Extension of functions with small oscillation

by

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Abstract. A classical theorem of Kuratowski says that every Baire one function on a $G_δ$ subspace of a Polish (= separable completely metrizable) space $X$ can be extended to a Baire one function on $X$. Kechris and Louveau introduced a finer gradation of Baire one functions into small Baire classes. A Baire one function $f$ is assigned into a class in this hierarchy depending on its oscillation index $β(f)$. We prove a refinement of Kuratowski’s theorem: if $Y$ is a subspace of a metric space $X$ and $f$ is a real-valued function on $Y$ such that $β_Y(f) < ω^α$, $α < ω_1$, then $f$ has an extension $F$ to $X$ so that $β_X(F) ≤ ω^α$. We also show that if $f$ is a continuous real-valued function on $Y$, then $f$ has an extension $F$ to $X$ so that $β_X(F) ≤ 3$. An example is constructed to show that this result is optimal.

Let $X$ be a topological space. A real-valued function on $X$ belongs to Baire class one if it is the pointwise limit of a sequence of continuous functions. If $X$ is a Polish (= separable completely metrizable) space, then a classical theorem of Kuratowski [7] states that every Baire one function on a $G_δ$ subspace of $X$ can be extended to a Baire one function on $X$. In [6], Kechris and Louveau introduced a finer gradation of Baire one functions into small Baire classes using the oscillation index $β$, whose definition we now recall.

Let $X$ be a topological space and let $C$ denote the collection of all closed subsets of $X$. A derivation on $C$ is a map $D : C → C$ such that $D(P) ⊆ P$ for all $P ∈ C$. The oscillation index $β$ is associated with a family of derivations. Let $ε > 0$ and a function $f : X → \mathbb{R}$ be given. For any $P ∈ C$, let $D^0(f, ε, P) = P$ and $D^1(f, ε, P)$ be the set of all $x ∈ P$ such that for any neighborhood $U$ of $x$, there exist $x_1, x_2 ∈ P ∩ U$ such that $|f(x_1) − f(x_2)| ≥ ε$. The derivation $D^1(f, ε, ·)$ may be iterated in the usual manner. For all $α < ω_1$, let

$$D^{α+1}(f, ε, P) = D^1(f, ε, D^α(f, ε, P)).$$

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If $\alpha$ is a countable limit ordinal, set
\[
D^\alpha(f, \varepsilon, P) = \bigcap_{\gamma < \alpha} D^\gamma(f, \varepsilon, P).
\]
If $D^\alpha(f, \varepsilon, P) \neq \emptyset$ for all $\alpha < \omega_1$, let $\beta_X(f, \varepsilon) = \omega_1$. Otherwise, let $\beta_X(f, \varepsilon)$ be the smallest countable ordinal $\alpha$ such that $D^\alpha(f, \varepsilon, P) = \emptyset$. The oscillation index of $f$ is $\beta_X(f) = \sup_{\varepsilon > 0} \beta_X(f, \varepsilon)$.

The main result of §1 is that if $Y$ is a subspace of a metric space $X$ and $f : Y \to \mathbb{R}$ satisfies $\beta_Y(f) < \omega^\alpha$ for some $\alpha < \omega_1$, then $f$ can be extended to a function $F$ on $X$ with $\beta_X(F) \leq \omega^\alpha$. It follows readily from the Baire characterization theorem [2, 10.15] that a real-valued function on a Polish space is Baire one if and only if its oscillation index is countable (see, e.g., [6]). Also, a theorem of Aleksandrov says that a $G_\delta$ subspace of a Polish space is Polish [2, 10.18]. Hence our result refines Kuratowski’s theorem in terms of the oscillation index. Let us mention that if $X$ is a metric space, then every real-valued function with countable oscillation index on a closed subspace of $X$ may be extended to $X$ with preservation of the index [8, Theorem 3.6]. (Note that the proof of [8, Theorem 3.6] does not require the compactness of the ambient space.) More recent results on the extension of Baire one functions on general topological spaces are found in [5].

It is well known that if a function is continuous on a closed subspace of a metric space, then there exists a continuous extension to the whole space. §2 is devoted to the study of extensions of continuous functions from an arbitrary subspace of a metric space. It is shown that if $f$ is a continuous function on a subspace $Y$ of a metric space $X$, then $f$ has an extension $F$ to $X$ with $\beta_X(F) \leq 3$. An example is given to show that the result is optimal. The criteria for continuous extension from dense subspaces were studied by several authors (see, e.g., [1], [4]).

1. Functions of small oscillation. Given a real-valued function defined on a set $S$, let $\|f\|_S = \sup_{s \in S} |f(s)|$. Since we do not assume that the function $f$ is bounded, $\|f\|_S$ may take the value $+\infty$. For any topological space $X$, the support $\text{supp} f$ of a function $f : X \to \mathbb{R}$ is the closed set $\{x \in X : f(x) \neq 0\}$. A family $\{\varphi_\alpha : \alpha \in A\}$ of nonnegative, continuous real-valued functions on $X$ is called a partition of unity on $X$ if

(1) the supports of the $\varphi_\alpha$’s form a locally finite closed covering of $X$,
(2) $\sum_{\alpha \in A} \varphi_\alpha(x) = 1$ for all $x \in X$.

