \{1, \ldots, l\}$ we have an oriented knot $K_P$ in $S^3$ and there are orientation preserving embeddings $h_i : B^3 \to S^3$ ($i = 1, \ldots, l$) such that:

1. $h_i(B^3) \cap h_j(B^3) = \emptyset$ if $i \neq j$,
2. $K_P - \bigcup_{i=1}^{l} h_i(B^3) = K_{P'} - \bigcup_{i=1}^{l} h_i(B^3)$ for all $P, P' \subset \{1, \ldots, l\}$,
3. $(h_i^{-1}(K_0), h_i^{-1}(K_{\{1, \ldots, i\}}))$ is a $B_{k_i}$-move ($i = 1, \ldots, l$), and
4. $K_P \cap h_i(B^3) = \begin{cases} K_{\{1, \ldots, i\}} \cap h_i(B^3) & \text{if } i \in P, \\
K_{\emptyset} \cap h_i(B^3) & \text{otherwise.} \end{cases}$

Then we call the set $\{K_P \mid P \subset \{1, \ldots, l\}\}$ of oriented knots a singular knot of type $B(k_1, \ldots, k_l)$. Let $\mathcal{K}$ be the set of knots, $A$ an abelian group, and $\varphi : \mathcal{K} \to A$ an invariant. We say that $\varphi$ is a finite type invariant of type $B(k_1, \ldots, k_l)$ if for any singular knot $\{K_P \mid P \subset \{1, \ldots, l\}\}$ of type $B(k_1, \ldots, k_l)$,

$$\sum_{P \subset \{1, \ldots, l\}} (-1)^{|P|} \varphi(K_P) = 0.$$  

Since a $B_2$-move is realized by some crossing changes we see that an invariant $\varphi : \mathcal{K} \to A$ is a finite type invariant of type $B(2, \ldots, 2)$ if and only if it is a Vassiliev invariant of order $\leq l - 1$.

In order to prove Theorem 1, we need the following theorems.
**Theorem 2** (cf. [4, Theorem 5.4]). The set of $B_k$-equivalence classes, denoted by $\mathcal{K}/B_k$, of oriented knots in $S^3$ forms an abelian group under connected sum of oriented knots.

**Theorem 3** (cf. [12, Theorem 1.2]). Let $l \geq 2$ and $k_1, \ldots, k_l \geq 2$ be integers, and $k - 1 = (k_1 - 1) + \cdots + (k_l - 1)$. Then the projection $p_k : \mathcal{K} \to \mathcal{K}/B_k$ is a finite type invariant of type $B(k_1, \ldots, k_l)$.

**Remark.** Since a $C_k$-move is the same as a $B_{k+1}$-move, Theorem 2 follows from [4, Theorem 5.4]. Theorem 3 is similar to [12, Theorem 1.2]. In order to give a self-contained proof of Theorem 1, we will give self-contained proofs of Theorems 2 and 3. Although the reasonings given in the proofs of Theorems 2 and 3 are analogous to those in [4] (and also in [11] and [12]) we provide simpler and shorter arguments.

**2. Band description.** It is known that any knot can be expressed as a “band sum” of the trivial knot and a split union of some Hopf links [8], [13] (or Borromean rings [14]). K. Taniyama and the author showed that if two knots are $C_k$-equivalent, then one can be expressed as a band sum of the other and a split union of certain $(k + 1)$-component Brunnian links [11], [12]. By similar arguments to those in [11], we describe a relation between $B_k$-equivalence and a certain band sum.

Let $(T_1, T_2)$ be a $k$-component Brunnian local move. Let $T \subset B^3$ be the trivial $k$-string tangle illustrated in Figure 1, and let $D$ be the disjoint union of the $k$ disks bounded by $T$ and arcs in $\partial B^3$ (see Figure 2). Since $T_2$ is a trivial tangle, there is a tangle $S$ such that $(S, T)$ and $(T_1, T_2)$ are equivalent. Then the pair $(S, \partial D - T)$ is called a $B_k$-link model (see Figure 3).
Let \((\alpha_i, \beta_i)\) be \(B_{\varrho(i)}\)-link models \((i = 1, \ldots, l)\), and \(K\) an oriented knot (respectively a tangle). Let \(\psi_i : B^3 \to S^3\) (respectively \(\psi_i : B^3 \to \text{int } B^3\)) be an orientation preserving embedding for \(i = 1, \ldots, l\), and let \(b_{1,1}, \ldots, b_{1,\varrho(1)}; b_{2,1}, \ldots, b_{2,\varrho(2)}; \ldots; b_{l,1}, \ldots, b_{l,\varrho(l)}\) be mutually disjoint disks embedded in \(S^3\) (respectively \(B^3\)). Suppose that they satisfy the following conditions:

