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Reflexive families of closed sets

by

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Abstract. Let $S(X)$ denote the set of all closed subsets of a topological space X, and $C(X)$ the set of all continuous mappings $f : X \to X$. A family $\mathcal{A} \subseteq S(X)$ is called *reflexive* if there exists $\mathcal{F} \subseteq C(X)$ such that $\mathcal{A} = \{A \in S(X) : f(A) \subseteq A \text{ for every } f \in \mathcal{F}\}$. We investigate conditions ensuring that a family of closed subsets is reflexive.

Recall [3] that a collection $\mathcal{A}$ of closed subspaces of a Hilbert space H is called *reflexive* if there exists a collection $\mathcal{F}$ of continuous operators on H such that

$$\mathcal{A} = \mathrm{Lat}(\mathcal{F}) = \{A : A \text{ is a closed subspace of } H \text{ with } T(A) \subseteq A, \forall T \in \mathcal{F}\}.$$

Reflexive families of continuous operators are defined in a dual way. See [2–7] for characterizations of such families. In [8], the second author considered reflexive families in concrete categories. For the category **SET** of sets, he obtained complete characterizations for both reflexive families of sets and reflexive families of mappings. In the present paper we investigate reflexive families in the context of topological spaces.

Given a topological space X, let $S(X)$ be the set of all closed subsets of X and $C(X)$ be the set of all continuous mappings $f : X \to X$. For any $\mathcal{A} \subseteq S(X)$ and $\mathcal{F} \subseteq C(X)$ define

$$\mathrm{Alg}(\mathcal{A}) = \{f \in C(X) : f(A) \subseteq A \text{ for every } A \in \mathcal{A}\},$$
$$\mathrm{Lat}(\mathcal{F}) = \{A \in S(X) : f(A) \subseteq A \text{ for every } f \in \mathcal{F}\}.$$

The two mappings Alg and Lat form a Galois connection between the sets of all subsets of $S(X)$ and $C(X)$, respectively. Thus, for any $\mathcal{A} \subseteq S(X)$ and $\mathcal{F} \subseteq C(X)$ we have

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(i) $\mathrm{Lat}(\mathrm{Alg}(\mathcal{A})) \supseteq \mathcal{A}, \mathrm{Alg}(\mathrm{Lat}(\mathcal{F})) \supseteq \mathcal{F}$;
(ii) $\mathrm{Alg}(\mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))) = \mathrm{Alg}(\mathcal{A})$, $\mathrm{Lat}(\mathrm{Alg}(\mathrm{Lat}(\mathcal{F}))) = \mathrm{Lat}(\mathcal{F})$.

A family $\mathcal{A} \subseteq S(X)$ is called *reflexive* if $\mathcal{A} = \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$. Similarly, $\mathcal{F} \subseteq C(X)$ is *reflexive* if $\mathcal{F} = \mathrm{Alg}(\mathrm{Lat}(\mathcal{F}))$.

As in the general case [8], $\mathcal{A} \subseteq S(X)$ is reflexive if and only if there exists $\mathcal{F} \subseteq C(X)$ such that $\mathcal{A} = \mathrm{Lat}(\mathcal{F})$. Also by (i), $\mathcal{A}$ is reflexive if and only if

$$\mathrm{Lat}(\mathrm{Alg}(\mathcal{A})) \subseteq \mathcal{A}.$$

LEMMA 1. *If $\mathcal{A} \subseteq S(X)$ is reflexive, then*:

(a) $X, \emptyset \in \mathcal{A}$.
(b) $\mathcal{B} \subseteq \mathcal{A}$ *implies* $\bigcap \mathcal{B} \in \mathcal{A}$.
(c) $\mathcal{B} \subseteq \mathcal{A}$ *implies* $\mathrm{cl}(\bigcup \mathcal{B}) \in \mathcal{A}$.
(d) *If D is a connected component of $A \in \mathcal{A}$ and $B \subseteq D$ for some nonempty B in $\mathcal{A}$, then $D \in \mathcal{A}$.*

Proof. Only (d) needs verification. For any $f \in \mathrm{Alg}(\mathcal{A})$, $f(B) \subseteq B$ since $B \in \mathcal{A}$. Also $f(B) \subseteq f(D) \subseteq f(A) \subseteq A$, and $f(D) \cap D \supseteq f(B) \cap B = f(B) \neq \emptyset$. Thus $f(D)$ is a connected set contained in A and has nonempty intersection with the connected component D of A, hence $f(D) \subseteq D$. Therefore $D \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{A})) = \mathcal{A}$. ■

If X is a space with the discrete topology, then $S(X)$ is the set of all subsets of X and $C(X)$ is the set of all mappings from X to X. By [8, Theorem 1], a family $\mathcal{A}$ of subsets of X is reflexive if and only if $\mathcal{A}$ is closed under arbitrary unions and intersections, so every family $\mathcal{A}$ of closed subsets of a discrete space satisfying conditions (a)–(c) in Lemma 1 is reflexive. A natural question is: besides the discrete topological spaces, what other spaces also have this property?