If $\{U_\beta : \beta \in \mathcal{B}\}$ is an open covering of $X$, we say that a partition of unity $\{\varphi_\beta : \beta \in \mathcal{B}\}$ on $X$ is subordinate to $\{U_\beta : \beta \in \mathcal{B}\}$ if the support of each $\varphi_\beta$ lies in the corresponding $U_\beta$. It is well known that if $X$ is paracompact (in particular, if $X$ is a metric space [3, Theorem IX.5.3]), then for each open
covering \{U_\beta : \beta \in B\} of X there is a partition of unity on X subordinate to \{U_\beta : \beta \in B\} (see, for example, [3, Theorem VIII.4.2]).

**Proposition 1.** Let X be a metric space and Y be a subspace of X. Suppose that \(f : Y \to \mathbb{R}\) is a function such that \(\beta_Y(f, \varepsilon) \leq \alpha\) for some \(\varepsilon > 0\), \(\alpha < \omega_1\). Then there exists a function \(\tilde{f} : X \to \mathbb{R}\) such that \(\beta_X(\tilde{f}) \leq \alpha + 1\), \(\|\tilde{f}\|_X \leq \|f\|_Y\) and \(\|f - \tilde{f}\|_Y \leq \varepsilon\).

In the following, denote \(D^\beta(f, \varepsilon, Y)\) by \(Y^\beta\) for all \(\beta < \omega_1\). Proposition 1 is proved by working on each of the pieces \(Y^\beta \setminus Y^{\beta+1}\), \(\beta < \alpha\), and gluing together the results.

**Lemma 2.** For all \(0 \leq \beta < \alpha\), there exist an open set \(Z_\beta\) in X such that \(Y^\beta \setminus Y^{\beta+1} \subseteq Z_\beta \subseteq (Y^{\beta+1})^\complement\), and a continuous function \(f_\beta : Z_\beta \to \mathbb{R}\) such that \(\|f - f_\beta\|_{Y^\beta \setminus Y^{\beta+1}} \leq \varepsilon\) and \(\|f_\beta\|_{Z_\beta} \leq \|f\|_Y\).

**Proof.** If \(0 \leq \beta < \alpha\) and \(y \in Y^\beta \setminus Y^{\beta+1}\), there exists a set \(U_y\) that is an open neighborhood of \(y\) in X so that \(U_y\) is disjoint from \(Y^{\beta+1}\) and that \(f(U_y \cap Y^\beta) \subseteq (f(y) - \varepsilon, f(y) + \varepsilon)\). Let

\[
Z_\beta = \bigcup_{y \in Y^\beta \setminus Y^{\beta+1}} U_y.
\]

Each \(Z_\beta\) is open in X. Clearly, \(Y^\beta \setminus Y^{\beta+1} \subseteq Z_\beta \subseteq (Y^{\beta+1})^\complement\). There exists a partition of unity \((\varphi_y)_{y \in Y^\beta \setminus Y^{\beta+1}}\) on \(Z_\beta\) subordinate to the open covering \(U = \{U_y : y \in Y^\beta \setminus Y^{\beta+1}\}\). Define \(f_\beta : Z_\beta \to \mathbb{R}\) by

\[
f_\beta(z) = \sum_{y \in Y^\beta \setminus Y^{\beta+1}} f(y)\varphi_y(z).
\]

Then \(f_\beta\) is well defined, continuous and \(\|f_\beta\|_{Z_\beta} \leq \|f\|_Y\). If \(x \in Y^\beta \setminus Y^{\beta+1}\), set \(V_x = \{y \in Y^\beta \setminus Y^{\beta+1} : \varphi_y(x) \neq 0\}\). Then \(\sum_{y \in V_x} \varphi_y(x) = 1\). If \(y \in V_x\), then \(x \in U_y\); thus \(|f(x) - f(y)| < \varepsilon\). Hence

\[
|f(x) - f_\beta(x)| = \left| \sum_{y \in V_x} (f(x) - f(y))\varphi_y(x) \right| \leq \sum_{y \in V_x} |f(x) - f(y)|\varphi_y(x) \leq \varepsilon.
\]

Therefore, \(\|f - f_\beta\|_{Y^\beta \setminus Y^{\beta+1}} \leq \varepsilon\), as required. \(\blacksquare\)

**Proof of Proposition 1.** Define a function \(\tilde{f} : X \to \mathbb{R}\) by

\[
\tilde{f}(x) = \begin{cases} f_\beta(x) & \text{if } x \in Z_\beta \setminus \bigcup_{\gamma < \beta} Z_\gamma, \beta < \alpha, \\ 0 & \text{if } x \notin \bigcup_{\gamma < \beta} Z_\gamma. \end{cases}
\]

