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Symplectic groups are N-determined 2-compact groups

by

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Abstract. We show that for $n \geq 3$ the symplectic group $Sp(n)$ is as a 2-compact group determined up to isomorphism by the isomorphism type of its maximal torus normalizer. This allows us to determine the integral homotopy type of $Sp(n)$ among connected finite loop spaces with maximal torus.

1. Introduction. The advent of p-compact groups in the celebrated work of Dwyer and Wilkerson [10] is the culmination of a research program that can be traced back to the work of Hopf and Serre on H-spaces and loop spaces, and fits within the philosophy of Hilbert's Fifth Problem: which are the non-differential (here homotopy-theoretical) properties that characterize compact Lie groups?

A *p-compact group* is a loop space (X, BX, e), i.e. $e : X \simeq \Omega(BX)$ for a pointed space BX, such that $H^*(X; \mathbb{F}_p)$ is finite and BX is p-complete in the sense of Bousfield and Kan [5]. As expected, examples of p-compact groups are given by p-completion of compact Lie groups G for which $\pi_0 G$ is a p-group, since $G^\wedge_p$ is homotopy equivalent to $\Omega(BG^\wedge_p)$. In this way a p-compact torus T of rank n is the p-completion of an ordinary torus, hence BT is the Eilenberg–MacLane space $K((\mathbb{Z}_p)^{\oplus n}, 2)$. Further examples are given by the realization of polynomial algebras, i.e. loop spaces ΩBX, where BX is p-complete and has polynomial mod p cohomology ([1], [6], [12], [28], [33], [38]). The importance of p-compact groups lies in a dictionary (reviewed in Section 2) that translates much of the rich internal algebraic structure of compact Lie groups to the homotopy-theoretical setting of p-compact

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groups, so the challenge is then to give homotopy-theoretical proofs of classical algebraic Lie group theory results.

One of those challenges quoted above is the following: p-compact groups admit maximal tori, Weyl groups and maximal torus normalizers in a way that extends the classical concepts in Lie group theory [10, Theorem 8.13 and Proposition 9.5], so can we "reprove" the Lie group theoretical Curtis–Wiederhold–Williams theorem [7] in the setting of p-compact groups? Recall that Curtis–Wiederhold–Williams' theorem states that two compact connected semisimple Lie groups are isomorphic if and only if their maximal torus normalizers are isomorphic, hence we are led to the following conjecture [8, Conjecture 5.3]:

Conjecture 1.1. *Let X be a connected p-compact group with maximal torus T_X. Then X is determined up to equivalence by the loop space NT_X.*

We shall say that a p-compact group X is *N-determined* if X satisfies Conjecture 1.1 even if the "connected" hypothesis is dropped, i.e. X is N-determined if every p-compact group Y, with the normalizer of a maximal torus isomorphic to that of X, is isomorphic to X.

Given an odd prime $p > 2$, p-compact groups are known to be N-determined [2], which leads to the classification of p-compact groups for p odd. But the situation is quite different at $p = 2$: there exist 2-compact groups which are not N-determined. For example $O(n)^\wedge_2$ and $SO(n+1)^\wedge_2$ are nonisomorphic 2-compact groups that have isomorphic maximal torus normalizers. So at $p = 2$ we cannot drop the "connected" hypothesis in Conjecture 1.1.

We say that a 2-compact group X is *weakly N-determined* if every 2-compact group Y for which there exists a homotopy equivalence $BN_X \simeq BN_Y$ between the maximal torus normalizers of X and Y, inducing an isomorphism $\pi_0 X \cong \pi_0 Y$, is isomorphic to X. From the definitions it follows that an N-determined 2-compact group is also weakly N-determined.

It has been shown that the 2-compact groups $O(n)^\wedge_2$, $SO(2n+1)^\wedge_2$ and $Spin(2n+1)^\wedge_2$ [26] are weakly N-determined 2-compact groups (and they are not N-determined), and that $U(n)^\wedge_2$ for $n \neq 2$ [24], $(G_2)^\wedge_2$ [35], $(F_4)^\wedge_2$ [34], and $DI(4)$ [27] are N-determined ($U(2)^\wedge_2$ is only weakly N-determined, because the normalizer N of a maximal torus of $U(2)^\wedge_2$ is also a 2-compact group but N is not isomorphic to $U(2)^\wedge_2$). In this paper we prove that the symplectic groups $Sp(n)^\wedge_2$ are N-determined 2-compact groups for $n \geq 3$.

Theorem 1.2. *Let $n \geq 3$ and let X be a 2-compact group with the maximal torus normalizer $f_N \colon N \to X$ isomorphic to that of $Sp(n)^\wedge_2$. Then X and $Sp(n)^\wedge_2$ are isomorphic 2-compact groups.*

Proof. First in Section 3 we prove that X is connected. In Section 4 we show that the mod 2 cohomology of BX is isomorphic to that of $BSp(n)$ as algebras over the Steenrod algebra, which implies that the Quillen categories associated to X and $Sp(n)$ are isomorphic. In Section 5 we describe the 2-stubborn decomposition of the group $Sp(n)$, which allows us to define a map from $BSp(n)^\wedge_2$ to BX that happens to be an equivalence. This is done in Section 6. ∎

Notice that the hypothesis $n \geq 3$ is necessary as $Sp(1)^\wedge_2 = SU(2)^\wedge_2$ and $Sp(2)^\wedge_2 = Spin(5)^\wedge_2$ are only weakly N-determined 2-compact groups.

The combination of the results in [2] and Theorem 1.2 shows that if G is a connected compact Lie group, then BG is in the adic genus of $BSp(n)$ if and only if $G = Sp(n)$, which in view of [31] characterizes the integral homotopy type of $BSp(n)$ as a loop space. Thus our final result is

THEOREM 1.3. *Let L be a connected finite loop space with a maximal torus normalizer isomorphic to that of $Sp(n)$. Then BL is homotopy equivalent to $BSp(n)$.*

Notation. Here all spaces are assumed to have the homotopy type of a CW-complex. Completion means Bousfield–Kan completion [5]. For a given space X, we write H^*X for the mod 2 cohomology $H^*(X;\mathbb{F}_2)$. For a prime p, we write $X^\wedge_p$ for the Bousfield–Kan p-completion ($(\mathbb{Z}_p)_\infty$-completion in the terminology of Bousfield and Kan) of the space X. We assume that the reader is familiar with Lannes' theory [19].

2. The dictionary. As announced in the introduction, this section is devoted to a brief review of the dictionary translating constructions and arguments from the algebraic theory of groups to the homotopical setting of p-compact groups. The aim of the minimalist style of this section is to ease the search of concepts by the reader who will find a more detailed exposition in the original [10], or the reviews [8], [22] and [29] if needed.

Along this section X and Y denote p-compact groups whose classifying spaces are BX and BY respectively. By T we denote a p-compact torus, i.e. $BT \simeq K(\mathbb{Z}_p^n, 2)$ where n is the rank of T. Finally, we define:

- *Homomorphisms* [10, §3.1]: A *homomorphism* $X \xrightarrow{f} Y$ of p-compact groups is a pointed map $BX \xrightarrow{Bf} BY$. The homomorphism f is an isomorphism if Bf is a homotopy equivalence. It is a monomorphism if the homotopy fiber Y/X of Bf is $\mathbb{F}_p$-finite or equivalently if $H^*(BX, \mathbb{F}_p)$ is a finitely generated module over $H^*(BY, \mathbb{F}_p)$ via Bf^*.
- *Centralizers* [10, §3.4]: For a homomorphism $Y \xrightarrow{f} X$ of p-compact groups, the *centralizer* $C_X(f(Y))$ is defined by the equation $BC_X(f(Y)) := \mathrm{Map}(BY, BX)_{Bf}$.

