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Kato decomposition of linear pencils

by

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Abstract. T. Kato [5] found an important property of semi-Fredholm pencils, now called the Kato decomposition. M. A. Kaashoek [3] introduced operators having the property $P(S:k)$ as a generalization of semi-Fredholm operators. In this work, we study this class of operators. We show that it is characterized by a Kato-type decomposition. Other properties are also proved.

1. Introduction. Throughout this paper, we shall denote by X, Y two Banach spaces. Let $\mathcal{B}(X,Y)$ be the set of all bounded linear operators from X to Y. For an operator A in $\mathcal{B}(X,Y)$, we denote by $N(A)$ and $R(A)$ its kernel and range, respectively.

We will write $\mathbb{N}=\{0,1,2,\ldots\}$.

Let $T,S\in\mathcal{B}(X,Y)$. The operator T is said to be *semi-Fredholm* if $R(T)$ is closed and $\min\{\dim N(T),\operatorname{codim} R(T)\}$ is finite. M. A. Kaashoek introduced the $P(S:k)$ property (see [3] and Section 2 below) as a generalization of semi-Fredholm operators.

1.1. DEFINITION. The couple (T,S) is said to have a *Kato decomposition of finite type* (KDF) if there exist closed subspaces X_1,X_2 of X and Y_1,Y_2 of Y such that:

(1) $X=X_1\oplus X_2$ and $Y=Y_1\oplus Y_2$,
(2) $\dim X_1<\infty$,
(3) $SX_i\subset Y_i$ and $TX_i\subset Y_i$, for $i=1,2$,
(4) $T|_{X_2}$ has the property $P(S|_{X_2}:0)$, whose definition is recalled in Section 2,
(5) $S:X_1\to Y_1$ is bijective,
(6) $S^{-1}T|_{X_1}$ is nilpotent.

In that case, we will say that (X_2,X_1,Y_2,Y_1) is a KDF associated to the couple (T,S).

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In [5, Theorem 4], T. Kato proved that if T is semi-Fredholm and S arbitrary, then (T, S) has a Kato decomposition of finite type.

The aim of this paper is to characterize couples of operators having a KDF. In particular, we show that (T, S) has a Kato decomposition of finite type if and only if T has the property $P(S : k)$ for some integer k. In the particular case of $X = Y$ and $S = I$, we recover known results (see [7], [8]) about s-regular and essentially s-regular operators (also [1], [3]–[5], [6], [9]).

2. Characterization of the class $\mathcal{P}(S)$. Throughout this paper, we will consider T, S in $\mathcal{B}(X, Y)$ such that $R(T)$ is closed.

T. Kato introduced sequences of subspaces of X, $(D_n(T : S))_{n\geq 0}$ and $(N_n(T : S))_{n\geq 0}$. They play an important role in perturbation theory (see [5]). A few years later, M. A. Kaashoek added other sequences of subspaces of Y, $(R_n(T : S))_{n\geq 0}$ and $(M_n(T : S))_{n\geq 0}$ (see [3]). Let us recall their definition:

$$\begin{cases} D_0(T : S) = X, \;\; R_0(T : S) = Y, \\ R_{n+1}(T : S) = TD_n(T : S), \;\; D_{n+1}(T : S) = S^{-1}R_{n+1}(T : S) \quad \text{for } n \geq 0, \\ N_0(T : S) = \{0\}, \;\; M_0(T : S) = \{0\}, \\ N_{n+1}(T : S) = T^{-1}M_n(T : S), \;\; M_{n+1}(T : S) = SN_{n+1}(T : S) \text{ for } n \geq 0. \end{cases}$$

If it is not ambiguous, we will write D_n, N_n, R_n and M_n for the corresponding subspaces. Clearly, we have $D_{n+1} = S^{-1}TD_n, N_{n+1} = T^{-1}SN_n, R_{n+1} = TS^{-1}R_n, M_{n+1} = ST^{-1}M_n$. Moreover, the sequences $(D_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ are decreasing, and the sequences $(N_n)_{n\geq 0}$ and $(M_n)_{n\geq 0}$ are increasing. Let

$$D(T : S) = \bigcap_{n=0}^{\infty} D_n(T : S), \qquad N(T : S) = \bigcup_{n=0}^{\infty} N_n(T : S),$$
$$R(T : S) = \bigcap_{n=0}^{\infty} R_n(T : S), \qquad M(T : S) = \bigcup_{n=0}^{\infty} M_n(T : S).$$

We can easily see that $TD(T : S) \subset R(T : S), S^{-1}R(T : S) = D(T : S)$ and for every complex number $\lambda \neq 0, N(T + \lambda S) \subset D(T : S)$.

We notice that in the particular case when $X = Y$ and $S = I$, we have $D_n = R_n = R(T^n)$ and $N_n = M_n = N(T^n)$.

Before giving the definition of the operators we will study in this paper, let us recall a notation. For two subspaces M and N of X, we write $M \subset_{\mathrm{e}} N$ if there exists a finite-dimensional subspace F of X such that $M \subset N + F$, i.e. $\dim[M/(M \cap N)] < \infty$. Notice that we can assume that F is a subset of M. Now, we can introduce the notion we are interested in. Let k be a positive integer. The operator T is said to *have the property* $P(S : k)$ if

$$\dim[N(T)/(D(T : S) \cap N(T))] = k$$

and $R(T)$ is closed. We will write $T \in \mathcal{P}(S)$ if $R(T)$ is closed and $N(T) \subset_{\mathrm{e}} D(T:S)$, i.e. if there exists k such that T has the property $P(S:k)$.

In this section, we will find other ways to characterize the property $P(S:k)$. First, let us give a condition (necessary and sufficient) for the operator T to belong to $\mathcal{P}(S)$.