Clearly, \(\|\tilde{f}\|_X = \sup_{\beta < \alpha} \|f_\beta\|_{Z_\beta} \leq \|f\|_Y\). If \(x \in Y\), then \(x \in Y^\beta \setminus Y^{\beta+1}\) for some \(\beta < \alpha\). In particular, \(x \in Z_\beta \setminus \bigcup_{\gamma < \beta} Z_\gamma\). Hence \(|f(x) - \tilde{f}(x)| = |f(x) - f_\beta(x)| \leq \|f - f_\beta\|_{Y^\beta \setminus Y^{\beta+1}} \leq \varepsilon\) according to Lemma 2. Since this is true for all \(x \in Y\), we have \(\|f - \tilde{f}\|_Y \leq \varepsilon\).
It remains to show that $\beta_X(\tilde{f}) \leq \alpha + 1$. To this end, we claim that $\mathcal{D}_X(\tilde{f}, \delta, X) \cap Z_\gamma = \emptyset$ for all $\delta > 0$, $\gamma < \beta \leq \alpha$. We prove the claim by induction. Let $\delta > 0$. Since $f_0$ is continuous on the open set $Z_0$, we have $\mathcal{D}^1(\tilde{f}, \delta, X) \cap Z_0 = \emptyset$. Suppose that the claim holds for all ordinals less than $\beta$. By the inductive hypothesis, $\mathcal{D}_X(\tilde{f}, \delta, X) \cap \bigcup_{\gamma < \xi} Z_\gamma = \emptyset$ for all $\xi < \beta$. Therefore,

$$\mathcal{D}_X(\tilde{f}, \delta, X) \cap \left[ Z_\xi \setminus \bigcup_{\gamma < \xi} Z_\gamma \right] = \mathcal{D}_X(\tilde{f}, \delta, X) \cap Z_\xi.$$

Now $\tilde{f} = f_\xi$ is continuous on this set, which is open in $\mathcal{D}_X(\tilde{f}, \delta, X)$. Therefore $\mathcal{D}_X^1(\tilde{f}, \delta, X) \cap Z_\xi = \emptyset$. Also, since $\mathcal{D}_X(\tilde{f}, \delta, X) \subseteq \mathcal{D}_X^{\gamma+1}(\tilde{f}, \delta, X)$ for all $\gamma < \beta$,

$$\mathcal{D}_X(\tilde{f}, \delta, X) \cap Z_\gamma = \emptyset$$

for all $\gamma < \beta$. This proves the claim. It follows from the claim that

$$\mathcal{D}_X^\alpha(\tilde{f}, \delta, X) \subseteq \left( \bigcup_{\gamma < \alpha} Z_\gamma \right)^c$$

for any $\delta > 0$. Since $\tilde{f} = 0$ on the latter set, $\mathcal{D}_X^{\alpha+1}(\tilde{f}, \delta, X) = \emptyset$. ■

In order to iterate Proposition 1 to obtain an extension of $f$, we need the following result.

**Proposition 3.** Let $Y$ be a subspace of a metric space $X$. If $\beta_Y(f) < \omega^\xi$ and $\beta_Y(g) < \omega^\xi$, then $\beta_Y(f + g) < \omega^\xi$.

Proposition 3 is proved by the method used in [6, Lemma 5]. This requires a slight modification in the derivation $\mathcal{D}$ associated with the index $\beta$.

Given a real-valued function $f : Y \rightarrow \mathbb{R}$, $\varepsilon > 0$, and a closed subset $P$ of $Y$, define $G(f, \varepsilon, P)$ to be the set of all $y \in P$ such that for every neighborhood $U$ of $y$, there exists $y' \in P \cap U$ such that $|f(y) - f(y')| \geq \varepsilon$. Let $G^0(f, \varepsilon, P) = P$ and

$$G^1(f, \varepsilon, P) = \overline{G(f, \varepsilon, P)},$$

where the closure is taken in $Y$. This defines a derivation $G$ on the closed subsets of $Y$ which may be iterated in the usual manner. If $\alpha < \omega_1$, let

$$G^{\alpha+1}(f, \varepsilon, P) = G^1(f, \varepsilon, G^\alpha(f, \varepsilon, P)).$$

If $\alpha < \omega_1$ is a limit ordinal, let

$$G^\alpha(f, \varepsilon, P) = \bigcap_{\alpha' < \alpha} G^{\alpha'}(f, \varepsilon, P).$$

Clearly, the derivation $G$ is closely related to $\mathcal{D}$. The precise relationship between $\mathcal{D}$ and $G$ is given in part (c) of the next lemma.

**Lemma 4.** If $f$ and $g$ are real-valued functions on $Y$, $\varepsilon > 0$, and $P, Q$ are closed subsets of $Y$, then
(a) $G^1(f + g, \varepsilon, P) \subseteq G^1(f, \varepsilon/2, P) \cup G^1(g, \varepsilon/2, P)$,
(b) $G^1(f, \varepsilon, P \cup Q) \subseteq G^1(f, \varepsilon, P) \cup G^1(f, \varepsilon, Q),$
(c) $D^1(f, 2\varepsilon, P) \subseteq G^1(f, \varepsilon, P) \subseteq D^1(f, \varepsilon, P)$.

