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Brunnian local moves of knots and Vassiliev invariants

by

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Dedicated to Professor Takao Matumoto for his 60th birthday

Abstract. K. Habiro gave a neccesary and sufficient condition for knots to have the same Vassiliev invariants in terms of C_k-moves. In this paper we give another geometric condition in terms of Brunnian local moves. The proof is simple and self-contained.

1. Introduction. We will define local moves via tangles. Our definition follows [11], [12]. A *tangle* T is a disjoint union of properly embedded arcs in the unit 3-ball B^3. Here T contains no closed arcs. A tangle T is *trivial* if there exists a properly embedded disk in B^3 that contains T. A *local move* is a pair of trivial tangles (T_1, T_2) with $\partial T_1 = \partial T_2$ such that for each component t of T_1 there exists a component u of T_2 with $\partial t = \partial u$. Two local moves (T_1, T_2) and (U_1, U_2) are *equivalent*, denoted by $(T_1, T_2) \cong (U_1, U_2)$, if there is an orientation preserving self-homeomorphism $\psi : B^3 \to B^3$ such that $\psi(T_i)$ and U_i are ambient isotopic in B^3 relative to ∂B^3 for $i = 1, 2$. A local move (T_1, T_2) is *trivial* if (T_1, T_2) is equivalent to the local move (T_1, T_1). Note that (T_1, T_2) is trivial if and only if T_1 and T_2 are ambient isotopic in B^3 relative to ∂B^3.

Let (T_1, T_2) be a local move, and let $t_1, \ldots, t_k$ and $u_1, \ldots, u_k$ be the components of T_1 and T_2 respectively with $\partial t_i = \partial u_i$ $(i = 1, \ldots, k)$. We call (T_1, T_2) a *k-component Brunnian local move* $(k \geq 2)$, or *B_k-move*, if each local move $(T_1 - t_i, T_2 - u_i)$ is trivial $(i = 1, \ldots, k)$ [10]. If (T_1, T_2) is Brunnian, then (T_2, T_1) is also Brunnian. For example, a crossing change is a B_2-move, the delta-move defined in [7] is a B_3-move, and a C_k-move defined in [3], [4] is a B_{k+1}-move.

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Let K_1 and K_2 be oriented knots in the three-sphere S^3 with a fixed orientation. We say that K_2 is *obtained from* K_1 *by a local move* (T_1, T_2) if there is an orientation preserving embedding $h : B^3 \to S^3$ such that $(h^{-1}(K_1), h^{-1}(K_2)) \cong (T_1, T_2)$ and $K_1 - h(B^3) = K_2 - h(B^3)$ as oriented tangles. Two oriented knots K_1 and K_2 are B_k-*equivalent* if K_2 is obtained from K_1 by a finite sequence of B_k-moves and ambient isotopies. This relation is an equivalence relation on knots.

We have the following geometric condition for knots to have the same value of Vassiliev invariant.

THEOREM 1 (cf. Goussarov–Habiro Theorem [4], [2]). *Two knots* K_1 *and* K_2 *are* B_{l+1}*-equivalent if and only if their values of any Vassiliev invariant of order* $\leq l-1$ *are equal.*

REMARK. The authors of [5] and [6] showed independently that B_{l+1}- and C_l-equivalence classes coincide. Therefore, the theorem above and the Goussarov–Habiro Theorem are consequences of each other. Although Theorem 1 can be obtained as a corollary of the Goussarov–Habiro Theorem the author believes that a new and self-contained proof is worth presenting. Moreover, the arguments used in our proof are shorter and simpler compared to those given in [4], [2] and [12] for the proof of the Goussarov–Habiro Theorem.