DEFINITION 1. A topological space X is called *s-reflexive* if every family $\mathcal{A}$ of closed subsets of X satisfying conditions (a)–(c) in Lemma 1 is reflexive.

The main results in this paper are: every strongly zero-dimensional complete metric space is s-reflexive; every countable metric space is s-reflexive; every Hausdorff s-reflexive space is hereditarily disconnected. From these it is deduced that a locally compact metric space is s-reflexive if and only if it is zero-dimensional.

LEMMA 2. *For every topological space X, the following conditions are equivalent*:

(1) *For any nonempty proper closed subset B and any finite subset D of X, both $B \setminus D$ and $B \cup D$ are closed.*

(2) *For any nonempty finite set $D \subseteq X$ and any element $b \in X$, the mapping $f_{D,b}$ which sends D to b and is the identity on $X \setminus D$, is continuous.*
(3) *For any points $a, b \in X$, the above mapping $f_{\{a\},b}$ is continuous.*

Remark 1. If X satisfies the equivalent conditions in Lemma 2 with $|X| \neq 2$ and there exists one closed singleton $\{a\}$, then X is T_1. The assumption that $|X| \neq 2$ is essential.

Let $\mathcal{A} \subseteq S(X)$ be a collection of closed subsets of X satisfying conditions (a)–(c) in Lemma 1. For each $Y \in S(X)$, let $\phi_{\mathcal{A}}(Y) = \bigcap\{A \in \mathcal{A} : Y \subseteq A\}$. If no confusion occurs, we simply write $\phi(Y)$ for $\phi_{\mathcal{A}}(Y)$. The following lemma can be verified easily.

Lemma 3. *Let $\mathcal{A} \subseteq S(X)$ satisfy conditions* (a)–(c)*.*

(1) *For any $Y \subseteq X$, $\phi(Y) \in \mathcal{A}$. And $Y \in \mathcal{A}$ if and only if $Y = \phi(Y)$.*
(2) *For any $B \in S(X)$, $B \in \mathcal{A}$ if and only if $\phi(\{x\}) \subseteq B$ for all $x \in B$.*
(3) *For any $B \in S(X)$, $B \in \mathcal{A}$ if and only if $B = \bigcup\{\phi(\{x\}) : x \in B\}$.*

Lemma 4. *Let $\mathcal{A} \subseteq S(X)$ satisfy conditions* (a)–(c)*. For each $U \subseteq X$ define $\kappa(U) = \{x \in X : \phi(\{x\}) \cap U \neq \emptyset\}$. Then*

(1) *$\kappa(V) \supseteq V$ for all $V \subseteq X$.*
(2) *$\kappa(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} \kappa(U_i)$.*
(3) *If U is an open subset of X, then $\kappa(U)$ is also open.*

Proof. We only prove (3). Let U be an open subset of X and $(x_\lambda)_{\lambda\in D}$ be a net in $X \setminus \kappa(U)$ which converges to a point $a \in X$. For each $\lambda \in D$, $\phi(\{x_\lambda\}) \cap U = \emptyset$. Thus $\mathrm{cl}(\bigcup_{\lambda\in D} \phi(\{x_\lambda\})) \cap U = \emptyset$. By (c), the set $B = \mathrm{cl}(\bigcup_{\lambda\in D} \phi(\{x_\lambda\}))$ is in $\mathcal{A}$. Now $a \in B$, so $\phi(\{a\}) \subseteq B \subseteq X \setminus U$. Hence $\phi(\{a\}) \cap U = \emptyset$, which implies $a \in X \setminus \kappa(U)$. Therefore, $\kappa(U)$ is open. ■

Theorem 1. *If a space X satisfies the equivalent conditions in Lemma* 2*, then X is s-reflexive.*

Proof. Suppose $\mathcal{A} \subseteq S(X)$ satisfies (a)–(c). Let B be a closed set not in $\mathcal{A}$. By Lemma 3(2), there is $a \in B$ such that $\phi(\{a\}) \not\subseteq B$. Choose a point $b \in \phi(\{a\}) \setminus B$. The mapping $f_{\{a\},b}$ defined in Lemma 2 is continuous. For every $A \in \mathcal{A}$, if $a \in A$ then $\phi(\{a\}) \subseteq A$, so $f(a) = b \in \phi(\{a\}) \subseteq A$, hence $f(A) \subseteq A$. If $a \notin A$, then $f(A) = A$. Thus $f \in \mathrm{Alg}(\mathcal{A})$. But $f(B) \not\subseteq B$, so $\mathcal{A} \supseteq \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$ and hence $\mathcal{A} = \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$, which shows that $\mathcal{A}$ is reflexive. ■

Example 1. (1) Every discrete and every anti-discrete space satisfies the conditions in Lemma 2, so they are s-reflexive.