We leave the simple proofs to the reader. Note that it follows from part (c) that for all $\alpha < \omega$

(d) $D^\alpha(f, 2\varepsilon, P) \subseteq G^\alpha(f, \varepsilon, P) \subseteq D^\alpha(f, \varepsilon, P)$.

Proof of Proposition 3. Parts (a) and (b) of Lemma 4 correspond to (\star) and (\star\star) in [6, Lemma 5] respectively. From the proof of that result we obtain, for all $n \in \mathbb{N}$ and $\zeta < \omega$,

$$G^\omega \cdot 2n(f + g, \varepsilon, Y) \subseteq G^\omega \cdot n(f, \varepsilon/2, Y) \cup G^\omega \cdot n(g, \varepsilon/2, Y).$$

Since $\beta_Y(f) < \omega^\xi$ and $\beta_Y(g) < \omega^\xi$, there exist $\zeta < \xi$ and $n \in \mathbb{N}$ such that $\beta_Y(f) < \omega^\xi \cdot n$ and $\beta_Y(g) < \omega^\xi \cdot n$. By (d), for any $\varepsilon > 0$,

$$G^\omega \cdot n(f, \varepsilon/2, Y) = G^\omega \cdot n(g, \varepsilon/2, Y) = \emptyset.$$

By (d) and (1),

$$D^\omega \cdot 2n(f + g, 2\varepsilon, Y) = \emptyset.$$

Since this is true for all $\varepsilon > 0$, we have

$$\beta_Y(f + g) \leq \omega^\xi \cdot 2n < \omega^\xi.$$

Theorem 5. Let $X$ be a metric space and let $Y$ be an arbitrary subspace of $X$. If $f : Y \to \mathbb{R}$ satisfies $\beta_Y(f) < \omega^\alpha$ for some $\alpha < \omega_1$, then there exists $F : X \to \mathbb{R}$ with $\beta_X(F) \leq \omega^\alpha$ and $F|_Y = f$.

Proof. Applying Proposition 1 to $f : Y \to \mathbb{R}$ with $\varepsilon = 1/2$, we obtain $g_1 : X \to \mathbb{R}$ with $\|f - g_1\|_Y \leq 1/2$ and $\beta_X(g_1) < \omega^\alpha$. By Proposition 3 we see that $\beta_Y(f - g_1) < \omega^\alpha$. Now apply Proposition 1 to $(f - g_1)|_Y$ with $\varepsilon = 1/2^2$. We obtain $g_2 : X \to \mathbb{R}$, with $\|g_2\|_X \leq \|f - g_1\|_Y \leq 1/2$, $\|f - g_1 - g_2\|_Y \leq 1/2^2$, and $\beta_X(g_2) < \omega^\alpha$. Continuing in this way, we obtain a sequence $(g_n)$ of real-valued functions on $X$ such that for all $n \in \mathbb{N},$

(i) $\|g_{n+1}\|_X \leq \|f - \sum_{i=1}^{n} g_i\|_Y \leq 1/2^n$,
(ii) $\beta_X(g_n) < \omega^\alpha$.

Let $F = \sum_{n=1}^{\infty} g_n$. Note that the series converges uniformly on $X$ and $g|_Y = f$ by (i). Finally, suppose that $\varepsilon > 0$. Choose $N$ such that $\sum_{n=N+1}^{\infty} g_n \cdot 2/2^N < \varepsilon/4$. Then

$$D^{\omega^\alpha}((F, \varepsilon, X) \subseteq D^{\omega^\alpha}(\sum_{n=1}^{N} g_n, \varepsilon/2, X) = \emptyset,$$

since $\beta_X(\sum_{n=1}^{N} g_n) < \omega^\alpha$ by Proposition 3. Thus $\beta_X(F) \leq \omega^\alpha$. ■
Corollary 6 (Kuratowski [7, §31, VI]). Let $X$ be a Polish space and $Y$ be a $G_δ$ subset of $X$. Then every real-valued function of Baire class one on $Y$ can be extended to a function of Baire class one on $X$.

Remarks. 1. Kuratowski’s theorem holds for functions with arbitrary Polish ranges. We do not know if our theorem is true in this more general context.

2. In general, the condition $β_Y(f) < ω_1$ implies that $f$ is of Baire class one on $Y$, but not vice versa. Indeed, if $Y$ is a subspace of a metric space $X$, then $β_Y(f) < ω_1$ if and only if $f$ has an extension $f'$ to a $G_δ$ subset $Y'$ of $X$ such that $β_{Y'}(f') = β_Y(f)$. The two conditions coincide if $Y$ is Polish.

3. Theorem 5 may be viewed as follows: For any $β < ω_1$, there exists $σ(β) < ω_1$ such that if $f$ is a real-valued function defined on a subspace $Y$ of a metric space $X$ with $β_Y(f) = β$, then there exists $F : X → ℝ$ with $β_X(F) ≤ σ(β)$ and $F|_Y = f$. (In fact, Theorem 5 shows that if $β = ω^α$, then $σ(β) = ω^{α+1}$ works.) A natural question is to ask for the optimal (i.e., minimal) value of $σ(β)$. Theorem 14 and Example 15 together show that $σ(1) = 3$ is optimal. We do not know the optimal value of $σ(β)$ for $1 < β < ω_1$.

