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C-Distribution semigroups

by

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Abstract. A class of C-distribution semigroups unifying the class of (quasi-) distribution semigroups of Wang and Kunstmann (when $C = I$) is introduced. Relations between C-distribution semigroups and integrated C-semigroups are given. Dense C-distribution semigroups as well as weak solutions of the corresponding Cauchy problems are also considered.

1. Introduction. Distribution semigroups and their generators were introduced by J.-L. Lions [14] in 1960. A new definition of distribution semigroups, covering in particular non-densely defined generators, was given by P. C. Kunstmann in [11]. He proved that a closed linear operator A generates a distribution semigroup if and only if A is the generator of a local integrated semigroup ([3]). Similar results were independently obtained in [17], where S. Wang defined the same class of semigroups and called it the quasi-distribution semigroups.

On the other hand, local integrated C-semigroups were analyzed by M. Li, F. Huang and Q. Zheng in [13] as a generalization of local integrated semigroups. The present paper can be viewed as a unification of the concepts of distribution semigroups and integrated C-semigroups. We will prove that every generator of a global integrated C-semigroup is also the generator of a so-called C-distribution semigroup. As we will see in Section 4, relations between C-distribution semigroups and local integrated C-semigroups seem to be more complicated, which is primarily caused by local properties of distribution spaces.

The paper is organized as follows. In Section 2, we recall some basic concepts necessary for further investigations. In Section 3, we introduce C-distribution semigroups and we analyze their structural properties. Our main results are presented in Section 4 where we clarify relations between C-distribution semigroups and integrated C-semigroups. In the same sec-

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tion, we introduce exponential C-distribution semigroups and relate them to exponentially bounded integrated C-semigroups. Section 5 is devoted to the study of dense C-distribution semigroups.

2. Preliminaries. Let E, F be Banach spaces. Denote by $L(E,F)$ the space of bounded linear operators from E into F. We write $L(E) = L(E,E)$. We refer to [3], [5], [15] and [20] for the notions of (local) C-semigroups and (local) integrated semigroups.

The spaces of test functions $\mathcal{D} = C_0^\infty(\mathbb{R})$ and $\mathcal{E} = C^\infty(\mathbb{R})$ ([1], [16]) carry the usual inductive limit topologies while the topology of the space $\mathcal{S}$ of rapidly decreasing functions is defined by the system of seminorms $p_{m,n}(\psi) = \sup_{x\in\mathbb{R}} |x^m\psi^{(n)}(x)|$ for $\psi \in \mathcal{S}$ and $m, n \in \mathbb{N}_0$. We denote by $\mathcal{D}_0$ the subspace of $\mathcal{D}$ which consists of the elements supported by $[0,\infty)$. Moreover, $\mathcal{D}'(L(E)) = L(\mathcal{D}, L(E))$, $\mathcal{E}'(L(E)) = L(\mathcal{E}, L(E))$ and $\mathcal{S}'(L(E)) = L(\mathcal{S}, L(E))$; $\mathcal{D}'_0(L(E))$, $\mathcal{E}'_0(L(E))$ and $\mathcal{S}'_0(L(E))$ are the subspaces of $\mathcal{D}'(L(E))$, $\mathcal{E}'(L(E))$ and $\mathcal{S}'(L(E))$, respectively, containing the elements supported by $[0,\infty)$.

Denote by $\mathcal{B}$ the family of all bounded subsets of $\mathcal{D}$. Put $p_B(f) = \sup_{\varphi\in B} \|f(\varphi)\|$ for $f \in \mathcal{D}'(L(E))$ and $B \in \mathcal{B}$. Then each p_B is a seminorm on $\mathcal{D}'(L(E))$ and the system $(p_B)_{B\in\mathcal{B}}$ defines the topology on $\mathcal{D}'(L(E))$. The topology on $\mathcal{E}'(L(E))$, resp., $\mathcal{S}'(L(E))$, is defined similarly.

Let $\varrho \in \mathcal{D}$ satisfy $\int_{-\infty}^{\infty} \varrho(t)\,dt = 1$ and $\operatorname{supp}\varrho \subset [0,1]$. By a *regularizing sequence* we mean a sequence (ϱ_n) in $\mathcal{D}_0$ defined by $\varrho_n(t) := n\varrho(nt)$, $t \in \mathbb{R}$.

If $\varphi, \psi : \mathbb{R} \to \mathbb{C}$ are measurable functions, we use the convolutions $\varphi * \psi$ and $\varphi *_0 \psi$ defined as follows:

$$\varphi * \psi(t) = \int_{-\infty}^{\infty} \varphi(t-s)\psi(s)\,ds, \qquad t \in \mathbb{R},$$

and

$$\varphi *_0 \psi(t) = \int_0^t \varphi(t-s)\psi(s)\,ds, \qquad t \in \mathbb{R}.$$

Notice that $\varphi*\psi = \varphi*_0\psi$ if $\varphi, \psi \in \mathcal{D}_0$. We refer to [15] for further information concerning the convolution of (vector-valued) distributions.

For a closed linear operator A on E, its domain, range, null space and resolvent set are denoted by $D(A), \operatorname{Im} A, \operatorname{Ker} A$ and $\varrho(A)$, respectively; $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm. We will assume in what follows that $C \in L(E)$ is an injective operator.

Let $n \in \mathbb{N}$ and $\tau \in (0,\infty]$. We refer to [13, Definition 2.1] for the definition of a (local) n-times integrated C-semigroup $(T_n(t))_{t\in[0,\tau)}$. We assume that it is nondegenerate. Let us recall that an operator family $(T(t))_{t\in[0,\tau)} \subset L(E)$ is *nondegenerate* if $T(t)x = 0$ for all $t \in [0,\tau)$ implies $x = 0$. The (*integral*) *generator* A of a nondegenerate (local) n-times integrated C-semigroup

$(T_n(t))_{t\in[0,\tau)}$ is defined by

$$\Big\{(x,y)\in E\times E : T_n(t)x-\frac{t^n}{n!}Cx=\int_0^t T_n(s)y\,ds \text{ for all } t\in[0,\tau)\Big\},$$

and it is a closed linear operator. If $(T_0(t))_{t\ge 0}$ is a C-regularized semigroup, we may define its (*infinitesimal*) *generator* by

$$A=\Big\{(x,y)\in E\times E : \lim_{t\to 0+}\frac{T_0(t)x-Cx}{t}=Cy\Big\},$$

which is a closed linear operator satisfying $C^{-1}AC=A$.

Let $n\in\mathbb{N}_0$. The Cauchy problem

$$C_{n+1}(\tau):\begin{cases} u\in C([0,\tau);[D(A)])\cap C^1([0,\tau);E),\\ u'(t)=Au(t)+\dfrac{t^n}{n!}Cx, \quad t\in[0,\tau),\\ u(0)=0,\end{cases}$$

is C-*well posed* if for any $x\in E$ it has a unique solution.

3. Basic properties of C-distribution semigroups. Assume that $\mathcal{G}\in\mathcal{D}_0'(L(E))$ satisfies $C\mathcal{G}=\mathcal{G}C$. If it also satisfies

$$(C\text{-D.S.1})\qquad \mathcal{G}(\varphi *_0\psi)C=\mathcal{G}(\varphi)\mathcal{G}(\psi),\quad \varphi,\psi\in\mathcal{D},$$

then $\mathcal{G}$ is called a *pre*-(C-DS), and if additionally

$$(C\text{-D.S.2})\qquad \mathcal{N}(\mathcal{G}):=\bigcap_{\varphi\in\mathcal{D}_0}\operatorname{Ker}\mathcal{G}(\varphi)=\{0\},$$

then $\mathcal{G}$ is called a *C-distribution semigroup*, (C-DS) for short. If moreover

$$(C\text{-D.S.3})\qquad \mathcal{R}(\mathcal{G}):=\bigcup_{\varphi\in\mathcal{D}_0}\operatorname{Im}\mathcal{G}(\varphi) \text{ is dense in } E,$$

then $\mathcal{G}$ is called a *dense* (C-DS).