1. \(\psi_i(B^3) \cap \psi_j(B^3) = \emptyset\) if \(i \neq j\),
2. \(\psi_i(B^3) \cap K = \emptyset\) for each \(i\),
3. \(b_{i,k} \cap K = \partial b_{i,k} \cap K\) is an arc for each \(i, k\),
4. \(b_{i,k} \cap \bigcup_{j=1}^{l} \psi_j(B^3) = \partial b_{i,k} \cap \psi_i(B^3)\) is a component of \(\psi_i(\beta_i)\) for each \(i, k\).

Let \(J\) be an oriented knot (respectively a tangle) defined by

\[
J = K \cup \left( \bigcup_{i,k} \partial b_{i,k} \right) \cup \left( \bigcup_{i=1}^{l} \psi_i(\alpha_i) \right) - \bigcup_{i,k} \text{int}\(\partial b_{i,k} \cap K\) - \bigcup_{i=1}^{l} \psi_i(\text{int } \beta_i),
\]

where the orientation of \(J\) coincides with that of \(K\) on \(K - \bigcup_{i,k} b_{i,k}\) if \(K\) is oriented. We call each \(b_{i,k}\) a band. Each image \(\psi_i(B^3)\) is called a link ball. We set \(B_i = ((\alpha_i, \beta_i), \psi_i, \{b_{i,1}, \ldots, b_{i,\varrho(i)}\})\) and call \(B_i\) a \(B_{\varrho(i)}\)-chord. We denote \(J\) by \(\Omega(K; \{B_1, \ldots, B_l\})\), and say that \(J\) is a band sum of \(K\) and chords \(B_1, \ldots, B_l\), or a band sum of \(K\) and \(\{B_1, \ldots, B_l\}\).

From now on we consider knots up to ambient isotopy of \(S^3\) and tangles up to ambient isotopy of \(B^3\) relative to \(\partial B^3\) without explicit mention.

By the definitions of a \(B_k\)-move and a \(B_k\)-link model, we have:

**Sublemma 4** (cf. [12, Sublemmas 3.3 and 3.5]):

1. A local move \((T_1, T_2)\) is a \(B_k\)-move if and only if \(T_1\) is a band sum of \(T_2\) and a \(B_k\)-link model.
2. A knot \(J\) is obtained from a knot \(K\) by a single \(B_k\)-move if and only if \(K\) is a band sum of \(J\) and a \(B_k\)-link model. \(\blacksquare\)

Note that, by Sublemma 4(1), a set \(K\) of knots is a singular knot of type \(B(k_1, \ldots, k_l)\) if and only if there is a knot \(K\) and a band sum \(J = \Omega(K; \{B_1, \ldots, B_l\})\) of \(K\) and \(B_{k_i}\)-chords \(B_i\) \((i = 1, \ldots, l)\) such that

\[
K = \left\{ \Omega \left( K; \bigcup_{i \in P} B_i \right) \bigg| P \subset \{1, \ldots, l\} \right\}.
\]

**Sublemma 5** (cf. [12, Sublemma 3.5]). Let \(K, J\) and \(I\) be oriented knots (or tangles). Suppose that \(J = \Omega(K; \{B_1, \ldots, B_l\})\) for some chords \(B_1, \ldots, B_l\) and \(I = \Omega(J; \{B_i\})\) for some \(B_{k_i}\)-chord \(B_i\). Then there is a \(B_k\)-chord \(B'\) such that \(I = \Omega(K; \{B_1, \ldots, B_i, B'\})\). Moreover, if for a subset \(P\) of \(\{1, \ldots, l\}\) the link ball or the bands of \(B\) intersect either the link ball or the bands of \(B_i\) only when \(i \in P\), then \(\Omega(\Omega(K; \bigcup_{i \in P} B_i); \{B\}) = \Omega(K; (\bigcup_{i \in P} B_i) \cup \{B'\})\).
**Proof.** If the bands and the link ball of $\mathcal{B}$ are disjoint from those of $\mathcal{B}_1, \ldots, \mathcal{B}_l$ then $I = \Omega(K; \{\mathcal{B}_1, \ldots, \mathcal{B}_l, \mathcal{B}\})$. If not, then we deform $I$ up to ambient isotopy as follows. By thinning and shrinking the bands and the link ball of $\mathcal{B}$ respectively, we may assume that the link ball of $\mathcal{B}$ intersects neither the bands nor the link balls of $\mathcal{B}_1, \ldots, \mathcal{B}_l$. And by sliding the bands of $\mathcal{B}$ along $J$, we may also assume that the intersection of the bands with $J$ is disjoint from the bands and the link balls of $\mathcal{B}_1, \ldots, \mathcal{B}_l$. Then we sweep the bands of $\mathcal{B}$ out of the link balls of $\mathcal{B}_1, \ldots, \mathcal{B}_l$. Note that this is always possible since the tangles of a local move are trivial. Finally, we sweep the intersection of the bands of $\mathcal{B}$ and the bands of $\mathcal{B}_1, \ldots, \mathcal{B}_l$ out of the intersection of the bands of $\mathcal{B}_1, \ldots, \mathcal{B}_l$ and $K$. Let $\mathcal{B}'$ be the result of the deformation of $\mathcal{B}$ described above. Then it is not hard to see that $\mathcal{B}'$ is the desired chord. ■