2. Extension of continuous functions. In this section, we study the extension of a continuous function on a subspace of a metric space to the whole space. To begin with, we consider the extension of a continuous function from a dense subspace.

Consider a metric space $X$ with a dense subspace $Y$. Suppose that $f : Y → ℝ$ is continuous on $Y$. An obvious way of extending $f$ to $X$ (if $f$ is locally bounded) is to consider the upper limit (or lower limit) of $f$, i.e.,

$$\tilde{f}(x) = \limsup_{y → x, y ∈ Y} f(y) = \inf_{δ > 0} \sup_{d(x, y) < δ, y ∈ Y} f(y).$$

The extended function, which is upper semicontinuous (lower semicontinuous in the case of lim inf), is not optimal as far as the oscillation index is concerned. In fact, the lim sup extension $\tilde{f}$ of the continuous function $f$ in Example 15 below has oscillation index $β_X(\tilde{f}) = ω$. The following is an alternative algorithm that produces an extension with the smallest possible oscillation index. If $A ⊆ \text{dom } f$, then osc($f, A$) is defined to be $\sup\{ |f(x) - f(x')| : x, x' ∈ A \}$. If $x$ belongs to the closure of dom $f$, then define

$$\osc(f, x) = \lim_{δ → 0} \osc(f, B(x, δ) \cap \text{dom } f).$$

We first define layers of approximate extensions inductively. Precisely, for each $k \geq 0$, we will choose open sets $S_k$ and $X_k$ such that $Y ⊆ S_k ⊆ X_k$, nonnegative integers $(n_k(s))_{s ∈ S_k}$ and a function $F_k : X_k → ℝ$. Let $S_0 = X$.
and $n_0(s) = 0$ for all $s \in S_0$. Assume that $S_k$ has been chosen and $n_k(s)$ is defined for all $s \in S_k$. Let $U_k = \{B(s, 2^{-n_k(s)}) : s \in S_k\}$ and $X_k = \bigcup U_k$. Choose a partition of unity $(\varphi^k_s)_{s \in S_k}$ on $X_k$ subordinate to $U_k$. For each $s \in S_k$, choose $y^k_s \in Y \cap B(s, 2^{-n_k(s)})$. Define $F_k : X_k \to \mathbb{R}$ by $F_k(x) = \sum_{s \in S_k} \varphi^k_s(x)f(y^k_s)$. For each $x \in X_k$, let $S_k(x) = \{s \in S_k : x \in \text{supp } \varphi^k_s\}$ and $l_k(x) = \max\{n_k(s) : s \in S_k(x)\} + 1$. Note that $S_k(x)$ is a finite set since $(\text{supp } \varphi^k_s)_{s \in S_k}$ is locally finite. Let $S_{k+1}$ be the set of all $x \in X_k$ such that $\text{osc}(f, x) < 2^{-l_k(x)}$. If $x \in S_{k+1}$, choose $n_{k+1}(x) \geq l_k(x)$ so that

1. $\text{osc}(f, B(x, 2^{1-n_{k+1}(x)}) \cap Y) < 2^{-l_k(x)}$,
2. $B(x, 2^{-n_{k+1}(x)}) \subseteq B(s, 2^{-n_k(s)})$ for all $s \in S_k(x)$,
3. $B(x, 2^{1-n_{k+1}(x)}) \cap \text{supp } \varphi^k_s = \emptyset$ if $s \in S_k \setminus S_k(x)$.

The extension $F$ (defined after Lemma 8) is obtained by pasting the layers $(F_k)$ one after another. Observe that $X_{k+1} \subseteq X_k$ because of condition (2).

**Lemma 7.** Suppose that $s \in S_k$, $t \in S_m$ for some $m > k$, and that $\text{supp } \varphi^k_s \cap \text{supp } \varphi^m_t \neq \emptyset$. Then $B(t, 2^{-m(t)}) \subseteq B(s, 2^{-n_k(s)})$.

**Proof.** Let $x \in \text{supp } \varphi^k_s \cap \text{supp } \varphi^m_t$. Then $x \in X_j$ for all $j \leq m$. In particular, if $m > j > k$, then there exists $s_j \in S_j$ such that $x \in \text{supp } \varphi^j_s$. Thus it suffices to prove the lemma for $m = k+1$. Assume that $x \in \text{supp } \varphi^k_s \cap \text{supp } \varphi^{k+1}_s$. Note that $s \in S_k(t)$. For otherwise, $B(t, 2^{1-n_{k+1}(t)}) \cap \text{supp } \varphi^k_s = \emptyset$ by (3). Since $x$ belongs to this set, we have reached a contradiction. It now follows from (2) that $B(t, 2^{-n_{k+1}(t)}) \subseteq B(s, 2^{-n_k(s)})$.