This definition, with $C=I$, was introduced in [11], where P. C. Kunstmann defined a distribution semigroup, (DS) for short. It is clear that if $\mathcal{G}$ is a pre-(C-DS), then $\mathcal{G}(\varphi)\mathcal{G}(\psi)=\mathcal{G}(\psi)\mathcal{G}(\varphi)$ for all $\varphi,\psi\in\mathcal{D}$. Also, in this case, $\mathcal{N}(\mathcal{G})$ is a closed subspace of E.

Recall that the polars of nonempty sets $M\subset E$ and $N\subset E^*$ are defined as follows: $M^\circ=\{y\in E^* : |y(x)|\le 1 \text{ for all } x\in M\}$ and $N^\circ=\{x\in E : |y(x)|\le 1 \text{ for all } y\in N\}$.

Repeating literally the arguments given in [11], one can prove the following assertion.

PROPOSITION 3.1. *Let $\mathcal{G}$ be a pre-(C-DS). Then, with $N=\mathcal{N}(\mathcal{G})$ and G_1 being the restriction of $\mathcal{G}$ to N ($G_1=\mathcal{G}_{|N}$), there exists a unique set of* *operators* $T_0, T_1, \ldots, T_m \in L(E)$ *such that*

$$G_1 = \sum_{j=1}^{m} \delta^{(j)} \otimes T_j, \qquad T_i C^i = (-1)^i T_0^{i+1}, \qquad i = 0, 1, \ldots, m-1,$$
$$T_0 T_m = T_0^{m+2} = 0.$$

In the next proposition we give some analogues of results known for distribution semigroups (cf. [11]).

Proposition 3.2. *Let* $\mathcal{G}$ *be a pre-*(C-DS), $F = E/\mathcal{N}(\mathcal{G})$ *and* q *be the canonical mapping* $q : E \to E/\mathcal{N}(\mathcal{G})$.

- (i) *Let* $H \in L(\mathcal{D}, L(F))$ *be defined by* $q\mathcal{G}(\varphi) = H(\varphi)q$ *for all* $\varphi \in \mathcal{D}$ *and let* $\widetilde{C}$ *be the linear operator in* F *defined by* $\widetilde{C}q = qC$. *Then* $\widetilde{C} \in L(F)$ *and it is injective. Moreover*, H *is a* ($\widetilde{C}$-DS) *in* F.
- (ii) $C(\overline{\langle \mathcal{R}(\mathcal{G}) \rangle}) \subset \overline{\mathcal{R}(\mathcal{G})}$, *where* $\langle \mathcal{R}(\mathcal{G}) \rangle$ *is the linear span of* $\mathcal{R}(\mathcal{G})$.
- (iii) *Assume that* $\mathcal{G}$ *is not dense and* $\overline{C\mathcal{R}(\mathcal{G})} = \overline{\mathcal{R}(\mathcal{G})}$. *Put* $R = \overline{\mathcal{R}(\mathcal{G})}$ *and* $H = \mathcal{G}_{|R}$. *Then* H *is a dense pre-*(C_1-DS) *on* R *with* $C_1 = C_{|R}$.
- (iv) *Assume* $\overline{\operatorname{Im} C} = E$. *Then the dual* $\mathcal{G}(\cdot)^*$ *is a pre-*(C^*-DS) *on* E^* *and* $\mathcal{N}(\mathcal{G}^*) = \overline{\mathcal{R}(\mathcal{G})}^{\circ}$.
- (v) *If* E *is reflexive and* $\overline{\operatorname{Im} C} = E$, *then* $\mathcal{N}(\mathcal{G}) = \overline{\mathcal{R}(\mathcal{G}^*)}^{\circ}$.
- (vi) *Suppose* $\overline{\operatorname{Im} C} = E$. *Then* $\mathcal{G}^*$ *is a* (C^*-DS) *in* E^* *if and only if* $\mathcal{G}$ *is a dense pre-*(C-DS). *If* E *is reflexive, then* $\mathcal{G}^*$ *is a dense pre-*(C^*-DS) *in* E^* *if and only if* $\mathcal{G}$ *is a* (C-DS).

Proof. We only give the detailed proof of (i). First, let us show that the definition of $\widetilde{C}(q(x))$ does not depend on the representative of the class $q(x)$. The assumptions $q(x) = q(y)$, i.e., $\mathcal{G}(\varphi)(x - y) = 0$ for all $\varphi \in \mathcal{D}_0$, and $C\mathcal{G} = \mathcal{G}C$, imply $\mathcal{G}(\varphi)(Cx - Cy) = 0$ for $\varphi \in \mathcal{D}_0$, and $\widetilde{C}(q(x)) = \widetilde{C}(q(y))$. It is clear that $\widetilde{C}$ is a linear operator in F.

We will show that $\widetilde{C}$ is continuous. Assume $x \in E$. Then $\|\widetilde{C}(q(x))\| = \inf_{y \in \mathcal{N}(\mathcal{G})} \|Cx + y\|$. Fix $y \in \mathcal{N}(\mathcal{G})$. Applying again $C\mathcal{G} = \mathcal{G}C$, we see that $Cy \in \mathcal{N}(\mathcal{G})$. Thus, $\|\widetilde{C}(q(x))\| \le \|Cx + Cy\| \le \|C\|\,\|x + y\|$; this implies $\|\widetilde{C}(q(x))\| \le \|C\|\,\|q(x)\|$, so $\widetilde{C} \in L(F)$ and $\|\widetilde{C}\| \le \|C\|$.

Suppose $\widetilde{C}(q(x)) = 0$. Then $Cx \in \mathcal{N}(\mathcal{G})$ and $C\mathcal{G}(\varphi)x = 0$ for all $\varphi \in \mathcal{D}_0$. Since C is injective, one has $x \in \mathcal{N}(\mathcal{G})$ and $q(x) = 0$. Therefore, $\widetilde{C} \in L(F)$ and it is injective. One sees directly that H satisfies ($\widetilde{C}$-D.S.1) and $\widetilde{C}H = H\widetilde{C}$. Suppose $H(\varphi)q(x) = 0$ for all $\varphi \in \mathcal{D}_0$, i.e., $\mathcal{G}(\varphi)x \in \mathcal{N}(\mathcal{G})$ for all $\varphi \in \mathcal{D}_0$. Then $\mathcal{G}(\psi)\mathcal{G}(\varphi)x = 0$, $C\mathcal{G}(\varphi * \psi)x = 0$ and $\mathcal{G}(\varphi * \psi)x = 0$ for all $\varphi, \psi \in \mathcal{D}_0$. Choose a regularizing sequence (ϱ_n) to obtain $\mathcal{G}(\varphi)x = \lim_{n\to\infty} \mathcal{G}(\varphi * \varrho_n)x = 0$ for all $\varphi \in \mathcal{D}_0$; this gives $q(x) = 0$ and ends the proof. ■

Let $\mathcal{G}$ be a (C-DS) and let $T \in \mathcal{E}'_0(\mathbb{C})$, i.e., T is a scalar-valued distribution with compact support in $[0,\infty)$. Define $G(T)$ on a subset of E by

$$y = G(T)x \quad \text{if and only if} \quad \mathcal{G}(T * \varphi)x = \mathcal{G}(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0.$$

Denote its domain by $D(G(T))$. By (C-D.S.2), $G(T)$ is a function. Moreover, $G(T)$ is a closed linear operator and $G(\delta) = I$.

