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On multipliers of Hilbert modules over pro-C^*-algebras

by

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Abstract. We investigate the structure of the multiplier module of a Hilbert module over a pro-C^*-algebra and the relationship between the set of all adjointable operators from a Hilbert A-module E to a Hilbert A-module F and the set of all adjointable operators from the multiplier module $M(E)$ to $M(F)$.

1. Introduction. The notion of Hilbert C^*-module is a generalization of the notion of Hilbert space by allowing the inner product to take values in a C^*-algebra. Hilbert modules over commutative C^*-algebras were used by I. Kaplansky [8] to show that derivations of type I AW^*-algebras are inner. The research on Hilbert modules over arbitrary C^*-algebras began in the 70's in [10, 14]. Hilbert C^*-modules are useful tools in the theory of operator algebras, operator K-theory, KK-theory of C^*-algebras, group representation theory, the C^*-algebraic theory of quantum groups and the theory of operator spaces. In applications, one often assumes that Hilbert modules are over C^*-algebras with countable approximate unit, because for a given C^*-algebra A, the Hilbert C^*-modules A and H_A (the Hilbert C^*-module of all sequences $(a_n)_n$ in A such that $\sum_n a_n^* a_n$ converges in the C^*-algebra A) are countably generated if and only if A has a countable approximate unit. In [13], I. Raeburn and S. J. Thompson considered a more general notion of countably generated module in which the generators are multipliers of the module. With their definition, A and H_A are countably generated.

In this paper, we investigate the multipliers of Hilbert modules over pro-C^*-algebras. Pro-C^*-algebras are generalizations of C^*-algebras. Instead of being given by a single C^*-norm, the topology on a pro-C^*-algebra is defined by a directed family of C^*-seminorms. Clearly, any C^*-algebra is a pro-C^*-algebra. The set $C_{cc}([0,1])$ of all complex-valued continuous functions on

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$[0, 1]$ with the topology of uniform convergence on countable compact subsets of $[0, 1]$ is a pro-C^*-algebra which is not topologically isomorphic to any C^*-algebra [3]. In [11, §1] other nice examples of pro-C^*-algebras are presented. Besides their intrinsic interest as topological algebras, pro-C^*-algebras provide an important tool in investigation of certain aspects of C^*-algebras (like multipliers of the Pedersen ideal, the tangent algebra of a C^*-algebra, crossed product and K-theory, as well as non-commutative algebraic topology) and quantum field theory. In the literature, pro-C^*-algebras have been given different names, such as b^*-algebras (C. Apostol), LMC^*-algebras (G. Lassner, K. Schmüdgen) or locally C^*-algebras (A. Inoue, M. Fragoulopoulou).

Let A be a pro-C^*-algebra and let E be a Hilbert A-module. A *multiplier* of E is an adjointable operator from A to E. The set $M(E)$ of all multipliers of E is a Hilbert module over the multiplier algebra $M(A)$ of A in a natural way. We show that $M(E)$ is an inverse limit of multiplier modules of Hilbert C^*-modules and E can be identified with a closed submodule of $M(E)$ which is strictly dense in $M(E)$ (Theorem 3.3). For a countable family $\{E_n\}_n$ of Hilbert A-modules, the multiplier module $M(\bigoplus_n E_n)$ can be identified with the set of all sequences $(t_n)_n$ with $t_n \in M(E_n)$ such that $\sum_n t_n^* \circ t_n$ converges strictly in $M(A)$ (Theorem 3.5). This is a generalization of a result of Bakic and Guljas [2] which sates that $M(H_A)$ is the set of all sequences $(m_n)_n$ in $M(A)$ such that the series $\sum_n m_n^* m_n a$ and $\sum_n a m_n^* m_n$ converge in A for all a in A.

Section 4 is devoted to the study of the connection between the set of all adjointable operators between two Hilbert A-modules E and F and the set of all adjointable operators between the respective multiplier modules $M(E)$ and $M(F)$. We show that any adjointable operator from $M(E)$ to $M(F)$ is strictly continuous (see Definition 3.2) and the locally convex space $L_A(E, F)$ of all adjointable operators from E to F is isomorphic to the locally convex space $L_{M(A)}(M(E), M(F))$ of all adjointable operators from $M(E)$ to $M(F)$ (Theorem 4.1). In particular the pro-C^*-algebras $L_A(E)$ and $L_{M(A)}(M(E))$ are isomorphic. The last result is a generalization of a result of Bakic and Guljas [2] which states that the C^*-algebra of all adjointable operators on a full Hilbert C^*-module is isomorphic to the C^*-algebra of all adjointable operators on the multiplier module. Also we show that E and F are unitarily equivalent if and only if $M(E)$ and $M(F)$ are unitarily equivalent (Corollary 4.2).

2. Preliminaries. A *pro-C^*-algebra* is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_i\}_{i\in I}$ converges to 0 in A if and only if the net $\{p(a_i)\}_i$ converges to 0 for all continuous C^*-seminorms p on A. From now on, we denote the set of all such seminorms by $S(A)$.

Here, we recall some facts about pro-C^*-algebras from [3, 4, 7, 11, 12]. Let A be a pro-C^*-algebra.

A *multiplier* on A is a pair (l, r) of linear maps from A to A such that $l(ab) = l(a)b$, $r(ab) = ar(b)$ and $al(b) = r(a)b$ for all $a, b \in A$. The set $M(A)$ of all multipliers of A is a pro-C^*-algebra with respect to the topology determined by the family of C^*-seminorms $\{p_{M(A)}\}_{p\in S(A)}$, where $p_{M(A)}(l, r) = \sup\{p(l(a)); \, p(a) \le 1\}$.

An *approximate unit* for A is an increasing net $\{e_i\}_{i\in I}$ of positive elements in A such that $p(e_i) \le 1$ for all $p \in S(A)$ and $i \in I$, and $p(ae_i - a) \to 0$ and $p(e_i a - a) \to 0$ for all $p \in S(A)$ and $a \in A$. Any pro-C^*-algebra has an approximate unit.

An element $a \in A$ is *bounded* if $\|a\|_\infty = \sup\{p(a); \, p \in S(A)\} < \infty$. The set $b(A)$ of all bounded elements in A is dense in A and it is a C^*-algebra in the C^*-norm $\|\cdot\|_\infty$.

By a morphism of pro-C^*-algebras we always mean a continuous morphism. Two pro-C^*-algebras A and B are isomorphic if there is a bijective map $\Phi : A \to B$ such that Φ and Φ^{-1} are morphisms of pro-C^*-algebras.