By repeated applications of Sublemmas 4 and 5 we immediately have the following lemma.

**Lemma 6** (cf. [12, Lemma 3.6]). Let $k$ be a positive integer and let $K$ and $J$ be oriented knots (or tangles). Then $K$ and $J$ are $B_k$-equivalent if and only if $J$ is a band sum of $K$ and some $B_k$-link models. ■

Since the local moves illustrated in Figures 4 and 5 are a $B_{k+1}$-move and $B_{j+k-1}$-move respectively, the following two lemmas follow from Sublemma 5.

![Fig. 4](image1.png)

**Fig. 4**

![Fig. 5](image2.png)

**Fig. 5**

**Lemma 7** (cf. [12, Lemma 3.8]). Let $K$, $J = \Omega(K; \{\mathcal{B}_1, \ldots, \mathcal{B}_l, \mathcal{B}_0\})$ and $I = \Omega(K; \{\mathcal{B}_1, \ldots, \mathcal{B}_l, \mathcal{B}'_0\})$ be oriented knots, where $\mathcal{B}_1, \ldots, \mathcal{B}_l$ are chords and $\mathcal{B}_0, \mathcal{B}'_0$ are $B_k$-chords. Suppose that $J$ and $I$ differ locally as illustrated in Figure 4, i.e., $I$ is obtained from $J$ by a crossing change between $K$ and a band.
of \( B_0 \). Then \( I \) is obtained from \( J \) by a \( B_{k+1} \)-move. Moreover, there is a \( B_{k+1} \)-chord \( B \) such that \( \Omega(K; \bigcup_{i \in P} \{ B_i \} \cup \{ B_0 \}) = \Omega(K; \bigcup_{i \in P} \{ B_i \} \cup \{ B'_0, B \}) \) for any subset \( P \) of \( \{ 1, \ldots, l \} \).

**Lemma 8** (cf. [12, Lemma 3.9]). Let \( K, J = \Omega(K; \{ B_1, \ldots, B_l, B_{0j}; B_{0k} \}) \) and \( I = \Omega(K; \{ B_1, \ldots, B_l, B'_{0j}, B'_{0k} \}) \) be oriented knots, where \( B_1, \ldots, B_l \) are chords and \( B_{0j}, B'_{0j} \) (respectively \( B_{0k}, B'_{0k} \)) are \( B_j \)-chords (respectively \( B_k \)-chords). Suppose that \( J \) and \( I \) differ locally as illustrated in Figure 5. Then \( I \) is obtained from \( J \) by a \( B_{j+k-1} \)-move. Moreover, there is a \( B_{j+k-1} \)-chord \( B \) such that \( \Omega(K; \bigcup_{i \in P} \{ B_i \} \cup \{ B_{0j}, B_{0k} \}) = \Omega(K; \bigcup_{i \in P} \{ B_i \} \cup \{ B'_{0j}, B'_{0k}, B \}) \) for any subset \( P \) of \( \{ 1, \ldots, l \} \).

We call the change from \( J \) to \( I \) in Lemma 8 a **band exchange**.

For a \( C_k \)-move, “band description” is also defined, and Sublemmas 4, 5, Lemmas 6, 7 and 8 hold [12]. However, the proofs given in [12] are not as obvious as ours. In fact, more complicated arguments are needed. In contrast, we need some arguments to prove the following lemma, which is trivial for a \( C_k \)-move.

**Lemma 9.** Let \( (T_1, T_2) \) be a \( B_k \)-move. For any integer \( l \) \(( \leq k) \), \( T_2 \) is obtained from \( T_1 \) by \( B_l \)-moves. In particular, \( B_k \)-equivalent knots are \( B_l \)-equivalent.