- *Maximal tori* [10, Definition 8.9]: A monomorphism $T \rightarrowtail X$ of a p-compact torus into a p-compact group X is a *maximal torus* if $C_X(T)$ is a p-compact toral group and $C_X(T)/T$ is homotopically discrete. Every p-compact group admits maximal tori [10, Theorem 8.13].
- *Weyl group* [10, Definition 9.2]: Let $BT_X \xrightarrow{Bf_T} BX$ be a maximal torus of a p-compact group X. Assume that Bf_T is already a fibration and treat $\mathcal{W}_X$ as the space of self-maps of BT_X over BX. Composition gives $\mathcal{W}_X$ the structure of an associative topological monoid. It is shown [10, Proposition 9.5] that $\mathcal{W}_X$ is homotopically discrete and therefore $W_X := \pi_0\mathcal{W}_X$ is a (finite) group. Moreover, if X is connected, the action of W_X on BT_X induces a faithful representation
$$W_X \rightarrowtail \mathrm{GL}(H^*(BT_X;\mathbb{Z}) \otimes \mathbb{Q}_p) \cong \mathrm{GL}_n(\mathbb{Q}_p^\wedge)$$
whose image is generated by pseudoreflections (elements of finite order which fix a codimension 1 subspace of $(\mathbb{Q}_p^\wedge)^n$), i.e. W_X is a pseudoreflection group [10, Theorem 9.7].
- *Maximal torus normalizers* [10, Definition 9.8]: Let $BT_X \xrightarrow{Bf_T} BX$ be a maximal torus of a p-compact group X. The *normalizer* of T_X, denoted by NT_X, or simply by N_X or N, is the loop space such that BNT_X is the Borel construction associated to the action of $\mathcal{W}_X$ on BT_X.

All these concepts generalize the classical algebraic definitions. In particular, if G is a compact Lie group such that $\pi_0 G$ is a p-group, $i\colon T \to G$ is a maximal torus of G, W is the Weyl group of G, and N is the normalizer of the maximal torus T, then the p-completion $i_p^\wedge\colon T_p^\wedge \to G_p^\wedge$ is a maximal torus of the p-compact group $G_p^\wedge$. The Weyl group W is naturally isomorphic to the Weyl group $W_{G_p^\wedge}$. The classifying space BN of the normalizer N sits in the fibration $BT \to BN \to BW$, and a normalizer of the maximal torus $T_p^\wedge$ of $G_p^\wedge$ is isomorphic to the fiberwise p-completion BN_p° by [24, Proposition 1.8], or [35, Lemma 6.1].

3. Connectedness. In this section we proceed with the first step in the proof of Theorem 1.2 by proving the following proposition.

Proposition 3.1. *Let X be a 2-compact group with the normalizer of a maximal torus isomorphic to that of $Sp(n)_2^\wedge$, where $n \geq 3$. Then X is connected.*

The proof requires calculating the Weyl group of some centralizer in the connected component of X. This is done by means of the technics developed by Dwyer and Wilkerson in [9] that we recall now.

An *extended p-discrete torus* P is an extension of a p-discrete torus $(\mathbb{Z}/p^\infty)^n$ by a finite group. There is a unique normal p-discrete torus T

in P such that P/T is finite. We will denote this unique p-discrete torus by P_0. A *discrete approximation* for an extended p-compact torus P is a homomorphism $f\colon \breve{P} \to P$, where $\breve{P}$ is an extended p-discrete torus and f induces an isomorphism $\breve{P}/\breve{P}_0 \to \pi_0 P$ and an isomorphism $H^*BP_0 \to H^*B\breve{P}_0$. Every extended p-compact torus has a discrete approximation [9, Proposition 3.13].

Definition 3.2 ([9, Definition 7.3]). Let $W \subset \mathrm{GL}_r(\mathbb{Q}_p^\wedge)$ be a pseudoreflection group. If $s \in W$ is a pseudoreflection of order $\mathrm{ord}(s)$, then

(1) the *fixed point set* $F(s)$ of s is the fixed point set of the action of x on $\breve{T}$ by conjugation, where $x \in \breve{N}(T)$ is an element which projects on s by the natural projection $\breve{N}(T) \to W$,
(2) the *singular hyperplane* $H(s)$ of s is the maximal divisible subgroup of $F(s)$ (so $H(s) \cong (\mathbb{Z}/p^\infty)^{r-1}$),
(3) the *singular coset* $K(s)$ of s is the subset of $\breve{T}$ given by elements of the form $x^{\mathrm{ord}(s)}$, as x runs through elements of $\breve{N}(T)$, which project to s in W,
(4) the *singular set* $\sigma(s)$ of s is the union $\sigma(s) = H(s) \cup K(s)$.

Notice that there are inclusions $H(s) \subset \sigma(s) \subset F(s)$ [9, Remark 7.7].

Let $A \subset \breve{T}$ be a subgroup. Let $W_X(A)$ denote the Weyl group of $C_X(A)$, and $W_X(A)_1$ the Weyl group of the unit component $C_X(A)_0$ of $C_X(A)$. There are inclusions $W_X(A)_1 \subset W_X(A) \subset W$, where the last follows from [9, §4]. The next theorem tells us how to calculate $W_X(A)$ and $W_X(A)_1$.

Theorem 3.3 ([9, Theorem 7.6]). *Let X be a connected p-compact group with maximal torus T and Weyl group W. Suppose that $A \subset \breve{T}$ is a subgroup. Then*

(1) *$W_X(A)$ is the subgroup of W consisting of the elements which, under the conjugation action of W on $\breve{T}$, pointwise fix the subgroup A,*
(2) *$W_X(A)_1$ is the subgroup of $W_X(A)$ generated by those elements $s \in W_X(A)$ such that $s \in W$ is a reflection and $A \subset \sigma(s)$.*

Now we have all the ingredients needed for the proof of Proposition 3.1:

Proof of Proposition 3.1. Let X_0 be the unit component of X, and let W_{X_0} be the Weyl group of X_0. Then W_{X_0} is a normal subgroup of W_X of index a power of 2 [23, Proposition 3.8]. The minimal normal subgroup of W_X of 2 power index, usually denoted by $O^2(W_X)$, equals $(\mathbb{Z}/2)^{n-1} \rtimes A_n$, i.e. the sequence

$$(\mathbb{Z}/2)^{n-1} \rtimes A_n = O^2(W_X) \rightarrowtail (\mathbb{Z}/2)^n \rtimes \Sigma_n \overset{\pi}{\twoheadrightarrow} (\mathbb{Z}/2)^2,$$

where A_n is the alternating group, is exact. The group $(\mathbb{Z}/2)^2$ has five subgroups: the trivial subgroup 1, the first and the second factor Z_1 and Z_2, the diagonal D, and the whole group $(\mathbb{Z}/2)^2$. Hence, there are five normal subgroups of W_X of index a power of 2:

(1) $\pi^{-1}(1) = (\mathbb{Z}/2)^{n-1} \rtimes A_n$,
(2) $\pi^{-1}(Z_1) = (\mathbb{Z}/2)^n \rtimes A_n$,
(3) $\pi^{-1}(Z_2) = (\mathbb{Z}/2)^{n-1} \rtimes \Sigma_n$,
(4) $\pi^{-1}(D)$,
(5) $\pi^{-1}((\mathbb{Z}/2)^2) = W_X$.

Because W_{X_0} is the Weyl group of a connected 2-compact group, W_{X_0} is a pseudoreflection group. According to the Clark–Ewing list [6], only the cases (3) and (5) may be pseudoreflection groups (note that $n \geq 3$). We complete the proof by showing that the case $W_{X_0} = (\mathbb{Z}/2)^{n-1} \rtimes \Sigma_n$ is not possible.