The set $S(A)$ of all continuous C^*-seminorms on A is directed by the order $p \ge q$ if $p(a) \ge q(a)$ for all $a \in A$. For each $p \in S(A)$, $\ker p = \{a \in A; \, p(a) = 0\}$ is a two-sided $*$-ideal of A and the quotient algebra $A/\ker p$, denoted by A_p, is a C^*-algebra in the C^*-norm induced by p (see, for example, [1]). The canonical map from A to A_p is denoted by π_p. For $p, q \in S(A)$ with $p \ge q$ there is a canonical surjective morphism of C^*-algebras $\pi_{pq} : A_p \to A_q$ such that $\pi_{pq}(\pi_p(a)) = \pi_q(a)$ for all $a \in A$, which extends to a morphism of C^*-algebras $\pi''_{pq} : M(A_p) \to M(A_q)$. Then $\{A_p; \, \pi_{pq}\}_{p,q\in S(A), p\ge q}$ and $\{M(A_p); \, \pi''_{pq}\}_{p,q\in S(A), p\ge q}$ are inverse systems of C^*-algebras, and moreover, the pro-C^*-algebras A and $M(A)$ are isomorphic to $\varprojlim_{p\in S(A)} A_p$ and $\varprojlim_{p\in S(A)} M(A_p)$, respectively.

Hilbert modules over pro-C^*-algebras are generalizations of Hilbert C^*-modules by allowing the inner product to take values in a pro-C^*-algebra rather than in a C^*-algebra. Here, we recall some facts about Hilbert modules over pro-C^*-algebras from [5, 6, 7, 11, 15].

Definition 2.1. A *pre-Hilbert A-module* is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ which is $\mathbb{C}$- and A-linear in its second variable and satisfies the following relations:

(i) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;
(ii) $\langle \xi, \xi \rangle \ge 0$ for every $\xi \in E$;
(iii) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

We say that E is a *Hilbert A-module* if E is complete with respect to the topology determined by the family of seminorms $\{\overline{p}_E\}_{p\in S(A)}$, where $\overline{p}_E(\xi) = \sqrt{p(\langle \xi, \xi\rangle)}$, $\xi \in E$.

An element $\xi \in E$ is *bounded* if $\sup\{\overline{p}_E(\xi);\ p \in S(A)\} < \infty$. The set $b(E)$ of all bounded elements in E is a Hilbert $b(A)$-module which is dense in E, where $b(A) = \{a \in A;\ \sup\{p(a);\ p \in S(A)\} < \infty\}$ is the so-called *bounded part* of A and it is a C^*-subalgebra of A (see, for example, [7, 11, 15]).

Any pro-C^*-algebra A is a Hilbert A-module in a natural way.

A Hilbert A-module E is *full* if the linear space $\langle E, E\rangle$ generated by $\{\langle \xi, \eta\rangle;\ \xi, \eta \in E\}$ is dense in A.

Let E be a Hilbert A-module. For $p \in S(A)$, $\ker \overline{p}_E = \{\xi \in E;\ \overline{p}_E(\xi) = 0\}$ is a closed submodule of E and $E_p = E/\ker \overline{p}_E$ is a Hilbert A_p-module with $(\xi + \ker \overline{p}_E)\pi_p(a) = \xi a + \ker \overline{p}_E$ and $\langle \xi + \ker \overline{p}_E, \eta + \ker \overline{p}_E\rangle = \pi_p(\langle \xi, \eta\rangle)$ (see, for example, [7, 11, 15]). The canonical map from E onto E_p is denoted by σ_p^E. For $p, q \in S(A)$ with $p \geq q$ there is a canonical morphism of vector spaces σ_{pq}^E from E_p onto E_q such that $\sigma_{pq}^E(\sigma_p^E(\xi)) = \sigma_q^E(\xi)$ for $\xi \in E$. Then $\{E_p; A_p; \sigma_{pq}^E, \pi_{pq}\}_{p,q\in S(A),\, p\geq q}$ is an inverse system of Hilbert C^*-modules in the following sense: $\sigma_{pq}^E(\xi_p a_p) = \sigma_{pq}^E(\xi_p)\pi_{pq}(a_p)$ for $\xi_p \in E_p$ and $a_p \in A_p$; $\langle \sigma_{pq}^E(\xi_p), \sigma_{pq}^E(\eta_p)\rangle = \pi_{pq}(\langle \xi_p, \eta_p\rangle)$ for $\xi_p, \eta_p \in E_p$; $\sigma_{pp}^E(\xi_p) = \xi_p$ for $\xi_p \in E_p$; and $\sigma_{qr}^E \circ \sigma_{pq}^E = \sigma_{pr}^E$ if $p \geq q \geq r$; moreover, $\varprojlim_{p\in S(A)} E_p$ is a Hilbert A-module which can be identified with E.

We say that an A-module morphism $T : E \to F$ is *adjointable* if there is an A-module morphism $T^* : F \to E$ such that $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$ for every $\xi \in E$ and $\eta \in F$. Any adjointable A-module morphism $T : E \to F$ is continuous (that is, for any $p \in S(A)$, there is $M_p > 0$ such that $\overline{p}_F(T(\xi)) \leq M_p \overline{p}_E(\xi)$ for all $\xi \in E$). The set $L_A(E, F)$ of all adjointable A-module morphisms from E into F is a complete locally convex space with the topology defined by the family of seminorms $\{\widetilde{p}_{L_A(E,F)}\}_{p\in S(A)}$, where $\widetilde{p}_{L_A(E,F)}(T) = \|(\pi_p^{E,F})_*(T)\|_{L_{A_p}(E_p,F_p)}$ for $T \in L_A(E, F)$ and $(\pi_p^{E,F})_*(T)(\sigma_p^E(\xi)) = \sigma_p^F(T\xi)$ for $\xi \in E$. Moreover, $\{L_{A_p}(E_p, F_p); (\pi_{pq}^{E,F})_*\}_{p,q\in S(A),\, p\geq q}$, where $(\pi_{pq}^{E,F})_* : L_{A_p}(E_p, F_p) \to L_{A_q}(E_q, F_q)$,

$$(\pi_{pq}^{E,F})_*(T_p)(\sigma_q^E(\xi)) = \sigma_{pq}^F(T_p(\sigma_p^E(\xi))),$$

is an inverse system of Banach spaces, and $\varprojlim_{p\in S(A)} L_{A_p}(E_p, F_p)$ can be identified with $L_A(E, F)$. Thus topologized, $L_A(E, E)$ becomes a pro-C^*-algebra, and we write $L_A(E)$ for $L_A(E, E)$.