(2) Let λ be an infinite cardinal and X be any nonempty set. Define τ to be the topology consisting of all subsets whose complements are either

X or have cardinality less than λ. Then (X, τ) satisfies the conditions in Lemma 2, so it is s-reflexive.

(3) Suppose E is a subset of a nonempty set X. Define τ_E to be the topology consisting of those $A \subseteq X$ such that either $A = X$, or $A \setminus E$ is finite. Then (X, τ_E) satisfies the conditions in Lemma 2, and so it is s-reflexive.

(4) The Euclidean interval $X = [-1, 1]$ is not s-reflexive. As a matter of fact, let $\mathcal{A} = \{A \in S(X) : A \subseteq [0, 1] \text{ or } A \ni 1\}$. Then $\mathcal{A}$ satisfies conditions (a)–(c) in Lemma 1. For every $f \in \mathrm{Alg}(\mathcal{A})$ and every $x \in X$, since $\{x, 1\} \in \mathcal{A}$, we have $f(\{x, 1\}) \subseteq \{x, 1\}$, so either $f(x) = x$ or $f(x) = 1$. Let $A = \{x \in [-1, 1) : f(x) = x\}$ and $B = \{x \in [-1, 1) : f(x) = 1\}$. Then A and B are two disjoint closed sets in $[-1, 1)$ and $A \cup B = [-1, 1)$. Hence $A = \emptyset$ or $B = \emptyset$ because $[-1, 1)$ is connected. For every $x \in [0, 1]$, since $\{x\} \in \mathcal{A}$ we have $f(x) = x$. It follows that $A \supseteq [0, 1) \neq \emptyset$. This implies $A = [-1, 1)$. Note that $f(1) = 1$ for every $f \in \mathrm{Alg}(\mathcal{A})$ because $\{1\} \in \mathcal{A}$. Thus $\mathrm{Alg}(\mathcal{A}) = \{\mathrm{id}_X\}$. But $\mathrm{Lat}(\{\mathrm{id}_X\}) = S(X) \neq \mathcal{A}$. Hence $\mathcal{A}$ is not reflexive.

Since the interval $[-1, 1]$ equipped with the discrete topology is s-reflexive and the Euclidean interval is a continuous image of it, it follows that continuous images of s-reflexive spaces need not be s-reflexive.

Let $\mathcal{U}$ be an open cover of a strongly zero-dimensional metric space X and $\varepsilon > 0$. Then there exists a locally finite open refinement $\mathcal{W}$ of $\mathcal{U}$ such that for every $W \in \mathcal{V}$, the diameter $\mathrm{diam}(W)$ of W is less than ε. It then follows from [1, Theorems 7.3.2 and 7.2.4] that there exists a refinement $\mathcal{V}$ of $\mathcal{W}$ consisting of pairwise disjoint clopen sets. Thus we have the following lemma.

Lemma 5. *Let $\mathcal{U}$ be an open cover of a strongly zero-dimensional metric space X. Then for any $\varepsilon > 0$, there is a refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ consists of pairwise disjoint clopen sets and* $\mathrm{diam}(V) < \varepsilon$ *for all $V \in \mathcal{V}$.*

Theorem 2. *Every strongly zero-dimensional complete metric space is s-reflexive.*

Proof. Let (X, d) be a strongly zero-dimensional complete metric space and let $\mathcal{A}$ be a family of closed sets in X satisfying conditions (a)–(c) in Lemma 1. Suppose $B \in S(X)$ and $B \notin \mathcal{A}$. We shall define an $f \in \mathrm{Alg}(\mathcal{A})$ so that $f(B) \nsubseteq B$. By Lemma 3(2), there is $b \in B$ such that $\phi(\{b\}) \nsubseteq B$. Choose $c \in \phi(\{b\}) \setminus B$ and a clopen set U_0 with $\mathrm{diam}(U_0) < 1$ and $c \in U_0 \subseteq X \setminus B$. By Lemma 4(3), $\kappa(U_0)$ is open and $b \in \kappa(U_0)$, there is a clopen set V_0 such that $\mathrm{diam}(V_0) < 1$, $b \in V_0 \subseteq \kappa(U_0)$ and $V_0 \cap U_0 = \emptyset$.