Suppose that X is disconnected, and let X_0 be the unit component. By the arguments above W_{X_0} is $(\mathbb{Z}/2)^{n-1} \rtimes \Sigma_n$. Let V be the subgroup of the maximal torus T of X (and also of X_0) generated by the elements $(-1,-1,1,\ldots)$, $(1,1,-1,-1,1,\ldots)$, and so on. Then V is an elementary abelian 2-group of rank $m = [n/2]$. Write $n = 2m + r$ where r is 0 or 1, and let C denote the centralizer $C_{X_0}(V)$. By Theorem 3.3(1), we get

$$W_C := W_{X_0}(V) = \{s \in W_{X_0} \mid s|_V = \mathrm{id}_V\} = (\mathbb{Z}/2)^{n-1} \rtimes (\mathbb{Z}/2)^m,$$

where the subgroup $(\mathbb{Z}/2)^m \subset \Sigma_n$ is generated by the transpositions $\tau_{2i-1,2i}$ for $i = 1,\ldots,m$. Let C_0 be the unit component of C. By Theorem 3.3(2), the Weyl group of C_0 is

$$W_{C_0} := W_{X_0}(V)_1 = \langle s \in W_C \mid s \text{ is a reflection and } V \subset \sigma(s)\rangle.$$

An element $s \in W_C = (\mathbb{Z}/2)^{n-1} \rtimes (\mathbb{Z}/2)^m$ is a reflection if and only if s equals $((1,\ldots,1),\tau_{2i-1,2i})$ or $((1,\ldots,1,-1,-1,1,\ldots,1),\tau_{2i-1,2i})$ for some i, where the two "-1" entries are in the $(2i-1)$th and $(2i)$th positions. We analyze both cases:

- If $s = ((1,\ldots,1),\tau_{2i-1,2i})$, then $F(s) = \{(x_1,\ldots,x_n) \in (\mathbb{Z}/2^\infty)^n \mid x_{2i-1} = x_{2i}\}$ and $H(s) = F(s)$. Therefore $\sigma(s) = F(s)$.
- If $s = ((1,\ldots,1,-1,-1,1,\ldots,1),\tau_{2i-1,2i})$, then $F(s) = \{(x_1,\ldots,x_n) \in (\mathbb{Z}/2^\infty)^n \mid x_{2i-1} = x_{2i}^{-1},\ i = 1,\ldots,m\}$. Hence also in this case $H(s) = F(s) = \sigma(s)$.

Since $(-1)^{-1} = -1 \in \mathbb{Z}/2^\infty$, the group V is a subgroup of $\sigma(s)$ in both cases, and by Theorem 3.3, we get

$$W_{C_0} = \langle s \in W_C \mid s \text{ is a reflection}\rangle = ((\mathbb{Z}/2)^2)^m = (\mathbb{Z}/2)^{2m}.$$

Hence the normalizer of a maximal torus of C_0 has the form M^m, where M is the subgroup of the normalizer of the maximal torus of $Sp(2)^\wedge_2$ corresponding to the subgroup $\langle((1,1),\tau_{1,2}),((-1,-1),\tau_{1,2})\rangle < (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/2 = W_{Sp(2)^\wedge_2}$. By [13, Theorem 6.1] and [11, Theorem 0.5B(5)], the 2-compact group C_0

splits into a product $C_0 \cong X_1 \times \cdots \times X_m$, where each X_i is isomorphic to $(SU(2)^2/E_i)^\wedge_2$ for some subgroup $E_i < Z(SU(2)^2) = (\mathbb{Z}/2)^2$, and M is isomorphic to the maximal torus normalizer of X_i. Among the five possibilities for each X_i:

(1) $(SU(2) \times SU(2))^\wedge_2 = Spin(4)^\wedge_2$,
(2) $(SU(2)/(\mathbb{Z}/2) \times SU(2))^\wedge_2 \cong (SO(3) \times SU(2))^\wedge_2$,
(3) $(SU(2) \times SU(2)/(\mathbb{Z}/2))^\wedge_2 \cong (SU(2) \times SO(3))^\wedge_2$,
(4) $(SU(2) \times_{\mathbb{Z}/2} SU(2))^\wedge_2 \cong SO(4)^\wedge_2$,
(5) $((SU(2) \times SU(2))/(\mathbb{Z}/2)^2)^\wedge_2 \cong (SO(3) \times SO(3))^\wedge_2$,

only $SO(4)$ produces a pseudoreflection group which is equivalent to that given by M. But while the maximal torus normalizer of $SO(4)$ is a split extension $T : (\mathbb{Z}/2 \times \mathbb{Z}/2)$, M is not. Therefore there is no 2-compact group X_i whose maximal torus normalizer is M, which contradicts our initial assumption of X being disconnected. ∎

4. Mod 2 cohomology of the 2-compact group X. In this section we calculate the mod 2 cohomology of a 2-compact group X whose maximal torus normalizer is isomorphic to that of $Sp(n)^\wedge_2$. This is done under the induction hypothesis that $Sp(m)^\wedge_2$ is N-determined for $2 < m < n$. Notice that we already know that $Sp(1)$ and $Sp(2)$ are weakly N-determined.

First we need some information about the centralizers of elementary abelian subgroups in $Sp(n)$. It is well known that theses centralizers are isomorphic to products $Sp(n_1) \times \cdots \times Sp(n_k)$, where $n_1 + \cdots + n_k = n$. The next lemma shows that they are N-determined if each of their factors is.

LEMMA 4.1. *Let $X = Sp(n_1)^\wedge_2 \times \cdots \times Sp(n_k)^\wedge_2$.*

(1) *If all factors $Sp(n_i)^\wedge_2$ are N-determined, then so is X.*
(2) *If all factors $Sp(n_i)^\wedge_2$ are weakly N-determined, then so is X.*

Proof. Let Y be a 2-compact group with maximal torus normalizer N_Y isomorphic to that of $Sp(n_1)^\wedge_2 \times \cdots \times Sp(n_k)^\wedge_2$. If at least one factor is only weakly N-determined, assume that Y is connected. Since N_Y is a product $N_1 \times \cdots \times N_k$, where N_i is the normalizer of a maximal torus of $Sp(n_i)^\wedge_2$, the space Y is by [13, Theorem 6.1] isomorphic to a product $Y_1 \times \cdots \times Y_k$, where N_i is the normalizer of a maximal torus of Y_i. If $Sp(n_i)^\wedge_2$ is N-determined, the 2-compact group Y_i is isomorphic to $Sp(n_i)^\wedge_2$. If $Sp(n_i)^\wedge_2$ is only weakly N-determined, the space Y is connected by assumption and then also Y_i is connected. Hence Y_i is isomorphic to $Sp(n_i)^\wedge_2$. Therefore Y is isomorphic to $Sp(n_1)^\wedge_2 \times \cdots \times Sp(n_k)^\wedge_2$. So $Sp(n_1)^\wedge_2 \times \cdots \times Sp(n_k)^\wedge_2$ is (weakly) N-determined if all factors are (weakly) N-determined. ∎

As X and $Sp(n)^\wedge_2$ "share" the same maximal torus normalizer N, they both "share" the same maximal torus T. Let $E_T < T$ be the maximal toral elementary abelian 2-group of both X and $Sp(n)^\wedge_2$. Let f_{E_T} be the monomorphism $E_T \rightarrowtail X$. The next lemma shows that E_T is in fact the maximal elementary abelian subgroup of X (up to conjugation).

LEMMA 4.2. *Let $g\colon E \rightarrowtail X$ be an elementary abelian subgroup of X. Then g factors through f_{E_T}.*

Proof. If $g\colon E \rightarrowtail X$ is central, then by [23, Lemma 4.1] or [9, Theorem 1.2] the map g factors through f_{E_T} (recall that X is connected by Proposition 3.1).

Now assume that $g\colon E \rightarrowtail X$ is not central, thus there exists a subgroup $V < E$ of rank 1 which is noncentral. By [20, Proof of Theorem 1.3] there exists $\tilde{g}\colon E \rightarrowtail N$ such that $Bg \simeq f_N B\tilde{g}$, the centralizer $C_N(\tilde{g})$ is the maximal torus normalizer of $C_X(g)$, and $\tilde{g}|_V$ factors through f_{E_T}. Because V is a toral subgroup, the centralizer $C_N(V)$ is the maximal torus normalizer of both $C_{Sp(n)^\wedge_2}(V)$ and $C_X(V)$ [20, Theorem 1.3]. So the calculation of $W_X(V)$ and $W_X(V)_1$ by means of Theorem 3.3 amounts to the calculation of $W_{Sp(n)^\wedge_2}(V)$ and $W_{Sp(n)^\wedge_2}(V)_1$, which implies that $C_X(V)$ is connected, and since by induction, the centralizer $C_{Sp(n)^\wedge_2}(V) = Sp(m)^\wedge_2 \times Sp(n-m)^\wedge_2$, $m > 0$, is weakly N-determined (Lemma 4.1), $C_X(V)$ is isomorphic to $C_{Sp(n)^\wedge_2}(V)$.