An element T in $L_A(E, F)$ is said to be *bounded* in $L_A(E, F)$ if $\|T\|_\infty = \sup\{\widetilde{p}_{L_A(E,F)}(T);\ p \in S(A)\} < \infty$. The set $b(L_A(E, F))$ of all bounded elements in $L_A(E, F)$ is a Banach space with respect to the norm $\|\cdot\|_\infty$, which is isometrically isomorphic to $L_{b(A)}(b(E), b(F))$.

For $\xi \in E$ and $\eta \in F$ we consider the rank one homomorphism $\theta_{\eta,\xi}$ from E into F defined by $\theta_{\eta,\xi}(\zeta) = \eta\langle\xi,\zeta\rangle$. Clearly, $\theta_{\eta,\xi} \in L_A(E,F)$ and $\theta^*_{\eta,\xi} = \theta_{\xi,\eta}$. The closed linear subspace of $L_A(E,F)$ spanned by $\{\theta_{\eta,\xi};\ \xi \in E, \eta \in F\}$ is denoted by $K_A(E,F)$, and we write $K_A(E)$ for $K_A(E,E)$. Moreover, $K_A(E,F)$ may be identified with $\varprojlim_{p\in S(A)} K_{A_p}(E_p,F_p)$.

We say that the Hilbert A-modules E and F are *unitarily equivalent* if there is a unitary element U in $L_A(E,F)$ (i.e., $U^*U = \mathrm{id}_E$ and $UU^* = \mathrm{id}_F$).

Given a countable family $\{E_n\}_n$ of Hilbert A-modules, the set $\bigoplus_n E_n$ of all sequences $(\xi_n)_n$ with $\xi_n \in E_n$ such that $\sum_n \langle\xi_n,\xi_n\rangle$ converges in A is a Hilbert A-module with the action of A on $\bigoplus_n E_n$ defined by $(\xi_n)_n a = (\xi_n a)_n$ and the inner product defined by $\langle(\xi_n)_n,(\eta_n)_n\rangle = \sum_n\langle\xi_n,\eta_n\rangle$. For each $p \in S(A)$, the Hilbert A_p-modules $\bigoplus_n (E_n)_p$ and $(\bigoplus_n E_n)_p$ are unitarily equivalent and so the Hilbert A-modules $\bigoplus_n E_n$ and $\varprojlim_{p\in S(A)} \bigoplus_n (E_n)_p$ are unitarily equivalent. If $E_n = A$ for any n, the Hilbert A-module $\bigoplus_n A$ is denoted by H_A.

3. Multiplier modules. Let A be a pro-C^*-algebra and E a Hilbert A-module. It is not difficult to check that $L_A(A,E)$ is a Hilbert $L_A(A)$-module with the action of $L_A(A)$ on $L_A(A,E)$ defined by $t \cdot m = t \circ m$ for $t \in L_A(A,E)$ and $m \in L_A(A)$, and with the $L_A(A)$-valued inner product defined by $\langle s,t\rangle_{L_A(A)} = s^* \circ t$. Moreover, since

$$\widetilde{p}_{L_A(A)}(s^* \circ s) = \widetilde{p}_{L_A(A,E)}(s)^2$$

for all $s \in L_A(A,E)$ and $p \in S(A)$, the topology on $L_A(A,E)$ induced by the inner product coincides with the topology determined by the family of seminorms $\{\widetilde{p}_{L_A(A,E)}\}_{p\in S(A)}$. Therefore $L_A(A,E)$ is a Hilbert $L_A(A)$-module, and since $L_A(A)$ can be identified with the multiplier algebra $M(A)$ of A (see, for example, [11]), $L_A(A,E)$ becomes a Hilbert $M(A)$-module.

Definition 3.1. The Hilbert $M(A)$-module $L_A(A,E)$ is called the *multiplier module* of E, and denoted by $M(E)$.

Definition 3.2. The *strict topology* on $M(E)$ is the one generated by the family of seminorms $\{\|\cdot\|_{p,a,\xi}\}_{(p,a,\xi)\in S(A)\times A\times E}$, where $\|t\|_{p,a,\xi} = \overline{p}_E(t(a)) + p(t^*(\xi))$.

Theorem 3.3. *Let A be a pro-C^*-algebra and E a Hilbert A-module.*

(i) $\{M(E_p); M(A_p); (\pi^{A,E}_{pq})_*; \pi''_{pq}\}_{p,q\in S(A),\, p\geq q}$ *is an inverse system of Hilbert C^*-modules.*
(ii) *The Hilbert $M(A)$-modules $M(E)$ and $\varprojlim_{p\in S(A)} M(E_p)$ are unitarily equivalent.*

(iii) *The isomorphism of* (ii) *identifies the strict topology on* E *with the topology on* $\varprojlim_{p\in S(A)} M(E_p)$ *obtained by taking the inverse limit of the strict topologies on the* $M(E_p)$*'s.*

(iv) $M(E)$ *is complete with respect to the strict topology.*

(v) *The map* $i_E : E \to M(E)$ *defined by* $i_E(\xi)(a) = \xi a$, $a \in A$, *embeds* E *as a closed submodule of* $M(E)$*. Moreover, if* $t \in M(E)$ *then* $t \cdot a = i_E(t(a))$ *for all* $a \in A$ *and* $\langle t, i_E(\xi)\rangle_{M(E)} = t^*(\xi)$ *for all* $\xi \in E$*.*

(vi) *The image of* i_E *is dense in* $M(E)$ *with respect to the strict topology.*

Proof. (i) Let $p, q \in S(A)$ with $p \geq q$, $t, t_1, t_2 \in M(E_p)$, $b \in M(A_p)$. Then

$$\begin{aligned}(\pi_{pq}^{A,E})_*(t \cdot b)(\pi_q(a)) &= \sigma_{pq}^E((t \cdot b)(\pi_p(a))) = \sigma_{pq}^E(t(b\pi_p(a)))\\ &= (\pi_{pq}^{A,E})_*(t)(\pi_{pq}(b\pi_p(a)))\\ &= (\pi_{pq}^{A,E})_*(t)(\pi''_{pq}(b)\pi_q(a))\\ &= ((\pi_{pq}^{A,E})_*(t) \cdot \pi''_{pq}(b))(\pi_q(a))\end{aligned}$$

and

$$\begin{aligned}\langle(\pi_{pq}^{A,E})_*(t_1), (\pi_{pq}^{A,E})_*(t_2)\rangle_{M(E_q)}(\pi_q(a)) &= ((\pi_{pq}^{A,E})_*(t_1))^*(\sigma_{pq}^E(t_2(\pi_p(a))))\\ &= (\pi_{pq}^{E,A})_*(t_1^*)(\sigma_{pq}^E(t_2(\pi_p(a))))\\ &= \pi_{pq}((t_1^* \circ t_2)(\pi_p(a)))\\ &= (\pi_{pq}^{A,A})_*(t_1^* \circ t_2)(\pi_q(a))\\ &= (\pi_{pq}^{A,A})_*(\langle t_1, t_2\rangle_{M(E_p)})(\pi_q(a))\end{aligned}$$

for all $a \in A$. From these relations we deduce (i).