2.1. PROPOSITION. (1) *If T has the property $P(S:k)$, then there exists a subspace M of X such that $S^{-1}TM = M$ and the map $\widehat{T} : X/M \to Y/TM$ defined by $\widehat{T}(x+M) = Tx + TM$ is such that $n(\widehat{T}) := \dim N(\widehat{T}) = k$.*

(2) *If there exists a subspace M of X such that $S^{-1}TM = M$ and $n(\widehat{T}) = k$, with $\widehat{T}$ the map defined in* (1), *then T has the property $P(S:k')$ for some $k' \le k$.*

Proof. (1) Let $M = D(T:S)$. Then, by [3, Theorem 3.1], $M = S^{-1}TM$ and M, TM are closed. Let $\widehat{T} : X/M \to Y/TM$ be as in the statement. Then $N(\widehat{T}) = \{x + M; Tx \in TM\} = N(T) + M$. Define $\varphi : N(T) \to N(\widehat{T})$ by $\varphi(x) = x + M$. Clearly, φ is surjective. Further, $N(\varphi) = N(T) \cap M$, so $\varphi : N(T)/[N(T) \cap M] \to N(\widehat{T})$ is bijective. Hence $n(\widehat{T}) = k$.

(2) As $S^{-1}TM = M$, we have $M \subset D(T:S)$. Let $\varphi : N(T) \to N(\widehat{T})$ be as above. Again, $\varphi : N(T)/[N(T) \cap M] \to N(\widehat{T})$ is bijective. Therefore $\dim[N(T)/(N(T) \cap M)] = \dim N(\widehat{T}) = k$. Since $M \subset D(T:S)$, it follows that $\dim[N(T)/(N(T) \cap D(T:S))] \le k$. Thus T has the property $P(S:k')$ for some $k' \le k$, as $R(T)$ is supposed to be closed.

REMARK. In both parts of Proposition 2.1, M and TM are closed and $\widehat{T}$ has closed range.

Proof. As $N(\widehat{T}) = N(T) + M$ is finite-dimensional, it is closed. Moreover, T has closed range. So, by [5, Lemma 331], $TN(\widehat{T}) = TM$ is closed. As $M = S^{-1}TM$, the subspace M is also closed. Let $\Pi : Y \to Y/TM$ be the canonical projection. Since $N(\Pi) = TM \subset R(T)$ and $R(T)$ is closed, $R(\widehat{T}) = \Pi R(T)$ is closed, by [5, Lemma 331].

2.2. COROLLARY. *$T \in \mathcal{P}(S)$ if and only if there exists a closed subspace M of X such that $S^{-1}TM = M$ and the map $\widehat{T} : X/M \to Y/TM$ defined by $\widehat{T}(x+M) = Tx + TM$ is upper semi-Fredholm.*

The main result of this work is the following theorem which allows us to characterize operators having the property $\mathcal{P}(S)$ in terms of a Kato decomposition of finite type.

2.3. THEOREM. *Let $T, S \in \mathcal{B}(X,Y)$ be such that $R(T)$ is closed. The following are equivalent*:

(1) $T \in \mathcal{P}(S)$.
(2) $N(T:S) \subset_{\mathrm{e}} S^{-1}R(T)$.

(3) $N(T:S) \subset_{\mathrm{e}} D(T:S)$.

(4) *The couple* (T,S) *has a Kato decomposition of finite type.*

The following lemmas will allow us to prove Theorem 2.3.

2.4. Lemma. *Let* U,V,W *be subspaces of* X *and* M,N *be subspaces of* Y*.*

(1) $T[U\cap V] \subset TU \cap TV$, *with equality if* $N(T) \subset U$.

(2) $S^{-1}M + S^{-1}N \subset S^{-1}[M+N]$, *with equality if* $N \subset R(S)$.

(3) *Let* $A \in \mathcal{B}(X,Y)$ *be a closed range operator. If* $U + N(A)$ *is closed, then* AU *is closed.*

(4) *If* $U \subset W$, *then* $[U+V]\cap W = U + [V\cap W]$.

Proof. (1) The inclusion is clear. Assume that $N(T) \subset U$ and let $y \in TU \cap TV$. There exist $u \in U$ and $v \in V$ such that $y = Tu = Tv$. Then $v - u \in N(T) \subset U$, and so $v \in U \cap V$. Thus $y \in T[U\cap V]$.

(2) Let $x \in S^{-1}M + S^{-1}N$. Then there exist $m \in S^{-1}M$ and $n \in S^{-1}N$ such that $x = m + n$. Thus $Sx \in M + N$, and so $x \in S^{-1}[M+N]$.

Now, assume that $N \subset R(S)$ and let $x \in S^{-1}[M+N]$. Then there exist $a \in M$ and $b \in N \subset R(S)$ such that $Sx = a + b$. Moreover, there exists $c \in X$ such that $b = Sc$, and so $c \in S^{-1}N$. Thus $S(x-c) = a \in M$, and $x - c \in S^{-1}M$. Hence $x \in S^{-1}N + S^{-1}M$.

(3) [5, Lemma 331].

(4) [2, Lemma 2.1].

2.5. Lemma. *The following are equivalent*:

(1) *For all* $n \in \mathbb{N}$, $N_n(T:S) \subset S^{-1}R(T)$, *i.e.* $N(T:S) \subset S^{-1}R(T)$.

(2) $N(T) \subset D(T:S)$, *i.e.* T *has the property* $P(S:0)$.

(3) *For all* $(n,k) \in \mathbb{N}^2$, $N_n(T:S) \subset D_k(T:S)$, *i.e.* $N(T:S) \subset D(T:S)$.

Proof. (3)⇒(1): As $D(T:S) \subset D_1(T:S) = S^{-1}R(T)$, we obtain $N(T:S) \subset S^{-1}R(T)$.