The map $g\colon E \to X$ has a lift to a map $g'\colon E \to C_V(X) \cong Sp(m)^\wedge_2 \times Sp(n-m)^\wedge_2$. Up to conjugacy every elementary abelian subgroup of $Sp(m) \times Sp(n-m)$ is toral. Hence g is toral, i.e. factors through f_{E_T}. ∎

We can calculate the centralizer of E_T in X:

LEMMA 4.3. *The centralizer $C_X(E_T)$ is isomorphic to the 2-compact group $(Sp(1)^n)^\wedge_2$.*

Proof. As E_T is toral, the centralizer $C_N(E_T)$ is the maximal torus normalizer of both $C_{Sp(n)^\wedge_2}(E_T)$ and $C_X(E_T)$ [21, Proposition 3.4(3)]. So the calculation of $W_X(E_T)$ and $W_X(E_T)_1$ by means of Theorem 3.3 amounts to the calculation of $W_{Sp(n)^\wedge_2}(E_T)$ and $W_{Sp(n)^\wedge_2}(E_T)_1$ which implies that $C_X(E_T)$ is connected. Since $C_{Sp(n)^\wedge_2}(E_T) = (Sp(1)^n)^\wedge_2$ is weakly N-determined, the centralizer $C_X(E_T)$ is isomorphic to $C_{Sp(n)^\wedge_2}(E_T)$ by Lemma 4.1, hence $C_X(E_T) \cong (Sp(1)^n)^\wedge_2$. ∎

The action of $\Sigma_n < W_{Sp(n)} = W_X$ on BE_T induces an action of Σ_n on $BC_X(E_T) = \mathrm{Map}(BE_T, BX)_{Bf_{E_T}} \cong (Sp(1)^\wedge_2)^n$ that permutes the copies $Sp(1)^\wedge_2$. Define $BY = BC_X(E_T) \times_{\Sigma_n} E\Sigma_n$ and consider the diagram

$$
\begin{tikzcd}
(BSp(1)^n)^\wedge_2 \arrow[r] & BY \arrow[r] & B\Sigma_n \\
(BT)^\wedge_2 \arrow[u] \arrow[r] \arrow[d, equal] & \mathrm{Map}(BT, BX)_{Bf_T} \times_{\Sigma_n} E\Sigma_n \arrow[u] \arrow[r] \arrow[d] & B\Sigma_n \arrow[u, equal] \arrow[d] \\
(BT)^\wedge_2 \arrow[r] & \mathrm{Map}(BT, BX)_{Bf_T} \times_{W_{Sp(n)}} EW_{Sp(n)} \arrow[r] & BW_{Sp(n)}
\end{tikzcd}
$$

where all rows are fibrations. The space $\mathrm{Map}(BT, BX)_{Bf_T} \times_{W_{Sp(n)}} EW_{Sp(n)}$ is the normalizer of the maximal torus T in X, so it is isomorphic to $B(N_{Sp(1)}T \rtimes \Sigma_n)$. Therefore the space $\mathrm{Map}(BT, BX)_{Bf_T} \times_{\Sigma_n} E\Sigma_n$ is isomorphic to $B(T \rtimes \Sigma_n)$. This means that the middle row has a section, and hence also the top row has a section. It follows that BY is homotopic to $B((Sp(1)_2^\wedge)^n \rtimes \Sigma_n)$.

PROPOSITION 4.4. *The cohomology H^*BX is detected by elementary abelian 2-subgroups.*

Proof. The cohomology $H^*BSp(1)^n$ is detected by elementary abelian 2-subgroups, hence by [16], H^*BY is detected by elementary abelian subgroups. The normalizer Bf_N factors through the map Bf_Y. According to [20, Theorem 1.2 and Lemma 3.1], the cohomology $H^*(Sp(n)/N)$ is finite and the Euler characteristic $\chi(Sp(n)/N)$ equals 1. Therefore the transfer argument [10, Theorem 9.13] shows that Bf_N^* is a monomorphism. So also Bf_Y^* is a monomorphism. Hence H^*BX is detected by elementary abelian 2-subgroups. ∎

We can now identify the algebra H^*BX:

PROPOSITION 4.5. *The cohomology H^*BX is isomorphic to $H^*BSp(n)$ as an algebra over the mod 2 Steenrod algebra.*

Proof. By Proposition 4.4, the cohomology H^*BX is detected by elementary abelian 2-subgroups, and by Lemma 4.2, every elementary abelian subgroup of X factors through E_T. Therefore H^*BX injects into H^*BE_T and therefore into $H^*BC_X(E_T)$. If we take trivial action of Σ_n on X, the inclusion $C_X(E_T) \to X$ is a Σ_n-equivariant map. Hence the cohomology H^*BX is a subalgebra of $(H^*BSp(1)^n)^{\Sigma_n} = H^*BSp(n)$. But $H^*(BX;\mathbb{Q}) = H^*(BT;\mathbb{Q})^{W_X} = \mathbb{Q}[x_4,\ldots,x_{4n}]$, hence the Bockstein spectral sequence associated to $H^*BX \subset H^*Sp(n) = \mathbb{F}_2[x_4,\ldots,x_{4n}]$ converges to $\mathbb{F}_2[x_4,\ldots,x_{4n}]$, and therefore $H^*BX \cong H^*BSp(n)$. ∎

Recall that the *Quillen category* $Q_p(G)$ of a group G at a prime p is the category with objects (V,α), where V is a nontrivial elementary abelian p-group and $\alpha\colon V \to G$ is a G-conjugacy class of monomorphisms, and $\mathrm{Mor}_{Q_p(G)}((V,\alpha),(V',\alpha'))$ is the set of group morphisms $f\colon V \to V'$ such that $\alpha = \alpha' \circ f$. By Lannes' theory ([19]) and the Dwyer–Zabrodsky theorem ([15] and [30]), the set of G-conjugacy classes of monomorphisms $\alpha\colon V \to G$ is in one-to-one correspondence with the set of morphisms $B\alpha^*\colon H^*BG \to H^*BV$ of unstable algebras over the Steenrod algebra $\mathcal{A}_p$ such that H^*BV is a finitely generated module over $B\alpha^*(H^*BG)$. Hence, there is an equivalent description of the Quillen category which can be used also for p-compact groups [14, §2]: If X is a p-compact group, then $Q_p(X)$ is the category with objects (V,α), where V is a nontrivial elementary abelian p-group and

$\alpha\colon H^*BX \to H^*BV$ is a morphism of unstable algebras over the Steenrod algebra $\mathcal{A}_p$ such that H^*BV is a finitely generated module over $\alpha^*(H^*BX)$, and $\mathrm{Mor}_{Q_p(G)}((V,\alpha),(V',\alpha'))$ is the set of group morphisms $f\colon V \to V'$ such that $\alpha = Bf^*\alpha'$. If X is the p-completion of a compact Lie group then both definitions agree [14, Proposition 2.2].

Using the "cohomological" definition of the Quillen category and Proposition 4.4 we obtain the following result.