**Proof.** Let \( t_1, \ldots, t_k \) and \( u_1, \ldots, u_k \) be the components with \( \partial t_i = \partial u_i \) \((i = 1, \ldots, k) \) of \( T_1 \) and \( T_2 \) respectively. We may assume that \( (T_1, T_2) \) has a diagram in the unit disk such that \( T_1 - t_1 \) and \( T_2 \) have no crossings.

Since \( (T_1 - t_2, T_2 - u_2) \) is a trivial local move, \( T_2 \) is obtained from \( T_1 \) by \( B_2 \)-moves that correspond to crossing changes between \( t_1 \) and \( t_2 \). By Lemma 6, \( T_1 \) is a band sum, \( \Omega(T_2; B_2) \), of \( T_2 \) and a set \( B_2 \) of \( B_2 \)-chords. Note that no band of \( B_2 \)-chords intersects \( T_2 - (u_1 \cup u_2) \).

Since \( \Omega(T_2; B_2) - t_3, T_2 - u_3 \) is a trivial local move, \( T_2 \) is obtained from \( T_1 \) by \( B_3 \)-moves that correspond to crossing changes between \( t_3 \) and some bands of \( B_2 \)-chords. By Lemma 6, \( T_1 \) is a band sum \( \Omega(T_2; B_3) \) of \( T_2 \) and a set \( B_3 \) of \( B_3 \)-chords. Note that no bands of \( B_3 \)-chords intersects \( T_2 - (u_1 \cup u_2 \cup u_3) \).

Continuing this process we obtain the conclusion.

3. **Proofs of Theorems 1, 2 and 3**

**Proof of Theorem 3.** Let \( K_0 \) be a knot and \( K_1 \) a band sum of \( K_0 \) and \( B_{kj} \)-chords \( B_{kj} \) \((j = 1, \ldots, l) \). It is sufficient to show that

\[
\sum_{P \subset \{ 1, \ldots, l \}} (-1)^{|P|} \left[ \Omega \left( K_0; \bigcup_{j \in P} \{ B_{kj,j} \} \right) \right] = 0 \in \mathcal{K}/B_k,
\]

where \([K]\) is the \( B_k \)-equivalence class which contains the knot \( K \).
Set

\[ K_P = \Omega \left( K_0; \bigcup_{j \in P} \{ B_{k,j} \} \right). \]

**Claim.** The knot \( K_1 (= K_{\{1, \ldots, l\}}) \) is \( B_k \)-equivalent to a band sum of \( K_0 (= K_0) \) and a set \( \bigcup_{i,j} \{ B_{i,j} \} \) of local chords such that

1. \( B_{i,j} \) is a \( B_i \)-chord \( (i < k) \) and it has an associated subset \( \omega(B_{i,j}) \subset \{1, \ldots, l\} \) with \( \sum_{t \in \omega(B_{i,j})}(k_t - 1) \leq i - 1 \),
2. for each \( P \subset \{1, \ldots, l\} \),

\[ [K_P] = \left[ \Omega \left( K_0; \bigcup_{\omega(B_{i,j}) \subset P} \{ B_{i,j} \} \right) \right]. \]

Here a chord \( B_{i,j} \) is called a local chord if there is a 3-ball \( B \) such that \( B \) contains all the bands and the link ball of \( B_{i,j} \), \( B \) does not intersect any other bands or link balls, and \( (B, B \cap K_0) \) is a trivial ball-arc pair.

Before proving the Claim, we will finish the proof of Theorem 3. Suppose \( K_1 \) is \( B_k \)-equivalent to a band sum of \( K_0 \) and some local chords \( B_{i,j} \). Each \( B_{i,j} \) represents a knot \( K_{i,j} \) which is connected summed with \( K_0 \). So the band sum is a connected sum of \( K_0 \) and \( K_{i,j} \)'s. Then we have

\[
\sum_{P \subset \{1, \ldots, l\}} (-1)^{|P|} \left[ \Omega \left( K_0; \bigcup_{\omega(B_{i,j}) \subset P} \{ B_{i,j} \} \right) \right] = \sum_{P \subset \{1, \ldots, l\}} (-1)^{|P|} \left( [K_0] + \sum_{\omega(B_{i,j}) \subset P} [K_{i,j}] \right) = \sum_{P \subset \{1, \ldots, l\}} (-1)^{|P|} [K_0] + \sum_{P \subset \{1, \ldots, l\}} (-1)^{|P|} \left( \sum_{\omega(B_{i,j}) \subset P} [K_{i,j}] \right) = 0 + \sum_{i,j} \left( \sum_{P \subset \{1, \ldots, l\}, \omega(B_{i,j}) \subset P} (-1)^{|P|} [K_{i,j}] \right).
\]