(ii) By (i), $\varprojlim_{p\in S(A)} M(E_p)$ is a Hilbert $\varprojlim_{p\in S(A)} M(A_p)$-module, and since $\varprojlim_{p\in S(A)} M(A_p)$ can be identified with $M(A)$, we can suppose that $\varprojlim_{p\in S(A)} M(E_p)$ is a Hilbert $M(A)$-module. The linear map $U : M(E) \to \varprojlim_{p\in S(A)} M(E_p)$ defined by $U(t) = ((\pi_p^{A,E})_*(t))_p$ is an isomorphism of locally convex spaces [11, Proposition 4.7]. Moreover,

$$\begin{aligned}\langle U(t), U(t)\rangle_{M(A)} &= (\langle(\pi_p^{A,E})_*(t), (\pi_p^{A,E})_*(t)\rangle_{M(A_p)})_p\\ &= ((\pi_p^{A,E})_*(t)^*(\pi_p^{A,E})_*(t))_p\\ &= ((\pi_p^{A,A})_*(t^* \circ t))_p = \langle t, t\rangle_{M(A)}\end{aligned}$$

for all $t \in M(E)$. From [5, Proposition 3.3], we now deduce that U is a unitary operator from $M(E)$ to $\varprojlim_{p\in S(A)} M(E_p)$. Therefore the Hilbert modules $M(E)$ and $\varprojlim_{p\in S(A)} M(E_p)$ are unitarily equivalent.

(iii) We will show that the connecting maps $(\pi_{pq}^{A,E})_*$, $p, q \in S(A)$ with $p \geq q$, are strictly continuous. Indeed, from

$$\begin{aligned}\|(\pi_{pq}^{A,E})_*(t)\|_{E_q,\pi_q(a),\sigma_q^E(\xi)} &= \|(\pi_{pq}^{A,E})_*(t)(\pi_q(a))\|_{E_q} \\ &\quad + \|((\pi_{pq}^{A,E})_*(t))^*(\sigma_q^E(\xi))\|_{A_q} \\ &= \|\sigma_{pq}^E(t(\pi_p(a)))\|_{E_q} + \|\pi_{pq}(t^*(\sigma_p^E(\xi)))\|_{A_q} \\ &\leq \|t(\pi_p(a))\|_{E_p} + \|t^*(\sigma_p^E(\xi))\|_{A_p} = \|t\|_{E_p,\pi_p(a),\sigma_p^E(\xi)}\end{aligned}$$

for all $a \in A$, $\xi \in E$, and $t \in M(E_p)$, we deduce that $(\pi_{pq}^{A,E})_*$ is strictly continuous. Clearly, the net $\{t_i\}_{i\in I}$ converges strictly in $M(E)$ if and only if the net $\{(\pi_p^{A,E})_*(t_i)\}_{i\in I}$ converges strictly in $M(E_p)$ for each $p \in S(A)$.

(iv) Since for each $p \in S(A)$, $M(E_p)$ is strictly complete, $\varprojlim_{p\in S(A)} M(E_p)$ is strictly complete, and then by (iii), so is $M(E)$.

(v) Let $p \in S(A)$. The map $i_{E_p} : E_p \to M(E_p)$ defined by $i_{E_p}(\xi_p)(a_p) = \xi_p a_p$ for $a_p \in A_p$ and $\xi_p \in E_p$ embeds E_p in $M(E_p)$ (see, for example, [13]). It is not difficult to check that $\sigma_{pq}^E \circ i_{E_p} = i_{E_q} \circ (\pi_{pq}^{A,E})_*$ for all $p, q \in S(A)$ with $p \geq q$. Therefore $\{i_{E_p}\}_p$ is an inverse system of isometric linear maps. Let $i_E = \varprojlim_{p\in S(A)} i_{E_p}$. Identifying E with $\varprojlim_{p\in S(A)} E_p$ and $M(E)$ with $\varprojlim_{p\in S(A)} M(E_p)$, we can suppose that i_E is a linear map from E to $M(E)$. It is not difficult to check that $i_E(\xi)(a) = \xi a$, $i_E(\xi a) = i_E(\xi) \cdot a$ and $\langle i_E(\xi), i_E(\xi)\rangle_{M(A)} = \langle \xi, \xi\rangle$ for all $a \in A$ and $\xi \in E$. Moreover, if $t \in M(E)$, $a \in A$ and $\xi \in E$, then

$$(t \cdot a)(c) = t(ac) = t(a)c = i_E(t(a))(c)$$

and

$$\langle t, i_E(\xi)\rangle_{M(A)}(c) = t^*(\xi c) = t^*(\xi)c = t^*(\xi)(c)$$

for all $c \in A$.

(vi) Let $\{e_i\}_{i\in I}$ be an approximate unit for A and let $t \in M(E)$. By (v), $\{t \cdot e_i\}_{i\in I}$ is a net in E. Let $p \in S(A)$, $a \in A$, $\xi \in E$. Then

$$\begin{aligned}\|t \cdot e_i - t\|_{p,a,\xi} &= \overline{p}_E((t \cdot e_i - t)(a)) + p((t \cdot e_i - t)^*(\xi)) \\ &= \overline{p}_E(t(e_i a - a)) + p(e_i t^*(\xi) - t^*(\xi)) \\ &\leq \overline{p}_{M(E)}(t) p(e_i a - a) + p(e_i t^*(\xi) - t^*(\xi)).\end{aligned}$$

Since $\{e_i\}_{i\in I}$ is an approximate unit for A, we have $p(e_i a - a) \to 0$ and $p(e_i t^*(\xi) - t^*(\xi)) \to 0$. Therefore $\{t \cdot e_i\}_{i\in I}$ converges strictly to t. ∎

Remark 3.4. Let A be a pro-C^*-algebra and E a Hilbert A-module.

(i) The multiplier module $M(A)$ coincides with the Hilbert $M(A)$-module $M(A)$.

(ii) According to Theorem 3.3(v), E can be identified with a closed submodule of $M(E)$. Thus, the image of an element ξ under i_E will also be denoted by ξ.

(iii) According to Theorem 3.3(v), $EA \subseteq M(E)A \subseteq E$. Since EA is dense in E, we conclude that $M(E)A$ is dense in E.