The infinitesimal generator of a (C-DS) $\mathcal{G}$ *is defined by* $A := G(-\delta')$.

Note that since for $\psi \in \mathcal{D}$, $\psi_+ := \psi \mathbf{1}_{[0,\infty)} \in \mathcal{E}'_0(\mathbb{C})$ ($\mathbf{1}_Z$ is the characteristic function of Z), $G(\psi_+)$ is defined. Moreover, as C does not appear in the definition of $G(T)$, one might think that the notion of $G(T)$ is misleading. However, it just simplifies the definition of A. Namely, define $G_C(T)$ (for $T \in \mathcal{E}'_0(\mathbb{C})$) by $G_C(T) = \{(x,y) \in E \times E : \mathcal{G}(T * \varphi)Cx = \mathcal{G}(\varphi)y$ for all $\varphi \in \mathcal{D}_0\}$. It can be easily seen that $G_C(T)$ is a closed linear operator. We have $G_C(\delta) = C$ and $G(T)C = G_C(T)$, $T \in \mathcal{E}'_0(\mathbb{C})$. Moreover, if $\mathcal{G}$ is a (C-DS), $T \in \mathcal{E}'_0(\mathbb{C})$ and $\varphi \in \mathcal{D}$, then it is straightforward to see that $G(\varphi)G(T) \subset G(T)\mathcal{G}(\varphi)$, $CG(T) \subset G(T)C$ and $\mathcal{R}(\mathcal{G}) \subset D(G(T))$.

If $f : \mathbb{R} \to \mathbb{C}$, let $(\tau_t f)(s) := f(s-t)$ for $s,t \in \mathbb{R}$. Note that if $\mathcal{G}$ is a pre-(C-DS) and $\varphi, \psi \in \mathcal{D}$, then the assumption $\varphi(t) = \psi(t)$ for all $t \geq 0$ implies $\mathcal{G}(\varphi) = \mathcal{G}(\psi)$. Indeed, put $\eta = \varphi - \psi$. Then $\eta \in \mathcal{D}_{(-\infty,0]}$ and the continuity of $\mathcal{G}$ implies $\lim_{h\to 0-} \mathcal{G}(\tau_h \eta)x = \mathcal{G}(\eta)x = 0$, $x \in E$. Now we state:

Proposition 3.3. *If* $\mathcal{G}$ *is a* (C-DS), *then* $G(\psi_+)C = \mathcal{G}(\psi)$ *for all* $\psi \in \mathcal{D}$.

Proof. Let $x \in E$ and $\psi \in \mathcal{D}$. Then $G(\psi_+)Cx = \mathcal{G}(\psi)x$ iff $\mathcal{G}(\psi_+ * \varphi)Cx = \mathcal{G}(\varphi)\mathcal{G}(\psi)x$ for all $\varphi \in \mathcal{D}_0$ iff $\mathcal{G}(\psi_+ * \varphi)Cx = \mathcal{G}(\varphi *_0 \psi)Cx$ for all $\varphi \in \mathcal{D}_0$. The last statement is true since for every fixed $\varphi \in \mathcal{D}_0$, we have $\varphi *_0 \psi = \psi_+ * \varphi$. ∎

Using the same arguments as in [11, Lemma 3.6] we have:

Proposition 3.4. *Let* $S, T \in \mathcal{E}'_0$, $\varphi \in \mathcal{D}_0$, $\psi \in \mathcal{D}$ *and* $x \in E$. *Then*

(i) $(\mathcal{G}(\varphi)x,\ \mathcal{G}(\underbrace{T * \cdots * T}_{m} * \varphi)x) \in G(T)^m$, $m \in \mathbb{N}$.

(ii) $G(S)G(T) \subset G(S*T)$ *with* $D(G(S)G(T)) = D(G(S*T)) \cap D(G(T))$, *and* $G(S) + G(T) \subset G(S+T)$.

(iii) $(\mathcal{G}(\psi)x, \mathcal{G}(-\psi')x - \psi(0)Cx) \in G(-\delta')$.

(iv) *If* $\mathcal{G}$ *is dense its generator is densely defined.*

Examples 3.5. (a) Let A be the infinitesimal generator of a C-semigroup $(T(t))_{t\geq 0}$ and $\mathcal{G}(\varphi) := \int_0^\infty \varphi(t)T(t)\,dt$ for $\varphi \in \mathcal{D}$. Then $\mathcal{G}$ is a (C-DS) with generator A.

Proof. We will only prove that A is the generator of $\mathcal{G}$. It is well known that $C^{-1}AC = A$, $T(t)C = CT(t)$ and $T(t)A \subset AT(t)$ for all $t \geq 0$. Suppose now $(x,y) \in C^{-1}AC = A$. Then $A\int_0^t T(s)Cx\,ds = T(t)Cx - C^2x$ and

$\int_0^t T(s)ACx\,ds = T(t)Cx - C^2x$ for all $t \geq 0$. Hence, $\int_0^t T(s)Cy\,ds = CT(t)x - C^2x$ and $\int_0^t T(s)y\,ds = T(t)x - Cx$ for all $t \geq 0$. We have to prove $-\int_0^\infty \varphi'(t)T(t)x\,dt = \int_0^\infty \varphi(t)T(t)y\,dt$ for all $\varphi \in \mathcal{D}_0$. Indeed,

$$\int_0^\infty \varphi(t)T(t)y\,dt = -\int_0^\infty \varphi'(t)\int_0^t T(s)y\,ds\,dt = -\int_0^\infty \varphi'(t)[T(t)x - Cx]\,dt$$
$$= -\int_0^\infty \varphi'(t)T(t)x\,dt.$$

Consequently, $(x, y) \in B$, where B is the generator of $\mathcal{G}$.

Conversely suppose $(x, y) \in B$. Then $-\int_0^\infty \varphi'(t)T(t)x\,dt = \int_0^\infty \varphi(t)T(t)y\,dt$ and $\int_0^\infty \varphi'(t)T(t)x\,dt = \int_0^\infty \varphi'(t)\int_0^t T(s)y\,ds\,dt$ for all $\varphi \in \mathcal{D}_0$. Thus, $T(t)x - \int_0^t T(s)y\,ds = \text{const}$ and $\int_0^t T(s)y\,ds = T(t)x - Cx$ for all $t \geq 0$. Hence, $A\int_0^t T(s)x\,ds = \int_0^t T(s)y\,ds$ for all $t \geq 0$. Since A is closed, we obtain $T(t)x \in D(A)$ and $AT(t)x = T(t)y$ for all $t \geq 0$. Accordingly, $(x, y) \in C^{-1}AC = A$. ∎

(b) If $\mathcal{G}$ is a (DS) with generator A and if $\mathcal{G}C = C\mathcal{G}$, then $\mathcal{G}C$ is a (C-DS) with generator A.

(c) ([9]) Let P be a bounded projector on E with $PC = CP$. Define $\mathcal{G}(\varphi) := \int_0^\infty \varphi(t)\,dt\,PC$ for $\varphi \in \mathcal{D}$. Then $\mathcal{G}$ is a pre-(C-DS). Moreover, $\mathcal{N}(\mathcal{G}) = \operatorname{Ker} P$.

4. Connections with integrated C-semigroups. In this section we investigate relations between C-distribution semigroups and the corresponding $C_{n+1}(\tau)$ problems with (local) integrated C-semigroups.