Let l be a positive integer and let $k_1, \ldots, k_l$ (≥ 2) be integers. Suppose that for each $P \subset \{1, \ldots, l\}$ we have an oriented knot K_P in S^3 and there are orientation preserving embeddings $h_i : B^3 \to S^3$ $(i = 1, \ldots, l)$ such that:

(1) $h_i(B^3) \cap h_j(B^3) = \emptyset$ if $i \neq j$,

(2) $K_P - \bigcup_{i=1}^{l} h_i(B^3) = K_{P'} - \bigcup_{i=1}^{l} h_i(B^3)$ for all $P, P' \subset \{1, \ldots, l\}$,

(3) $(h_i^{-1}(K_\emptyset), h_i^{-1}(K_{\{1,\ldots,l\}}))$ is a B_{k_i}-move $(i = 1, \ldots, l)$, and

(4) $K_P \cap h_i(B^3) = \begin{cases} K_{\{1,\ldots,l\}} \cap h_i(B^3) & \text{if } i \in P, \\ K_\emptyset \cap h_i(B^3) & \text{otherwise.} \end{cases}$

Then we call the set $\{K_P \mid P \subset \{1, \ldots, l\}\}$ of oriented knots a *singular knot of type* $B(k_1, \ldots, k_l)$. Let $\mathcal{K}$ be the set of knots, A an abelian group, and $\varphi : \mathcal{K} \to A$ an invariant. We say that φ is a *finite type invariant of type* $B(k_1, \ldots, k_l)$ if for any singular knot $\{K_P \mid P \subset \{1, \ldots, l\}\}$ of type $B(k_1, \ldots, k_l)$,

$$\sum_{P \subset \{1,\ldots,l\}} (-1)^{|P|} \varphi(K_P) = 0.$$

Since a B_2-move is realized by some crossing changes we see that an invariant $\varphi : \mathcal{K} \to A$ is a finite type invariant of type $B(\underbrace{2, \ldots, 2}_{l})$ if and only if it is a Vassiliev invariant of order $\leq l-1$.

In order to prove Theorem 1, we need the following theorems.

Theorem 2 (cf. [4, Theorem 5.4]). *The set of B_k-equivalence classes, denoted by $\mathcal{K}/B_k$, of oriented knots in S^3 forms an abelian group under connected sum of oriented knots.*

Theorem 3 (cf. [12, Theorem 1.2]). *Let l (≥ 2) and $k_1, \ldots, k_l$ (≥ 2) be integers, and $k-1 = (k_1-1)+\cdots+(k_l-1)$. Then the projection $p_k : \mathcal{K} \to \mathcal{K}/B_k$ is a finite type invariant of type $B(k_1, \ldots, k_l)$.*

Remark. Since a C_k-move is the same as a B_{k+1}-move, Theorem 2 follows from [4, Theorem 5.4]. Theorem 3 is similar to [12, Theorem 1.2]. In order to give a self-contained proof of Theorem 1, we will give self-contained proofs of Theorems 2 and 3. Although the reasonings given in the proofs of Theorems 2 and 3 are analogous to those in [4] (and also in [11] and [12]) we provide simpler and shorter arguments.

2. Band description. It is known that any knot can be expressed as a "band sum" of the trivial knot and a split union of some Hopf links [8], [13] (or Borromean rings [14]). K. Taniyama and the author showed that if two knots are C_k-equivalent, then one can be expressed as a band sum of the other and a split union of certain $(k+1)$-component Brunnian links [11], [12]. By similar arguments to those in [11], we describe a relation between B_k-equivalence and a certain band sum.

Let (T_1, T_2) be a k-component Brunnian local move. Let $T \subset B^3$ be the trivial k-string tangle illustrated in Figure 1, and let D be the disjoint union of the k disks bounded by T and arcs in ∂B^3 (see Figure 2). Since T_2 is a trivial tangle, there is a tangle S such that (S, T) and (T_1, T_2) are equivalent. Then the pair $(S, \partial D - T)$ is called a B_k-*link model* (see Figure 3).

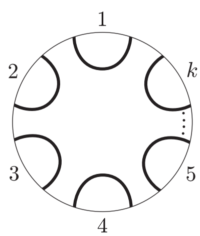


Fig. 1

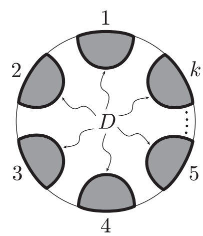
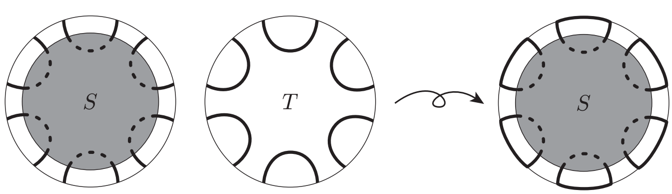