Now we construct two sequences $\{\mathcal{U}_n\}_{n=1}^{\infty}$ and $\{\mathcal{V}_n\}_{n=1}^{\infty}$ of collections of pairwise disjoint nonempty clopen sets, and a mapping $\alpha_n : \mathcal{V}_n \to \mathcal{U}_n$ for each n, such that the following conditions are satisfied:

(i) $\bigcup \mathcal{U}_n = U_0$, $\bigcup \mathcal{V}_n = V_0$ for each n;
(ii) $\operatorname{diam}(U) \le 1/n$ for every $U \in \mathcal{U}_n$;
(iii) $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$, and $\mathcal{V}_{n+1}$ is a refinement of $\mathcal{V}_n$;
(iv) $V \subseteq \kappa(\alpha_n(V))$ for every $V \in \mathcal{V}_n$;
(v) if $m \le n$ and $V \in \mathcal{V}_m$, $W \in \mathcal{V}_n$ then either $W \subseteq V$ or $V \cap W = \emptyset$;
(vi) if $m \le n$, $W \in \mathcal{V}_n, V \in \mathcal{V}_m$ and $W \subseteq V$, then $\alpha_n(W) \subseteq \alpha_m(V)$.

First, let $\mathcal{U}_1 = \{U_0\}$, $\mathcal{V}_1 = \{V_0\}$ and $\alpha_1(V_0) = U_0$. Then the above six conditions are satisfied.

Now suppose for each $i \le k$, $\mathcal{U}_i, \mathcal{V}_i$ and $\alpha_i : \mathcal{V}_i \to \mathcal{U}_i$ have been defined and satisfy (i)–(vi). For any $U \in \mathcal{U}_k$, by Lemma 5 there exists a family $\mathcal{U}_{k+1}(U)$ of pairwise disjoint clopen sets in X such that $\bigcup \mathcal{U}_{k+1}(U) = U$ and $\operatorname{diam}(W) \le 1/(k+1)$ for every $W \in \mathcal{U}_{k+1}(U)$. Let $\mathcal{U}_{k+1} = \bigcup\{\mathcal{U}_{k+1}(U) : U \in \mathcal{U}_k\}$. Obviously $\mathcal{U}_{k+1}$ is a refinement of $\mathcal{U}_k$. Next, for each $V \in \mathcal{V}_k$, by (iv) we have $V \subseteq \kappa(\alpha_k(V))$. Note that κ preserves unions by Lemma 4(2); it follows that $\{\kappa(W) \cap V : W \in \mathcal{U}_{k+1}(\alpha_k(V))\}$ is an open cover of V. Again, by Lemma 5, there is a cover $\mathcal{V}_{k+1}(V)$ of V consisting of pairwise disjoint clopen sets and $\mathcal{V}_{k+1}(V)$ is finer than $\{\kappa(W) \cap V : W \in \mathcal{U}_{k+1}(\alpha_k(V))\}$. Put $\mathcal{V}_{k+1} = \bigcup\{\mathcal{V}_{k+1}(V) : V \in \mathcal{V}_k\}$. Then $\mathcal{V}_{k+1}$ is a refinement of $\mathcal{V}_k$.

To define α_{k+1}, for each $A \in \mathcal{V}_{k+1}$, there is a unique $V \in \mathcal{V}_k$ such that $A \in \mathcal{V}_{k+1}(V)$. Then $A \subseteq \kappa(E) \cap V$ for some $E \in \mathcal{U}_{k+1}(\alpha_k(V)) \subseteq \mathcal{U}_{k+1}$. Such a set E need not be unique. Choose any of them and let $\alpha_{k+1}(A) = E$. Thus we have defined a mapping $\alpha_{k+1} : \mathcal{V}_{k+1} \to \mathcal{U}_{k+1}$.

Conditions (i)–(iv) for $k+1$ follow immediately from the construction of these objects. To show (v) for $k+1$, let $m \le k+1$; then $\mathcal{V}_{k+1}$ is a refinement of $\mathcal{V}_m$. If $V \in \mathcal{V}_m$, $W \in \mathcal{V}_{k+1}$ and $W \not\subseteq V$, then $W \subseteq V'$ for some $V' \in \mathcal{V}_m$, where $V' \ne V$, thus $W \cap V \subseteq V' \cap V = \emptyset$. To prove (vi) it is enough to check the case where $m = k$ and $n = k+1$. Let $V \in \mathcal{V}_k$, $W \in \mathcal{V}_{k+1}$ and $W \subseteq V$. By the definition of α_{k+1}, $\alpha_{k+1}(W) = E$ for some $E \in \mathcal{U}_{k+1}(\alpha_k(V))$, hence $\alpha_{k+1}(W) = E \subseteq \alpha_k(V)$.

By induction we have defined the sequences $\{\mathcal{U}_n\}_{n=1}^{\infty}$, $\{\mathcal{V}_n\}_{n=1}^{\infty}$, and the mapping α_n for each n.