(iv) If A is unital, then E is complete with respect to the strict topology and so $E = M(E)$.

(v) If $K_A(E)$ is unital, then, for each $p \in S(A)$, $K_{A_p}(E_p)$ is unital and by [2, Proposition 2.8], $M(E_p) = E_p$. From Theorem 3.3(ii) we now deduce that $E = M(E)$.

(vi) The map $\Phi : b(L_A(A,E)) \to L_{b(A)}(b(A), b(E))$ defined by $\Phi(t) = t|_{b(A)}$, where $t|_{b(A)}$ denotes the restriction of t to $b(A)$, is an isometric isomorphism of Banach spaces [6, Theorem 3.7]. Since

$$\Phi(t \cdot b)(a) = (t \cdot b)|_{b(A)}(a) = t(ba)$$

and

$$(\Phi(t) \cdot b)(a) = (t|_{b(A)} \cdot b)(a) = t(ba)$$

for all $t \in b(L_A(A,E))$, $b \in M(b(A))$, and $a \in b(A)$, Φ is a unitary operator from $b(L_A(A,E))$ to $L_{b(A)}(b(A), b(E))$ [9]. Therefore the Hilbert $M(b(A))$-modules $b(M(E))$ and $M(b(E))$ are unitarily equivalent.

Let $\{E_n\}_n$ be a countable family of Hilbert A-modules and let

$$\text{str.-}\bigoplus_n M(E_n) = \{(t_n)_n;\ t_n \in M(E_n) \text{ and } \textstyle\sum_n t_n^* \circ t_n \text{ converges strictly in } M(A)\}.$$

If α is a complex number and $(t_n)_n \in \text{str.-}\bigoplus_n M(E_n)$, then clearly $(\alpha t_n)_n \in \text{str.-}\bigoplus_n M(E_n)$.

Let $(t_n)_n \in \text{str.-}\bigoplus_n M(E_n)$ with $t = \text{str.-}\lim_n \sum_{k=1}^n t_k^* \circ t_k$. Clearly, $\{\sum_{k=1}^n t_k^* \circ t_k\}_n$ is an increasing sequence of positive elements in $M(A)$. Thus for any $a \in A$ and $p \in S(A)$, $\{p(\sum_{k=1}^n a^* t_k^*(t_k(a)))\}_n$ is an increasing sequence of positive numbers which converges to $p(a^* t(a))$. If $\{e_i\}_i$ is an approximate unit for A, then

$$\begin{aligned}\widetilde{p}_{L_A(A)}\Big(\sum_{k=1}^n t_k^* \circ t_k\Big) &= \sup\Big\{p\Big(\sum_{k=1}^n t_k^*(t_k(a))\Big);\ a \in A,\ p(a) \le 1\Big\}\\ &= \sup\Big\{\lim_i p\Big(\sum_{k=1}^n e_i t_k^*(t_k(e_i a))\Big);\ a \in A,\ p(a) \le 1\Big\}\\ &\le \lim_i p\Big(\sum_{k=1}^n e_i t_k^*(t_k(e_i))\Big) \le \lim_i p(e_i t(e_i)) \le \widetilde{p}_{L_A(A,E)}(t).\end{aligned}$$

Let $(t_n)_n, (s_n)_n \in \text{str.-}\bigoplus_n M(E_n)$ with

$$t = \text{str.-}\lim_n \sum_{k=1}^{n} t_k^* \circ t_k, \qquad s = \text{str.-}\lim_n \sum_{k=1}^{n} s_k^* \circ s_k,$$

and $a \in A$ and $p \in S(A)$. Then

$$\begin{aligned}
p\Big(\sum_{k=n}^{m} s_k^*(t_k(a))\Big) &= p\Big(\sum_{k=n}^{m} \langle s_k, t_k\rangle_{M(A)}(a)\Big) = p\Big(\sum_{k=n}^{m} \langle s_k, t_k\rangle_{M(A)} \cdot a\Big) \\
&= \widetilde{p}_{L_A(A)}\Big(\sum_{k=n}^{m} \langle s_k, t_k \cdot a\rangle_{M(A)}\Big) = \widetilde{p}_{L_A(A)}(\langle (s_k)_{k=n}^{m}, (t_k \cdot a)_{k=n}^{m}\rangle_{M(A)}) \\
&\le \widetilde{p}_{L_A(A)}\Big(\sum_{k=n}^{m} \langle s_k, s_k\rangle_{M(A)}\Big)^{1/2} \widetilde{p}_{L_A(A)}\Big(\sum_{k=n}^{m} \langle t_k \cdot a, t_k \cdot a\rangle_{M(A)}\Big)^{1/2} \\
&\qquad\qquad \text{(Cauchy–Schwarz inequality)} \\
&\le \widetilde{p}_{L_A(A)}(s)^{1/2} \widetilde{p}_{L_A(A)}\Big(\sum_{k=n}^{m} (t_k^* \circ t_k)(a)\Big)^{1/2} p(a)^{1/2}
\end{aligned}$$

and

$$p\Big(\sum_{k=n}^{m} t_k^*(s_k(a))\Big) \le \widetilde{p}_{L_A(A)}(t)^{1/2} \widetilde{p}_{L_A(A)}\Big(\sum_{k=n}^{m} (s_k^* \circ s_k)(a)\Big)^{1/2} p(a)^{1/2}$$

for all positive integers n and m with $m \ge n$. Hence $\{\sum_{k=1}^{n} s_k^* \circ t_k\}_n$ converges strictly in $M(A)$ and so $(t_n + s_n)_n \in \text{str.-}\bigoplus_n M(E_n)$, since

$$\begin{aligned}
p\Big(\sum_{k=n}^{m} (t_k + s_k)^*((t_k + s_k)(a))\Big) &\le p\Big(\sum_{k=n}^{m} t_k^*(t_k(a))\Big) + p\Big(\sum_{k=n}^{m} s_k^*(s_k(a))\Big) \\
&\quad + p\Big(\sum_{k=n}^{m} t_k^*(s_k(a))\Big) + p\Big(\sum_{k=n}^{m} s_k^*(t_k(a))\Big)
\end{aligned}$$

for all positive integers n and m with $n \ge m$. It is not difficult to check that $\text{str.-}\bigoplus_n M(E_n)$ with the above addition and multiplication by complex scalars is a complex vector space.