Fig. 2



B_k-move B_k-link model

Fig. 3

Let (α_i, β_i) be $B_{\varrho(i)}$-link models ($i = 1, \ldots, l$), and K an oriented knot (respectively a tangle). Let $\psi_i : B^3 \to S^3$ (respectively $\psi_i : B^3 \to \operatorname{int} B^3$) be an orientation preserving embedding for $i = 1, \ldots, l$, and let $b_{1,1}, \ldots, b_{1,\varrho(1)}$, $b_{2,1}, \ldots, b_{2,\varrho(2)}, \ldots, b_{l,1}, \ldots, b_{l,\varrho(l)}$ be mutually disjoint disks embedded in S^3 (respectively B^3). Suppose that they satisfy the following conditions;

(1) $\psi_i(B^3) \cap \psi_j(B^3) = \emptyset$ if $i \neq j$,
(2) $\psi_i(B^3) \cap K = \emptyset$ for each i,
(3) $b_{i,k} \cap K = \partial b_{i,k} \cap K$ is an arc for each i, k,
(4) $b_{i,k} \cap \bigcup_{j=1}^{l} \psi_j(B^3) = \partial b_{i,k} \cap \psi_i(B^3)$ is a component of $\psi_i(\beta_i)$ for each i, k.

Let J be an oriented knot (respectively a tangle) defined by

$$J = K \cup \Big(\bigcup_{i,k} \partial b_{i,k}\Big) \cup \Big(\bigcup_{i=1}^{l} \psi_i(\alpha_i)\Big) - \bigcup_{i,k} \operatorname{int}(\partial b_{i,k} \cap K) - \bigcup_{i=1}^{l} \psi_i(\operatorname{int} \beta_i),$$

where the orientation of J coincides with that of K on $K - \bigcup_{i,k} b_{i,k}$ if K is oriented. We call each $b_{i,k}$ a *band*. Each image $\psi_i(B^3)$ is called a *link ball*. We set $\mathcal{B}_i = ((\alpha_i, \beta_i), \psi_i, \{b_{i,1}, \ldots, b_{i,\varrho(i)}\})$ and call $\mathcal{B}_i$ a $B_{\varrho(i)}$-*chord*. We denote J by $J = \Omega(K; \{\mathcal{B}_1, \ldots, \mathcal{B}_l\})$, and say that J is a *band sum* of K and chords $\mathcal{B}_1, \ldots, \mathcal{B}_l$, or a *band sum* of K and $\{\mathcal{B}_1, \ldots, \mathcal{B}_l\}$.

From now on we consider knots up to ambient isotopy of S^3 and tangles up to ambient isotopy of B^3 relative to ∂B^3 without explicit mention.

By the definitions of a B_k-move and a B_k-link model, we have:

SUBLEMMA 4 (cf. [12, Sublemmas 3.3 and 3.5]).

(1) *A local move (T_1, T_2) is a B_k-move if and only if T_1 is a band sum of T_2 and a B_k-link model.*
(2) *A knot J is obtained from a knot K by a single B_k-move if and only if K is a band sum of J and a B_k-link model.* ■

Note that, by Sublemma 4(1), a set $\mathbf{K}$ of knots is a singular knot of type $B(k_1, \ldots, k_l)$ if and only if there is a knot K and a band sum $J = \Omega(K; \{\mathcal{B}_1, \ldots, \mathcal{B}_l\})$ of K and B_{k_i}-chords $\mathcal{B}_i$ ($i = 1, \ldots, l$) such that

$$\mathbf{K} = \Big\{ \Omega\Big(K; \bigcup_{i \in P} \{\mathcal{B}_i\}\Big) \,\Big|\, P \subset \{1, \ldots, l\} \Big\}.$$