Lemma 4.1. *Let $\mathcal{G}$ be a* (C-DS) *generated by A. Then $C^{-1}AC = A$.*

Proof. Let $(x, y) \in A$. Then $\mathcal{G}(-\varphi')x = \mathcal{G}(\varphi)y$, so $C\mathcal{G}(-\varphi')x = C\mathcal{G}(\varphi)y$ and hence $\mathcal{G}(-\varphi')Cx = \mathcal{G}(\varphi)Cy$ for all $\varphi \in \mathcal{D}_0$. So $(Cx, Cy) \in A$ and $A \subset C^{-1}AC$.

Conversely, assume $(x, y) \in C^{-1}AC$. Then $ACx = Cy$ and $\mathcal{G}(-\varphi')Cx = \mathcal{G}(\varphi)Cy$ for all $\varphi \in \mathcal{D}_0$. Since $C\mathcal{G} = \mathcal{G}C$ and C is injective, we have $\mathcal{G}(-\varphi')x = \mathcal{G}(\varphi)y$ for all $\varphi \in \mathcal{D}_0$. Thus, $(x, y) \in A$ and $C^{-1}AC = A$. ∎

Theorem 4.2. *Let $\mathcal{G}$ be a* (C-DS) *generated by A. Then there exist $\tau > 0$, $n \in \mathbb{N}$ and a nondegenerate operator family $(W(t))_{t\in[0,\tau)}$ such that*:

(i) $A\int_0^t W(s)x\,ds = W(t)x - (t^n/n!)Cx$ for all $t \in [0, \tau)$ and $x \in E$,
(ii) $CA \subset AC$ and $W(t)A \subset AW(t)$ and $CW(t) = W(t)C$ for all $t \in [0, \tau)$,
(iii) $(W(t))_{t\in[0,\tau)}$ *is a local n-times integrated C-semigroup generated by A. Moreover*, *the problem $C_{n+1}(\tau)$ is C-well posed for A*, *and*

for all $\tau' > 0$, there exist $n', l \in \mathbb{N}$ such that the problem $C_{n'}(\tau')$ is C^l-well posed for A.

Proof. We have $A\mathcal{G}(\varphi)x = -\mathcal{G}(\varphi')x - \varphi(0)Cx$, for every $\varphi \in \mathcal{D}$ and $x \in E$. This implies that $\mathcal{G}$ is a continuous linear mapping from $\mathcal{D}$ into $L(E, [D(A)])$. As in [3, Theorem 7.2] and [17, Theorem 3.8] we can conclude that there exist $\tau > 0$, $n \in \mathbb{N}$ and a continuous function $W : [-\tau, \tau] \to L(E, [D(A)])$ such that

$$\mathcal{G}(\varphi)x = (-1)^n \int_{-\tau}^{\tau} \varphi^{(n)}(t)W(t)x\,dt, \qquad x \in E,\ \varphi \in \mathcal{D}_{(-\tau,\tau)}.$$

Moreover, $\operatorname{supp} W \subset [0, \tau]$. We have

$$(-1)^n \int_0^{\tau} \varphi^{(n)}(t)AW(t)x\,dt = A\mathcal{G}(\varphi)x = \mathcal{G}(-\varphi')x - \varphi(0)Cx$$

$$= (-1)^{n+1} \int_0^{\tau} \varphi^{(n+1)}(t)W(t)x\,dt - \varphi(0)Cx,$$

and hence

$$\int_0^{\tau} \varphi^{(n+1)}(t)\Big[\int_0^t AW(s)x\,ds - W(t)x\Big]\,dt = 0, \qquad x \in E,\ \varphi \in \mathcal{D}_{[0,\tau)}.$$

This implies

$$\int_0^t AW(s)x\,ds - W(t)x = \sum_{j=0}^{n} t^j B_j x, \qquad t \in [0, \tau),$$

for some operators $B_j \in L(E)$, $j = 0, 1, \dots, n$. As in [17, Theorem 3.8] we obtain $B_j = 0$ for $j = 0, 1, \dots, n-1$ and $B_n = -(t^n/n!)C$, which gives

$$\int_0^t AW(s)x\,ds = W(t)x - \frac{t^n}{n!}Cx, \qquad t \in [0, \tau),\ x \in E. \tag{1}$$

Since $C\mathcal{G} = \mathcal{G}C$ and A commutes with C (see Lemma 4.1) and with $\mathcal{G}(\varphi)$ for all $\varphi \in \mathcal{D}$, the rest of (ii) may be obtained in a similar way, and (iii) is a simple consequence of [13, Proposition 2.4, Theorems 2.5 & 4.1]. ∎

REMARK 4.3. Assume that $\mathcal{G}$ is a (C-DS) generated by A. Then we saw that for all $\tau' > 0$, there exist $n', l \in \mathbb{N}$ such that the problem $C_{n'}(\tau')$ is C^l-well posed for A. Moreover, $C^{-1}AC = A$, and consequently $C^{-n}AC^n = A$ for all $n \in \mathbb{N}_0$. Applying again [13, Proposition 2.4, Theorem 2.5], we infer that for all $\tau' > 0$, there exist $n'', l \in \mathbb{N}$ such that $A = C^{-l}AC^l$ is the generator of an n''-times integrated C^l-semigroup on $[0, \tau')$.

THEOREM 4.4. *Suppose that there exists a sequence $\langle(p_k, \tau_k)\rangle$ ($p_k \in \mathbb{N}_0$, $\tau_k \in (0, \infty)$; $k \in \mathbb{N}_0$) such that $\lim_{k\to\infty} \tau_k = \infty$ and $C_{p_k+1}(\tau_k)$ is C-well posed for A. If $CA \subset AC$, then $C^{-1}AC$ generates a* (C-DS).

Proof. Clearly, we may assume $\tau_k < \tau_{k+1}$ and $p_k \geq 2$ for all $k \in \mathbb{N}_0$. Let $(W_{p_k}(t))_{t\in[0,\tau_k)}$ be the local p_k-times integrated C-regularized semigroup generated by $C^{-1}AC$. Here W_{p_k} is given by [13, Theorem 2.5]. Because every local integrated C-semigroup is uniquely determined by its generator (cf. [12, Proposition 1.3]), the following definition is independent of $k \in \mathbb{N}_0$. Let $\varphi \in \mathcal{D}_{(-\infty,\tau_k)}$. Define

$$\mathcal{G}(\varphi)x := (-1)^{p_k} \int_0^\infty \varphi^{(p_k)}(t) W_{p_k}(t)x\,dt, \quad x \in E.$$