(1)⇒(3): We argue by induction on $k \in \mathbb{N}$. As the cases $k = 0,1$ are true, assume that the property holds for some $k \geq 1$. Let $n \in \mathbb{N}$. We have $N_{n+1}(T:S) \subset D_k(T:S)$. As $N_{n+1}(T:S) = T^{-1}SN_n(T:S)$, it follows that

$$SN_n(T:S) \cap R(T) \subset TD_k(T:S).$$

Then $[N_n(T:S) + N(S)] \cap S^{-1}R(T) \subset S^{-1}TD_k(T:S) = D_{k+1}(T:S)$. By Lemma 2.4, as $N(S) \subset S^{-1}R(T)$, we have

$$N_n(T:S) \cap S^{-1}R(T) + N(S) \subset D_{k+1}(T:S).$$

Since $N_n(T:S) \subset S^{-1}R(T)$, we obtain the result.

(3)⇒(2): As $N(T) = N_1(T:S) \subset N(T:S)$, we obtain $N(T) \subset D(T:S)$.

(2)⇒(3): We reason by induction on $n \in \mathbb{N}$. As the cases $n = 0,1$ are true, assume that the property holds for some $n \geq 1$. Let $k \in \mathbb{N}$. We have

$N_n(T:S) \subset D_{k+1}(T:S)$. As $D_{k+1}(T:S) = S^{-1}TD_k(T:S)$, it follows that

$$SN_n(T:S) \subset TD_k(T:S) \cap R(S) \subset TD_k(T:S).$$

Then $N_{n+1}(T:S) = T^{-1}SN_n(T:S) \subset D_k(T:S) + N(T) = D_k(T:S)$, as $N(T) \subset D_k(T:S)$.

2.6. LEMMA. *Consider the following properties*:

(a) $N(T) \subset_{\mathrm{e}} D(T:S)$, *i.e.* $T \in \mathcal{P}(S)$.
(b) $N(T:S) \subset_{\mathrm{e}} S^{-1}R(T)$.
(c) *For all* $(n,p) \in \mathbb{N}^2, N_n(T:S) \subset_{\mathrm{e}} D_p(T:S)$.

We have the following implications: (a)⇒(c) *and* (b)⇒(c).

Proof. (a)⇒(c): Assuming (a), we show (c) by induction on n. Since $D(T:S) \subset D_p(T:S)$ for every $p \in \mathbb{N}$, the cases $n = 0,1$ are clear. Let $n \geq 1$ and assume that for all $m \leq n$ and all $p \in \mathbb{N}$, there exists a finite-dimensional subspace $F_{m,p} \subset N_m(T:S)$ such that $N_m(T:S) \subset D_p(T:S) + F_{m,p}$.

Let $p \in \mathbb{N}$. Then

$$\begin{aligned} N_{n+1}(T:S) &= T^{-1}SN_n(T:S) \subset T^{-1}S[D_{p+1}(T:S) + F_{n,p+1}] \\ &= T^{-1}[SD_{p+1}(T:S) + SF_{n,p+1}] \\ &= T^{-1}[R_{p+1}(T:S) \cap R(S) + SF_{n,p+1}] \\ &\subset T^{-1}[R_{p+1}(T:S) + SF_{n,p+1}] \\ &= T^{-1}[TD_p(T:S) + SF_{n,p+1}] \\ &= D_p(T:S) + N(T) + T^{-1}SF_{n,p+1} \quad \text{by Lemma 2.4.} \end{aligned}$$

As $SF_{n,p+1} \cap R(T)$ is finite-dimensional, we can find $F'_{n+1,p} \subset X$ such that

$$\begin{cases} TF'_{n+1,p} = SF_{n,p+1} \cap R(T), \\ \dim F'_{n+1,p} = \dim[SF_{n,p+1} \cap R(T)]. \end{cases}$$

Thus $N_{n+1}(T:S) \subset D_p(T:S) + F'_{n+1,p} + N(T)$. Further $N(T) \subset F_{1,p} + D_p(T:S)$. Defining $F_{n+1,p} = F_{1,p} + F'_{n+1,p}$, a finite-dimensional subspace, we have $N_{n+1}(T:S) \subset D_p(T:S) + F_{n+1,p}$.

(b)⇒(c): Assuming (b), we show (c) by induction on p. As $N_n(T:S) \subset N(T:S)$ for all $n \in \mathbb{N}$, the cases $p = 0,1$ are clear. Let $p \geq 1$ and assume that for all $q \leq p$ and all $n \in \mathbb{N}$, there exists a finite-dimensional subspace $F_{n,q} \subset N_n(T:S)$ such that $N_n(T:S) \subset D_q(T:S) + F_{n,q}$.

Let $n \in \mathbb{N}$. We have

$$\begin{aligned} TN_{n+1}(T:S) &\subset T[D_p(T:S) + F_{n+1,p}] = TD_p(T:S) + TF_{n+1,p} \\ &= R_{p+1}(T:S) + TF_{n+1,p}. \end{aligned}$$

As $F_{n+1,p} \subset N_{n+1}(T:S)$, we obtain

$$TF_{n+1,p} \subset TN_{n+1}(T:S) \subset M_n(T:S) = SN_n(T:S) \subset R(S).$$

Thus

$$\begin{aligned} S^{-1}TN_{n+1}(T:S) &\subset S^{-1}R_{p+1}(T:S) + S^{-1}TF_{n+1,p} \quad \text{by Lemma 2.4} \\ &= D_{p+1}(T:S) + S^{-1}TF_{n+1,p}. \end{aligned}$$

Since $TN_{n+1}(T:S) = SN_n(T:S) \cap R(T)$, we have

$$\begin{aligned} S^{-1}TN_{n+1}(T:S) &= [N_n(T:S) + N(S)] \cap S^{-1}R(T) \\ &= [N_n(T:S) \cap S^{-1}R(T)] + N(S) \quad \text{by Lemma 2.4} \\ &\supset N_n(T:S) \cap D_1(T:S). \end{aligned}$$

Moreover, $N_n(T:S) \subset D_1(T:S) + F_{n,1}$ with $F_{n,1} \subset N_n(T:S)$. So

$$\begin{aligned} N_n(T:S) &\subset D_1(T:S) \cap N_n(T:S) + F_{n,1} \subset S^{-1}TN_{n+1}(T:S) + F_{n,1} \\ &\subset D_{p+1}(T:S) + S^{-1}TF_{n+1,p} + F_{n,1}. \end{aligned}$$

