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Connection graphs

by

Piotr Bartłomiejczyk (Gdańsk)

Abstract. We introduce connection graphs for both continuous and discrete dynamical systems. We prove the existence of connection graphs for Morse decompositions of isolated invariant sets.

Introduction. In [4] Conley and Zehnder presented some generalization of the classical Morse theory to flows. Using the Conley index theory they proved the existence of periodic solutions of some Hamiltonian systems. Their main idea is to study isolated invariant sets by decomposing them into invariant subsets (Morse sets) and connecting orbits between them. This structure is called a Morse decomposition of an isolated invariant set. A filtration of index pairs associated with a Morse decomposition can be used to find connections between different Morse sets. The principal tools for this purpose are connection matrices and graphs. In [1, 2, 6–9] the connection matrix theory was developed for flows and homeomorphisms. In [5] Fiedler and Mischaikow introduced connection graphs for flows. The connection graph is a simplified version of the connection matrix. The vertices of these graphs correspond to the homological Conley indices of the Morse sets. The connection graphs provide some information on the structure of the Morse decomposition. For example, the edges of the graphs give a condition for the existence of connecting orbits between different Morse sets. Furthermore, the homological Conley index for the total invariant set is given by the vertices with no edges.

In this paper we prove the existence of connection graphs for both continuous and discrete dynamical systems. The existence of such a graph in the case of a flow was proved in [5]. Fiedler and Mischaikow deduced the existence of connection graphs from the existence of connection matrices. Our purpose is to present a direct construction of the connection graph. For

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that reason our proof makes no appeal to the connection matrix theory. It is based only on some simple ideas from linear algebra and algebraic topology.

The organization of the paper is as follows. Section 1 contains some preliminaries. In Sections 2 and 3 we study the properties of attractor-repeller pairs, Morse decompositions and index filtrations. In Section 4 our main result, the theorem on the existence of connection graphs is stated. Sections 5 and 6 contain the material from linear algebra and algebraic topology needed in Section 7, in which our main result is proved. Section 8 contains examples, which illustrate how connection graphs can be computed and used to detect connecting orbits. We admit that our examples are not convincing applications of the theory, as one of the referees rightly observed. Our goal was just to present the form of the connection graph for some well known and quite simple dynamical systems.

Besides [3] and [5], the works of of Mischaikow [13], Mrozek [14, 15], Reineck [16, 17] and Robbin and Salamon [18] are important references for the index theory presented here.

1. Preliminaries. Let (X,d) be a locally compact metric space and let

$$\mathbb{T}\times X\to X:(t,x)\mapsto f^t(x)$$

be a dynamical system on X with discrete time ($\mathbb{T}=\mathbb{Z}$) or continuous time ($\mathbb{T}=\mathbb{R}$). Let $f:=f^1$ denote the time-one map. Since we consider only one fixed dynamical system, we will use the convenient notation $xt:=f^t(x)$ for $x\in X$ and $t\in\mathbb{T}$. If $A\subset X$ and $\Delta\subset\mathbb{T}$ then $A\Delta:=\{xt\mid x\in A \text{ and } t\in\Delta\}$. For a given subset $N\subset X$ the set $\operatorname{Inv}(N):=\{x\in X\mid x\mathbb{T}\subset N\}$ is called the *invariant part* of N. We say that $S\subset X$ is *invariant* if $\operatorname{Inv}(S)=S$.

Recall that given a set $Y\subset X$ the *omega limit set* of Y is

$$\omega(Y):=\bigcap_{t>0}\operatorname{cl}(Y[t,\infty))$$

and the *alpha limit set* of Y is

$$\alpha(Y):=\bigcap_{t<0}\operatorname{cl}(Y(-\infty,t]).$$

Let S be a compact invariant set. A subset $A\subset S$ is called an *attractor* in S if there exists a neighbourhood U of A in S such that $\omega(U)=A$. A *repeller* is an attractor for the time-reversed dynamical system. For given subsets A, B of S we define the *connecting orbit set* by

$$C(A,B;S):=\{x\in S\mid \alpha(x)\subset A,\ \omega(x)\subset B\}.$$

A compact set $N\subset X$ is called an *isolating neighbourhood* if $\operatorname{Inv}(N)\subset\operatorname{int}(N)$. A set S is called an *isolated invariant set* if $S=\operatorname{Inv}(N)$ for some isolating neighbourhood N. A subset $A\subset L$ is said to be *positively invariant* in L if given $x \in A$ and $x[0,t] \subset L$, we have $x[0,t] \subset A$. A subset A of L is called an *exit set* for L if given $x \in L$ such that $x[0,\infty) \not\subset L$, there exists $t \geq 0$ such that $x[0,t] \subset L$ and $xt \in A$.

Let S be an isolated invariant set. A pair (N^1, N^0) of compact sets is called an *index pair* for S if:

(i) $S = \mathrm{Inv}(\mathrm{cl}(N^1 \setminus N^0)) \subset \mathrm{int}(N^1 \setminus N^0)$,
(ii) N^0 is positively invariant in N^1,
(iii) N^0 is an exit set for N^1.

In the case of a flow the *homological Conley index* is defined by

$$CH_*(S) := H_*(N^1/N^0, [N^0]) \approx H_*(N^1, N^0),$$

where (N^1, N^0) is any index pair for S and H_* stands for the singular homology with field coefficients. Unfortunately, it is not true that for any index pair $H_*(N^1/N^0, [N^0]) \approx H_*(N^1, N^0)$. Therefore, we need either an extra assumption on the (co)homology or an extra condition (regularity) on the index pair. In the first case, it is convenient to use the Alexander–Spanier cohomology functor with its strong excision property. We are convinced that all results of this paper concerning connection graphs can be reformulated in terms of the cohomological Conley index. The cohomological approach suffers only one disadvantage. Namely, since the cohomology functor is contravariant, the arrow of time in the phase portrait and the arrow (directed edge) in the connection graph point in opposite directions, which may be, in our opinion, misleading. Hence we prefer to assume that we are working with regular index pairs and index filtrations. In the discrete case the definition of the index is a little more complicated.