Now define the mapping $f : X \to X$ as follows:

$$\{f(x)\} = \begin{cases} \{x\} & \text{if } x \in X \setminus V_0, \\ \bigcap_{n=1}^{\infty} \{\alpha_n(V_n) : x \in V_n \in \mathcal{V}_n\} & \text{if } x \in V_0. \end{cases}$$

First, f is well defined. As a matter of fact, if $x \in V_0$, then for each n, there is a unique $V_n \in \mathcal{V}_n$ with $x \in V_n$. If $x \in V_n \in \mathcal{V}_n$, $x \in V_m \in \mathcal{V}_m$ and $m \le n$, then it follows from (v) and (vi) that $\emptyset \ne \alpha_n(V_n) \subseteq \alpha_m(V_m)$. Thus $\{\alpha_n(V_n) : n \in \mathbb{N}\}$ is a sequence of closed sets whose every finite subfamily has a nonempty intersection. Furthermore, $\operatorname{diam}(\alpha_n(V_n)) \le 1/n$ for each n and X is complete, so the set $\bigcap_{n=1}^{\infty} \alpha_n(V_n)$ is a singleton.

The mapping f is clearly continuous on $X \setminus V_0$. For any $x \in V_0$ and every $\varepsilon > 0$, there exists V_n such that $x \in V_n \in \mathcal{V}_n$ and $\alpha_n(V_n) \subseteq B(f(x), \varepsilon)$. For each $y \in V_n$, by the definition of f, $f(y) \in \alpha_n(V_n)$, hence $f(y) \in B(f(x), \varepsilon)$. This shows that f is also continuous on $X \setminus V_0$.

For any $x \in X$, we show $f(x) \in \phi(\{x\})$. If $x \notin V_0$, $f(x) = x \in \phi(\{x\})$. If $x \in V_0$, then for each n, there is a unique $V_n \in \mathcal{V}_n$ such that $x \in V_n$. Moreover, $V_n \subseteq \kappa(\alpha_n(V_n))$, so $\phi(\{x\}) \cap \alpha_n(V_n) \neq \emptyset$. Choose any $x_n \in \phi(\{x\}) \cap \alpha_n(V_n)$. Then the sequence $\{x_n\}$ converges to $f(x)$, so $f(x) \in \phi(\{x\})$ because $\phi(\{x\})$ is closed.

Now for each $A \in \mathcal{A}$, if $x \in A$ and $x \in X \setminus V_0$, then $f(x) = x \in A$; if $x \in A \cap V_0$, then $f(x) \in \phi(\{x\}) \subseteq A$, so again $f(x) \in A$. Thus $f \in \mathrm{Alg}(\mathcal{A})$. However, $b \in V_0 \cap B$, $f(b) \in \alpha_1(V_0) = U_0$, and $U_0 \cap B = \emptyset$, so $f(b) \notin B$. Hence $f(B) \nsubseteq B$, which implies $B \notin \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$. The proof is complete. ■

Note that in constructing the sequences $\{\mathcal{U}_n\}_{n=1}^{\infty}$, $\{\mathcal{V}_n\}_{n=1}^{\infty}$ and mappings α_n we did not make use of the completeness of X.

By [1, Corollary 6.2.8] every countable metric space is strongly zero-dimensional but is not necessarily complete.

THEOREM 3. *Every countable metric space is s-reflexive.*

Proof. Let X be a countable metric space. Then X is strongly zero-dimensional. We show that X is s-reflexive. The proof is similar to that of Theorem 2. Again, let $\mathcal{A}$ be a family of closed sets in X satisfying (a)–(c) in Lemma 1, and $B \notin \mathcal{A}$. Let $b \in B$ such that $\phi(\{b\}) \nsubseteq B$. Choose $c \in \phi(\{b\}) \setminus B$ and a clopen set U_0 with $\mathrm{diam}(U_0) < 1$ and $c \in U_0 \subseteq X \setminus B$. Since $\kappa(U_0)$ is open and $b \in \kappa(U_0)$ and $b \notin U_0$, there is a clopen set V_0 such that $\mathrm{diam}(V_0) < 1$, $b \in V_0 \subseteq \kappa(U_0)$ and $V_0 \cap U_0 = \emptyset$.

In the following we assume that V_0 is an infinite set. If V_0 has only n elements, we stop our inductive constructions in the nth step. The rest of the arguments will be the same as in the infinite case.

Arrange V_0 as $\{x_1, x_2, \ldots\}$, where $x_1 = b$.