Notice that $TF_{n+1,p} \cap R(S)$ is finite-dimensional. Let $F'_{n+1,p}$ be such that

$$\begin{cases} SF'_{n+1,p} = TF_{n+1,p} \cap R(S), \\ \dim F'_{n+1,p} = \dim(TF_{n+1,p} \cap R(S)). \end{cases}$$

Thus $N_n(T:S) \subset D_{p+1}(T:S) + F_{n,1} + F'_{n+1,p}$, as $N(S) \subset D_{p+1}(T:S)$. Setting $F_{n,p+1} = F_{n,1} + F'_{n+1,p}$, a finite-dimensional subspace, we obtain

$$N_n(T:S) \subset D_{p+1}(T:S) + F_{n,p+1}.$$

2.7. Lemma. *For all* $(m,d) \in \mathbb{N}^2$,

$$(S^{-1}T)^d N_{m+d}(T:S) = N_m(T:S) \cap D_d(T:S) + N_d(S:T).$$

Notice that in the last term, the roles of T and S are reversed. This will happen at some other places as well.

Proof. We prove by induction on $d \in \mathbb{N}$ that the stated equality holds for all integers m. As the case $d = 0$ is clear, assume that the equality holds for some $d \geq 0$ and let $m \in \mathbb{N}$. Thanks to Lemma 2.4, we have

$$\begin{aligned} (S^{-1}T)^{d+1}N_{m+d+1}(T:S) &= (S^{-1}T)[(S^{-1}T)^d N_{(m+1)+d}(T:S)] \\ &= (S^{-1}T)[N_{m+1}(T:S) \cap D_d(T:S) + N_d(S:T)] \\ &= S^{-1}[M_m(T:S) \cap R_{d+1}(T:S) + M_d(S:T)] \\ &= S^{-1}[SN_m(T:S) \cap R_{d+1}(T:S) + M_d(S:T)] \\ &= [N_m(T:S) + N(S)] \cap S^{-1}R_{d+1}(T:S) + S^{-1}M_d(S:T) \\ &= N_m(T:S) \cap D_{d+1}(T:S) + N(S) + N_{d+1}(S:T) \\ &= N_m(T:S) \cap D_{d+1}(T:S) + N_{d+1}(S:T). \end{aligned}$$

2.8. Lemma. *Assume that* $N_k(T:S) \subset D_1(T:S)$ *for all* $k \in \{0,\dots,p\}$. *Then* $N_{p-j}(T:S) \subset D_{j+1}(T:S)$ *for each* $j \in \{0,\dots,p\}$.

Proof. We prove this by induction on j. The result is clear for $j = 0$. Assume that the property is true for some $j \in \{0, \ldots, p-1\}$. As

$$T^{-1}SN_{p-(j+1)}(T:S) = N_{p-j}(T:S) \subset D_{j+1}(T:S),$$

we have $SN_{p-j-1}(T:S) \cap R(T) \subset TD_{j+1}(T:S)$. So

$$[N_{p-j-1}(T:S) + N(S)] \cap S^{-1}R(T) \subset D_{j+2}(T:S).$$

Since $N_{p-j-1}(T:S)$ and $N(S)$ are included in $D_1(T:S) = S^{-1}R(T)$, we obtain

$$N_{p-(j+1)}(T:S) \subset N_{p-j-1}(T:S) + N(S) \subset D_{j+2}(T:S).$$

2.9. Lemma. *Assume that $N(T) \subset_{\mathrm{e}} D_k(T:S)$ for all $k \in \mathbb{N}$. Then, for each $j \in \mathbb{N}, D_j(T:S)$ and $R_j(T:S)$ are closed.*

Proof. As $D_0(T:S)$ and $R_0(T:S)$ are closed, assume that the property is true for some $j \in \mathbb{N}$. As $\dim[N(T)/(N(T) \cap D_j(T:S))] < \infty$, there exists a finite-dimensional subspace N of $N(T)$ such that $N \oplus D_j(T:S) = N(T) + D_j(T:S)$ and $N \cap D_j(T:S) = \{0\}$.

Let $M = D_j(T:S) \oplus N = D_j(T:S) + N(T)$. The subspace M is closed and $TM = R_{j+1}(T:S)$. Since $N(T) \subset M$ and $R(T)$ and M are closed, $R_{j+1}(T:S)$ is closed, by Lemma 2.4. As S is continuous, $S^{-1}R_{j+1}(T:S) = D_{j+1}(T:S)$ is also closed.

2.10. Lemma. *Assume that there exists $n \in \mathbb{N} \setminus \{0\}$ such that $N_n(T:S) \not\subset S^{-1}R(T)$ and $N_{n-1}(T:S) \subset S^{-1}R(T)$. Let $y \in N_n(T:S) \setminus S^{-1}R(T)$. Then $y, (S^{-1}T)y, \ldots, (S^{-1}T)^{n-1}y$ are independent modulo $D_n(T:S)$.*

Proof. First, we prove that

$$(1) \qquad (S^{-1}T)^j y \subset N_{n-j}(T:S) + N_j(S:T) \quad \text{for } j = 0, \ldots, n.$$

This is clear for $j = 0$. Assume that it is true for some $j \in \{0, \ldots, n-1\}$. Then $(S^{-1}T)^j y \subset N_{n-j}(T:S) + N_j(S:T) = T^{-1}SN_{n-j-1}(T:S) + N_j(S:T)$. So

$$\begin{aligned} T(S^{-1}T)^j y &\subset SN_{n-j-1}(T:S) \cap R(T) + TN_j(S:T) \\ &\subset SN_{n-j-1}(T:S) + TN_j(S:T). \end{aligned}$$

Thus, by Lemma 2.4, we have

$$\begin{aligned} (S^{-1}T)^{j+1} y &\subset N_{n-j-1}(T:S) + N(S) + N_{j+1}(S:T) \\ &= N_{n-j-1}(T:S) + N_{j+1}(S:T). \end{aligned}$$