**Lemma 8.** Suppose that $x \in X_m$ and $m > k \geq 1$. Then there exists $s \in S_k(x)$ such that $|F_k(x) - F_m(x)| < 2^{1-l_{k-1}(s)}$. Moreover, if $x \in Y$, then $|F_k(x) - f(x)| < 2^{-l_{k-1}(s)}$ for some $s \in S_k(x)$.

**Proof.** Denote by $S$ the set of all $t \in S_m$ such that $\varphi^m_t(x) > 0$ and choose a point $y \in \bigcap_{t \in S} B(t, 2^{-m(t)}) \cap Y$. Let $s$ be an element where $l_{k-1}(s)$ attains its minimum over $S_k(x)$. By Lemma 7, $B(t, 2^{-m(t)}) \subseteq B(s, 2^{-n_k(s)})$ for all $t \in S$. Hence $|f(y) - f(y^m_s)| < 2^{-l_{k-1}(s)}$ for any $t \in S$. By Lemma 7 again, $y \in B(t, 2^{-m(t)}) \subseteq B(s', 2^{-n_k(s')})$ for all $t \in S$ and all $s' \in S_k(x)$. Hence $|f(y) - f(y^k_{s'})| < 2^{-l_{k-1}(s')} < 2^{-l_{k-1}(s)}$ for all $s' \in S_k(x)$. Therefore

$$|F_k(x) - F_m(x)| \leq |F_k(x) - f(y)| + |f(y) - F_m(x)| < 2^{-l_{k-1}(s)} + 2^{-l_{k-1}(s)} = 2^{1-l_{k-1}(s)}.$$ 

Moreover, if $x \in Y$, then the above applies for $y = x$. Hence $|F_k(x) - f(x)| < 2^{-l_{k-1}(s)}$.

Observe that $l_k(s) \geq k+1$ for all $s \in S_k$, $k \geq 0$. It follows from Lemma 8 that $(F_k)$ converges pointwise on $\bigcap X_k$ and that the limit is $f$ on $Y$. Define
$F : X \to \mathbb{R}$ by

$$F(x) = \begin{cases} 
\lim_k F_k(x) & \text{if } x \in \bigcap X_k, \\
F_k(x) & \text{if } x \in X_k \setminus X_{k+1}, \ k \geq 0.
\end{cases}$$

Then $F$ is an extension of $f$ to $X$.

**Lemma 9.** Suppose that $x \in X_k$ for some $k \geq 1$. There exists an open neighborhood $U$ of $x$ and $s \in S_k(x)$ such that $|F(z) - F(x)| < 2^{3-l_{k-1}(s)}$ for all $z \in U$.

**Proof.** Let $s$ be an element where $l_{k-1}(s)$ attains its minimum over $S_k(x)$. Note that $F_k$ is continuous on the open set $X_k$. Hence there is an open neighborhood $U$ of $x$ such that

1. $\text{osc}(F_k, U) < 2^{-l_{k-1}(s)}$,
2. $U \subseteq X_k$,
3. $U \cap \text{supp} \varphi^k_s = \emptyset$ if $s \in S_k \setminus S_k(x)$.

We claim that $S_k(z) \subseteq S_k(x)$ for all $z \in U$. Indeed, if $z \in U$ and $s \in S_k(z) \setminus S_k(x)$, then $z \in U \cap \text{supp} \varphi^k_s = \emptyset$, a contradiction. Now if $z \in U$, then either $z \in X_m$ for all $m$ or $z \in X_m \setminus X_{m+1}$ for some $m \geq k$. In either case, $|F_k(z) - F(z)| \leq 2^{1-l_{k-1}(s)}$ by Lemma 8. Therefore,

$$|F(z) - F(x)| \leq |F(z) - F_k(z)| + |F_k(z) - F_k(x)| + |F_k(x) - F(x)|$$

$$< 2^{1-l_{k-1}(s)} + 2^{-l_{k-1}(s)} + 2^{1-l_{k-1}(s)} < 2^{3-l_{k-1}(s)}.$$

The next proposition is an immediate consequence of Lemma 9.

**Proposition 10.** Every $x \in \bigcap X_k$ is a point of continuity of $F$.

**Proposition 11.** If $x \in D^1(F, 2^{-m}, X) \cap X_k$, $k \geq 1$, then there exists $s \in S_k(x)$ such that $l_{k-1}(s) \leq m+3$.

**Proof.** Since $x \in X_k$, by Lemma 9, there exist an open neighborhood $U$ of $x$ and $s \in S_k(x)$ such that $|F(z) - F(x)| < 2^{3-l_{k-1}(s)}$ for all $z \in U$. Hence $|F(z_1) - F(z_2)| < 2^{4-l_{k-1}(s)}$ for all $z_1, z_2 \in U$. As $x \in D^1(F, 2^{-m}, X)$, we see that $-m < 4 - l_{k-1}(s)$. Thus $l_{k-1}(s) \leq m+3$.

**Proposition 12.** Suppose that $x \in X_k \cap D^2(F, 2^{-m}, X)$, $k \geq 0$. Then $n_k(s) \leq m+2$ for all $s \in S_k$ such that $\varphi^k_s(x) > 0$.