We now define by induction two sequences $\{\mathcal{U}_n\}_{n=1}^{\infty}$ and $\{\mathcal{V}_n\}_{n=1}^{\infty}$ of pairwise disjoint clopen sets of X, a map $\alpha_n : \mathcal{V}_n \to \mathcal{U}_n$ for each n, and $f(x_n)$ ($n \in \mathbb{N}$), such that conditions (i)–(vi) in the proof of Theorem 2 hold and moreover:

(vii) for any $1 \le i < j \le n$, if $x_i \in V \in \mathcal{V}_n$ and $x_j \in V' \in \mathcal{V}_n$, then $V \cap V' = \emptyset$;

(viii) if $i \le n$ and $x_i \in V \in \mathcal{V}_n$, then $f(x_i) \in \alpha_n(V) \cap \phi(x_i)$.

For $n = 1$ we let $\mathcal{U}_1 = \{U_0\}, \mathcal{V}_1 = \{V_0\}$, $\alpha_1(V_0) = U_0$, and $f(x_1) = c$.

Suppose for each $i \le n$, $\mathcal{U}_i, \mathcal{V}_i$, α_i and $f(x_i)$ have been defined and they satisfy the required conditions. To define these objects for $n + 1$, for each $U \in \mathcal{U}_n$ choose $\mathcal{U}_{n+1}(U)$ as a collection of pairwise disjoint clopen sets with diameter less than $1/(n+1)$ and $\bigcup \mathcal{U}_{n+1}(U) = U$. Put $\mathcal{U}_{n+1} = \bigcup\{\mathcal{U}_{n+1}(U) : U \in \mathcal{U}_n\}$. Let $V \in \mathcal{V}_n$. We consider three cases:

CASE A: $V \cap \{x_1, \ldots, x_n, x_{n+1}\} = \emptyset$ or $V \cap \{x_1, \ldots, x_n, x_{n+1}\} = \{x_{n+1}\}$. Then $\mathcal{V}_{n+1}(V)$ and the restriction of α_{n+1} to $\mathcal{V}_{n+1}(V)$ are defined as in the proof of Theorem 2. When the intersection is $\{x_{n+1}\}$, there is $W \in \mathcal{V}_{n+1}(V)$ with $x_{n+1} \in W$. Then define $f(x_{n+1})$ to be any point in $\alpha_{n+1}(W) \cap \phi(x_{n+1})$. Note that as $x_{n+1} \in W \subseteq \kappa(\alpha_{n+1}(W))$, such a point exists.

CASE B: $V \cap \{x_1, \ldots, x_n, x_{n+1}\} = \{x_i, x_{n+1}\}$ for some $i \leq n$. Choose two disjoint clopen sets C and D such that $x_i \in C$, $x_{n+1} \in D$, and $C \subseteq \kappa(U') \cap V$ for some $U' \in \mathcal{U}_{n+1}(\alpha_n(V))$ with $f(x_i) \in U'$ (note: by induction assumption $x_i \in V \in \mathcal{V}_n$ implies $f(x_i) \in \alpha_n(V) \cap \phi(x_i)$, so there is $U' \in \mathcal{U}_{n+1}(\alpha_n(V))$ with $f(x_i) \in U' \cap \phi(x_i)$; it follows that $U' \cap \phi(x_i) \neq \emptyset$ and hence $x_i \in \kappa(U')$). Also $D \subseteq \kappa(W') \cap V$ for some $W' \in \mathcal{U}_{n+1}(\alpha_n(V))$. Now choose a refinement $\mathcal{V}_{n+1}(V)$ of $\{\kappa(W) \cap V : W \in \mathcal{U}_{n+1}(\alpha_n(V))\}$ consisting of pairwise disjoint clopen sets and $\mathcal{V}_{n+1}(V)$ contains both C and D as members. Define α_{n+1} on $\mathcal{V}_{n+1}(V)$ by letting $\alpha_{n+1}(C) = U'$, and $\alpha_{n+1}(F)$ as before if $F \neq C$. Also define $f(x_{n+1})$ to be any point in $\alpha_{n+1}(D) \cap \phi(x_{n+1})$ (note: $\alpha_{n+1}(D) \cap \phi(x_{n+1}) \neq \emptyset$ because $x_{n+1} \in D \subseteq \kappa(\alpha_{n+1}(D))$).

CASE C: $V \cap \{x_1, \ldots, x_n, x_{n+1}\} = \{x_i\}$ for some $i \leq n$. Then $\mathcal{V}_{n+1}(V)$ is defined as in the proof of Theorem 2.

Finally, let $\mathcal{V}_{n+1} = \bigcup\{\mathcal{V}_{n+1}(V) : V \in \mathcal{V}_n\}$. Since there is a unique $V \in \mathcal{V}_n$ that contains x_{n+1} and satisfies the condition in either Case A or Case B, $f(x_{n+1})$ is defined.