Let $b \in M(A)$ and $(t_n)_n \in \text{str.-}\bigoplus_n M(E_n)$. From

$$\begin{aligned}
p\Big(\sum_{k=n}^{m} (t_k \cdot b)^*((t_k \cdot b)(a))\Big) &= p\Big(\sum_{k=n}^{m} b^* t_k^*(t_k(ba))\Big) \le p\Big(b^* \sum_{k=n}^{m} t_k^*(t_k(ba))\Big) \\
&\le p(b) p\Big(\sum_{k=n}^{m} t_n^*(t_n(ba))\Big)
\end{aligned}$$

for all $a \in A$, $p \in S(A)$, and $m \ge n$, we conclude that $\sum_n (t_n \cdot b)^* \circ (t_n \cdot b)$ converges strictly in $M(A)$ and so $(t_n \cdot b)_n \in \text{str.-}\bigoplus_n M(E_n)$.

THEOREM 3.5. *Let $\{E_n\}_n$ be a countable family of Hilbert A-modules. Then the vector space* str.-$\bigoplus_n M(E_n)$ *is a Hilbert $M(A)$-module with the module action defined by $(t_n)_n \cdot b = (t_n \cdot b)_n$ and the $M(A)$-valued inner product defined by*

$$\langle (t_n)_n, (s_n)_n \rangle_{M(A)} = \text{str.-}\lim_n \sum_{k=1}^{n} t_k^* \circ s_k.$$

Moreover, the Hilbert $M(A)$-modules str.-$\bigoplus_n M(E_n)$ *and $M(\bigoplus_n E_n)$ are unitarily equivalent.*

Proof. It is not difficult to check that str.-$\bigoplus_n M(E_n)$ with the above inner product and action of $M(A)$ is a pre-Hilbert $M(A)$-module. Let $(t_n)_n \in$ str.-$\bigoplus_n M(E_n)$ and $a \in A$. Since

$$p\Big(\sum_{k=n}^{m} \langle t_k(a), t_k(a)\rangle\Big) = p\Big(\sum_{k=n}^{m} a^* t_k{}^*(t_k(a))\Big) \le p(a) p\Big(\sum_{k=n}^{m} (t_k{}^* \circ t_k)(a)\Big)$$

for all $p \in S(A)$ and $m \ge n$, we have $(t_n(a))_n \in \bigoplus_n E_n$. It is not difficult to check that the map $U((t_n)_n)$ from A to $\bigoplus_n E_n$ defined by $U((t_n)_n)(a) = (t_n(a))_n$ is a module morphism. Let $(\xi_n)_n \in \bigoplus_n E_n$ and $p \in S(A)$. Since

$$\begin{aligned}
p\Big(\sum_{k=n}^{m} t_k^*(\xi_k)\Big) &= \sup\Big\{p\Big(\Big\langle \sum_{k=n}^{m} t_k^*(\xi_k), a\Big\rangle\Big);\, p(a) \le 1\Big\} \\
&= \sup\Big\{p\Big(\sum_{k=n}^{m} \langle \xi_k, t_k(a)\rangle\Big);\, p(a) \le 1\Big\} \\
&= \sup\{p(\langle (\xi_k)_{k=n}^m, (t_k(a))_{k=n}^m\rangle);\, p(a) \le 1\} \\
&= p\Big(\sum_{k=n}^{m} \langle \xi_k, \xi_k\rangle\Big)^{1/2} \sup\Big\{p\Big(\sum_{k=n}^{m} \langle a, t_k^*(t_k(a))\rangle\Big)^{1/2};\, p(a) \le 1\Big\} \\
&\qquad \text{(Cauchy–Schwarz inequality)} \\
&= p\Big(\sum_{k=n}^{m} \langle \xi_k, \xi_k\rangle\Big)^{1/2} \sup\Big\{p\Big(\sum_{k=n}^{m} a^* t_k^*(t_k(a))\Big)^{1/2};\, p(a) \le 1\Big\} \\
&\le p\Big(\sum_{k=n}^{m} \langle \xi_k, \xi_k\rangle\Big)^{1/2} \widetilde{p}_{L_A(A)}\Big(\sum_n t_k^* \circ t_k\Big)^{1/2}
\end{aligned}$$

for $m \ge n$, it follows that $\sum_n t_n^*(\xi_n)$ converges in A. Thus we can define a linear map $U((t_n)_n)^* : \bigoplus_n M(E_n) \to A$ by

$$U((t_n)_n)^*((\xi_n)_n) = \sum_n t_n^*(\xi_n).$$

Moreover, since

$$\begin{aligned}\langle U((t_n)_n)(a), (\xi_n)_n\rangle &= \langle (t_n(a))_n, (\xi_n)_n\rangle = \sum_n \langle t_n(a), \xi_n\rangle \\ &= \sum_n \langle a, t_n^*(\xi_n)\rangle = \langle a, U((t_n)_n)^*((\xi_n)_n)\rangle\end{aligned}$$

for all $a \in A$ and $(\xi_n)_n \in \bigoplus_n E_n$, we see that $U((t_n)_n) \in M(\bigoplus_n E_n)$. Thus, we have defined a map U from str.-$\bigoplus_n M(E_n)$ to $M(\bigoplus_n E_n)$. It is not difficult to check that U is a module morphism. Moreover,

$$\begin{aligned}\langle U((t_n)_n), U((s_n)_n)\rangle_{M(A)}(a) &= U((t_n)_n)^*(U((s_n)_n)(a)) \\ &= U((t_n)_n)^*((s_n(a))_n) = \sum_n t_n^*(s_n(a))) \\ &= \langle (t_n)_n, (s_n)_n\rangle_{M(A)}(a)\end{aligned}$$

for all $a \in A$ and $(t_n)_n, (s_n)_n \in$ str.-$\bigoplus_n M(E_n)$.

Now, we will show that U is surjective. Let m be a positive integer. Clearly, $P_m : \bigoplus_n E_n \to E_m$ defined by $P_m((\xi_n)_n) = \xi_m$ is in $L_A(\bigoplus_n E_n, E_m)$. Moreover, P_m^* is the embedding of E_m in $\bigoplus_n E_n$. Let $t \in M(\bigoplus_n E_n)$, and set $t_n = P_n \circ t$ for each integer n. Then $t_n \in M(E_n)$ for each n and $t(a) = (t_n(a))_n$ for all $a \in A$. Therefore $\sum_n a^* t_n^*(t_n(a))$ converges in A for all $a \in A$. Moreover, $\sum_n a^* t_n^*(t_n(a)) = a^* t^*(t(a))$ for all $a \in A$, and so