**Proof.** Choose an open neighborhood $U_1$ of $x$ such that $U_1 \subseteq \{\varphi^k_s > 0\}$ for all $s \in S_k$ such that $\varphi^k_s(x) > 0$. Note that, in particular, $U_1 \subseteq X_k$. Then choose an open neighborhood $U_2$ of $x$ such that $\text{osc}(F_k, U_2) < 2^{-m}$. Let $U = U_1 \cup U_2$. There exist $z_1, z_2 \in U \cap D^1(F, 2^{-m}, X)$ such that $|F_k(z_1) - F_k(z_2)| \geq 2^{-m}$. If $z_1, z_2 \notin X_{k+1}$, then $F(z_i) = F_k(z_i)$, $i = 1, 2$. This leads to a contradiction with the fact that $\text{osc}(F_k, U_2) < 2^{-m}$. Thus at least one of $z_1, z_2$ belongs to $X_{k+1}$. Denote it by $z$. By the previous proposition, there exists $t \in S_{k+1}(z)$ such that $l_k(t) \leq m+3$. Let $s \in S_k$ be such that $\varphi^k_s(x) > 0$. 
We claim that \( s \in S_k(t) \). For otherwise, \( B(t, 2^{1-n_{k+1}(t)}) \cap \text{supp} \varphi^k_s = \emptyset \). This is absurd since the intersection contains the point \( z \). It follows from the claim that \( l_k(t) \geq n_k(s) + 1 \). Hence \( n_k(s) \leq m + 2 \), as required. 

**Proposition 13.** \( \beta_X(F) \leq 3 \).

**Proof.** Suppose that \( x \in D^3(F, 2^{-m}, X) \) for some \( m \). Then there exists \( k \) such that \( x \in X_k \setminus X_{k+1} \). Choose a neighborhood \( U \) of \( x \) such that \( U \subseteq B(x, 2^{-m-2}) \cap X_k \) and \( \text{osc}(F_k, U) < 2^{-m} \). There exist \( z_1, z_2 \in U \cap D^2(F, 2^{-m}, X) \) such that \( |F(z_1) - F(z_2)| \geq 2^{-m} \). If \( z_1, z_2 \notin X_{k+1} \), then \( F(z_i) = F_k(z_i), i = 1, 2 \). This contradicts the fact that \( \text{osc}(F_k, U) < 2^{-m} \). Hence there exists \( z \in U \cap X_{k+1} \cap D^2(F, 2^{-m}, X) \). By Proposition 12, \( n_{k+1}(t) \leq m + 2 \) for all \( t \in S_{k+1} \) such that \( \varphi^k(z) > 0 \). Fix such a \( t \). Note that

\[
d(x, t) \leq d(x, z) + d(z, t) < 2^{-m-2} + 2^{-n_{k+1}(t)} \leq 2^{-n_{k+1}(t)}.
\]

Thus

\[
\text{osc}(f, x) \leq \text{osc}(f, B(t, 2^{1-n_{k+1}(t)}) \cap Y) < 2^{-l_k(t)}.
\]

We claim that \( S_k(x) \subseteq S_k(t) \). For otherwise, there exists \( s \in S_k(x) \setminus S_k(t) \). Then \( B(t, 2^{1-n_{k+1}(t)}) \cap \text{supp} \varphi^k_s = \emptyset \). This is absurd since the intersection contains the point \( x \). It follows from the claim that \( l_k(t) \geq l_k(x) \). Hence \( \text{osc}(f, x) < 2^{-l_k(x)} \). Then \( x \in S_{k+1} \subseteq X_{k+1} \), a contradiction.

**Theorem 14.** Let \( X \) be a metric space and \( Y \) be a subspace of \( X \). Every continuous function \( f \) on \( Y \) can be extended to a function \( F \) on \( X \) with \( \beta_X(F) \leq 3 \).

**Proof.** Applying the preceding lemmas and propositions, we obtain an extension \( \tilde{f} \) of \( f \) to \( \tilde{Y} \) such that \( \beta_{\tilde{Y}}(\tilde{f}) \leq 3 \). By [8, Theorem 3.6], there is a further extension \( F \) of \( \tilde{f} \) to \( X \) such that \( \beta_X(F) = \beta_{\tilde{Y}}(\tilde{f}) \leq 3 \). (Note that the proof of [8, Theorem 3.6] does not require the compactness of \( X \).)

The following example shows that Theorem 14 is optimal.

**Example 15.** There is a subspace \( Y \subseteq \{0, 1\}^\omega = X \) and a continuous real-valued function \( f \) on \( Y \) such that \( \beta_X(F) \geq 3 \) for any extension \( F \) of \( f \) to \( X \).