SUBLEMMA 5 (cf. [12, Sublemma 3.5]). *Let K, J and I be oriented knots (or tangles). Suppose that $J = \Omega(K; \{\mathcal{B}_1, \ldots, \mathcal{B}_l\})$ for some chords $\mathcal{B}_1, \ldots, \mathcal{B}_l$ and $I = \Omega(J; \{\mathcal{B}\})$ for some B_k-chord $\mathcal{B}$. Then there is a B_k-chord $\mathcal{B}'$ such that $I = \Omega(K; \{\mathcal{B}_1, \ldots, \mathcal{B}_l, \mathcal{B}'\})$. Moreover, if for a subset P of $\{1, \ldots, l\}$ the link ball or the bands of $\mathcal{B}$ intersect either the link ball or the bands of $\mathcal{B}_i$ only when $i \in P$, then $\Omega(\Omega(K; \bigcup_{i \in P} \{\mathcal{B}_i\}); \{\mathcal{B}\}) = \Omega(K; (\bigcup_{i \in P} \{\mathcal{B}_i\}) \cup \{\mathcal{B}'\})$.*

Proof. If the bands and the link ball of $\mathcal{B}$ are disjoint from those of $\mathcal{B}_1, \ldots, \mathcal{B}_l$ then $I = \Omega(K; \{\mathcal{B}_1, \ldots, \mathcal{B}_l, \mathcal{B}\})$. If not, then we deform I up to ambient isotopy as follows. By thinning and shrinking the bands and the link ball of $\mathcal{B}$ respectively, we may assume that the link ball of $\mathcal{B}$ intersects neither the bands nor the link balls of $\mathcal{B}_1, \ldots, \mathcal{B}_l$. And by sliding the bands of $\mathcal{B}$ along J, we may also assume that the intersection of the bands with J is disjoint from the bands and the link balls of $\mathcal{B}_1, \ldots, \mathcal{B}_l$. Then we sweep the bands of $\mathcal{B}$ out of the link balls of $\mathcal{B}_1, \ldots, \mathcal{B}_l$. Note that this is always possible since the tangles of a local move are trivial. Finally, we sweep the intersection of the bands of $\mathcal{B}$ and the bands of $\mathcal{B}_1, \ldots, \mathcal{B}_l$ out of the intersection of the bands of $\mathcal{B}_1, \ldots, \mathcal{B}_l$ and K. Let $\mathcal{B}'$ be the result of the deformation of $\mathcal{B}$ described above. Then it is not hard to see that $\mathcal{B}'$ is the desired chord. ∎

By repeated applications of Sublemmas 4 and 5 we immediately have the following lemma.

LEMMA 6 (cf. [12, Lemma 3.6]). *Let k be a positive integer and let K and J be oriented knots (or tangles). Then K and J are B_k-equivalent if and only if J is a band sum of K and some B_k-link models.* ∎

Since the local moves illustrated in Figures 4 and 5 are a B_{k+1}-move and B_{j+k-1}-move respectively, the following two lemmas follow from Sublemma 5.