First we recall the notion of the *Leray functor* introduced by Mrozek (see [14, 15]). Let $\mathcal{E}$ be the category of graded vector spaces and linear maps of degree zero. The full subcategory of $\mathcal{E}$ consisting of all objects with finite-dimensional components and their morphisms will be denoted by $\mathcal{E}_0$. We define a new category $\mathrm{Endo}(\mathcal{E})$ as follows. Its objects are pairs (A, a), where $A \in \mathcal{E}$ and $a \in \mathcal{E}(A, A)$. Morphisms from (A, a) to (B, b) are all maps $\varphi \in \mathcal{E}(A, B)$ such that $\varphi a = b\varphi$. $\mathrm{Auto}(\mathcal{E})$ is the full subcategory of $\mathrm{Endo}(\mathcal{E})$ consisting of graded vector spaces with a distinguished isomorphism. For $(A, a) \in \mathrm{Endo}(\mathcal{E})$ we define the *generalized kernel* of a as

$$\mathrm{gker}(a) := \bigcup \{\ker(a^n) \mid n \geq 1\}.$$

Note that the quotient map

$$a' : A' \ni [x] \mapsto [a(x)] \in A', \quad \text{where} \quad A' := A/\mathrm{gker}(a),$$

is a well defined monomorphism. Then we restrict a' to the subspace

$$A'' = \mathrm{gIm}(a') := \bigcap \{\mathrm{Im}\,(a')^n \mid n \geq 0\}$$

called the *generalized image* of a'. Since $a'(A'') \subset A''$, the restriction $a'' := a'|_{A''} : A'' \to A''$ is a well defined automorphism of A''. Assume $\varphi : (A, a) \to (B, b)$ is a morphism in $\mathrm{Endo}(\mathcal{E})$. Let $\varphi' : A/\mathrm{gker}(a) \ni [x] \mapsto [\varphi(x)] \in B/\mathrm{gker}(b)$ denote the induced morphism and $\varphi'' = \varphi'|_{A''} : A'' \to B''$ its restriction to A''. We put $L(A, a) = (L(A), L(a)) := (A'', a'')$ and $L(\varphi) := \varphi''$. Thus we have defined a covariant functor $L : \mathrm{Endo}(\mathcal{E}) \to \mathrm{Auto}(\mathcal{E})$ called the *Leray functor*. If $N = (N^1, N^0)$ is an index pair, then the map $f_N : N^1/N^0 \to N^1/N^0$ defined by

$$f_N([x]) = \begin{cases} [f(x)] & \text{if } x, f(x) \in N^1 \setminus N^0, \\ [N^0] & \text{otherwise,} \end{cases}$$

is continuous (see e.g. [21, Lemma 4.3]). Just as in the case of flows (see remarks above) we have to assume that $H_*(N^1/N^0, [N^0])$ is isomorphic to $H_*(N^1, N^0)$. Once more an extra assumption on the index pair or the (co)homology guarantees that this isomorphism holds and hence f_N induces an endomorphism $f_{N*} : H_*(N^1, N^0) \to H_*(N^1, N^0)$. Therefore $(H_*(N^1, N^0), f_{N*}) \in \mathrm{Endo}(\mathcal{E})$ and consequently

$$L(H_*(N^1, N^0), f_{N*}) = (LH_*(N^1, N^0), L(f_{N*})) \in \mathrm{Auto}(\mathcal{E}).$$

We now define the *homological Conley index* of an isolated invariant set S as

$$CH_*(S) := LH_*(N),$$

where N is any index pair for S. It is proved in [12] that this definition is independent of the choice of an index pair N. It turns out that if f comes from a flow then f_N is homotopic to the identity on N^1/N^0 and therefore

$$L(H_*(N), f_{N*}) = (H_*(N), \mathrm{id}|_{H_*(N)}).$$

This is why we will write $CH_*(S) = LH_*(N)$ also in the case of a flow.

Since in our paper we want to use methods of finite-dimensional linear algebra, we will need the following assumption throughout the paper: for every isolated invariant set in the phase space X there exist index pairs (N^1, N^0) such that $LH_*(N^1, N^0)$ is a finite dimensional graded vector space. This assumption is in particular satisfied if X is a compact ANR (see [14, 15]). Consequently, denoting by $\mathrm{Auto}_0(\mathcal{E})$ the full subcategory of $\mathrm{Auto}(\mathcal{E})$ consisting of objects with finite-dimensional components and their morphisms we have

$$L(H_*(N^1, N^0), f_{N*}) = (LH_*(N^1, N^0), L(f_{N*})) \in \mathrm{Auto}_0(\mathcal{E})$$

for any index pair N in X.

The next result will not be needed until Section 7.

Proposition 1.1. *Let f be a dynamical system. Assume that $N \subset P$ are index pairs for an isolated invariant set S. Then the inclusion of pairs*

$i : N \to P$ induces an isomorphism

$$L(i_*) : LH_*(N) \to LH_*(P).$$

The proof is straightforward.

Recall that a pair (V, E) is called a *directed graph* (*graph* for short) if V is a finite set and $E \subset \{(u, v) \in V \times V \mid u \neq v\}$. Elements of V are called *vertices* and elements of E *edges*. We say that a graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. A *filtered graph* is a collection $\mathcal{G} = \{G_s\}_{s=1}^n$ of graphs such that G_s is a subgraph of G_{s+1} and $E_s = E_n \cap (V_s \times V_s)$ for each $s = 1, \ldots, n-1$.

2. Attractor-repeller pairs. If A is an attractor in S, then the set $A^* := \{x \in S \mid \omega(x) \cap A = \emptyset\}$ is a repeller in S. It is called the *repeller dual* to A in S. We call such a pair (A, A^*) an *attractor-repeller pair* in S. It is easy to check that if S is an isolated invariant set then so are A and A^*.