Then $\mathcal{G} \in \mathcal{D}_0'(L(E))$ and $\mathcal{G}C = C\mathcal{G}$. Furthermore, for all $x \in E$ and all $\varphi, \psi \in \mathcal{D}_{(-\infty,\tau_k)}$ with $\operatorname{supp}\varphi + \operatorname{supp}\psi \subset (-\infty, \tau_k)$, we have

$$\begin{aligned}
&\mathcal{G}(\varphi)\mathcal{G}(\psi)x\\
&\quad= \int_0^\infty \varphi^{(p_k)}(t) \int_0^\infty \psi^{(p_k)}(s) W_{p_k}(t) W_{p_k}(s)x\,ds\,dt\\
&\quad= \int_0^\infty \varphi^{(p_k)}(t) \int_0^\infty \psi^{(p_k)}(s) \Big[\Big(\int_t^{t+s} - \int_0^s\Big) \frac{(t+s-r)^{p_k-1}}{(p_k-1)!} W_{p_k}(r)Cx\,dr\Big]\,ds\,dt\\
&\quad= -\int_0^\infty \varphi^{(p_k)}(t)\\
&\qquad\cdot \int_0^\infty \psi^{(p_k-1)}(s) \frac{d}{ds}\Big[\Big(\int_t^{t+s} - \int_0^s\Big) \frac{(t+s-r)^{p_k-1}}{(p_k-1)!} W_{p_k}(r)Cx\,dr\Big]\,ds\,dt\\
&\quad= -\int_0^\infty \varphi^{(p_k)}(t) \int_0^\infty \psi^{(p_k-1)}(s)\\
&\qquad\cdot \Big[\Big(\int_t^{t+s} - \int_0^s\Big) \frac{(t+s-r)^{p_k-2}}{(p_k-2)!} W_{p_k}(r)Cxdr - \frac{t^{p_k-1}}{(p_k-1)!} W_{p_k}(s)Cx\Big]\,ds\,dt\\
&\quad= -\int_0^\infty \varphi^{(p_k)}(t) \int_0^\infty \psi^{(p_k-1)}(s) \Big[\Big(\int_t^{t+s} - \int_0^s\Big) \frac{(t+s-r)^{p_k-2}}{(p_k-2)!} W_{p_k}(r)Cx\,dr\Big]\,ds\,dt\\
&\qquad+ (-1)^{p_k}\varphi(0) \int_0^\infty \psi^{(p_k-1)}(s) W_{p_k}(s)Cx\,ds.
\end{aligned}$$

Applying the same argument sufficiently many times, we obtain

$$\begin{aligned}
\mathcal{G}(\varphi)\mathcal{G}(\psi)x = (-1)^{p_k}\Big(&\int_0^\infty \varphi^{(p_k)}(t) \int_0^\infty \psi(s) W_{p_k}(t+s)Cx\,ds\,dt\\
&+ \sum_{j=0}^{p_k-1} \varphi^{(j)}(0) \int_0^\infty \psi^{(p_k-1-j)}(s) W_{p_k}(s)Cx\,ds\Big)
\end{aligned}$$

$$= (-1)^{p_k}\Big(\int_0^\infty \varphi^{(p_k)}(t)\int_t^\infty \psi(t-s)W_{p_k}(s)Cx\,ds\,dt + \sum_{j=0}^{p_k-1}\varphi^{(j)}(0)\int_0^\infty \psi^{(p_k-1-j)}(s)W_{p_k}(s)Cx\,ds\Big)$$

$$= (-1)^{p_k}\int_0^\infty \Big[(\varphi^{(p_k)} *_0 \psi)(s) + \sum_{j=0}^{p_k-1}\varphi^{(j)}(0)\psi^{(p_k-1-j)}(s)\Big]W_{p_k}(s)Cx\,ds$$

$$= (-1)^{p_k}\int_0^\infty (\varphi *_0 \psi)^{(p_k)}(s)W_{p_k}(s)Cx\,ds = \mathcal{G}(\varphi *_0 \psi)Cx,\ x \in E.$$

So (C-D.S.1) holds. Suppose $x \in E$ satisfies $\mathcal{G}(\varphi)x = 0$ for all $\varphi \in \mathcal{D}_{[0,\tau_k]}$, for some $k \in \mathbb{N}$. Then

$$W_{p_k}(t)x = \sum_{j=0}^{p_k-1} t^j z_j, \qquad t \in [0,\tau_k),$$

for some $z_j \in E$, $j = 0,1,\ldots,p_k-1$. Using the closedness of A and the relation

$$A\int_0^t W_{p_k}(s)x\,ds = W_{p_k}(t)x - \frac{t^{p_k}}{(p_k)!}\,Cx, \qquad t \in [0,\tau_k),$$

we easily get $z_j = 0$ for $j = 0,1,\ldots,p_k-1$. Hence, $x = 0$ and (C-D.S.2) holds.

Let us show that $C^{-1}AC$ is the generator of $\mathcal{G}$. Suppose $(x,y) \in C^{-1}AC$ and $\varphi \in \mathcal{D}_{[0,\tau_k)}$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned}\mathcal{G}(-\varphi')x &= (-1)^{p_k+1}\int_0^\infty \varphi^{(p_k+1)}(t)W_{p_k}(t)x\,dt\\ &= (-1)^{p_k+1}\int_0^\infty \varphi^{(p_k+1)}(t)\Big[\frac{t^{p_k}}{p_k!}\,Cx + \int_0^t W_{p_k}(s)y\,ds\Big]\,dt\\ &= (-1)^{p_k+1}\int_0^\infty \varphi^{(p_k+1)}(t)\int_0^t W_{p_k}(s)y\,ds\,dt = \mathcal{G}(\varphi)y,\end{aligned}$$

and $C^{-1}AC \subset B$, where B is the generator of $\mathcal{G}$. Conversely, assume $(x,y) \in B$. Then

$$\begin{aligned}(-1)^{p_k+1}\int_0^\infty \varphi^{(p_k+1)}(t)W_{p_k}(t)x\,dt &= (-1)^{p_k}\int_0^\infty \varphi^{(p_k)}(t)W_{p_k}(t)y\,dt\\ &= (-1)^{p_k+1}\int_0^\infty \varphi^{(p_k+1)}(t)\int_0^t W_{p_k}(s)y\,ds\,dt, \qquad \varphi \in \mathcal{D}_{[0,\tau_k)}.\end{aligned}$$

Consequently,

$$W_{p_k}(t)x - \int_0^t W_{p_k}(s)y\,ds = \sum_{j=0}^{p_k} t^j z_j, \qquad t \in [0, \tau_k), \tag{3}$$

for some $z_j \in E$, $j = 0, 1, \ldots, p_k$. We can take $t = 0$ to obtain $z_0 = 0$. Using (3) we have $\frac{d}{dt} W_{p_k}(t)x - W_{p_k}(t)y = \sum_{j=1}^{p_k} j t^{j-1} z_j$, and

$$AW_{p_k}(t)x + \frac{t^{p_k-1}}{(p_k-1)!}\,Cx - A\int_0^t W_{p_k}(s)y\,ds - \frac{t^{p_k}}{p_k!}\,Cy = \sum_{j=1}^{p_k} j t^{j-1} z_j, \qquad t \in [0, \tau_k).$$

Then we obtain

$$A\sum_{j=1}^{p_k} t^j z_j = \sum_{j=1}^{p_k} j t^{j-1} z_j - \frac{t^{p_k-1}}{(p_k-1)!}\,Cx + \frac{t^{p_k}}{p_k!}\,Cy, \qquad t \in [0, \tau_k). \tag{4}$$

Since A is closed, we can differentiate both sides of (4) sufficiently many times to get $z_j = 0$, $j = 1, \ldots, p_k - 1$, and $z_{p_k} = Cx/p_k!$. This implies

$$W_{p_k}(t)x - \int_0^t W_{p_k}(s)y\,ds = \frac{t^{p_k}}{p_k!}\,Cx, \qquad t \in [0, \tau_k),$$

and hence $(x, y) \in C^{-1}AC$. Now the proof is complete. ∎

Remark 4.5. If $C = I$, then the wellposedness of $C_{k+1}(\tau)$ for some $k \in \mathbb{N}$ and $\tau > 0$ implies that A generates a (DS) (see [11] and [17]). This fact follows directly from Theorem 4.4 and an additional observation that the wellposedness of $C_{k+1}(\tau)$ implies the wellposedness of $C_{2k+1}(2\tau)$ (cf. [3, Theorem 4.1]). Due to [13, Theorem 4.1], the C-wellposedness of $C_{k+1}(\tau)$ implies the C^2-wellposedness of $C_{2k+1}(2\tau)$.