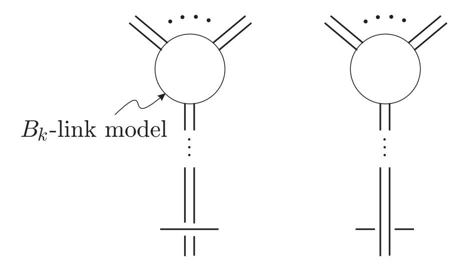


Fig. 4

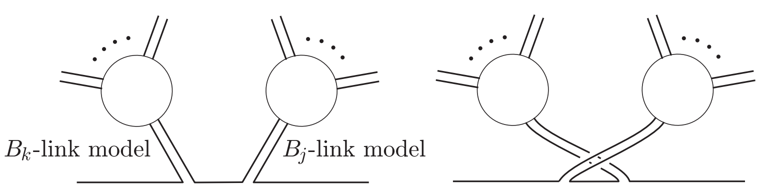


Fig. 5

LEMMA 7 (cf. [12, Lemma 3.8]). *Let K, $J = \Omega(K; \{\mathcal{B}_1, \ldots, \mathcal{B}_l, \mathcal{B}_0\})$ and $I = \Omega(K; \{\mathcal{B}_1, \ldots, \mathcal{B}_l, \mathcal{B}'_0\})$ be oriented knots, where $\mathcal{B}_1, \ldots, \mathcal{B}_l$ are chords and $\mathcal{B}_0, \mathcal{B}'_0$ are B_k-chords. Suppose that J and I differ locally as illustrated in Figure 4, i.e., I is obtained from J by a crossing change between K and a band* *of* $\mathcal{B}_0$. *Then* I *is obtained from* J *by a* B_{k+1}*-move. Moreover, there is a* B_{k+1}*-chord* $\mathcal{B}$ *such that* $\Omega(K;(\bigcup_{i\in P}\{\mathcal{B}_i\})\cup\{\mathcal{B}_0\}) = \Omega(K;(\bigcup_{i\in P}\{\mathcal{B}_i\})\cup\{\mathcal{B}'_0,\mathcal{B}\})$ *for any subset* P *of* $\{1,\dots,l\}$. ■

LEMMA 8 (cf. [12, Lemma 3.9]). *Let* K, $J = \Omega(K;\{\mathcal{B}_1,\dots,\mathcal{B}_l,\mathcal{B}_{0j},\mathcal{B}_{0k}\})$ *and* $I = \Omega(K;\{\mathcal{B}_1,\dots,\mathcal{B}_l,\mathcal{B}'_{0j},\mathcal{B}'_{0k}\})$ *be oriented knots, where* $\mathcal{B}_1,\dots,\mathcal{B}_l$ *are chords and* $\mathcal{B}_{0j},\mathcal{B}'_{0j}$ (*respectively* $\mathcal{B}_{0k},\mathcal{B}'_{0k}$) *are* B_j*-chords* (*respectively* B_k*-chords*). *Suppose that* J *and* I *differ locally as illustrated in Figure* 5. *Then* I *is obtained from* J *by a* B_{j+k-1}*-move. Moreover, there is a* B_{j+k-1}*-chord* $\mathcal{B}$ *such that* $\Omega(K;(\bigcup_{i\in P}\{\mathcal{B}_i\})\cup\{\mathcal{B}_{0j},\mathcal{B}_{0k}\}) = \Omega(K;(\bigcup_{i\in P}\{\mathcal{B}_i\})\cup\{\mathcal{B}'_{0j},\mathcal{B}'_{0k},\mathcal{B}\})$ *for any subset* P *of* $\{1,\dots,l\}$. ■

We call the change from J to I in Lemma 8 a *band exchange.*

For a C_k-move, "band description" is also defined, and Sublemmas 4, 5, Lemmas 6, 7 and 8 hold [12]. However, the proofs given in [12] are not as obvious as ours. In fact, more complicated arguments are needed. In contrast, we need some arguments to prove the following lemma, which is trivial for a C_k-move.

LEMMA 9. *Let* (T_1,T_2) *be a* B_k*-move. For any integer* l $(\le k)$, T_2 *is obtained from* T_1 *by* B_l*-moves. In particular,* B_k*-equivalent knots are* B_l*-equivalent.*

Proof. Let $t_1,\dots,t_k$ and $u_1,\dots,u_k$ be the components with $\partial t_i = \partial u_i$ $(i = 1,\dots,k)$ of T_1 and T_2 respectively. We may assume that (T_1,T_2) has a diagram in the unit disk such that $T_1 - t_1$ and T_2 have no crossings.

Since $(T_1 - t_2, T_2 - u_2)$ is a trivial local move, T_2 is obtained from T_1 by B_2-moves that correspond to crossing changes between t_1 and t_2. By Lemma 6, T_1 is a band sum, $\Omega(T_2;\mathbf{B}_2)$, of T_2 and a set $\mathbf{B}_2$ of B_2-chords. Note that no band of B_2-chords intersects $T_2 - (u_1\cup u_2)$.

Since $(\Omega(T_2;\mathbf{B}_2) - t_3, T_2 - u_3) = (T_1 - t_3, T_2 - u_3)$ is a trivial local move, T_2 is obtained from T_1 by B_3-moves that correspond to crossing changes between t_3 and some bands of B_2-chords. By Lemma 6, T_1 is a band sum $\Omega(T_2;\mathbf{B}_3)$ of T_2 and a set $\mathbf{B}_3$ of B_3-chords. Note that no bands of B_3-chords intersects $T_2 - (u_1\cup u_2\cup u_3)$.