THEOREM 2.1. *Let S be an isolated invariant set and (A, A^*) be an attractor-repeller pair in S. Then there exists a triple $N^0 \subset N^1 \subset N^2$ of compact sets such that*:

(a) (N^2, N^0) *is an index pair for S*,
(b) (N^1, N^0) *is an index pair for A*,
(c) (N^2, N^1) *is an index pair for A^**.

The detailed proof in the case of a flow can be found in [19] and in the case of a homeomorphism in [15, 2].

If (A, A^*) is an attractor-repeller pair in S such that $CH_*(S)$, $CH_*(A)$, $CH_*(A^*)$ are graded vector spaces with finite-dimensional components, then we can construct a long exact sequence relating the homology indices of S, A and A^*. Namely, there is a long exact sequence

$$\cdots \to H_q(N^1, N^0) \xrightarrow{i} H_q(N^2, N^0) \xrightarrow{j} H_q(N^2, N^1) \xrightarrow{\partial} H_{q-1}(N^1, N^0) \to \cdots$$

where i, j are induced by inclusions and (N^2, N^1, N^0) is the triple given by Theorem 2.1. Applying the Leray functor we obtain an exact sequence of homological Conley indices

$$\cdots \to CH_q(A) \to CH_q(S) \to CH_q(A^*) \xrightarrow{\partial} CH_{q-1}(A) \to \cdots .$$

This sequence, called the *homology index sequence of the attractor-repeller pair*, provides an algebraic condition for the existence of connecting orbits. The map $\partial : CH_*(A^*) \to CH_*(A)$ is called the *connection map*.

THEOREM 2.2. *If the connection map ∂ is nontrivial*, *then $C(A^*, A; S) \neq \emptyset$.*

Proof. If $C(A^*, A; S) = \emptyset$ then $S = A \cup A^*$. From additivity of the Conley index $CH_*(S) \simeq CH_*(A) \oplus CH_*(A^*)$ and so $\partial = 0$. ∎

3. Morse decompositions and index filtrations. Let $(P,<)$ be a finite partially ordered set. A subset $I \subset P$ is called an *interval* if $p, q \in I$ and $p < r < q$ implies $r \in I$. The set of intervals will be denoted by $\mathcal{I}(<)$. An interval $I \subset P$ is called an *attracting interval* if $p \in I$ and $q < p$ implies that $q \in I$. The set of attracting intervals is written as $\mathcal{A}(<)$. $\mathcal{A}(<)$ is easily seen to be a lattice of sets. Two elements $p, q \in P$ are called *adjacent* if $\{p, q\} \in \mathcal{I}(<)$. Similarly, a pair (I, J) of disjoint intervals is called adjacent if

(i) $I \cup J \in \mathcal{I}(<)$,
(ii) $p \in I$, $q \in J$ implies $q \not< p$.

We write IJ instead of $I \cup J$. The collection of adjacent pairs of intervals is denoted by $\mathcal{I}_2(<)$.

Definition 3.1. A finite collection
$$\mathcal{M} = \{M(p) \mid p \in P\}$$
of mutually disjoint compact invariant subsets of an isolated invariant set S is called a *Morse decomposition* if there is a partial order $<$ on the indexing set P such that for every $x \in S \setminus \bigcup_{p \in P} M(p)$ there are $p, q \in P$ with $p < q$ such that $\omega(x) \subset M(p)$ and $\alpha(x) \subset M(q)$.

The sets $M(p)$ are called *Morse sets*. Observe that we do not assume that the above order on P is unique. Any such ordering on P is called *admissible*. Of all the admissible orderings for a Morse decomposition, there is a unique minimal order (one with the fewest relations) called the *dynamical system defined order* and denoted by $<_D$. This order is the transitive closure of the relation $p <_D q$ if $C(M(q), M(p); S) \neq \emptyset$. All other admissible orderings are extensions of $<_D$. For each $I \in \mathcal{I}(<)$ we define
$$M(I) := \Big(\bigcup_{p \in I} M(p)\Big) \cup \Big(\bigcup_{p,q \in I} C(M(p), M(q); S)\Big)$$
One can show

Proposition 3.2.

(i) *$M(I)$ is an isolated invariant set,*
(ii) *if $(I, J) \in \mathcal{I}_2(<)$, then $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$.*

Definition 3.3. An *index filtration* for the admissible ordering of the Morse decomposition $\mathcal{M}(<) = \{M(p) \mid p \in (P,<)\}$ is a collection of compact sets $\mathcal{N}(<) = \{N(\alpha) \mid \alpha \in \mathcal{A}(<)\}$ such that:

(1) for each $\alpha \in \mathcal{A}(<)$, $(N(\alpha), N(\emptyset))$ is an index pair for the attractor $M(\alpha)$,
(2) for each $\alpha, \beta \in \mathcal{A}(<)$, $N(\alpha \cap \beta) = N(\alpha) \cap N(\beta)$ and $N(\alpha \cup \beta) = N(\alpha) \cup N(\beta)$.

Let $I \in \mathcal{I}(<)$. Then there are $\alpha, \beta \in \mathcal{A}(<)$ such that $(\alpha, I) \in \mathcal{I}_2(<)$ and $\alpha \cup I = \beta$. It is easy to check that this implies that $(N(\beta), N(\alpha))$ is an index pair for $M(I)$. Thus the index filtration defines an index pair for each $M(I)$ where $I \in \mathcal{I}(<)$.

The following theorem proved by Salamon [19] for flows and by Richeson [6] for homeomorphisms gives the existence of index filtrations.

THEOREM 3.4. *For any given admissible ordering of the Morse decomposition there exists an index filtration.*

4. Connection graphs. We are now ready to introduce the notion of a connection graph, following [5]. Let $\mathcal{M} = \{M(i) \mid i \in P\}$ be a Morse decomposition of an isolated invariant set S.