**Proof.** For any integer \( n \), denote \( n \) (mod 2) by \( \hat{n} \). Let

\[
Y = \{(\varepsilon_i) \in X : \varepsilon_i = 0 \text{ for infinitely many } i \text{'s}\}.
\]

We denote elements in \( X \) of the form

\[
(1, \ldots, 1, 0, 1, \ldots, 1, 0, \ldots, 1, \ldots, 1, 0, \ldots)
\]
by \((1^{n_1}, 0, 1^{n_2}, 0, \ldots, 1^{n_k}, 0, \ldots)\). Also write \((\varepsilon_1, \ldots, \varepsilon_k, \varepsilon, \varepsilon, \ldots)\) as \((\varepsilon_1, \ldots, \\
\varepsilon_k, \varepsilon^\omega)\), \(\varepsilon, \varepsilon \in \{0, 1\}\). Define \(g : Y \to X\) by

\[
g(1^{n_1}, 0, 1^{n_2}, 0, \ldots, 1^{n_k}, 0, \ldots) = (\hat{n}_1, \hat{n}_2, \ldots), \quad n_1, n_2, \ldots \in \mathbb{N} \cup \{0\},
\]

and let \(h : X \to \mathbb{R}\) be the canonical embedding of \(X\) into \(\mathbb{R}\), \(h(\varepsilon_1, \varepsilon_2, \ldots) = \sum_{k=1}^\infty 2\varepsilon_k/3^k\). Then the function \(f = h \circ g : Y \to \mathbb{R}\) is continuous. Suppose that \(F\) is an extension of \(f\) to \(X\) such that \(\beta_X(F) \leq 2\). First observe that for any \(n_1, \ldots, n_k \in \mathbb{N} \cup \{0\}\) and all \(n \in \mathbb{N}\),

\[
|F(1^{n_1}, 0, \ldots, 1^{n_k}, 0, 1^{2n}, \omega) - F(1^{n_1}, 0, \ldots, 1^{n_k}, 0, 1^{2n-1}, 0, 1, 0, 1, \ldots)| = \frac{1}{3^n}.
\]

Hence \((1^{n_1}, 0, \ldots, 1^{n_k}, 0, 1^\omega) \in D^1(F, 1/3^k, X)\). Let \(F(1^\omega) = a\). Either \(|a| \geq 1/2\) or \(|1 - a| \geq 1/2\). We assume the former; the proof for the latter case is similar. Since \((1^\omega) \notin D^2(F, 1/3, X)\), there exists a neighborhood \(U\) of \((1^\omega)\) such that \(|F(x) - a| < 1/3\) if \(x \in U \cap D^1(F, 1/3, X)\). In particular, there exists \(n_1 \in \mathbb{N}\) such that

\[
|F(1^{2n_1}, 0, 1^\omega) - a| = \frac{1}{3} - \delta \quad \text{for some } \delta > 0.
\]

Similarly, using the fact that \((1^{2n_1}, 0, 1^\omega) \notin D^2(F, 1/3^2, X)\), we obtain some \(n_2 \in \mathbb{N}\) such that

\[
|F(1^{2n_1}, 0, 1^{2n_2}, 0, 1^\omega) - F(1^{2n_1}, 0, 1^\omega)| < \frac{1}{3^2}.
\]

Continuing, we choose \(n_1, n_2, \ldots \in \mathbb{N}\) such that

\[
|F(1^{2n_1}, 0, \ldots, 1^{2n_{k+1}}, 0, 1^\omega) - F(1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega)| < \frac{1}{3^{k+1}}, \quad k \in \mathbb{N}.
\]

In particular,

\[
|F(1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega) - a| \leq \frac{1}{3} + \frac{1}{3^2} + \ldots + \delta = \frac{1}{2} - \delta, \quad k \in \mathbb{N}.
\]

Since \(|a| \geq 1/2\), we have \(|F(1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega)| \geq \delta\) for all \(k \in \mathbb{N}\). But

\[
F(1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega) = f(1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^{2n}, 0^\omega) = 0
\]

for all \(n \in \mathbb{N}\). Hence \((1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega) \in D^1(F, \delta, X)\) for all \(k \in \mathbb{N}\). However, note that the sequence \(((1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega))_{k \in \mathbb{N}}\) converges to the point \((1^{2n_1}, 0, \ldots, 1^{2n_j}, 0, 1^{2n_{j+1}}, 0, \ldots)\) and

\[
|F(1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega) - F(1^{2n_1}, 0, \ldots, 1^{2n_j}, 0, 1^{2n_{j+1}}, 0, \ldots)|
\]

\[
= |F(1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega) - f(1^{2n_1}, 0, \ldots, 1^{2n_j}, 0, 1^{2n_{j+1}}, 0, \ldots)|
\]

\[
= |F(1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega)| \geq \delta
\]

for all \(n \in \mathbb{N}\). Therefore, \((1^{2n_1}, 0, \ldots, 1^{2n_j}, 0, 1^{2n_{j+1}}, 0, \ldots) \in D^2(F, \delta, X)\), contrary to the assumption that \(\beta_X(F) \leq 2\). ■
Remark. With regard to the question raised in Remark 3 of §1, we have been able to show that if $Y$ is a subspace of a countable ordinal $X$ (not necessarily compact), and $f : Y \to \mathbb{R}$ satisfies $\beta_Y(f) \leq 3$, then there is an extension $F : X \to \mathbb{R}$ of $f$ such that $\beta_X(F) \leq \beta_Y(f) + 1$.

References


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