Let $g : X \to X$ be defined by $g(x) = x$ for $x \notin V_0$ and $g(x_i) = f(x_i)$ ($i = 1, 2, \ldots$). From the above construction it is clear that $g(x) \in \phi(x)$ for all $x \in X$, thus $g(A) \subseteq A$ for all $A \in \mathcal{A}$. In addition $g(b) = f(b) = c \notin B$, so $g(B) \nsubseteq B$. To complete the proof we only need to verify that g is continuous, and for this it is enough to show that f is continuous on V_0. For any $x = x_n \in V_0$ and any $\varepsilon > 0$, choose $m \geq n$ with $1/m < \varepsilon$. Let $x_n \in V \in \mathcal{V}_m$. Then $f(x_n) \in \alpha_m(V)$. If $i \geq m$ and $x_i \in V$, then there is $W \in \mathcal{V}_i$ with $x_i \in W \subseteq V$. Then $f(x_i) \in \alpha_i(W) \subseteq \alpha_m(V)$. Thus $d(f(x_n), f(x_i)) < \operatorname{diam}(V) < 1/m < \varepsilon$. Hence f is continuous at x_n. ∎

A space is called *hereditarily disconnected* if it does not contain any connected subset of cardinality larger than one.

LEMMA 6. *Every T_1 connected space with more than two elements contains a proper connected subset with more than one element.*

LEMMA 7. *Every s-reflexive Hausdorff space is hereditarily disconnected.*

Proof. Let X be an s-reflexive Hausdorff space. If X has at most two elements, it is clearly hereditarily disconnected. Now assume X has more than two elements. Suppose X is not hereditarily disconnected. If X is not connected, then as it is not hereditarily disconnected, one of its connected components, say B, is a proper non-singleton connected subset. If X is connected, by Lemma 6 there also exists a proper non-singleton connected subset B. Choose $b_1, b_2 \in B$ with $b_1 \neq b_2$ and $x_0 \in X \setminus B$. Let

$$\mathcal{A} = \{A \in S(X) : A = \{b_1\} \text{ or } A \ni x_0\} \cup \{\emptyset\}.$$

Then $\mathcal{A}$ satisfies conditions (a)–(c) in Lemma 1, and $\{b_2\} \notin \mathcal{A}$. However, $\{b_2\} \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$, which implies that X is not s-reflexive. In fact, if $f \in \mathrm{Alg}(\mathcal{A})$ and $b \in B$, from $\{b, x_0\} \in \mathcal{A}$ it follows that $f(b) = b$ or $f(b) = x_0$. Thus B is the union of the disjoint closed sets $E = \{b \in B : f(b) = b\}$ and $K = \{b \in B : f(b) = x_0\}$. Trivially, $b_1 \in E$, thus $E = B$ because B is connected. In particular, $f(b_2) = b_2$. We are done. ∎

THEOREM 4. *A locally compact metric space is s-reflexive if and only if it is zero-dimensional* (*or*, *equivalently*, *if and only if it is strongly zero-dimensional or hereditarily disconnected*).

Proof. The equivalence of all the above conditions except s-reflexivity follows from [1, Theorem 6.2.9]. Since the s-reflexivity is topological and every locally compact metrizable space is completely metrizable, the result follows from Theorem 2 and Lemma 7. ∎

In the following we construct a locally compact countable complete metric space which has a one-point extension that is not s-reflexive. We shall define a family of closed sets in the one-point extension space which satisfies all the conditions (a)–(d) in Lemma 1 and is not reflexive.

EXAMPLE 2. Let

$$Y_1 = \{(1/n, m) : n, m = 1, 2, \ldots\},$$
$$Y_0 = \{(1/n, 0) : n = 1, 2, \ldots\} \cup \{(0, 0)\}.$$

As a subspace of $\mathbb{R}^2$, $Y = Y_0 \cup Y_1$ is a locally compact countable complete space. Let $X = Y \cup \{p\}$, where p is an element not in Y. Define a local base at p as follows: for every map $g : \mathbb{N} \setminus D \to \mathbb{N} \cup \{0\}$, where D is a finite subset of $\mathbb{N}$, let

$$U(g) = \{p\} \cup \{(1/n, m) : n \in \mathbb{N} \setminus D,\ m > g(n)\}.$$

Thus $(1/n, m) \notin U(g)$ for any $n \in D$ and $m \in \mathbb{N} \cup \{0\}$. Note that $U(g) \cap U(h) = U(\max\{g, h\})$, where $g : \mathbb{N} \setminus D_1 \to \mathbb{N} \cup \{0\}$, $h : \mathbb{N} \setminus D_2 \to \mathbb{N} \cup \{0\}$ and $\max\{g, h\} : \mathbb{N} \setminus (D_1 \cup D_2) \to \mathbb{N} \cup \{0\}$ is defined by $\max\{g, h\}(n) = \max\{f(n), g(n)\}$ for every $n \in \mathbb{N} \setminus (D_1 \cup D_2)$. Thus all $U(g)$'s form a local base at p. Assuming that Y is an open subspace of X, we have thus defined a topology on X.