Continuing this process we obtain the conclusion. ■

3. Proofs of Theorems 1, 2 and 3

Proof of Theorem 3. Let K_0 be a knot and K_1 a band sum of K_0 and B_{k_j}-chords $\mathcal{B}_{k_j,j}$ $(j = 1,\dots,l)$. It is sufficient to show that

$$\sum_{P\subset\{1,\dots,l\}} (-1)^{|P|}\Big[\Omega\Big(K_0;\bigcup_{j\in P}\{\mathcal{B}_{k_j,j}\}\Big)\Big] = 0 \in \mathcal{K}/B_k,$$

where $[K]$ is the B_k-equivalence class which contains the knot K.

Set

$$K_P = \Omega\Big(K_0; \bigcup_{j\in P}\{\mathcal{B}_{k_j,j}\}\Big).$$

CLAIM. *The knot K_1 $(= K_{\{1,\dots,l\}})$ is B_k-equivalent to a band sum of K_0 $(= K_\emptyset)$ and a set $\bigcup_{i,j}\{\mathcal{B}_{i,j}\}$ of local chords such that*

(1) *$\mathcal{B}_{i,j}$ is a B_i-chord $(i < k)$ and it has an associated subset $\omega(\mathcal{B}_{i,j}) \subset \{1,\dots,l\}$ with $\sum_{t\in\omega(\mathcal{B}_{i,j})}(k_t - 1) \le i - 1$,*

(2) *for each $P \subset \{1,\dots,l\}$,*

$$[K_P] = \Big[\Omega\Big(K_0; \bigcup_{\omega(\mathcal{B}_{i,j})\subset P}\{\mathcal{B}_{i,j}\}\Big)\Big].$$

Here a chord $\mathcal{B}_{i,j}$ is called a *local chord* if there is a 3-ball B such that B contains all the bands and the link ball of $\mathcal{B}_{i,j}$, B does not intersect any other bands or link balls, and $(B, B\cap K_0)$ is a trivial ball-arc pair.

Before proving the Claim, we will finish the proof of Theorem 3. Suppose K_1 is B_k-equivalent to a band sum of K_0 and some local chords $\mathcal{B}_{i,j}$. Each $\mathcal{B}_{i,j}$ represents a knot $K_{i,j}$ which is connected summed with K_0. So the band sum is a connected sum of K_0 and $K_{i,j}$'s. Then we have

$$\begin{aligned}
&\sum_{P\subset\{1,\dots,l\}}(-1)^{|P|}\Big[\Omega\Big(K_0; \bigcup_{\omega(\mathcal{B}_{i,j})\subset P}\{\mathcal{B}_{i,j}\}\Big)\Big]\\
&\qquad= \sum_{P\subset\{1,\dots,l\}}(-1)^{|P|}\Big([K_0] + \sum_{\omega(\mathcal{B}_{i,j})\subset P}[K_{i,j}]\Big)\\
&\qquad= \sum_{P\subset\{1,\dots,l\}}(-1)^{|P|}[K_0] + \sum_{P\subset\{1,\dots,l\}}(-1)^{|P|}\Big(\sum_{\omega(\mathcal{B}_{i,j})\subset P}[K_{i,j}]\Big)\\
&\qquad= 0 + \sum_{i,j}\Big(\sum_{P\subset\{1,\dots,l\},\,\omega(\mathcal{B}_{i,j})\subset P}(-1)^{|P|}\Big)[K_{i,j}].
\end{aligned}$$

We consider the coefficient of $[K_{i,j}]$. Since $\sum_{t\in\omega(\mathcal{B}_{i,j})}(k_t - 1) < k-1$, $\omega(\mathcal{B}_{i,j})$ is a proper subset of $\{1,\dots,l\}$. We may assume that $\omega(\mathcal{B}_{i,j})$ does not contain $a\in\{1,\dots,l\}$. Then

$$\begin{aligned}
\sum_{P\subset\{1,\dots,l\},\,\omega(\mathcal{B}_{i,j})\subset P}(-1)^{|P|} &= \sum_{P\subset\{1,\dots,l\}\setminus\{a\},\,\omega(\mathcal{B}_{i,j})\subset P}(-1)^{|P|}\\
&\quad+ \sum_{P\subset\{1,\dots,l\}\setminus\{a\},\,\omega(\mathcal{B}_{i,j})\subset P}(-1)^{|P\cup\{a\}|} = 0.
\end{aligned}$$

Thus, we have the conclusion. ∎

Now we will show the Claim.