We now prove that

$$(2) \qquad (S^{-1}T)^j y \cap D_{j+1}(T:S) = \emptyset \quad \text{for } j = 0, \ldots, n-1.$$

The case $j = 0$ is clear. Assume that the property is true for some $j \in \{0, \ldots, n-2\}$ and that there exists x in $(S^{-1}T)^{j+1}y \cap D_{j+2}(T:S)$. Then

$Sx = Tz$ for some z in $(S^{-1}T)^j y$. We have $z \in T^{-1}Sx \subset T^{-1}SD_{j+2}(T:S)$. As $D_{j+2}(T:S) = S^{-1}TD_{j+1}(T:S)$, it follows that

$$SD_{j+2}(T:S) = TD_{j+1}(T:S) \cap R(S) \subset TD_{j+1}(T:S),$$

and $T^{-1}SD_{j+2}(T:S) \subset D_{j+1}(T:S) + N(T)$. By Lemma 2.8, as $N_k(T:S) \subset D_1(T:S)$ for all $k \in \{0,\dots,j+1\}$ $(j+1 \leq n-1)$, we have $N(T) \subset D_{j+1}(T:S)$. So $T^{-1}SD_{j+2}(T:S) \subset D_{j+1}(T:S)$. Thus $z \in D_{j+1}(T:S) \cap (S^{-1}T)^j y$, a contradiction.

Now, we can prove that $y, (S^{-1}T)y, \dots, (S^{-1}T)^{n-1}y$ are independent modulo $D_n(T:S)$. Let $z_j \in (S^{-1}T)^j y$ and $\alpha_j \in \mathbb{C}$ for $j = 0,\dots,n-1$. Assume that $\alpha_0 z_0 + \dots + \alpha_{n-1} z_{n-1}$ belongs to $D_n(T:S)$. Then, by Lemma 2.4, applying $(S^{-1}T)^{n-1}$, we obtain

$$\begin{aligned}\alpha_0(S^{-1}T)^{n-1}z_0 + \dots + \alpha_{n-1}(S^{-1}T)^{n-1}z_{n-1} &\subset (S^{-1}T)^{n-1}D_n(T:S)\\ &= D_{2n-1}(T:S).\end{aligned}$$

For every $k \in \mathbb{N}$, $N(S) \subset D_k(T:S)$, so $N_{j+1}(S:T) = (S^{-1}T)^j N(S) \subset D_{k+j}(T:S)$ for all $(k,j) \in \mathbb{N}^2$. Since the sequence $(D_k(T:S))_{k\geq 0}$ is decreasing, we have

$$N_j(S:T) \subset D_k(T:S) \quad \text{for each couple } (k,j) \in \mathbb{N}^2. \tag{3}$$

For $j \in \{0,\dots,n-1\}$, we have

$$\begin{aligned} z_j \in (S^{-1}T)^j y &\subset N_{n-j}(T:S) + N_j(S:T) \quad \text{by (1)}\\ &\subset D_j(T:S) + N_j(S:T) \quad \text{by Lemma 2.8}\\ &\subset D_j(T:S) \quad \text{by (3).}\end{aligned}$$

So $(S^{-1}T)^{n-1}z_j \subset (S^{-1}T)^{n-1}D_j(T:S) = D_{n+j-1}(T:S)$ for $j=1,\dots,n-1$. As the sequence $(D_k(T:S))_{k\geq 0}$ is decreasing, we obtain

$$\alpha_1(S^{-1}T)^{n-1}z_1 + \dots + \alpha_{n-1}(S^{-1}T)^{n-1}z_{n-1} \subset D_n(T:S).$$

Hence $\alpha_0(S^{-1}T)^{n-1}z_0 = \alpha_0(S^{-1}T)^{n-1}y \subset D_n(T:S)$.

If $\alpha_0 \neq 0$, we have found an element belonging to $D_n(T:S)$ and to $(S^{-1}T)^{n-1}y$, which contradicts (2).

So $\alpha_0 = 0$. Thus $\alpha_1 z_1 + \dots + \alpha_{n-1}z_{n-1} \in D_n(T:S)$. Applying $(S^{-1}T)^{n-2}$, we show that $\alpha_1 = 0$. Step by step, we conclude that $\alpha_i = 0$ for $i = 0,\dots,n-1$.

Proof of Theorem 2.3. The implications (3)⇒(1) and (3)⇒(2) are clear since $N(T) = N_1(T:S) \subset N(T:S)$ and $D(T:S) \subset D_1(T:S) = S^{-1}R(T)$.

(4)⇒(3): Let (X_2, X_1, Y_2, Y_1) be a KDF associated to the couple (T,S). We have

$$N(T:S) = N(T_2:S_2) \oplus N(T_1:S_1) \subset N(T_2:S_2) + X_1 \subset D(T_2:S_2) + X_1,$$

by Lemma 2.5, since $T|_{X_2}$ has the property $P(S|_{X_2}:0)$. As $S^{-1}T|_{X_1}$ is nilpotent of index d, $D_n(T:S) = D_n(T_1:S_1) \oplus D_n(T_2:S_2) = D_n(T_2:S_2)$

for all $n \geq d$. Therefore $D(T : S) = D(T_2 : S_2)$. Moreover X_1 is finite-dimensional, so we have $N(T : S) \subset_{\rm e} D(T : S)$.

We now prove that (1) or (2) implies (4). In these two cases, (c) of Lemma 2.6 is satisfied. Thus, for all positive integers n, we have $N_n(T : S) \subset_{\rm e} D_1(T : S) = S^{-1}R(T)$.

If $N_n(T : S) \subset S^{-1}R(T)$ for all $n \geq 0$, then by Lemma 2.5, T has the property $P(S : 0)$, and $(X, \{0\}, Y, \{0\})$ is a Kato decomposition of finite type associated to (T, S).