Observe that there are typographical errors in the proof of [13, Theorem 4.1]. Let $\tau_0 \in (0, \tau)$. The main formula (see [13]) should read

$$T_{2k}(t) = S_k(\tau_0)S_k(t - \tau_0) + \sum_{m=0}^{k-1} \frac{1}{m!}(\tau_0^m T_{2k-m}(t - \tau_0) + (t - \tau_0)^m T_{2k-m}(\tau_0)),$$

and in the other formula, "$0 \leq m \leq k$" should read "$0 \leq m < k$". Let us explain this in more detail. Put $T_k(t) = S_k(t)C$ for $t \in [0, \tau_0]$. Note that if $x \in E$ and $t \in [0, \tau_0]$, then

$$A\int_0^t T_i(s)x\,ds = T_i(t)x - \frac{t^i}{i!}\,C^2x, \qquad i = k, \ldots, 2k,$$

$$AT_{2k-m}(t)x = T_{2k-m-1}(t)x - \frac{t^{2k-m-1}}{(2k-m-1)!}\,C^2x, \qquad m = 0, \ldots, k-1,$$

and $A\int_0^t S_k(s)x\,ds = S_k(t)x-(t^k/k!)Cx$. Suppose now $t\in[\tau_0,2\tau_0]$. Proceeding as in the proof of [3, Theorem 4.1], one can see that

$$A\int_0^t T_{2k}(s)x = T_{2k}(t)x - \frac{1}{(2k)!}\,(\tau_0+(t-\tau_0))^{2k}C^2x.$$

The rest of the proof of [13, Theorem 4.1] is clear.

By the foregoing we have the next result.

PROPOSITION 4.6.

(i) *Let $\mathcal{G}$ be a $(C\text{-DS})$. Then there exist $\tau>0$, $n\in\mathbb{N}$ and a local n-times integrated C-semigroup $(W_n(t))_{t\in[0,\tau)}$ such that*

$$\mathcal{G}(\varphi)=(-1)^n\int_0^\infty \varphi^{(n)}(t)W_n(t)\,dt,\qquad \varphi\in\mathcal{D}_{(-\infty,\tau)}.$$

(ii) *Let $n\in\mathbb{N}_0$ and let $(W_n(t))_{t\ge0}$ be an n-times integrated C-semigroup generated by A. Put $\mathcal{G}(\varphi):=(-1)^n\int_0^\infty\varphi^{(n)}(t)W_n(t)\,dt$ for $\varphi\in\mathcal{D}$. Then $\mathcal{G}$ is a $(C\text{-DS})$ generated by A.*

The following lemma can be proved as in the case of integrated semigroups.

LEMMA 4.7. *Let $(S(t))_{t\in[0,\tau)}$ be an n-times integrated C-semigroup generated by A, $0<\tau\le\infty$, $n\in\mathbb{N}$. If $x\in D(A^k)$ for some $k\in\mathbb{N}$ with $k\le n$, then*

$$\frac{d^k}{dt^k}\,S(t)x = S(t)A^kx+\sum_{i=0}^{k-1}\frac{t^{n-i-1}}{(n-i-1)!}\,CA^{k-i-1}x,\qquad t\in[0,\tau).$$

Recall that an n-times integrated C-semigroup $(W_n(t))_{t\ge0}$ is *exponentially bounded* if there exist $M>0$ and $\omega\in\mathbb{R}$ such that $\|W_n(t)\|\le Me^{\omega t}$ for all $t\ge0$. Next, if $\mathcal{G}\in\mathcal{D}'(E)$ and $\varepsilon\in\mathbb{R}$, define $e^{-\varepsilon t}\mathcal{G}$ by $e^{-\varepsilon t}\mathcal{G}(\varphi):=\mathcal{G}(e^{-\varepsilon\cdot}\varphi)$ for $\varphi\in\mathcal{D}$. Clearly, $e^{-\varepsilon t}\mathcal{G}\in\mathcal{D}'(E)$.

DEFINITION 4.8. A $(C\text{-DS})$ $\mathcal{G}$ is said to be an *exponential C-distribution semigroup* if there exists $\varepsilon\in\mathbb{R}$ such that $e^{-\varepsilon t}\mathcal{G}\in\mathcal{S}'(L(E))$.

In what follows, if $\mathcal{G}\in\mathcal{D}'(E)$ and $\varphi\in\mathcal{D}$, then we also write $\langle\mathcal{G},\varphi\rangle$ for $\mathcal{G}(\varphi)$. Now we state the relation between exponential C-distribution semigroups and exponentially bounded integrated C-semigroups.

THEOREM 4.9. *Let A be a closed linear operator. Then the following conditions are equivalent*:

(a) *A is the generator of an exponential C-distribution semigroup $\mathcal{G}$.*
(b) *There exists $n\in\mathbb{N}$ such that A is the generator of an exponentially bounded n-times integrated C-semigroup $(W_n(t))_{t\ge0}$.*

Proof. (b)⇒(a). Let A be the generator of a semigroup $(W_n(t))_{t\geq 0}$ satisfying $\|W_n(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Put $\mathcal{G}(\varphi) := (-1)^n \int_0^\infty \varphi^{(n)}(t)W_n(t)\,dt$ for $\varphi \in \mathcal{D}$. By Proposition 4.6, $\mathcal{G}$ is a (C-DS) generated by A. For any $\varepsilon > 0$ and $\varphi \in \mathcal{D}$ we have

$$\begin{aligned}\|\langle e^{-(\omega+\varepsilon)t}\mathcal{G},\varphi\rangle\| &\leq M \int_0^\infty e^{\omega t}|(e^{-(\omega+\varepsilon)\cdot}\varphi)^{(n)}(t)|\,dt\\ &\leq M\,2^n \int_0^\infty e^{\omega t}\sum_{i=0}^n |\omega+\varepsilon|^{n-i}e^{-(\omega+\varepsilon)t}|\varphi^{(i)}(t)|\,dt\\ &\leq M_1\int_0^\infty e^{-\varepsilon t}\sum_{i=0}^n |\varphi^{(i)}(t)|\,dt \leq \frac{M_1}{\varepsilon}\sum_{i=0}^n p_{0,i}(\varphi),\end{aligned}$$

for a suitable constant M_1 independent of φ, where $p_{0,i}(\psi) = \sup_{x\in\mathbb{R}}|\psi^{(i)}(x)|$ for $\psi \in \mathcal{S}$ is a continuous seminorm on $\mathcal{S}$. This implies $e^{-(\omega+\varepsilon)t}\mathcal{G} \in \mathcal{S}'(L(E))$ if $\varepsilon > 0$.