DEFINITION 4.1. A finite directed graph G is called a *connection graph* for the Morse decomposition $\mathcal{M}$ if:

(1) the set of vertices of G has the form

$$\bigcup_{i\in P} \bigcup_{k=0}^{\infty} \operatorname{basis}_k(i),$$

where $\operatorname{basis}_k(i)$ is a basis for $CH_k(M(i))$ (elements of this basis will be denoted by e_k^i),

(2) each vertex has 1 edge or 0 edges (a vertex with no edges is called *free*),

(3) any edge has the form $e_k^i \to e_{k-1}^j$, where $i \neq j$,

(4) the set of vertices with no edges determines a basis for $CH_*(S)$, i.e. there is a monomorphism $\varphi : CH_*(S) \to \bigoplus_{i\in P} CH_*(M(i))$ such that $\operatorname{Im} \varphi$ is spanned by the set of free vertices of G,

(5) the relation $<_G$ defined as the transitive closure of $i <_G j$ iff there exists an edge in G of the form $CH_*(M(j)) \ni e \to e' \in CH_*(M(i))$, is a partial order on P (this partial order is called the *connection graph defined order*),

(6) $<_D$ extends $<_G$.

Observe that the vertices with no edges and the ones with edges provide complementary information about the Morse decomposition. Namely, the free vertices form the Conley index of the total invariant set, while the vertices with edges yield the existence of connecting orbits between different Morse sets in the decomposition.

We can now formulate the main result of this paper, which will be proved in Section 7.

THEOREM 4.2. *For any Morse decomposition of an isolated invariant set there exists a connection graph.*

The significance of the above result comes from the fact that it allows one to detect connecting orbits by combining the general properties of connection graphs with information about the Conley indices of Morse sets.

5. Algebraic lemma. Most of this section will be devoted to the proof of Lemma 5.1. This result may be viewed as a generalization of Theorem 2 in Kostrikin and Manin [11, Ch. 1, Sec. 8]. Roughly speaking, our lemma enables us to choose bases best fitting the structure of linear maps. More precisely, in matrix language, our result ensures the existence of bases for vector spaces in a sequence of linear maps such that all matrices of these maps are diagonal with entries in $\{0,1\}$. So it is not surprising that our proof is similar in spirit to one of the proofs of the Jordan decomposition theorem (see for instance [10, 11, 20]).

But first we have to introduce the following simple notion. Let A, B, C be vector spaces and let $A = B \oplus C$. We say that a basis $b(A)$ of A *agrees* with the decomposition $B \oplus C$ if for every $e \in b(A)$, either $e \in B$ or $e \in C$. We can now formulate the main result of this section.

LEMMA 5.1. *Let*

$$A^1 \xrightarrow{L_1} A^2 \xrightarrow{L_2} \cdots \xrightarrow{L_{n-2}} A^{n-1} \xrightarrow{L_{n-1}} A^n$$

be a sequence of linear maps of finite-dimensional vector spaces. Then for each $1 \le p \le n$ there exist subspaces V^p, $\widetilde{V}^p$ of A^p and a basis $b(A^p)$ such that:

- (i) $A^p = \operatorname{Im} L_{p-1} \oplus V^p$,
- (ii) $A^p = \ker L_p \oplus \widetilde{V}^p$,
- (iii) $b(A^p)$ *agrees with both of the above decompositions of* A^p,
- (iv) *if* (a^p_{ij}) *is the matrix of* L_p *with respect to the bases* $b(A^p)$ *and* $b(A^{p+1})$, *then* $a^p_{ii} = 1$ *for* $1 \le i \le r(p)$ *and* $a^p_{ij} = 0$ *for other* i, j.

REMARK 5.2. By part (iv) of the above lemma, $L_p(e) \in b(A^{p+1})$ or $L_p(e) = 0$ for any $e \in b(A^p)$ and $p = 1, \ldots, n-1$. Consequently, for every $e \in b(A^p)$ there exists a unique element a such that $a \in b(A^s) \setminus \operatorname{Im} L_{s-1}$ and $e = (L_{p-1} \circ \cdots \circ L_s)(a)$ for some $s \le p$. The element a is called the *origin* of e. In particular, if $e \notin \operatorname{Im} L_{p-1}$, then the origin of e is e itself.

Proof. For convenience of notation, we write

$$E^p_q := \operatorname{Im}(L_{p-1} \circ \cdots \circ L_{q+1} \circ L_q)$$

for $1 \le q < p \le n$. Observe that for each $p = 2, \ldots, n$ we have the filtration

$$E^p_1 \subset E^p_2 \subset \cdots \subset E^p_{p-1} \subset A^p.$$

The basic idea of the proof is to find bases for A^p best fitting the above filtrations and consistent with each other for different p. Using them we will define the subspaces V^p and $\widetilde{V}^p$. For clarity, the proof will be divided into four steps. The first step is based on the proof of Jordan's theorem given in [10].

STEP 1. We choose any basis for E_1^n and denote by $a_1^n(k)$ its elements. Since $a_1^n(k) \in E_1^n$, we have

$$a_1^n(k) = (L_{n-1} \circ \cdots \circ L_2 \circ L_1)(a_1^1(k))$$

for some $a_1^1(k) \in A^1$. We define

$$a_1^p(k) = (L_{p-1} \circ \cdots \circ L_2 \circ L_1)(a_1^1(k))$$

for each $p = 2, \ldots, n$. Next we extend the vectors $a_1^n(k)$ to a basis for E_2^n by adding vectors $a_2^n(k)$. We write $a_2^n(k) = (L_{n-1} \circ \cdots \circ L_2)(a_2^2(k))$ and define $a_2^p(k) = (L_{p-1} \circ \cdots \circ L_2)(a_2^2(k))$ for $p = 3, \ldots, n$. Once again we extend the vectors $a_1^n(k)$, $a_2^n(k)$ to a basis for E_3^n by adding vectors $a_3^n(k)$. A schematic view of this whole procedure is presented in Figure 1.