PROPOSITION 4.6. *The categories* $Q_2(Sp(n))$ *and* $Q_2(X)$ *are isomorphic.*

5. The 2-stubborn decomposition of $Sp(n)$**.** A 2-*stubborn subgroup* of a Lie group G is a 2-toral group P such that $N_G(P)/P$ is a finite group which has no nontrivial normal 2-subgroup. Let $\mathcal{R}_2(Sp(n))$ be the 2-*stubborn category* of $Sp(n)$, which is the full subcategory of the orbit category of $Sp(n)$ with objects $Sp(n)/P$, where $P \subset Sp(n)$ is a 2-stubborn subgroup. Then the natural map

$$\operatorname*{hocolim}_{Sp(n)/P\in\mathcal{R}_2(Sp(n))} ESp(n)/P \to BSp(n)$$

induces an isomorphism in homology with $\mathbb{Z}_{(2)}$-coefficients [17, Theorem 4]. Therefore, although homotopy colimits are not colimits in a categorical sense, in order to define a map $f\colon BSp(n)^\wedge_2 \to X$ it is enough to define a family of compatible maps $\{f_P\colon ESp(n)/P \simeq BP \to X \mid Sp(n)/P \in \mathrm{ob}(\mathcal{R}_2(Sp(n)))\}$.

We now proceed to recall the 2-stubborn subgroups of $Sp(n)$ which are calculated in [32]. Let the permutations $\sigma_0,\ldots,\sigma_{k-1}$ in Σ_{2^k} be defined by

$$\sigma_r(s) = \begin{cases} s+2^r, & s \equiv 1,\ldots,2^r \bmod 2^{r+1},\\ s-2^r, & s \equiv 2^r+1,\ldots,2^{r+1} \bmod 2^{r+1}.\end{cases}$$

Let $A_0,\ldots,A_{k-1} \in Sp(2^k)$ be diagonal matrices with

$$(A_r)_{ss} = (-1)^{[(s-1)/2^r]},$$

where $[-]$ denotes greatest integer, and let $B_0,\ldots,B_{k-1}$ be the permutation matrices for $\sigma_0,\ldots,\sigma_{k-1}$.

DEFINITION 5.1. For every $k \ge 0$, the subgroups $E_{2^k} \subset \Sigma_{2^k}$ and $\Gamma_{2^k}, \overline{\Gamma}_{2^k} \subset Sp(2^k)$ are defined by

$$\begin{aligned} E_{2^k} &= \langle \sigma_0,\ldots,\sigma_{k-1}\rangle \cong (\mathbb{Z}/2)^k,\\ \Gamma_{2^k} &= \langle uI, A_r, B_r \mid u \in Q(8),\ 0 \le r < k\rangle,\\ \overline{\Gamma}_{2^k} &= \langle uI, A_r, B_r \mid u \in S^1(j),\ 0 \le r < k\rangle,\end{aligned}$$

where $Q(8) = \{\pm1,\pm i,\pm j,\pm k\}$ is the quaternion group and $S^1(j) = \{a+bi, aj+bk \mid a^2+b^2=1\}$ is the normalizer of the maximal torus in $Sp(1) = S^3$.

REMARK 5.2. Let P be Γ_{2^k} (resp. $\overline{\Gamma}_{2^k}$), and P_D be the subgroup of all diagonal matrices in P. Then P_D is $Q(8) \times E_{2^k}$ (resp. $S^1(j) \times E_{2^k}$) and the extension $P_D \to P \to (\mathbb{Z}/2)^k$ splits.

A subgroup $P \subset Sp(n)$ is called *irreducible* if the induced P-representation in $\mathbb{H}^n$ is irreducible. We have

THEOREM 5.3 ([32, Theorem 3]).

(1) *An irreducible subgroup $P \subset Sp(n)$ is a 2-stubborn subgroup if and only if it is conjugate to either*

$$P = \Gamma_{2^k} \wr E_{2^{r_1}} \wr \cdots \wr E_{2^{r_s}} \quad \text{or} \quad P = \overline{\Gamma}_{2^k} \wr E_{2^{r_1}} \wr \cdots \wr E_{2^{r_s}}$$

and $n = 2^{k+r_1+\cdots+r_s}$.

(2) *An arbitrary subgroup $P \subset Sp(n)$ is a 2-stubborn subgroup if and only if it is conjugate to $P_1 \times \cdots \times P_s$, where P_i is an irreducible 2-stubborn subgroup of $Sp(n_i)$ and $n = n_1 + \cdots + n_s$.*

Let $\widetilde{\mathcal{R}}_2(Sp(n))$ be the full subcategory of $\mathcal{R}_2(Sp(n))$ with objects $Sp(n)/P$, where P is one of the representative 2-stubborn subgroups from the previous theorem. The category $\widetilde{\mathcal{R}}_2(Sp(n))$ is equivalent to $\mathcal{R}_2(Sp(n))$, so the natural map

$$\operatorname*{hocolim}_{Sp(n)/P\in\widetilde{\mathcal{R}}_2(Sp(n))} ESp(n)/P \to BSp(n)$$

is also a homotopy equivalence up to 2-completion.

PROPOSITION 5.4. *Let $Sp(n)/P \in \widetilde{\mathcal{R}}_2(Sp(n))$ and define $P_D = P \cap Sp(1)^n$ and $P_T = P \cap T_{Sp(n)}$. Then*

(1) $C_{Sp(n)}(P_T) = T_{Sp(n)}$ *and* $C_{Sp(n)}(P_D) = (\mathbb{Z}/2)^n$,
(2) *for any extension $\alpha\colon P \to Sp(n)$ of $i\colon P_T \to Sp(n)$, we have $C_{Sp(n)}(\alpha(P)) = Z(P)$,*
(3) *the canonical map*

$$\pi_0(\mathrm{Map}(BP, BSp(n)^\wedge_2)_{B\alpha|_{BP_T}=Bi_{P_T}}) \to \mathrm{Hom}(H^*BSp(n), H^*BP)$$

is an injection.

REMARK 5.5. By $\mathrm{Map}(BP, BSp(n)^\wedge_2)_{B\alpha|_{BP_T}=Bi_{P_T}}$ we denote the components of the mapping space $\mathrm{Map}(BP, BSp(n)^\wedge_2)$ given by maps $B\alpha\colon BP \to BSp(n)^\wedge_2$ such that $B\alpha|_{BP_T} \simeq Bi_{P_T}$.

Proof. Part (1) is obvious for $P = \Gamma_{2^n}$ and $P = \overline{\Gamma}_{2^n}$. If $P = Q \wr E_{2^r}$, where Q is an irreducible 2-stubborn subgroup of $Sp(2^{n-r})$, then $C_{Sp(2^n)}(P_T) = C_{Sp(2^{n-r})}(Q_T)^{2^r}$, which is, by induction, $(T_{Sp(2^{n-r})})^{2^r} = T_{Sp(2^n)}$. If $P = \prod_{i=1}^s P_i$ is a product of irreducible 2-stubborn subgroups, then $C_{Sp(n)}(P_T) = \prod_{i=1}^s C_{Sp(n_1)}((P_i)_T) = \prod_{i=1}^s T_{Sp(n_i)} = T_{Sp(n)}$. Analogously we prove that $C_{Sp(n)}(P_D) = (\mathbb{Z}/2)^n$.

Let P be an irreducible 2-stubborn subgroup of $Sp(2^n)$ and let $B\alpha\colon BP \to BSp(2^n)^\wedge_2$ be a homomorphism such that $B\alpha|_{BP_T} = Bi_{P_T}$. The extensions $B\alpha|_{BP_D}\colon BP_D \to BSp(2^n)^\wedge_2$ of Bi_{P_T} are classified by obstruction classes lying in

$$H^m(P_D/P_T; \pi_m(\mathrm{Map}(BP_T, BSp(2^n)^\wedge_2)_{Bi_{P_T}})).$$

By [15] and [30], $\mathrm{Map}(BP_T, BSp(2^n)^\wedge_2)_{Bi_{P_T}}$ is homotopy equivalent to $BC_{Sp(2^n)}(P_T)^\wedge_2$, which is isomorphic to $((BS^1)^\wedge_2)^{2^n}$, by part (1). Then

$$H^m(P_D/P_T; \pi_m(\mathrm{Map}(BP_T, BSp(2^n)^\wedge_2)_{Bi_{P_T}})) = H^m(P_D/P_T; \pi_m(BS^1)^{2^n})$$

and the only possible nontrivial group appears when $m = 2$. And

$$H^2(P_D/P_T; \pi_2(\mathrm{Map}(BP_T, BSp(2^n)^\wedge_2)_{Bi_{P_T}})) = H^2(\mathbb{Z}/2; (\mathbb{Z}^\wedge_2)^{2^n}),$$

where the group $\mathbb{Z}/2$ acts on $(\mathbb{Z}^\wedge_2)^{2^n}$ by reflection on each component; this action can be seen as a diagonal action of the Weyl group of $Sp(1)$, i.e. $\mathbb{Z}/2$, on 2^n copies of the maximal torus S^1. By Shapiro's lemma [4, III, Proposition 6.2], the group $H^2(\mathbb{Z}/2; (\mathbb{Z}^\wedge_2)^{2^n})$ is trivial, so all obstruction classes vanish. Hence if $B\alpha|_{BP_T} = Bi_{P_T}$ then $B\alpha|_{BP_D} = Bi_{P_D}$.