Assume that there exists $n \geq 1$ such that $N_n(T : S) \not\subseteq S^{-1}R(T)$ and $N_{n-1}(T : S) \subset S^{-1}R(T)$. Let $y \in N_n(T : S) \setminus S^{-1}R(T)$. By Lemma 2.10, $y, (S^{-1}T)y, \ldots, (S^{-1}T)^{n-1}y$ are independent modulo $D_n(T : S)$. As $y \in N_n(T : S)$, there exist $x_0, \ldots, x_n$ such that $Sx_{i+1} = Tx_i$ for $i = 0, \ldots, n-1$, $x_0 = y$ and $x_n = 0$. Let $z_i = Sx_i$ for $i = 0, \ldots, n-1$. We prove that $z_0, \ldots, z_{n-1}$ are independent. If $\alpha_0 z_0 + \ldots + \alpha_{n-1}z_{n-1} = 0$, then $\alpha_0 x_0 + \ldots + \alpha_{n-1}x_{n-1} \in N(S) \subset D_n(T : S)$. As $y, (S^{-1}T)y, \ldots, (S^{-1}T)^{n-1}y$ are independent modulo $D_n(T : S), \alpha_i = 0$ for $i = 0, \ldots, n-1$. Notice that $x_0, \ldots, x_{n-1}$ are also independent.

Step 1. We will define two projections P and Q which will allow us to find a Kato decomposition. To do so, we first define a functional $f \in X^*$ which will be useful in the construction of P and Q.

As (c) of Lemma 2.6 holds, by Lemma 2.9, $D_j(T : S)$ is closed for all positive integers j. In particular, $D_n(T : S)$ is closed. So, by the Hahn–Banach Theorem, there exists $f \in D_n(T : S)^{\perp}$ such that $f(x_{n-1}) = 1$ and $f(x_j) = 0$ for $j = 0, \ldots, n-2$.

Let us prove that

$$f|_{(S^{-1}T)^j y} = \delta_{j,n-1} \quad \text{for } j = 0, \ldots, n-1.$$

This is clear for $j = 0$. Let $j \in \{1, \ldots, n-1\}$ and $a_j \in (S^{-1}T)^j y$. There exist $u_0, \ldots, u_j, v_0, \ldots, v_j$ such that $u_0 = x_j, v_0 = a_j, u_j = v_j = y, Su_i = Tu_{i+1}$ and $Sv_i = Tv_{i+1}$. Then $u_{j-1} - v_{j-1} \in N(S)$. Step by step, we obtain $u_0 - v_0 \in N_j(S : T) \subset D_n(T : S)$. As $f \in D_n(T : S)^{\perp}$, we find that $f(u_0) = f(v_0)$, i.e. $f(x_j) = f(a_j)$.

Let us show some properties of f. Set $K_{i,j} = (S^{-1}T)^i(S^{-1}T)^{n-j-1}y$ for $0 \leq j \leq n-1$.

We prove that

$$f|_{K_{i,j}} = \delta_{ij} \quad \text{for } 0 \leq i, j \leq n-1.$$

If $i = j$, then $K_{i,i} = (S^{-1}T)^{n-1}y$, so $f|_{K_{i,i}} = 1$.

If $i > j$, then $K_{i,j} = (S^{-1}T)^{n+(i-j-1)}y \subset (S^{-1}T)^{n+(i-j-1)}N_n(T : S)$. Moreover, by Lemma 2.7, for all positive integers k, we have

$$(S^{-1}T)^p N_k(T:S) \\ = \begin{cases} N_{k-p}(T:S) \cap D_p(T:S) + N_p(S:T) & \text{if } p \in \{0,\dots,k\}, \\ N_p(S:T) & \text{if } p \geq k. \end{cases}$$

Thus $K_{i,j} \subset N_{n+i-j-1}(S:T) \subset D_n(T:S)$, so $K_{i,j} \subset D_n(T:S)$. Hence $f|_{K_{i,j}} = 0$, as $f \in D_n(T:S)^\perp$.

If $i < j$, then $0 \leq (n-1)-(j-i) \leq n-2$, hence also $f|_{K_{i,j}} = 0$.

We now prove that

$$\forall x \in X,\ \forall j \in \{0,\dots,n-1\}, \qquad f \text{ is constant on } H_j(x) = (s^{-1}T)^j x.$$

Let $x \in X$, $j \in \{0,\dots,n-1\}$ and consider z_1, z_2 in $H_j(x)$. For $i = 1,2$, there exist $h_i^0,\dots,h_i^j$ such that $h_i^0 = z_i, h_i^j = x$ and $Sh_i^{p-1} = Th_i^p$ for $1 \leq p \leq j$. We can easily show that $h_1^{j-p} - h_2^{j-p} \in N_p(S:T)$ for $p \in \{0,\dots,j\}$. So $z_1 - z_2 \in N_j(S:T) \subset D_n(T:S)$, hence $f(z_1) = f(z_2)$. This will allow us to define the projection P.

Next, we show that

$$\forall z \in Y,\ \forall j \in \{0,\dots,n-1\}, \qquad f \text{ is constant on } H_j(S^{-1}z),$$

which will allow us to construct Q. Let $z \in Y$, $j \in \{0,\dots,n-1\}$ and z_1, z_2 be in $H_j(S^{-1}z)$. For $i = 1,2$, there exist $h_i^0,\dots,h_i^j$ such that $h_i^0 = z_i, Sh_i^j = z$ and $Sh_i^{p-1} = Th_i^p$ for $1 \leq p \leq j$. We can easily show that $h_1^{j-p} - h_2^{j-p} \in N_{p+1}(S:T)$ for $p \in \{0,\dots,j\}$. So $z_1 - z_2 \in N_{j+1}(S:T) \subset D_n(T:S)$, and hence $f(z_1) = f(z_2)$.