$$\begin{array}{cccccc}
a_1^1(k) & & & & & \\
\downarrow & & & & & \\
a_1^2(k) & a_2^2(k) & & & & \\
\downarrow & \downarrow & & & & \\
a_1^3(k) & a_2^3(k) & a_3^3(k) & & & \\
\downarrow & \downarrow & \downarrow & & & \\
\vdots & \vdots & \vdots & & & \\
\downarrow & \downarrow & \downarrow & & & \\
a_1^{n-1}(k) & a_2^{n-1}(k) & a_3^{n-1}(k) & \cdots & a_{n-1}^{n-1}(k) & \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \\
a_1^n(k) & a_2^n(k) & a_3^n(k) & \cdots & a_{n-1}^n(k) & a_n^n(k)
\end{array}$$

Fig. 1. Step 1 in the proof of Lemma 5.1

It is easy to check that the pth row in Figure 1 (we write it $a_{i\le p}^p(k)$ for short) can be treated as a basis of $A^p/\ker L_p$ for $p = 1, \ldots, n-1$. Consequently, the pth row represents a linearly independent set of vectors in A^p. Moreover, the set $\{a_1^p(k), a_2^p(k), \ldots, a_{p-1}^p(k)\}$ (we denote it briefly by $\{a_{i<p}^p(k)\}$) can be seen as a basis of $\operatorname{Im} L_{p-1}/\operatorname{Im} L_{p-1} \cap \ker L_p$ for $p = 2, \ldots, n-1$. Similarly, the vectors $a_{i<n}^n(k)$ form a basis for $\operatorname{Im} L_{n-1}$ and the vectors $a_n^n(k)$ extend this basis to a basis for A^n.

STEP 2. For fixed $p = 2, \ldots, n-1$ let $b^p(k)$ be a basis for $\operatorname{Im} L_{p-1} \cap \ker L_p$. This basis can be extended to a basis for $\ker L_p$ by adding vectors $c^p(k)$. Similarly, for $p = 1$ let $c^1(k)$ form any basis for $\ker L_1$.

STEP 3. Finally, for $p = 2, \ldots, n-1$ we define $b(A^p)$ to consist of all the vectors $a^p_{i\leq p}(k)$, $b^p(k)$, $c^p(k)$, and we set

$$V^p := \operatorname{span}\{c^p(k), a^p_p(k)\}, \qquad \widetilde{V}^p := \operatorname{span}\{a^p_{i\leq p}(k)\}$$

as shown in Figure 2.

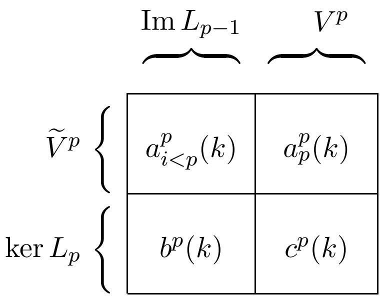

Fig. 2. Step 3 in the proof of Lemma 5.1

Since there is no image in A^1 and no kernel in A^n, the above definitions have to be changed slightly for $p = 1$ and $p = n$. Namely, let $b(A^1)$ consist of the vectors $a^1_1(k)$, $c^1(k)$ and let $b(A^n)$ consist of the vectors $a^n_{i\leq n}(k)$. Moreover, set

$$\widetilde{V}^1 := \operatorname{span}\{a^1_1(k)\}, \qquad V^n := \operatorname{span}\{a^n_n(k)\}.$$

We see at once that the bases $b(A^p)$ and the subspaces V^p, $\widetilde{V}^p$ satisfy the assertion of the lemma. ■

6. Topological filtrations and graphs. In this section we introduce filtered graphs for topological filtrations and present a general result on such graphs. These graphs are not directly connected to the Conley index theory. In fact, it is possible to give the definition of the filtered graph for the topological filtration even in the absence of any dynamical system. However, the material of this section will be needed in the next one devoted to connection graphs. For this reason, it seems preferable to take the presence of a flow or a homeomorphism into consideration when formulating the results of this section. That explains why we use the Leray functor instead of the usual homology.

Consider a topological filtration $\mathcal{N} = \{N^i\}_{i=0}^n$, i.e. a filtration $N^0 \subset N^1 \subset \cdots \subset N^n$ of topological spaces. Let us introduce the following notation for $i = 1, \ldots, n$ and $k = 0, 1, 2, \ldots$:

$$A^i := LH_*(N^i, N^0), \qquad B^i := LH_*(N^i, N^{i-1}),$$
$$A^i_k := LH_k(N^i, N^0), \qquad B^i_k := LH_k(N^i, N^{i-1}).$$

From now on we assume that all A^i and B^i are finite-dimensional linear spaces. The filtration $\mathcal{N}$ determines the following exact sequences:

$$\begin{tikzcd} A^s \arrow[rr, "i"] & & A^{s+1} \arrow[dl, "j"] \\ & B^{s+1} \arrow[ul, "\partial"] & \end{tikzcd} \tag{10}$$

for $s = 1, \ldots, n-1$.

DEFINITION 6.1. We say that $\mathcal{G} = \{G_s\}_{s=1}^n$ is a *filtered graph for the topological filtration* $\mathcal{N} = \{N_s\}_{s=0}^n$ if for every $s = 1, \ldots, n$:

(1) the set of vertices of G_s has the form
$$\bigcup_{i=1}^{s} \bigcup_{k=0}^{\infty} b(B_k^i),$$
where $b(B_k^i)$ is a basis for B_k^i (elements of this basis will be denoted by b_k^i),

(2) each vertex has 1 edge or 0 edges (a vertex with no edges is called *free*),

(3) there is in G_s an edge of the form $b_k^q \to b_l^p$ iff the following conditions are satisfied:
   - (3.1) $1 \le p < q \le s$,
   - (3.2) $l = k - 1$,
   - (3.3) there exists an element $a \in A^p$ such that $ja = b_l^p$ and $\partial b_k^q = i^{q-p-1} a \neq 0$,

(4) the set of vertices of G_s with no edges determines a basis of A^s, i.e. there is a monomorphism $\varphi : A^s \to \bigoplus_{i=1}^s B^i$ such that $\operatorname{Im}\varphi$ is spanned by the set of free edges of G_s.