$$\begin{aligned}\widetilde{p}_{L_A(A)}\Big(\sum_{k=n}^m t_k^* \circ t_k\Big) &= \sup\Big\{p\Big(\Big\langle\Big(\sum_{k=n}^m t_k^* \circ t_k\Big)(a), a\Big\rangle\Big);\, p(a) \le 1\Big\} \\ &= \sup\Big\{p\Big(\sum_{k=n}^m a^* t_k^*(t_k(a))\Big);\, p(a) \le 1\Big\} \\ &\le \sup\{p(a^* t^*(t(a)));\, p(a) \le 1\} \le \widetilde{p}_{L_A(A)}(t^* \circ t)\end{aligned}$$

for all $m \ge n$ and $p \in S(A)$. Let $a \in A$. From

$$\begin{aligned}p\Big(\sum_{k=n}^m t_k^*(t_k(a))\Big)^2 &= p\Big(\Big\langle \sum_{k=n}^m t_k^*(t_k(a)), \sum_{k=n}^m t_k^*(t_k(a))\Big\rangle\Big) \\ &= p\Big(\Big\langle\Big(\sum_{k=n}^m t_k^* \circ t_k\Big)(a), \Big(\sum_{k=n}^m t_k^* \circ t_k\Big)(a)\Big\rangle\Big) \\ &= \Big\|\Big\langle (\pi_p^{A,A})_*\Big(\sum_{k=n}^m t_k^* \circ t_k\Big)(\pi_p(a)), (\pi_p^{A,A})_*\Big(\sum_{k=n}^m t_k^* \circ t_k\Big)(\pi_p(a))\Big\rangle\Big\|_{A_p} \\ &\le \Big\|(\pi_p^{A,A})_*\Big(\sum_{k=n}^m t_k^* \circ t_k\Big)\Big\|_{L_{A_p}(A_p)}\Big\|\Big\langle \pi_p(a), \pi_p\Big(\Big(\sum_{k=n}^m t_k^* \circ t_k\Big)(a)\Big)\Big\rangle\Big\|_{A_p}\end{aligned}$$

([10, Proposition 2.6])

$$\leq \widetilde{p}_{L_A(A)}\Big(\sum_{k=n}^{m} t_k^* \circ t_k\Big) p\Big(\Big\langle a, \Big(\sum_{k=n}^{m} t_k^* \circ t_k\Big)(a)\Big\rangle\Big)$$

$$\leq \widetilde{p}_{L_A(A)}(t^* \circ t) p\Big(\sum_{k=n}^{m} a^* t_k^*(t_k(a))\Big)$$

for all $m \geq n$ and $p \in S(A)$, we conclude that $\sum_n t_n^*(t_n(a))$ converges in A. Therefore $(t_n)_n \in$ str.-$\bigoplus_n M(E_n)$. Moreover, $U((t_n)_n) = t$ and so U is surjective. As

$$\langle U((t_n)_n), U((t_n)_n)\rangle_{M(A)} = \langle (t_n)_n, (t_n)_n\rangle_{M(A)}$$

for all $(t_n)_n \in$ str.-$\bigoplus_n M(E_n)$, we conclude that str.-$\bigoplus_n M(E_n)$ is a Hilbert $M(A)$-module, and moreover U is a unitary operator [5, Proposition 3.3]. Therefore the Hilbert $M(A)$-modules str.-$\bigoplus_n M(E_n)$ and $M(\bigoplus_n E_n)$ are unitarily equivalent. ∎

REMARK 3.6. Let $\{E_n\}_n$ be a countable family of Hilbert A-modules. In general, $\bigoplus_n M(E_n)$ is a submodule of $M(\bigoplus_n E_n)$; they coincide when the pro-C^*-algebra A is also unital.

4. Operators on multiplier modules. Let E and F be Hilbert A-modules. If $T \in L_{M(A)}(M(E), M(F))$, then

$$T(E) \subseteq \overline{T(M(E)A)} = \overline{T(M(E))A} \subseteq \overline{M(F)A} = F.$$

Therefore $T(E) \subseteq F$. Clearly $T|_E : E \to F$ is a module morphism. Moreover, $T|_E \in L_A(E, F)$, since

$$\langle T|_E(\xi), \eta\rangle = \langle T(i_E(\xi)), i_E(\eta)\rangle_{M(A)} = \langle i_E(\xi), T^*(i_E(\eta))\rangle_{M(A)} = \langle \xi, T^*|_F(\eta)\rangle$$

for all $\xi \in E$ and $\eta \in F$.

THEOREM 4.1. *Let E and F be Hilbert A-modules.*

(i) *If $T \in L_{M(A)}(M(E), M(F))$, then T is strictly continuous.*
(ii) *The locally convex spaces $L_{M(A)}(M(E), M(F))$ and $L_A(E, F)$ are isomorphic.*
(iii) *The pro-C^*-algebras $L_{M(A)}(M(E))$ and $L_A(E)$ are isomorphic.*

Proof. (i) Let $\{s_i\}_{i\in I}$ be a net in $M(E)$ which converges strictly to 0. From

$$\overline{p}_F(T(s_i)(a)) = \overline{p}_F(T(s_i \cdot a)) = \overline{p}_F(T|_E(s_i(a))) \leq \widetilde{p}_{L_A(E,F)}(T|_E)\overline{p}_E(s_i(a))$$

and

$$p(T(s_i)^*(\xi)) = p(s_i^*(T^*(\xi)))$$

for all $p \in S(A)$, $a \in A$, $\xi \in F$ and $i \in I$, we conclude that $\{T(s_i)\}_{i\in I}$ converges strictly to 0. Therefore T is strictly continuous.

(ii) We show that the map $\Phi : L_{M(A)}(M(E), M(F)) \to L_A(E, F)$ defined by $\Phi(T) = T|_E$ is an isomorphism of locally convex spaces. Clearly, Φ is a linear map. Moreover, Φ is continuous, since

$$\widetilde{p}_{L_A(E,F)}(\Phi(T)) = \widetilde{p}_{L_A(E,F)}(T|_E) \le \widetilde{p}_{L_{M(A)}(M(E),M(F))}(T)$$

for all $T \in L_{M(A)}(M(E), M(F))$ and $p \in S(A)$. To show that Φ is injective, let $T \in L_{M(A)}(M(E), M(F))$ be such that $T|_E = 0$. Then

$$\begin{aligned}\overline{p}_{M(F)}(T(s)) &= \sup\{\overline{p}_F(T(s)(a));\ p(a) \le 1\}\\ &= \sup\{\overline{p}_F(T(s \cdot a));\ p(a) \le 1\} = 0\end{aligned}$$

for all $s \in M(E)$ and $p \in S(A)$. Therefore $T = 0$.