Now we can define the two projections P and Q. For $(x,z) \in X \times Y$, set

$$P(x) = \sum_{j=0}^{n-1} f(H_j(x)) x_{n-j-1}, \qquad Q(z) = \sum_{j=0}^{n-1} f(H_j(S^{-1}z)) z_{n-j-1}.$$

We prove that

$$P^2 = P \quad \text{and} \quad Q^2 = Q.$$

For $x \in X$,

$$P^2(x) = \sum_{j=0}^{n-1} f(H_j(P(x))) x_{n-j-1}.$$

We show that $f(H_j(P(x))) = f(H_j(x))$. Indeed,

$$H_j(P(x)) = (S^{-1}T)^j \Big[\sum_{i=0}^{n-1} f(H_i(x)) x_{n-i-1} \Big] \\ \supset \sum_{i=0}^{n-1} f(H_i(x)) H_j(x_{n-i-1}) \quad \text{by Lemma 2.4.}$$

Let $w \in \sum_{i=0}^{n-1} f(H_i(x))H_j(x_{n-i-1})$. As f is constant on each $H_j(x_{n-i-1})$ ($i \in \{0, \dots, n-1\}$), we have

$$f(w) = \sum_{i=0}^{n-1} f(H_i(x))f(H_j(x_{n-i-1})).$$

As $x_{n-i-1} \in (S^{-1}T)^{n-i-1}y$ and $f|_{K_{i,j}} = \delta_{ij}$, we have $f(H_j(x_{n-i-1})) = \delta_{i,j}$. Thus $f(w) = f(H_j(x))$. As f is constant on $H_j(P(x))$, it follows that $f(H_j(P(x))) = f(w) = f(H_j(x))$. Therefore $P^2 = P$.

For $z \in Y$,

$$Q^2(z) = \sum_{j=0}^{n-1} f(H_j(S^{-1}Q(z)))z_{n-j-1}.$$

We show that $f(H_j(S^{-1}Q(z))) = f(H_j(S^{-1}z))$. We have

$$H_j(S^{-1}Q(z)) \supset \sum_{i=0}^{n-1} f(H_i(S^{-1}z))H_j(S^{-1}z_{n-i-1}) \quad \text{by Lemma 2.4.}$$

Notice that $x_{n-i-1} \in S^{-1}z_{n-i-1}$. So $f[H_j(S^{-1}z_{n-i-1})] = f[H_j(x_{n-i-1})]$. Let $w \in \sum_{i=0}^{n-1} f(H_i(S^{-1}z))H_j(x_{n-i-1})$. As before, we find that $f(w) = f(H_j(S^{-1}z))$. Hence $f(H_j(S^{-1}Q(z))) = f(H_j(S^{-1}z))$.

Step 2. We study different properties of the two projections. Set

$$X_1 = R(P), \quad X_2 = N(P), \quad Y_1 = R(Q), \quad Y_2 = N(Q).$$

We have dim $X_1 = \dim Y_1 < \infty$, the equality following from the fact that $P(x_k) = x_k$ and $Q(z_k) = z_k$ for all $k \in \{0, \dots, n-1\}$.

We want to prove that

$$QS = SP \quad \text{and} \quad QT = TP.$$

Let $x \in X$. As $x \in S^{-1}Sx$ and f is constant on each $H_j(S^{-1}z)$, it follows that $f(H_jS^{-1}Sx) = f(H_jx)$. Thus, we have

$$\begin{aligned}
QS(x) &= \sum_{j=0}^{n-1} f(H_j(x))z_{n-j-1} = \sum_{j=0}^{n-1} f(H_j(x))Sx_{n-j-1} = SP(x),\\
QT(x) &= \sum_{j=0}^{n-1} f(H_jS^{-1}Tx)z_{n-j-1} = \sum_{j=0}^{n-1} f(H_{j+1}(x))z_{n-j-1}\\
&= \sum_{j=1}^{n-1} f(H_j(x))z_{n-j} \quad \text{as } f((S^{-1}T)^n x) = 0\\
&= TP(x) \quad \text{as } x_{n-1} \in N(T).
\end{aligned}$$

Hence $TX_i \subset Y_i$ and $SX_i \subset Y_i$ for $i = 1, 2$. Denote by S_i and T_i the restrictions of S and T to X_i, considered as operators from X_i to Y_i.

Let us prove that

$$N(T) \cap X_1 = \mathbb{C}x_{n-1}.$$

Let $x \in N(T) \cap X_1$. As $x = P(x)$, we have

$$0 = Tx = \sum_{j=1}^{n-1} f(H_j(x)) T x_{n-j-1} = \sum_{j=1}^{n-1} f(H_j(x)) S x_{n-j} = \sum_{j=1}^{n-1} f(H_j(x)) z_{n-j}.$$

As $z_0, \ldots, z_{n-1}$ are independent, $f(H_j(x)) = 0$ for each $j \in \{1, \ldots, n-1\}$. Thus $x = f(x)x_{n-1}$, and so $N(T) \cap X_1 \subset \mathbb{C}x_{n-1}$. Conversely, let $x = \lambda x_{n-1}$. Then $f(x) = \lambda f(x_{n-1}) = \lambda$, so $x = f(x)x_{n-1}$. Moreover, $f(x)x_{n-1} = P(x)$, so $x \in R(P) = X_1$. Further, $Tx = f(x)Tx_{n-1} = 0$.

We now prove that

$$N(S) \cap X_1 = \{0\}.$$

Let $x \in N(S) \cap X_1$. Then $x = P(x) = \sum_{j=0}^{n-1} f(H_j(x)) x_{n-j-1}$. We have

$$0 = Sx = SP(x) = \sum_{j=0}^{n-1} f(H_j(x)) S x_{n-j-1} = \sum_{j=0}^{n-1} f(H_j(x)) z_{n-j-1}.$$

So $f(H_j(x)) = 0$ for $j \in \{0, \ldots, n-1\}$, since $z_0, \ldots, z_{n-1}$ are independent. Hence $x = 0$.

Thus $S_1 : X_1 \to Y_1$ is injective. Moreover, $\dim X_1 = \dim Y_1 < \infty$. So $S_1 : X_1 \to Y_1$ is bijective. Thus, we can consider $S^{-1}T$ from X_1 to X_1.