First we will prove part (2) and (3) for the case of P being either Γ_{2^n} or $\overline{\Gamma}_{2^n}$. Let $B\alpha\colon BP \to BSp(2^n)^\wedge_2$ be a map such that $B\alpha|_{BP_T} = Bi_{BP_T}$. Then by the paragraph above, $B\alpha|_{BP_D}$ is homotopic to Bi_{P_D}. The extensions $B\alpha\colon BP \to BSp(2^n)^\wedge_2$ of Bi_{P_D} are classified by obstruction classes lying in $H^m(P/P_D; \pi_m(\mathrm{Map}(BP_D, BSp(2^n)^\wedge_2)_{Bi_{P_D}}))$. By [15], [30], and [3], the mapping space $\mathrm{Map}(BP_D, BSp(2^n)^\wedge_2)_{Bi_{P_D}}$ is homotopy equivalent to $BC_{Sp(2^n)}(P_D)^\wedge_2$, which is isomorphic to $(B\mathbb{Z}/2)^{2^n}$ (part (1)). Then the obstruction group

$$H^m(P/P_D; \pi_m(\mathrm{Map}(BP_D, BSp(2^n)^\wedge_2))_{Bi_{P_D}}) = H^m(P/P_D; \pi_m(B(\mathbb{Z}/2)^{2^n}))$$

is nontrivial only possibly for $m = 1$. The group P/P_D is isomorphic to the group generated by the permutation matrices $B_0, \ldots, B_{n-1}$ (Definition 5.1). So $P/P_D = (\mathbb{Z}/2)^n$ and the action of P/P_D on $\pi_1(B(\mathbb{Z}/2)^{2^n}) = (\mathbb{Z}/2)^{2^n}$ is given by the permutations $\sigma_0, \ldots, \sigma_{n-1}$ which define the matrices $B_0, \ldots, B_{n-1}$. By Shapiro's lemma [4, III, Proposition 6.2],

$$\begin{aligned} H^1(P/P_D; \pi_1(\mathrm{Map}(BP_D, BSp(2^n)^\wedge_2)_{Bi_{P_D}})) &= H^1(E_{2^n}; (\mathbb{Z}/2)^{2^n}) \\ &= H^1(1; \mathbb{Z}/2) = 1, \end{aligned}$$

so all obstruction groups vanish. Therefore $B\alpha$ is homotopic to Bi_P and $C_{Sp(n)}(\alpha)$ equals $Z(P)$.

Now we will prove parts (2) and (3) for an irreducible 2-stubborn subgroup P of $Sp(2^n)$. Write $P = Q \wr E_{2^r}$, where Q is an irreducible 2-stubborn subgroup of $Sp(2^{n-r})$. Let $\alpha, \beta\colon P \to Sp(2^n)$ be two homomorphisms such that $B\alpha^* = B\beta^*$ and $\alpha|_{BP_T} = i_{P_T} = \beta|_{BP_T}$. We have proved that $\alpha|_{BP_D} =$

$i_{P_D} = \beta|_{BP_D}$. Let $\bar{\alpha}, \bar{\beta}\colon Q^{2^r} \to Sp(2^n)$ be the restrictions of α and β. Because $Z(Q^{2^r}) = Z(Q)^{2^r} = (\mathbb{Z}/2)^{2^r}$, the homomorphisms $\bar{\alpha}$ and $\bar{\beta}$ factor through homomorphisms

$$\widetilde{\alpha}, \widetilde{\beta}\colon Q^{2^r} \to C_{Sp(2^n)}(Z(Q^{2^r})) = Sp(2^{n-r})^{2^r}.$$

The map $B\widetilde{\alpha}$ is homotopic to the map

$$\begin{aligned} BQ^{2^r} &\simeq \mathrm{Map}(BZ(Q^{2^r}), BQ^{2^r})_{Bi} \xrightarrow{B\bar{\alpha}_\sharp} \mathrm{Map}(BZ(Q^{2^r}), BSp(2^n)^\wedge_2)_{Bi} \\ &\simeq (BSp(2^{n-r})^{2^r})^\wedge_2, \end{aligned}$$

hence $B\widetilde{\alpha}^* = (B\bar{\alpha}_\sharp)^*$. An analogous argument shows that $B\widetilde{\beta}^* = (B\bar{\beta}_\sharp)^*$. By Lannes' theory [19],

$$(B\bar{\alpha}_\sharp)^* = T^{Z(Q^{2^r})}_{B\bar{\alpha}^*} = T^{Z(Q^{2^r})}_{B\bar{\beta}^*} = (B\bar{\beta}_\sharp)^*,$$

so $B\widetilde{\alpha}^* = B\widetilde{\beta}^*$.

The homomorphisms $\widetilde{\alpha}$ and $\widetilde{\beta}$ are matrices of dimension $2^r \times 2^r$, where the entries are $\widetilde{\alpha}_{i,j}, \widetilde{\beta}_{i,j}\colon Q_i \to Sp(2^{n-r})_j$. The indices i and j indicate the components in the products. By induction, $B\widetilde{\alpha}_{i,i}$ and $B\widetilde{\beta}_{i,i}$ are homotopic and therefore $\widetilde{\alpha}_{i,i}$ and $\widetilde{\beta}_{i,i}$ are conjugate [19, Théorème 3.4.5]. We can assume that $\widetilde{\alpha}_{i,i} = \widetilde{\beta}_{i,i}$. Because Q_i and Q_j commute for $i \neq j$, the homomorphisms $\widetilde{\alpha}_{i,j}$ and $\widetilde{\beta}_{i,j}$ factor through homomorphisms $\widehat{\alpha}_{i,j}, \widehat{\beta}_{i,j}\colon Q_i \to C_{Sp(2^{n-r})}(\widetilde{\alpha}_{j,j}(Q))$. By induction, the centralizer $C_{Sp(2^{n-r})}(\widetilde{\alpha}_{j,j}(Q))$ equals $Z(Q_j) = (\mathbb{Z}/2)_j$. Because $\widetilde{\alpha}|_{P_D} = \widetilde{\beta}|_{P_D}$, the homomorphism $\widetilde{\alpha}_{i,j} \cdot \widetilde{\beta}^{-1}_{i,j}\colon Q_i \to (\mathbb{Z}/2)_j$ factors through a homomorphism $\gamma_{i,j}\colon (Q/Q_D)_i \to (\mathbb{Z}/2)_j$. Then $\widetilde{\beta}_{i,j}$ equals the composition

$$Q_i \xrightarrow{\Delta} Q_i \times (Q/Q_D)_i \xrightarrow{\widetilde{\alpha}_{i,j} \times \gamma_{i,j}} Sp(2^{n-r})_j \times (\mathbb{Z}/2)_j \xrightarrow{\mu} Sp(2^{n-r})_j,$$

where Δ is the diagonal map composed with the quotient map and μ is the multiplication in $Sp(2^{n-r})$. Because $B\widetilde{\alpha}^*_{i,j} = B\widetilde{\beta}^*_{i,j}$, the map $B\gamma_{i,j}$ induces a trivial map in mod 2 cohomology. Because Q/Q_D is an iterated wreath product of elementary abelian groups, the map $\gamma_{i,j}$ is constant [25, Lemma 6.10]. Hence $\widetilde{\alpha}_{i,j} = \widetilde{\beta}_{i,j}$ and so $\bar{\alpha} = \bar{\beta}$, and the centralizer $C_{Sp(2^n)}(\alpha)$ is given by the fixed-point set $C_{Sp(2^n)}(\alpha) = (C_{Sp(2^{n-r})^{2^r}}(\widetilde{\alpha}))^{E_{2^r}} = ((C_{Sp(2^{n-r})}(Q))^{2^r})^{E_{2^r}} = ((\mathbb{Z}/2)^{2^r})^{E_{2^r}} = \mathbb{Z}/2 = Z(P)$, which proves part (2).