The following diagram explains the idea behind condition (3.3) of the above definition.

$$\begin{tikzcd} & A_{k-1}^p \arrow[r, "j"] \arrow[d, "i"] & B_{k-1}^p \ni b_l^p \\ & A_{k-1}^{p+1} \arrow[d, "i"] & \\ & \vdots \arrow[d, "i"] & \\ b_k^q \in B_k^q \arrow[r, "\partial"] & A_{k-1}^{q-1} & \end{tikzcd}$$

We can formulate the main result of this section.

THEOREM 6.2. *Let $\mathcal{N} = \{N^s\}_{s=0}^n$ be a topological filtration such that all A^i and B^i are finite-dimensional. Then there exists a filtered graph for $\mathcal{N}$.*

Proof. The proof will be divided into three steps.

STEP 1: *Construction of vertices of G_s.* Consider the following sequence of homomorphisms:

$$A^1 \xrightarrow{i} A^2 \xrightarrow{i} \cdots \xrightarrow{i} A^{n-1} \xrightarrow{i} A^n,$$

where the i are induced by inclusions. By Lemma 5.1, there exist subspaces V^s, $\widetilde{V}^s$ and bases $b(A^s)$ that agree with the decompositions

$$A^s = \ker i \oplus \widetilde{V}^s = \operatorname{Im} \partial \oplus \widetilde{V}^s, \qquad A^s = \operatorname{Im} i \oplus V^s = \ker j \oplus V^s.$$

Let W^s denote any subspace complementary to $\operatorname{Im} j = \ker \partial$ in B^s, i.e. $B^s = \operatorname{Im} j \oplus W^s = \ker \partial \oplus W^s$ for $s = 2, \ldots, n$. Observe that we have already chosen a basis in $B^1 = A^1$. Under the above notations, the maps

$$j|_{V^{s+1}} : V^{s+1} \to \operatorname{Im} j = \ker \partial, \qquad \partial|_{W^{s+1}} : W^{s+1} \to \operatorname{Im} \partial = \ker i,$$

which are isomorphisms for $s = 1, \ldots, n-1$, and the bases $b(A^s)$ uniquely define for $s = 2, \ldots, n$ bases $b(B^s)$ that agree with the decompositions $B^s = \operatorname{Im} j \oplus W^s$. Finally, we define the vertices of G_s to be $\bigcup_{i=1}^s b(B^i)$.

STEP 2: *Construction of edges of G_s.* Let each vertex in $b(W^q) \subset b(B^q)$ $(q = 2, \ldots, s)$ be the initial vertex of some edge. We will show how to find the end of this edge. All other vertices will be free. Let $b \in b(W^q) \subset b(B^q)$, and so $e = \partial b \in b(A^{q-1})$. If $a \in A^p$ for $p < q$ is the origin of e, as in Remark 5.2, then $c = ja$ is the end of the edge that begins at b. This ends the construction of a filtered graph $\mathcal{G}$. The above procedure shows that $\mathcal{G}$ satisfies conditions (1)–(3) of the definition. What is left is to prove (4).

STEP 3: *Construction of φ.* We define a linear map

$$\varphi : A^s \to \bigoplus_{i=1}^s B^s$$

by a procedure complementary to determining the edges of G_n. Let $b(A^i)$, $b(B^i)$ be the bases chosen in Step 1. It is enough to define φ on the elements of $b(A^s)$. Let $a \in b(A^s)$. Set

$$\varphi a = jc,$$

where $c \in b(A^{s-l})$ is the origin of a, as in Remark 5.2. By definition, φ provides the desired monomorphism. ∎

7. Proof of the main result. We recall that our main result states the existence of connection graphs for both continuous and discrete dynamical systems. Let $\mathcal{M} = \{M(i) \mid i \in P\}$ be a Morse decomposition of an isolated invariant set S. Consider the dynamical-system-defined ordering of the Morse decomposition $\mathcal{M}$, i.e.

$$\mathcal{M}(<_D) = \{M(i) \mid i \in (P, <_D)\}.$$

By Theorem 3.4, there exists an index filtration

$$\mathcal{N}(<_D) = \{N(I) \mid I \in \mathcal{A}(<_D)\}$$

for $\mathcal{M}(<_D)$. Let $\mathcal{M}(<_L)$ be any admissible linear ordering of the Morse decomposition $\mathcal{M}$. Observe that $<_L$ always exists and $<_L$ extends $<_D$. Furthermore, we will write elements of P as $1, 2, \dots, n$ according to $<_L$. Define $I_i := \{1, 2, \dots, i-1, i\}$ for $i = 0, 1, \dots, n$. In particular, $I_0 = \emptyset$. It follows that $\mathcal{A}(<_L) = \{I_i\}_{i=0}^n \subset \mathcal{A}(<_D)$. Now we define the following topological filtration:

$$\mathcal{N} := \{N(I) \in \mathcal{N}(<_D) \,|\, I \in \mathcal{A}(<_L)\}.$$

Writing $N^i := N(I_i)$ we obtain $\mathcal{N} = \{N^i\}_{i=0}^n$. Observe that $\mathcal{N}$ is an index filtration for $\mathcal{M}(<_L)$, but since $\mathcal{N}$ comes from $\mathcal{N}(<_D)$, $\mathcal{N}$ has some additional properties, which are not included in the definition of an index filtration for $\mathcal{M}(<_L)$. Applying Theorem 6.2 we get a filtered graph $\mathcal{G} = \{G_i\}_{i=1}^n$ for $\mathcal{N}$. Finally, we claim that G_n is a connection graph for $\mathcal{M}$. By the definition of the filtered graph, G_n satisfies conditions (1)–(5) of the definition of the connection graph. It remains to prove that $<_D$ extends $<_G$. We start with the observation that the last assertion is nothing but the statement that for any edge $e_k^j \to e_{k-1}^i$ we have $i <_D j$. To obtain a contradiction, suppose that there exists an edge $e_k^j \to e_{k-1}^i$ with $\neg(i <_D j)$. By the definition of the filtered graph, this implies $i <_L j$. Set