We now prove that

$$S^{-1}T : X_1 \to X_1 \text{ is nilpotent.}$$

Let $x \in X_1$. We have

$$TP(x) = \sum_{j=0}^{n-1} f(H_j(x)) T x_{n-j-1} = \sum_{j=1}^{n-1} f(H_j(x)) S x_{n-j},$$

and $S^{-1}TPx = \sum_{j=1}^{n-1} f(H_j(x)) x_{n-j}$, as $S_1 : X_1 \to Y_1$ is injective, as well as $(S^{-1}T)^2 Px = \sum_{j=2}^{n-1} f(H_j(x)) x_{n-j+1}$. Step by step, we obtain $(S^{-1}T)^n Px = 0$.

We now prove that

$$N(S) \subset X_2.$$

Let $x \in X = X_1 \oplus X_2, x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. Assume that $Sx = 0$, i.e. $Sx_1 = -Sx_2$. As $Sx_1 \in Y_1$ and $Sx_2 \in Y_2$, it follows that $Sx_1 = Sx_2 = 0$. We thus have $x_1 \in N(S) \cap X_1 = \{0\}$, hence $x_1 = 0$.

Therefore we can consider $S^{-1}T : X_2 \to X_2$. As $X = X_2 \oplus X_1$, $N(T) = N(T_2) \oplus \mathbb{C}x_{n-1}$.

Step 3. Now we finish the proof. We have to separate two cases. First, assume that (1) is true, i.e. $N(T) \subset_{\mathrm{e}} D(T : S)$. We have $N(T) = N(T_2) \oplus$

$\mathbb{C}x_{n-1}$ and $D(T:S) = D(T_2:S_2)$, as $S^{-1}T|_{X_1}$ is nilpotent. So

$$N(T)/[N(T)\cap D(T:S)] \cong N(T_2)/[N(T_2)\cap D(T_2:S_2)] \times \mathbb{C}x_{n-1}.$$

Hence

$$\dim(N(T_2)/[N(T_2)\cap D(T_2:S_2)]) = \dim(N(T)/[N(T)\cap D(T:S)]) - 1.$$

If $\dim(N(T)/[N(T)\cap D(T:S)]) = 1$, we stop. Otherwise, we reiterate the operation with X replaced by $N(P)$ and Y by $N(Q)$. We obtain two new projections P_1 and Q_1. Thus, in a finite number of steps, we will obtain

$$\begin{cases} X = \Big(\bigcap_{j=0}^{k-1} N(P_j)\Big) \oplus \Big(\bigoplus_{j=0}^{k-1} R(P_j)\Big), \\ Y = \Big(\bigcap_{j=0}^{k-1} N(Q_j)\Big) \oplus \Big(\bigoplus_{j=0}^{k-1} R(Q_j)\Big), \\ T_2 \text{ has the property } P(S_2:0). \end{cases}$$

Now, assume that (2) is true, i.e. $N(T:S) \subset_{\mathrm{e}} S^{-1}R(T)$. We have $N_k(T:S) = N_k(T_1:S_1)\oplus N_k(T_2:S_2)$ for each $k\in\mathbb{N}$. As $S^{-1}T|_{X_1}$ is nilpotent of index $\le n$, we get

$$N_k(T_1:S_1) = X_1 \quad \text{for every } k \ge n.$$

In fact, $N_k(T_1:S_1)$ is clearly a subset of X_1. Conversely, let $x\in X_1$. We want to prove that $x\in N_n(T_1:S_1)$, i.e. we want to find $a_0,\dots,a_n$ such that $a_0 = x, a_n = 0$ and $Ta_i = Sa_{i+1}$. We have $x = P(x) = \sum_{j=0}^{n-1} f(H_j(x))x_{n-j-1}$, so

$$Tx = \sum_{j=0}^{n-1} f(H_j(x))Tx_{n-j-1} = \sum_{j=1}^{n-1} f(H_j(x))Sx_{n-j}.$$

Hence we can put $a_1 = \sum_{j=1}^{n-1} f(H_j(x))x_{n-j}$. Then

$$Ta_1 = \sum_{j=2}^{n-1} f(H_j(x))Sx_{n-j+1},$$

and we can put $a_2 = \sum_{j=2}^{n-1} f(H_j(x))x_{n-j+1}$. Step by step, we find the desired a_i, and we prove that $x\in N_n(T_1:S_1)$.

So $N(T:S) = X_1\oplus N(T_2:S_2)$. Moreover, $D_1(T:S) = D_1(T_1:S_1)\oplus D_1(T_2:S_2)$. We have already shown that $D_1(T_1:S_1) = S^{-1}TPX = \langle x_i\rangle_{i=1}^{n-1}$. Thus

$$N(T:S)/[N(T:S)\cap D_1(T:S)] \cong N(T_2:S_2)/[N(T_2:S_2)\cap D_1(T_2:S_2)] \times X_1/\langle x_i\rangle_{i=1}^{n-1}.$$

As $X_1/\langle x_i\rangle_{i=1}^{n-1} = \mathbb{C}x_0$, we get

$$\dim(N(T_2 : S_2)/[N(T_2 : S_2) \cap D_1(T_2 : S_2)]) = \dim(N(T : S)/[N(T : S) \cap D_1(T : S)]) - 1.$$

By Lemma 2.5, T_2 has the property $P(S_2 : 0)$ if and only if $N(T_2 : S_2) \subset D_1(T_2 : S_2)$. Thus, we can complete the proof in the same way as in the first case.

2.9. Remarks. (1) In the particular case where $X = Y$ and $S = I$, we recover the result on the essentially s-regular operators proved by V. Müller and V. Rakočević (see [8, Theorem 3.1] and [9]).

(2) Under condition (4) of Theorem 2.3, as $S_1^{-1}T_1$ is nilpotent and $S_1 : X_1 \to Y_1$ is bijective, we have $\sigma(T_1 : S_1) = \{\lambda \in \mathbb{C}; T_1 - \lambda S_1$ is not invertible$\} = \{0\}$.

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