We show that X is not s-reflexive. Define a map $q : X \to Y_0$ by $q((1/n,m)) = (1/n,0)$, $q(p) = q((0,0)) = (0,0)$ for any $n \in \mathbb{N}$ and $m \in \{0\} \cup \mathbb{N}$. Clearly, q is continuous.

The family

$$\mathcal{A} = \{q^{-1}(A) : A \text{ is closed in } Y_0\}$$

satisfies conditions (a)–(c). In fact, (a) and (b) are clearly valid. To see that (c) is also satisfied, consider any family $\{A_i : i \in I\}$ of closed subsets of Y_0. Then $\mathrm{cl}(\bigcup_{i\in I} q^{-1}(A_i)) = \mathrm{cl}(q^{-1}(\bigcup_{i\in I} A_i)) = q^{-1}(\mathrm{cl}(\bigcup_{i\in I} A_i))$. But $\mathcal{A}$ is not reflexive. In fact, $\{(0,0)\} \notin \mathcal{A}$. For any $f \in \mathrm{Alg}(\mathcal{A})$, as $\{(0,0),p\} = q^{-1}(\{(0,0)\})$ is in $\mathcal{A}$, we have $f((0,0)) = (0,0)$ or $f((0,0)) = p$. If the latter holds, then $p = \lim_{n\to\infty} f((1/n,0))$. By the definition of $\mathcal{A}$, $f((1/n,0)) \in q^{-1}(\{(1/n,0)\})$; set $f((1/n,0)) = (1/n, g(n))$, $n \in \mathbb{N}$. This yields a mapping $g : \mathbb{N} \to \mathbb{N} \cup \{0\}$. But then $f((1/n,0)) \notin U(g)$ for all $n \in \mathbb{N}$, which contradicts $p = \lim_{n\to\infty} f((1/n,0))$. This contradiction indicates $f((0,0)) = (0,0)$, hence $\{(0,0)\} \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$. Thus $\mathcal{A}$ is not reflexive.

Note that the only nonempty connected subsets of X are singletons, so the family $\mathcal{A}$ constructed above also satisfies condition (d) in Lemma 1.

A closed subset A of a space X is called *reflexive* if $\{\emptyset, A, X\}$ is reflexive. Obviously both $\emptyset$ and X are reflexive.

LEMMA 8. *A closed subset A of a space X is reflexive if and only if for each $B \in S(X)$ with $A \subset B \neq X$, there exists $f \in C(X)$ such that $f(A) \subseteq A$ but $f(B) \not\subseteq B$.*

Proof. Assume that $A \neq \emptyset$ and $A \neq X$. The necessity is trivial. To show the sufficiency, suppose $B \in S(X)$ and $B \notin \{\emptyset, A, X\}$. If $A \not\subset B$, choose $a \in A \setminus B$ and define $f \in C(X)$ by $f(x) = a$ for all $x \in X$. Then $f(A) \subseteq A$ but $f(B) \not\subseteq B$. If $A \subset B$, it follows from the assumption that there exists $f \in C(X)$ such that $f(A) \subseteq A$ but $f(B) \not\subseteq B$. Thus $\{\emptyset, A, X\}$ is reflexive. ∎

PROPOSITION 1. *If each path-connected component of a Tikhonov space X is dense in X, then every closed subset of X is reflexive.*

Proof. Let $A, B \in S(X)$ with $\emptyset \neq A \subset B \neq X$. Choose $a \in A$ and a continuous mapping $g : X \to [0,1]$ such that $g(x) = 0$ for all $x \in A$ and $g(b) = 1$ for some $b \in B$. Since the path component of a intersects $X \setminus B$, there is a path $h : [0,1] \to X$ such that $h(0) = a$ and $h(1) \in X \setminus B$. Now $f = h \circ g \in C(X)$ and $f(A) \subseteq A$. However, $f(b) = h(1) \notin B$, so $f(B) \not\subseteq B$. By Lemma 8, A is reflexive. ∎

PROPOSITION 2. *Every closed subset of a zero-dimensional space is reflexive.*

Proof. Let X be a zero-dimensional space, and $A, B \in S(X)$ with $\emptyset \neq A \subset B \neq X$. Choose $a \in A$ and $x_0 \in X \setminus A$. There exists a clopen set $U \ni x_0$ such that $U \cap A = \emptyset$. Consider the mapping $f : X \to X$ defined by $f(x) = a$ if $x \notin U$ and $f(x) = x_0$ if $x \in U$. Then $f \in C(X)$ and $f(A) \subseteq A$ but $f(B) \nsubseteq B$. ∎

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