Let $T \in L(E, F)$. Then, for each $s \in M(E)$, $T \circ s \in M(F)$. Define $\widetilde{T} : M(E) \to M(F)$ by $\widetilde{T}(s) = T \circ s$. Clearly, $\widetilde{T}$ is linear. Moreover,

$$\widetilde{T}(s \cdot b)(a) = T((s \cdot b)(a)) = T(s(ba)) = \widetilde{T}(s)(ba) = (\widetilde{T}(s) \cdot b)(a)$$

and

$$\langle \widetilde{T}(s), r\rangle_{M(A)} = s^* \circ T^* \circ r = \langle s, T^* \circ r\rangle_{M(A)}$$

for all $s \in M(E)$, $r \in M(F)$, $b \in M(A)$, and $a \in A$. Hence $\widetilde{T}$ is an adjointable module morphism. Therefore $\widetilde{T} \in L_{M(A)}(M(E), M(F))$. It is not difficult to check that $\widetilde{T}|_E = T$. Thus Φ is surjective. Therefore it is a continuous bijective linear map from $L_{M(A)}(M(E), M(F))$ onto $L_A(E, F)$. Moreover, $\Phi^{-1}(T)(s) = T \circ s$ for all $s \in M(E)$ and $T \in L_A(E, F)$.

To show that Φ is an isomorphism of locally convex spaces it remains to prove that Φ^{-1} is continuous. Let $p \in S(A)$ and $T \in L_A(E, F)$. Then

$$\begin{aligned}\widetilde{p}_{L_{M(A)}(M(E),M(F))}(\Phi^{-1}(T)) &= \sup\{\overline{p}_{M(F)}(T \circ s);\ \overline{p}_{M(E)}(s) \le 1\}\\ &\le \sup\{\widetilde{p}_{L_A(E,F)}(T)\widetilde{p}_{L_A(A,E)}(s);\ \overline{p}_{M(E)}(s) \le 1\}\\ &\le \widetilde{p}_{L_A(E,F)}(T).\end{aligned}$$

Hence Φ^{-1} is continuous. Moreover, we showed that $\widetilde{p}_{L_{M(A)}(M(E),M(F))}(T) = \widetilde{p}_{L_A(E,F)}(T|_E)$ for all $p \in S(A)$.

(iii) We have shown that $\Phi : L_{M(A)}(M(E)) \to L_A(E)$ defined by $\Phi(T) = T|_E$ is an isomorphism of locally convex spaces. It is not difficult to check that also $\Phi(T_1T_2) = \Phi(T_1)\Phi(T_2)$ and $\Phi(T^*) = \Phi(T)^*$ for all $T, T_1, T_2 \in L_{M(A)}(M(E))$. Therefore Φ is an isomorphism of pro-C^*-algebras. ■

If E and F are unitarily equivalent full Hilbert C^*-modules, then the Hilbert C^*-modules $M(E)$ and $M(F)$ are unitarily equivalent [2, Proposition 1.7]. This is also valid for Hilbert modules over pro-C^*-algebras.

Corollary 4.2. *Let E and F be Hilbert A-modules. Then E and F are unitarily equivalent if and only if $M(E)$ and $M(F)$ are unitarily equivalent.*

Proof. Indeed, E and F are unitarily equivalent if and only if there is a unitary operator U in $L_A(E,F)$. But it is not difficult to check that $T \in L_{M(A)}(M(E), M(F))$ is unitary if and only if $T|_E$ is unitary in $L_A(E,F)$. This yields the assertion. ∎

Corollary 4.3. *If E is a Hilbert A-module, then $K_A(E)$ is isomorphic to an essential ideal of $K_{M(A)}(M(E))$.*

Proof. By the proof of Theorem 4.1, $\Phi^{-1}(K_A(E))$ is a pro-C^*-subalgebra of $L_{M(A)}(M(E))$. Moreover, the pro-C^*-algebras $K_A(E)$ and $\Phi^{-1}(K_A(E))$ are isomorphic. Clearly, $\Phi^{-1}(K_A(E))$ is a two-sided $*$-ideal of $K_{M(A)}(M(E))$. To show that $\Phi^{-1}(K_A(E))$ is essential, let $\xi, \eta \in E$. If $\Phi^{-1}(\theta_{\xi,\eta})\theta_{t_1,t_2} = 0$ for all $t_1, t_2 \in M(E)$, then

$$\theta_{\xi,\eta}((t_1 \circ t_2^* \circ t_3)(a)) = 0$$

for all $a \in A$ and $t_1, t_2, t_3 \in M(E)$. As $M(E)\langle M(E), M(E)\rangle_{M(A)}A$ is dense in E, we conclude that $\theta_{\xi,\eta} = 0$. ∎

References

[1] C. Apostol, *b^*-algebras and their representations*, J. London Math. Soc. 33 (1971), 30–38.

[2] D. Bakic and B. Guljas, *Extensions of Hilbert C^*-modules*, Houston J. Math. 30 (2004), 537–558.

[3] M. Fragoulopoulou, *Topological Algebras with Involution*, North-Holland Math. Stud. 200, Elsevier, Amsterdam, 2005.

[4] A. Inoue, *Locally C^*-algebras*, Mem. Faculty Sci. Kyushu Univ. Ser. A 25 (1971), 197–235.

[5] M. Joiţa, *On Hilbert-modules over locally C^*-algebras. II*, Period. Math. Hungar. 51 (2005), 27–36.

[6] —, *On bounded module maps between Hilbert modules over locally C^*-algebras*, Acta Math. Univ. Comenian. (N.S.) 74 (2005), 71–78.

[7] —, *Hilbert Modules over Locally C^*-Algebras*, Univ. of Bucharest Press, 2006.

[8] I. Kaplansky, *Modules over operator algebras*, Amer. J. Math. 75 (1953), 839-853.

[9] E. C. Lance, *Hilbert C^*-Modules. A Toolkit for Operator Algebraists*, London Math. Soc. Lecture Note Ser. 210, Cambridge Univ. Press, Cambridge, 1995.

[10] W. L. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. 182 (1973), 443–468.

[11] N. C. Phillips, *Inverse limits of C^*-algebras*, J. Operator Theory 19 (1988), 159–195.

[12] —, *Inner derivations on σ-C^*-algebras*, Math. Nachr. 176 (1995), 243–247.

[13] I. Raeburn and S. J. Thompson, *Countably generated Hilbert modules*, *the Kasparov stabilisation theorem*, *and frames with Hilbert modules*, Proc. Amer. Math. Soc. 131 (2003), 1557–1564.

[14] M. A. Rieffel, *Induced representations of C^*-algebras*, Adv. Math. 13 (1974), 176–256.

[15] Yu. I. Zhuraev and F. Sharipov, *Hilbert modules over locally C^*-algebras*, arXiv: math.OA/0011053.

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