(a)⇒(b). Suppose that $\mathcal{G}$ is the C-distribution semigroup generated by A and $e^{-\varepsilon t}\mathcal{G} \in \mathcal{S}'(L(E))$. Clearly, $e^{-\varepsilon t}\mathcal{G}$ is a (C-DS) generated by $A - \varepsilon I$ and Lemma 4.1 implies $C^{-1}(A-\varepsilon I)C = A - \varepsilon I$. For every $\varphi \in \mathcal{D}$ one has

$$A\langle e^{-\varepsilon t}\mathcal{G},\varphi\rangle x = \langle e^{-\varepsilon t}\mathcal{G}, -\varphi'\rangle x + \varepsilon\langle e^{-\varepsilon t}\mathcal{G},\varphi\rangle x - \varphi(0)Cx,$$

which shows that $e^{-\varepsilon t}\mathcal{G} \in \mathcal{S}'(L(E,[D(A)]))$. Now [15, Theorem 2.1.2] implies that there exist $n \in \mathbb{N}$, $r > 0$ and a continuous function $V : \mathbb{R} \to L(E,[D(A)])$ supported by $[0,\infty)$ such that

$$\langle e^{-\varepsilon t}\mathcal{G},\varphi\rangle x = (-1)^n \int_0^\infty V(t)\varphi^{(n)}(t)x\,dt$$

for all $\varphi \in \mathcal{D}$ and $x \in E$, and $|V(t)| \leq Mt^r$ for all $t \geq 0$. Since $e^{-\varepsilon t}\mathcal{G}$ is a C-distribution semigroup generated by $A - \varepsilon I$, arguing as in the proofs of statements (i) and (ii) of Theorem 4.2, one can conclude that

$$(A-\varepsilon I)\int_0^t V(s)x\,ds = V(t)x - \frac{t^n}{n!}Cx, \qquad t \geq 0,\ x \in E,$$

and $V(t)(A-\varepsilon I) \subset (A-\varepsilon I)V(t)$ and $CV(t) = V(t)C$ for all $t \geq 0$. The same arguments as in the proofs of [13, Proposition 2.4, Theorem 2.5], shows that $(V(t))_{t\geq 0}$ is an exponentially bounded, n-times integrated C-semigroup generated by $C^{-1}(A-\varepsilon I)C = A - \varepsilon I$. Define

$$W_n(t) := e^{\varepsilon t}V(t) + \int_0^t e^{\varepsilon s}p_n(t-s)V(s)\,ds, \qquad t \geq 0,$$

where p_n is the polynomial of degree $n-1$ such that

$$\sum_{i=1}^{n}\binom{n}{i}(-\varepsilon)^i\lambda^{-i}=\int_0^\infty e^{-\lambda t}p_n(t)\,dt,\qquad \lambda>0.$$

A standard perturbation argument (as in [3, Lemma 3.2]) shows that $(W_n(t))_{t\ge 0}$ is an exponentially bounded, n-times integrated C-semigroup generated by A. ∎

Remark 4.10. It is well known that if A is the (integral) generator of a (local) n-times integrated C-semigroup $(T_n(t))_{t\in[0,\tau)}$, $n\in\mathbb{N}_0$, then $C^{-1}AC=A$ (cf. [12]). Note also that we do not require $\overline{D(A)}=E$ in the previous theorem.

Following [19, Definition 1.2, Theorem 1.5], a strongly continuous operator family $(C_n(t))_{t\ge 0}$ is an *n-times integrated C-cosine function* with subgenerator A if and only if, for every $x\in E$ and $t\ge 0$, $C_n(t)A\subset AC_n(t)$,

$$\int_0^t (t-s)C_n(s)x\,ds\in D(A)\quad\text{and}\quad A\int_0^t (t-s)C_n(s)x\,ds=C_n(t)x-\frac{t^n}{n!}Cx.$$

Proposition 4.11. *Let A be a subgenerator of an n-times integrated C-cosine function $(C_n(t))_{t\ge 0}$, $n\in\mathbb{N}_0$. Then the operator $\mathcal{C}^{-1}\mathcal{A}\mathcal{C}$ generates a $\mathcal{C}$-distribution semigroup in E^2, where*

$$\mathcal{A}:=\begin{pmatrix}0 & I\\ A & 0\end{pmatrix}\quad\textit{and}\quad \mathcal{C}:=\begin{pmatrix}C & 0\\ 0 & C\end{pmatrix}.$$

Proof. First, recall that $C_n(t)C=CC_n(t)$ for all $t\ge 0$, and $CA\subset AC$ (cf. [19]). Thus, $\mathcal{C}\mathcal{A}\subset\mathcal{A}\mathcal{C}$, i.e., $\mathcal{C}^{-1}\mathcal{A}\mathcal{C}\supset\mathcal{A}$. Define

$$W_{n+1}(t):=\begin{pmatrix}\int_0^t C_n(s)\,ds & \int_0^t (t-s)C_n(s)\,ds\\ C_n(t)-\frac{t^n}{n!}C & \int_0^t C_n(s)\,ds\end{pmatrix},\qquad 0\le t<\infty.$$

It is easy to prove that $(W_{n+1}(t))_{t\ge 0}$ is a strongly continuous operator family satisfying $W_{n+1}(t)\mathcal{C}=\mathcal{C}W_{n+1}(t)$, $0\le t<\infty$. Let $x,y\in E$. Then

$$\begin{aligned}\mathcal{A}\int_0^t W_{n+1}(s)\binom{x}{y}ds&=\mathcal{A}\begin{pmatrix}\int_0^t (t-s)C_n(s)x\,ds+\int_0^t \frac{(t-s)^2}{2}C_n(s)y\,ds\\ \int_0^t C_n(s)x\,ds-\frac{t^{n+1}}{(n+1)!}Cx+\int_0^t (t-s)C_n(s)y\,ds\end{pmatrix}\\
&=\begin{pmatrix}\int_0^t C_n(s)x\,ds-\frac{t^{n+1}}{(n+1)!}Cx+\int_0^t (t-s)C_n(s)y\,ds\\ A\big[\int_0^t (t-s)C_n(s)xds+\int_0^t \frac{(t-s)^2}{2}C_n(s)y\,ds\big]\end{pmatrix}\\
&=\begin{pmatrix}\int_0^t C_n(s)x\,ds-\frac{t^{n+1}}{(n+1)!}Cx+\int_0^t (t-s)C_n(s)y\,ds\\ C_n(t)x-\frac{t^n}{n!}Cx+\int_0^t C_n(s)y\,ds-\frac{t^{n+1}}{(n+1)!}Cy\end{pmatrix}\\
&=W_{n+1}(t)\binom{x}{y}-\frac{t^{n+1}}{(n+1)!}\mathcal{C}\binom{x}{y},\qquad t\ge 0.\end{aligned}$$

Suppose $(x\ y)^T \in D(\mathcal{A})$. Then

$$W_{n+1}(t)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \int_0^t C_n(s)x\,ds + \int_0^t (t-s)C_n(s)y\,ds \\ C_n(t)x - \frac{t^n}{n!}Cx + \int_0^t C_n(s)y\,ds \end{pmatrix}$$

and $x \in D(A)$. Therefore, $C_n(t)x \in D(A)$ and $W_{n+1}(t)\binom{x}{y} \in D(\mathcal{A})$ for all $t \geq 0$, and

$$\begin{aligned} \mathcal{A}W_{n+1}(t)\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} C_n(t)x - \frac{t^n}{n!}Cx + \int_0^t C_n(s)y\,ds \\ \int_0^t C_n(s)Ax\,ds + A\int_0^t (t-s)C_n(s)y\,ds. \end{pmatrix} \\ &= \begin{pmatrix} C_n(t)x - \frac{t^n}{n!}Cx + \int_0^t C_n(s)y\,ds \\ \int_0^t C_n(s)Ax\,ds + C_n(t)y - \frac{t^n}{n!}Cy \end{pmatrix} \\ &= W_{n+1}(t)\begin{pmatrix} y \\ Ax \end{pmatrix} = W_{n+1}(t)\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

So $W_{n+1}(t)\mathcal{A} \subset \mathcal{A}W_{n+1}(t)$ for all $t \geq 0$. By [13, Proposition 2.4, Theorem 2.5], $(W_{n+1}(t))_{t\geq 0}$ is an $(n+1)$-times integrated $\mathcal{C}$-semigroup on E^2 generated by $\mathcal{C}^{-1}\mathcal{A}\mathcal{C}$. Proposition 4.6 ends the proof. ∎

Remark 4.12. Let A be a closed operator and let $\mathcal{A}$ and $\mathcal{C}$ be defined as above. Then $CA \subset AC \Leftrightarrow \mathcal{C}\mathcal{A} \subset \mathcal{A}\mathcal{C}$ and $C^{-1}AC = A \Leftrightarrow \mathcal{C}^{-1}\mathcal{A}\mathcal{C} = \mathcal{A}$.