Proof of Claim. We first set $\omega(\mathcal{B}_{k_j,j}) = \{j\}$ for $j = 1, \ldots, l$. Then we have $\sum_{t\in\omega(\mathcal{B}_{k_j,j})}(k_t - 1) = k_j - 1 < k - 1$ and

$$K_P = \Omega\Big(K_0; \bigcup_{\omega(\mathcal{B}_{k_j,j})\subset P} \{\mathcal{B}_{k_j,j}\}\Big).$$

Note that a crossing change between bands can be realized by crossing changes between K_0 and a band as illustrated in Figure 6. Therefore we can deform each chord into a local chord by (i) crossing changes between K_0 and bands, and (ii) band exchanges.

(i) When we perform a crossing change between K_0 and a B_p-band of a B_p-chord $\mathcal{B}_{p,q}$ with $p \leq k-2$, by using Lemma 7, we introduce a new B_{p+1}-chord $\mathcal{B}_{p+1,r}$ and we set $\omega(\mathcal{B}_{p+1,r}) = \omega(\mathcal{B}_{p,q})$ so that conditions (1) and (2) still hold. By Lemma 7, a crossing change between K_0 and a B_{k-1}-band is realized by a B_k-move and therefore does not change the B_k-equivalence class.

(ii) If we perform a band exchange between a B_p-chord $\mathcal{B}_{p,q}$ and a B_r-chord $\mathcal{B}_{r,s}$ with $p + r \leq k$, then, by using Lemma 8, we introduce a new B_{p+r-1}-chord $\mathcal{B}_{p+r-1,n}$ and set $\omega(\mathcal{B}_{p+r-1,n}) = \omega(\mathcal{B}_{p,q}) \cup \omega(\mathcal{B}_{r,s})$ so that conditions (1) and (2) still hold. By Lemmas 8 and 9, a band exchange between a B_p-chord $\mathcal{B}_{p,q}$ and a B_r-chord $\mathcal{B}_{r,s}$ with $p+r \geq k+1$ does not change the B_k-equivalence class. ∎

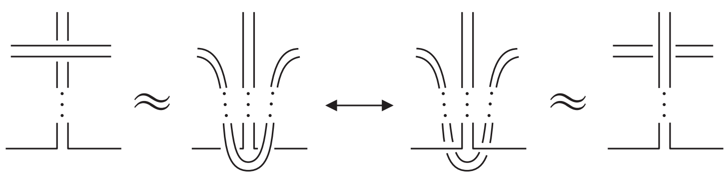


Fig. 6

Proof of Theorem 2. It is sufficient to show the existence of an inverse element for a given knot K. Suppose that there is a knot J such that $K \# J$ is B_k-equivalent to a trivial knot O. Then, by Lemma 6, O is a band sum of $K \# J$ and some B_k-chords. By using Lemma 7, we deform O up to B_{k+1}-equivalence so that the B_k-chords are local chords. Then the result is a connected sum of $K \# J$ and some knots $K_1, \ldots, K_n$ that correspond to local chords. Hence $K \# J \# K_1 \# \cdots \# K_n$ is B_{k+1}-equivalent to O. Thus $J \# K_1 \# \cdots \# K_n$ is the desired knot. ∎

Proof of Theorem 1. It is not hard to see that B_{l+1}-equivalent knots are also l-similar [9] (($l-1$)-equivalent [1]).

By Theorem 3, the projection $p_{l+1} : \mathcal{K} \to \mathcal{K}/B_{l+1}$ is a Vassiliev invariant of order $\leq l-1$. If two knots have the same values of any Vassiliev invariant of order $\leq l-1$, then they are B_{l+1}-equivalent. ∎

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