We consider the coefficient of \( [K_{i,j}] \). Since \( \sum_{t \in \omega(B_{i,j})}(k_t - 1) < k - 1 \), \( \omega(B_{i,j}) \) is a proper subset of \( \{1, \ldots, l\} \). We may assume that \( \omega(B_{i,j}) \) does not contain \( a \in \{1, \ldots, l\} \). Then

\[
\sum_{P \subset \{1, \ldots, l\}, \omega(B_{i,j}) \subset P} (-1)^{|P|} = \sum_{P \subset \{1, \ldots, l\} \setminus \{a\}, \omega(B_{i,j}) \subset P} (-1)^{|P|} + \sum_{P \subset \{1, \ldots, l\} \setminus \{a\}, \omega(B_{i,j}) \subset P} (-1)^{|P \cup \{a\}|} = 0.
\]

Thus, we have the conclusion. \( \blacksquare \)

Now we will show the Claim.
Proof of Claim. We first set $\omega(\mathcal{B}_{k,j}) = \{j\}$ for $j = 1, \ldots, l$. Then we have $\sum_{t \in \omega(\mathcal{B}_{k,j})} (k_t - 1) = k_j - 1 < k - 1$ and

$$K_P = \Omega\left(K_0; \bigcup_{\omega(\mathcal{B}_{k,j}) \subset P} \{\mathcal{B}_{k,j}\}\right).$$

Note that a crossing change between bands can be realized by crossing changes between $K_0$ and a band as illustrated in Figure 6. Therefore we can deform each chord into a local chord by (i) crossing changes between $K_0$ and bands, and (ii) band exchanges.

(i) When we perform a crossing change between $K_0$ and a $B_p$-band of a $B_p$-chord $\mathcal{B}_{p,q}$ with $p \leq k - 2$, by using Lemma 7, we introduce a new $B_{p+1}$-chord $\mathcal{B}_{p+1,r}$ and we set $\omega(\mathcal{B}_{p+1,r}) = \omega(\mathcal{B}_{p,q})$ so that conditions (1) and (2) still hold. By Lemma 7, a crossing change between $K_0$ and a $B_{k-1}$-band is realizing by a $B_k$-move and therefore does not change the $B_k$-equivalence class.

(ii) If we perform a band exchange between a $B_p$-chord $\mathcal{B}_{p,q}$ and a $B_r$-chord $\mathcal{B}_{r,s}$ with $p + r \leq k$, then, by using Lemma 8, we introduce a new $B_{p+r-1}$-chord $\mathcal{B}_{p+r-1,n}$ and set $\omega(\mathcal{B}_{p+r-1,n}) = \omega(\mathcal{B}_{p,q}) \cup \omega(\mathcal{B}_{r,s})$ so that conditions (1) and (2) still hold. By Lemmas 8 and 9, a band exchange between a $B_p$-chord $\mathcal{B}_{p,q}$ and a $B_r$-chord $\mathcal{B}_{r,s}$ with $p + r \geq k + 1$ does not change the $B_k$-equivalence class. 

\[\text{Fig. 6}\]

Proof of Theorem 2. It is sufficient to show the existence of an inverse element for a given knot $K$. Suppose that there is a knot $J$ such that $K \# J$ is $B_k$-equivalent to a trivial knot $O$. Then, by Lemma 6, $O$ is a band sum of $K \# J$ and some $B_k$-chords. By using Lemma 7, we deform $O$ up to $B_{k+1}$-equivalence so that the $B_k$-chords are local chords. Then the result is a connected sum of $K \# J$ and some knots $K_1, \ldots, K_n$ that correspond to local chords. Hence $K \# J \# K_1 \# \cdots \# K_n$ is $B_{k+1}$-equivalent to $O$. Thus $J \# K_1 \# \cdots \# K_n$ is the desired knot.

Proof of Theorem 1. It is not hard to see that $B_{l+1}$-equivalent knots are also $l$-similar [9] $((l-1)$-equivalent [1]).

By Theorem 3, the projection $p_{l+1} : K \to K/B_{l+1}$ is a Vassiliev invariant of order $\leq l - 1$. If two knots have the same values of any Vassiliev invariant of order $\leq l - 1$, then they are $B_{l+1}$-equivalent.
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