$$J := \{p \in P \,|\, p \leq_D j\}, \qquad J' := J \setminus \{j\}.$$

By definition, $i \notin J$ and $J, J' \in \mathcal{A}(<_D)$. From this we conclude that for $A \in \{I_i, I_{i-1}\} \subset \mathcal{A}(<_L) \subset \mathcal{A}(<_D)$ and $B \in \{J, J'\}$ we have $A \cup B \in \mathcal{A}(<_D)$. Consider a commutative diagram $(*)$ of Figure 3, in which every row corresponds to some exact sequence of a triple from the filtration $\mathcal{N}(<_D)$ and

$$\begin{array}{ccccc}
 & & LH_k(N(I_{i-1} \cup J), N(I_{i-1})) & \to & LH_{k-1}(N(I_{i-1}), N(\emptyset)) \\
 & & \updownarrow & & \downarrow \\
LH_k(N(I_i \cup J), N(\emptyset)) & \to & LH_k(N(I_i \cup J), N(I_i)) & \to & LH_{k-1}(N(I_i), N(\emptyset)) \\
\| & & \downarrow & & \downarrow \\
LH_k(N(I_i \cup J), N(\emptyset)) & \to & LH_k(N(I_i \cup J), N(I_i \cup J')) & \to & LH_{k-1}(N(I_i \cup J'), N(\emptyset)) \\
 & & \updownarrow & & \downarrow \\
 & & LH_k(N(I_j), N(I_{j-1})) & \to & LH_{k-1}(N(I_{j-1}), N(\emptyset))
\end{array}$$

Fig. 3. Diagram $(*)$

the vertical homomorphisms are induced by inclusions of pairs. Moreover, the maps denoted by $\updownarrow$ are isomorphisms since they correspond to inclusions of index pairs of the same isolated invariant sets ($M(i)$ and $M(J - I_i)$ respectively).

From the diagram $(*)$ and the definition of the filtered graph, we obtain a new commutative diagram $(**)$ (see Figure 4), in which the elements a^i, a^{j-1} are defined as in (3.3) of the above mentioned definition. The notation $x \to y$ means that x is mapped onto y in the diagram $(*)$.

$$\begin{array}{ccccc}
 & & \boxed{d} & \to & \boxed{e} \\
 & & \updownarrow & & \downarrow \\
h & \to & \boxed{c} & \to & a^i \\
\| & & \downarrow & & \downarrow \\
h & \to & \boxed{a} & \to & \boxed{b} \\
 & & \updownarrow & & \downarrow \\
 & & e^j_k & \to & a^{j-1}
\end{array}$$

Fig. 4. Diagram $(**)$

The following reasoning shows that $e \in LH_*(N(I_{i-1}), N(\emptyset))$ is nonzero. Since the vertical arrow over e^j_k corresponds to an isomorphism, $a \neq 0$. By the commutativity of the bottom square in $(*)$, we get $b \neq 0$. If c were zero, there would be h such that $h \to c$, which contradicts the commutativity of the second and third rows of $(*)$. Therefore $c \neq 0$. The vertical arrow over c corresponds to an isomorphism, and consequently, $d \neq 0$. Finally, by the commutativity of the top square in $(*)$, $e \neq 0$.

It remains to consider two cases. If $i = 1$, then $LH_{k-1}(N(I_{i-1}), N(\emptyset)) = LH_{k-1}(N(\emptyset), N(\emptyset)) = \{0\}$, a contradiction. If $i > 1$, we get the exact sequence

$$LH_{k-1}(N^{i-1}, N^0) \to LH_{k-1}(N^i, N^0) \to LH_{k-1}(N^i, N^{i-1})$$

with three nonzero elements such that $e \to a^i \to e^i_{k-1}$, which contradicts the exactness of the above sequence and completes the proof.

8. Examples. Let D^2 be the closed unit ball in $\mathbb{R}^2$ and let φ be a flow on the plane with the dynamics as in Figure 5.

Assume that $S = D^2$ is an isolated invariant set and that $M(0) = (0, 0)$, $M(1^\pm) = (\pm 1, 0)$, $M(2^\pm) = (0, \pm 1)$ form a Morse decomposition $\mathcal{M}$ of S with the flow-defined order $0 < 1^\pm < 2^\pm$. Moreover, a simple verification shows that the local Conley indices of the Morse sets are

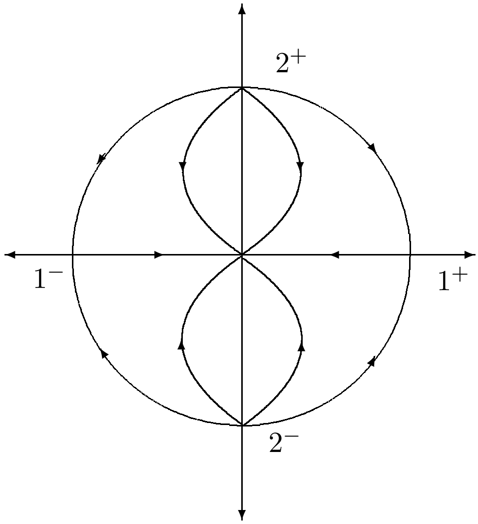

Fig. 5. The dynamics of φ

$$CH_k(M(0)) = \begin{cases} \mathbb{Q} & \text{if } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$CH_k(M(p^\pm)) = \begin{cases} \mathbb{Q} & \text{if } k = p, \\ 0 & \text{otherwise,} \end{cases}$$

and the total Conley index of the whole set is

$$CH_k(M(S)) = \begin{cases} \mathbb{Q} & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

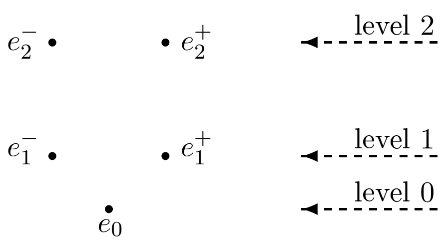

Fig. 6. Vertices of any connection graph for φ

Let $e_0, e_1^\pm, e_2^\pm$ be any basis vectors for $CH_0(M(0))$, $CH_1(M(1^\pm))$, $CH_2(M(2^\pm))$ respectively. Consequently, each connection graph for the Morse decompositions $\mathcal{M}$ has five vertices as in Figure 6. Three levels in Figure 6 correspond to the natural gradation in homology. Theorem 4.2 states that the set of connection graphs for $\mathcal{M}$ is nonempty, but we see at once that in this special case we have exactly four different connection graphs as in Figure 7.