The extensions $B\alpha\colon BP \to BSp(2^n)^\wedge_2$ of $B\bar{\alpha}$ are classified by the obstruction groups $H^m(P/Q^{2^r}; \pi_m(\mathrm{Map}(BQ^{2^r}, BSp(2^n)^\wedge_2)_{B\bar{\alpha}}))$. By [17], the mapping space $\mathrm{Map}(BQ^{2^r}, BSp(2^n)^\wedge_2)_{B\bar{\alpha}}$ is homotopy equivalent to $BC_{Sp(2^n)}(Q^{2^r})^\wedge_2 = (\mathbb{Z}/2)$. Hence the obstruction groups are

$$H^m(P/Q^{2^r}; \pi_m(\mathrm{Map}(BQ^{2^r}, BSp(2^n)^\wedge_2)_{B\bar{\alpha}})) = H^m(P/Q^{2^r}; \pi_m(B\mathbb{Z}/2)).$$

The only possible nontrivial obstruction group is for $m = 1$. The group $P/Q^{2^r} = E_{2^r}$ acts on $BC_{Sp(2^n)}(Q^{2^r}) = (B\mathbb{Z}/2)^{2^r}$ permuting the factors, hence $\mathrm{Ind}_1^{E_{2^r}}(\mathbb{Z}/2)^{2^r} = (\mathbb{Z}/2)$, by Shapiro's lemma [4, III, Proposition 6.2]. Therefore

$$H^1(K \wr E_{2^r}; (\mathbb{Z}/2)^{2^n}) = H^1(K; (\mathbb{Z}/2)^{2^{n-r}}).$$

Thus all obstruction groups vanish, so $B\alpha$ is homotopic to Bi_P.

Finally, let $P = P_1 \times \cdots \times P_s$, where P_i is an irreducible 2-stubborn subgroup of $Sp(n_i)$. Let $\alpha, \beta \colon P \to Sp(n)$ be two homomorphisms such that $B\alpha^* = B\beta^*$ and $\alpha|_{P_T} = \beta|_{P_T}$. Both homomorphisms factor through $\bar{\alpha}, \bar{\beta} \colon P \to C_{Sp(n)}(Z(P)) = C_{Sp(n)}((\mathbb{Z}/2)^s) = Sp(n_1) \times \cdots \times Sp(n_s)$. In the same way as in the case of P being an irreducible 2-stubborn group, we can show that $B\bar{\alpha}^* = B\bar{\beta}^*$. The maps $\bar{\alpha}$ and $\bar{\beta}$ are matrices of dimension $s \times s$ with entries maps $\bar{\alpha}_{i,j}, \bar{\beta}_{i,j} \colon P_i \to Sp(n_j)$. As before we can show that $\bar{\alpha}_{i,j} = \bar{\beta}_{i,j}$, so $B\alpha \simeq B\beta$. The equality $Z(P) = Z(P_1) \times \cdots \times Z(P_s)$ finishes the proof. ∎

6. The map from $Sp(n)_2^\wedge$ to X. For every object $Sp(n)/P$ in $\widetilde{\mathcal{R}}(Sp(n))$ we define a 2-compact group morphism $f_P \colon P \to X$ as the composition of the two inclusions $i_P \colon P \to N$ and $f_N \colon N \to X$. We will prove that for every morphism $c_g \colon Sp(n)/P \to Sp(n)/Q$ in $\widetilde{\mathcal{R}}(Sp(n))$, the diagram

(1)
$$\begin{tikzcd} BP \arrow[r, "Bi_P"] \arrow[d, "Bc_g"'] & BN \arrow[r, "Bf_N"] & BX \arrow[d, equal] \\ BQ \arrow[r, "Bi_Q"] & BN \arrow[r, "Bf_N"] & BX \end{tikzcd}$$

commutes up to homotopy.

Let us define $\alpha = f_N \circ i_P$ and $\beta = f_N \circ i_Q \circ c_g$. Then $B\alpha^* = B\beta^*$. The group $P_T = P \cap T_{Sp(n)}$ is 2-toral. The restrictions $\alpha|_{P_T}$ and $\beta|_{P_T}$ are conjugate in $Sp(n)$, and hence by [24, Proposition 4.1], they are also conjugate in the normalizer N of the maximal torus. So $B\alpha|_{BP_T} \simeq B\beta|_{BP_T}$. By the next proposition, $B\alpha \simeq B\beta$.

Let $K \to G \to H$ be an exact sequence of groups. Then H acts freely on $\widetilde{BK} = EG/K \simeq BK$, and $\widetilde{BK}/H$ equals BG. For any space BX with trivial action of the group H, we have

(2)
$$\begin{aligned} \mathrm{Map}(BG, BX) &= \mathrm{Map}(\widetilde{BK}/H, BX) = \mathrm{Map}_H(\widetilde{BK}, BX) \\ &\simeq \mathrm{Map}_H(EH \times \widetilde{BK}, BX) \\ &= \mathrm{Map}_H(EH, \mathrm{Map}(\widetilde{BK}, BX)) = \mathrm{Map}(\widetilde{BK}, BX)^{hH}. \end{aligned}$$

PROPOSITION 6.1. *For every $Sp(n)/P \in \mathrm{ob}(\widetilde{\mathcal{R}}(Sp(n)))$, the canonical map*

$$\pi_0(\mathrm{Map}(BP, BX)_{B\alpha|_{BP_T} = Bf_{P_T}}) \to \mathrm{Hom}(H^*BX, H^*BP)$$

is an injection.

Proof. Consider the diagram

$$\begin{tikzcd}[column sep=tiny] & \mathrm{Map}(\widetilde{BP_T}, BY)_{Bi_T} \arrow[dl] \arrow[dr] & \\ \mathrm{Map}(\widetilde{BP_T}, BSp(n)^\wedge_2)_{Bi_T} & & \mathrm{Map}(\widetilde{BP_T}, BX)_{Bf_T} \end{tikzcd} \tag{3}$$

By [30], the mapping space $\mathrm{Map}(\widetilde{BP_T}, BSp(n)^\wedge_2)_{Bi_T}$ is homotopy equivalent to $BC_{Sp(n)}(BP_T)^\wedge_2$ and by Proposition 5.4, the latter is homotopy equivalent to $(BT_{Sp(n)})^\wedge_2$. Analogously $\mathrm{Map}(\widetilde{BP_T}, BY)_{Bi_T}$ is homotopy equivalent to $(BT_{Sp(n)})^\wedge_2$. The mapping space $\mathrm{Map}(\widetilde{BP_T}, BX)_{Bf_T}$ is the classifying space of a 2-compact group [10]. Its Weyl group is $\mathrm{Iso}(Bf_N \circ Bi_{P_T}) = \{w \in W_X \mid w \circ Bf_N \circ Bi_{P_T} \simeq Bf_N \circ Bi_{P_T}\}$ [35, Proposition 4.3]. By the construction of the map f_N, the group $\mathrm{Iso}(Bf_N \circ Bi_{P_T})$ equals $\mathrm{Iso}(Bi_N \circ Bi_{P_T})$. Because $\mathrm{Iso}(Bi_N \circ Bi_{P_T})$ is the Weyl group of the mapping space $\mathrm{Map}(\widetilde{BP_T}, BSp(n)^\wedge_2)_{Bi_T} \simeq (BT_{Sp(n)})^\wedge_2$, the group $\mathrm{Iso}(Bf_N \circ Bi_{P_T})$ is trivial, hence $\mathrm{Map}(\widetilde{BP_T}, BX)_{Bf_T} \simeq (BT_{Sp(n)})^\wedge_2$. Therefore both maps in diagram (3) are homotopy equivalences.