So the examples in [19] can serve as examples of nonexponential C-distribution semigroups.

Problem. Does any generator A of a *local* integrated C-semigroup generate a ($\widetilde{C}$-DS) for some $\widetilde{C}$ which may be different from C?

5. Dense C-distribution semigroups. In this section we will consider some new conditions for $\mathcal{G} \in \mathcal{D}_0'(L(E))$:

(d_1) $\mathcal{G}(\varphi * \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi)$ for all $\varphi, \psi \in \mathcal{D}_0$,
(d_2) the same as (C-D.S.2),
(d_3) $\mathcal{R}(\mathcal{G})$ is dense in E,
(d_4) for every $x \in \mathcal{R}(\mathcal{G})$, there exists a function $u_x \in C([0,\infty); E)$ so that $u_x(0) = Cx$ and $\mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)u_x(t)\,dt$, $\varphi \in \mathcal{D}$,
(d_5) (d_2) holds and $G(\varphi_+)C = \mathcal{G}(\varphi)$ for all $\varphi \in \mathcal{D}$.

One can prove the following assertions:

1. Let $\mathcal{G} \in \mathcal{D}_0'(L(E))$ and $\mathcal{G}C = C\mathcal{G}$. Then $\mathcal{G}$ is a (C-DS) if and only if (d_1), (d_2) and (d_5) hold.
2. Let $\mathcal{G} \in \mathcal{D}_0'(L(E))$ satisfy (d_1)–(d_4). Then $\mathcal{G}$ is a (C-DS).

Remark 5.1. Even if $C = I$, (d_1), (d_2) and (d_4) do not imply (C-D.S.1) (see [11]).

PROPOSITION 5.2. *Let $\mathcal{G}$ be a (C-DS) generated by A. Then there exists $\varepsilon > 0$ such that for every $x \in D_\infty(A)$ there exists a function u_x satisfying*

$$\begin{cases} u_x \in C^\infty([0,\varepsilon];E), \\ \mathcal{G}(\varphi)x = \int_0^\infty \varphi(t)u_x(t)\,dt, \quad \varphi \in \mathcal{D}_{[0,\varepsilon]}, \\ u_x(0) = Cx. \end{cases}$$

Proof. Let $\mathcal{G}(\varphi) = (-1)^n \int_0^{s_0} \varphi^{(n)}(t)W_n(t)\,dt$ on $(-\infty, s_0)$, for some n-times integrated C-semigroup $(W_n(t))_{t\in[0,s_0)}$ generated by A, $n \in \mathbb{N}$. Define $\varepsilon := s_0/2$ and

$$u_x(t) := \frac{d^n}{dt^n}W_n(t)x, \quad 0 \le t \le \varepsilon.$$

More precisely, by Lemma 4.7 we have

$$u_x(t) = W_n(t)A^n x + \sum_{i=0}^{n-1} \frac{t^{n-i-1}}{(n-i-1)!} CA^{n-i-1}x, \quad 0 \le t \le \varepsilon.$$

Since $x \in D_\infty(A)$, one obtains $u_x \in C^\infty([0,\varepsilon],E)$. Moreover, $u_x(0) = Cx$ and the result follows by integration by parts. ∎

REMARK 5.3. Let $\mathcal{G}$ and A be as above. Then $CD_\infty(A) \subset \overline{\mathcal{R}(\mathcal{G})}$. This follows from Proposition 5.2 using the regularizing sequence (ϱ_n). Assume now that $\varrho(A) \neq \emptyset$, and $D(A)$ and $\operatorname{Im} C$ are dense in E. Then $\overline{D_\infty(A)} = E$ and consequently $\overline{CD_\infty(A)} = E$, which implies that $\mathcal{G}$ is dense. By similar methods to those in [11], this equivalent to the statement: $\mathcal{G}^*$ is a (C^*-DS) in E^*. Now we collect these observations in the following proposition.

PROPOSITION 5.4. *Let $\mathcal{G}$ be a (C-DS) generated by A. Then $CD_\infty(A) \subset \overline{\mathcal{R}(\mathcal{G})}$. Assume additionally $\varrho(A) \neq \emptyset$ and $\overline{\operatorname{Im} C} = E$. Then the following statements are equivalent*:

(a) *$\mathcal{G}$ is dense*,
(b) *A is densely defined*,
(c) *$\mathcal{G}^*$ is a (C^*-DS) in E^*.*

We now prove the following statement; the corresponding result for distribution semigroups is proved in [11, Remark 3.13].

PROPOSITION 5.5. *Let $\mathcal{G}$ be a (C-DS). Then $\mathcal{G}$ satisfies (d_4).*

Proof. Let $x = \mathcal{G}(\psi)y$ for some $\psi \in \mathcal{D}_0$ and $y \in E$. Then the continuity of $\mathcal{G}$ on $\mathcal{D}$ implies

$$\mathcal{G}(\varphi)x = \mathcal{G}(\varphi)\mathcal{G}(\psi)y = \mathcal{G}(\varphi *_0 \psi)Cy = \mathcal{G}\Big(\int_0^\infty \varphi(t)\tau_t\psi\,dt\Big)Cy$$
$$= \int_0^\infty \varphi(t)\mathcal{G}(\tau_t\psi)Cy\,dt, \quad \varphi \in \mathcal{D}.$$

The function $u_x : t \mapsto \mathcal{G}(\tau_t\psi)Cy$ has the desired properties. ∎

Finally, examples of global integrated C-semigroups can be used for the construction of C-distribution semigroups by virtue of Proposition 4.6. The analysis in Example 5.3(b) of [18] (cf. also [8]) gives an example of an exponential $(I + \Delta)^{-r}$-distribution semigroup (for some $r \geq 0$) on $L^p(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$.

It is proved in [21] that if a bounded analytic semigroup $(T(z))_{\mathrm{Re}z>0}$ on $L^2(\Omega)$ ($\Omega \subset \mathbb{R}^n$ is an open nonempty set) satisfies a Gaussian estimate of order m and A_p is the generator of its consistent semigroup on $L^p(\Omega)$ ($1 \leq p < \infty$), then iA_p generates an $(I - A_p)^{-\alpha}$-regularized semigroup $(S_p(t))_{t\geq 0}$ on $L^p(\Omega)$ satisfying $\|S_p(t)\| \leq M(1 + t^{2n|1/2-1/p|})$ for all $t \geq 0$, where $\alpha > 2n|1/2 - 1/p|$. Thus, iA_p generates an exponential $(I - A_p)^{-\alpha}$-distribution semigroup.

The interested reader may consult the monograph [8] for the theory of regularized semigroups and integrated semigroups and their applications. It is clear that regularized semigroups can serve as examples of C-distribution semigroups. In this sense, the analysis of the backwards heat equation on $L^p(\Omega)$, where Ω is an open nonempty bounded set in $\mathbb{R}^n$ with smooth boundary ([8]), as well as the analysis of the Petrovskiĭ and Shilov correct abstract parabolic systems of differential equations in a Banach space (cf. [8] and [22]) can be used for the construction of C-distribution semigroups.

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