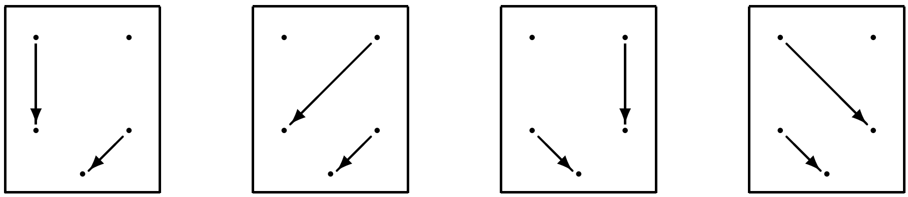

Fig. 7. All connection graphs for φ

Note that each of them contains the same information about the total Conley index (the set of free vertices consists of one element of homological level two) but completely different information about connecting orbits. For instance, the edges of the first connection graph imply the existence of connecting orbits from $M(2^-)$ to $M(1^-)$ and from $M(1^+)$ to $M(0)$, while the edges of the third one yield the existence of connections from $M(2^+)$ to $M(1^+)$ and from $M(1^-)$ to $M(0)$. Consequently, each connection graph provides only partial information about the structure of the Morse decomposition. On the other hand, the set of all connection graphs gives the full knowledge of connecting orbits in the Morse decomposition.

The second example is adapted from [15]. Let $D \subset \mathbb{R}^2$ be a square and let $f_0 : D \to D$ be a continuous map as indicated in Figure 8. Extend f_0 to a homeomorphism $f : S^2 \to S^2$ with a repelling point r outside D.

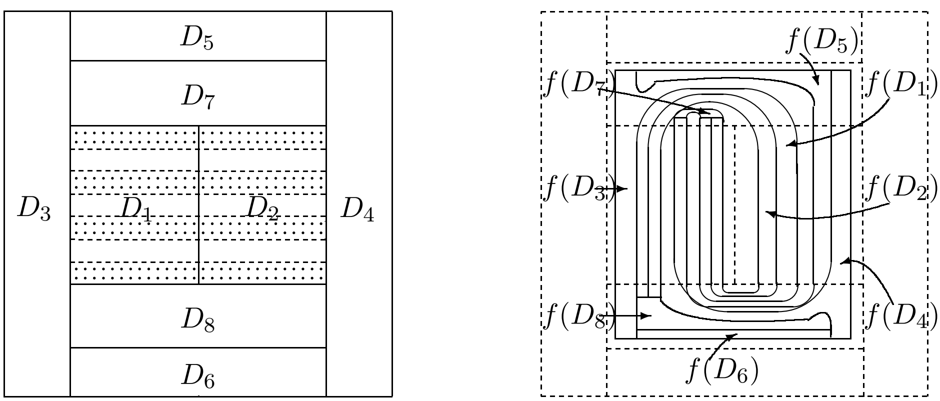

Fig. 8. The dynamics of f

Take $M(0) = \mathrm{Inv}(D_7 \cup D_8)$, $M(1) = \mathrm{Inv}(D_1 \cup D_2)$, $M(2) = \{r\}$. It is easy to check that $\mathcal{M} = \{M(0), M(1), M(2)\}$ is a Morse decomposition of $S = S^2$ with admissible ordering $0 < 1 < 2$ and $N^0 = \emptyset$, $N^1 = D_7 \cup D_8 \cup P$

(P is the union of dotted areas), $N^2 = D_1 \cup D_2 \cup D_7 \cup D_8$ is an index triple for $\mathcal{M}$. Moreover, an easy computation shows that

$$CH_k(M(0)) = \begin{cases} \mathbb{Q}^2 & \text{if } k = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$CH_k(M(p)) = \begin{cases} \mathbb{Q} & \text{if } k = p, \\ 0 & \text{otherwise}, \end{cases}$$

for $p = 1, 2$. We will denote by $\{e_0, e_0'\}$, $\{e_1\}$, $\{e_2\}$ bases of the vector spaces $CH_0(M(0))$, $CH_1(M(1))$ and $CH_2(M(2))$ respectively. Each connection graph for $\mathcal{M}$ has four vertices as in Figure 9.

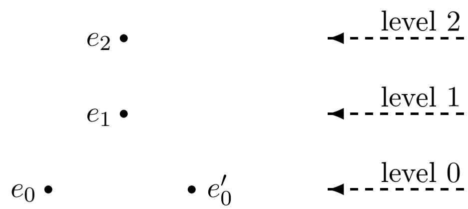

Fig. 9. Vertices of any connection graph for f

It is obvious that in this case we have two different connection graphs as in Figure 10.

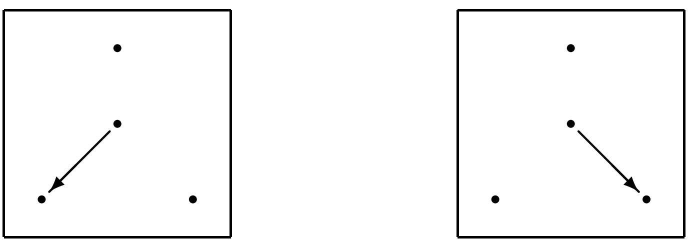

Fig. 10. All connection graphs for f

Both of them contain identical information:

- the set of free vertices forms the global Conley index of $S = S^2$,
- there are connecting orbits from $M(1)$ to $M(0)$.

Observe that none of them provides any information about connections from $M(2)$ to $M(1)$.

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Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: pb@math.univ.gda.pl

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