Taking homotopy fixed points we obtain the diagram

$$\begin{tikzcd}[column sep=tiny] & \mathrm{Map}(\widetilde{BP_T}, BY)^{h(P/P_T)}_{Bi_T} \arrow[dl] \arrow[dr] & \\ \mathrm{Map}(\widetilde{BP_T}, BSp(n)^\wedge_2)^{h(P/P_T)}_{Bi_T} & & \mathrm{Map}(\widetilde{BP_T}, BX)^{h(P/P_T)}_{Bf_T} \end{tikzcd}$$

where both maps are mod 2 equivalences, since an equivariant mod 2 equivalence between 1-connected spaces induces a mod 2 equivalence between the homotopy fixed-point sets.

By Proposition 5.4, the components of $\mathrm{Map}(\widetilde{BP_T}, BSp(n)^\wedge_2)^{h(P/P_T)}_{Bi_T}$ are distinguished by mod 2 cohomology. Any map in $\mathrm{Map}(\widetilde{BP_T}, BX)^{h(P/P_T)}_{Bf_T}$ has a lift to BN and therefore to BY. The obstruction group which classifies the extensions is

$$\begin{aligned} &H^2(P/P_T; \pi_2 \mathrm{Map}(\widetilde{BP_T}, BX)_{Bf_T}) \\ &\qquad\qquad \cong H^2(P/P_T; \pi_2 \mathrm{Map}(\widetilde{BP_T}, BSp(n)^\wedge_2)_{Bi_T}), \end{aligned}$$

so the components of $\mathrm{Map}(\widetilde{BP_T}, BX)^{h(P/P_T)}_{Bf_T} \simeq \mathrm{Map}(BP, BX)_{B\alpha|_{BP_T} = Bi_{P_T}}$ are also distinguished by mod 2 cohomology. ∎

Diagram (1) establishes a map from the 1-skeleton of the homotopy colimit $\{BP\}_{\widetilde{\mathcal{R}}_2(Sp(n))}$ to BX. The obstruction groups for extending a map defined on the 1-skeleton of the homotopy colimit to a map on the total homotopy colimit are

$$\varprojlim_{\widetilde{\mathcal{R}}_2(Sp(n))}{}^{i+1} \pi_i \operatorname{Map}(BP, BX)_{Bf_P}$$

for $i \geq 2$, where $\varprojlim^i$ is the ith derived functor of the inverse limit functor ([5] and [37]).

Let $\mathcal{A}b$ be the category of abelian groups and let

$$\mathit{\Pi}_j^X, \mathit{\Pi}_j^{Sp(n)} \colon \widetilde{\mathcal{R}}_2(Sp(n)) \to \mathcal{A}b$$

be functors defined by

$$\mathit{\Pi}_j^X(Sp(n)/P) = \pi_j \operatorname{Map}(BP, BX)_{Bf_P},$$
$$\mathit{\Pi}_j^{Sp(n)}(Sp(n)/P) = \pi_j \operatorname{Map}(BP, BSp(n)_2^\wedge)_{Bi_P}.$$

Note that $\operatorname{Map}(BP, BSp(n)_2^\wedge)_{Bi_P}$ is homotopic to $BZ(P)_2^\wedge$ [17, Theorem 3.2] and therefore $\mathit{\Pi}_1^{Sp(n)}(Sp(n)/P)$ is well defined. By the next proposition, also $\mathit{\Pi}_1^X(Sp(n)/P)$ is well defined.

Proposition 6.2. *There exists a natural transformation*

$$\mathcal{T} \colon \mathit{\Pi}_j^{Sp(n)} \to \mathit{\Pi}_j^X$$

which is an equivalence.

Proof. For every 2-stubborn group P we have homotopy equivalences

$$\operatorname{Map}(BP, BSp(n)_2^\wedge)_{Bi_P} \xleftarrow{\simeq} \operatorname{Map}(BP, BY)_{Bi_P} \xrightarrow{\simeq} \operatorname{Map}(BP, BX)_{Bf_P} \tag{4}$$

which depend on the chosen lift $Bi_P \colon BP \to BY$ of the map $Bi_P \colon BP \to BSp(n)_2^\wedge$. Denote by $P_\infty \leq P$ the subgroup of 2-elements. Because the inclusion $P_\infty \leq P$ induces a mod 2 equivalence, and $\operatorname{Rep}(P_\infty, Sp(n)) \to [BP, BSp(n)_2^\wedge]$ is a bijection [18, Theorem 1.1(i)], any two lifts differ by a conjugation Bc_g. Since $Bf_P \simeq Bf_P \circ Bc_g$, the equivalence (4) induces well defined isomorphisms

$$\mathit{\Pi}_j^{Sp(n)}(Sp(n)/P) \to \mathit{\Pi}_j^X(Sp(n)/P)$$

which commute with maps induced by morphisms in $\widetilde{\mathcal{R}}_2(Sp(n))$. ∎

Proposition 6.3. *For all $i, j \geq 1$,*

$$\varprojlim_{\widetilde{\mathcal{R}}_2(Sp(n))}{}^{i} \pi_j \operatorname{Map}(BP, BX)_{Bf_P} = 0.$$

Proof. By the previous lemma,

$$\varprojlim_{\widetilde{\mathcal{R}}_2(Sp(n))}{}^{i} \pi_j \operatorname{Map}(BP, BX)_{Bf_P} = \varprojlim_{\widetilde{\mathcal{R}}_2(Sp(n))}{}^{i} \pi_j \operatorname{Map}(BP, BSp(n)_2^\wedge)_{Bi_P}$$

and the right side is 0 [17, Theorem 4.8]. ∎

Because all obstructions vanish, there exists a map

$$f\colon \operatorname*{hocolim}_{\widetilde{R}_2(Sp(n))} BP \to BX.$$

By the construction of the map we have a commutative diagram

$$\begin{tikzcd}
 & & BY \arrow[dll] \arrow[d] \\
\operatorname*{hocolim}_{\widetilde{R}_2(Sp(n))} ESp(n)/P \arrow[rr, "f"] & & BX
\end{tikzcd}$$

where the diagonal maps induce monomorphisms in cohomology and therefore also the map f^* is a monomorphism. Since $H^*BSp(n) \cong H^*BX$, f^* is an isomorphism and therefore f is a homotopy equivalence.

7. $Sp(n)$ as a loop space. The normalizer conjecture can be stated also for finite loop spaces with maximal torus normalizers as a weak version of Wilkerson's conjecture (see [36]).

THEOREM 7.1. *Let L be a connected finite loop space with a maximal torus normalizer isomorphic to that of $Sp(n)$. Then BL is homotopy equivalent to $BSp(n)$.*

Proof. To prove $BL \simeq BSp(n)$ is equivalent to showing that BL and $BSp(n)$ lie in the same adic genus [31]. The loop spaces BL and $BSp(n)$ have the same rational genus. Since BL is finite and connected, $L_p^\wedge$ is a p-compact group. The maximal torus normalizer of $L_p^\wedge$ is just the fiberwise p-completion of N by the fibration $BT \to BN \to BW_L$. Hence $L_p^\wedge$ and $Sp(n)_p^\wedge$ have isomorphic normalizers of the maximal torus. By [2], $BSp(n)_p^\wedge$ is N-determined if p is an odd prime, and by the main theorem of this paper, $BSp(n)_2^\wedge$ is (weakly) N-determined. So $BL_p^\wedge$ and $BSp(n)_p^\wedge$ are homotopy equivalent. ■

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