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Structures of left n-invertible operators and their applications

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Abstract. We study left n-invertible operators introduced in two recent papers. We show how to construct a left n-inverse as a sum of a left inverse and a nilpotent operator. We provide refinements for results on products and tensor products of left n-invertible operators by Duggal and Müller (2013). Our study leads to improvements and different and often more direct proofs of results of Duggal and Müller (2013) and Sid Ahmed (2012). We make a conjecture about tensor products of left n-invertible operators and prove this conjecture in several cases. Finally, applications of these results are given to left n-invertible elementary operators and essentially left n-invertible operators.

1. Introduction. Let $B(X)$ be the algebra of all bounded operators on a Banach space X. Let

$$p(y,x)=\sum_{i,j=0}^{n} c_{ij}y^i x^j.$$

For $S,T\in B(X)$, we define the functional calculus $p(S,T)$ by

$$p(S,T)=p(y,x)|_{y=S,\,x=T}=\sum_{i,j=0}^{n} c_{ij}S^iT^j \tag{1}$$

where S is always on the left side of T. Let

$$\beta_n(y,x)=(yx-1)^n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}y^kx^k.$$

Then $\beta_n(S,T)$ is given by

$$\beta_n(S,T)=(yx-1)^n|_{y=S,\,x=T}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}S^kT^k.$$

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Recall that S is a *left inverse* of T (or T is a *right inverse* of S) if $ST = 1$, that is, $\beta_1(S,T) = ST - 1 = 0$. As in Sid Ahmed [25] and Duggal and Müller [17], S is a *left n-inverse* of T (or T is a *right n-inverse* of S) if

$$\beta_n(S,T) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}S^kT^k = 0.$$

Since $\beta_n(y,x)$ divides $\beta_m(y,x)$ for $m \geq n$, if S is a left n-inverse of T, then S is a left m-inverse of T for $m \geq n$. This fact also follows from the recursive formula (2) below. We say that S is a *strict* left n-inverse of T if S is a left n-inverse of T but not a left $(n-1)$-inverse of T. It is also clear that S is a left n-inverse of T if and only if T^* is a left n-inverse of S^*. Similarly we say S is an *n-inverse* of T if S is both a left n-inverse and a right n-inverse of T. We say T is *left n-invertible* if T has a left n-inverse, and T is *n-invertible* if T has an n-inverse.

The concept of left n-invertible operators is motivated by the m-isometries studied earlier in [2]–[6], [24] on Hilbert spaces and more recently in [9], [11]–[13], [15], [26] on Hilbert spaces and [7], [8], [10], [16], [22] on Banach spaces. An operator T on a Hilbert space is an *n-isometry* if $\beta_n(T^*,T) = 0$, that is, T^* is a left n-inverse of T.

Motivated by [9] and [19], in Section 2 we show that if S is a left m-inverse of T and Q is a nilpotent operator of order l commuting with S, then $S+Q$ is a left n-inverse of T where $n = m + l - 1$. We also discuss the converse of this result. In particular we show that a 2-inverse S of T is necessarily of the form $S = T^{-1} + Q$ where $QT = TQ$ and $Q^2 = 0$. A further study of this converse for tensor product operators is carried out in Section 4.

In Section 3 we give different proofs of some results on products and tensor products of left n-invertible operators by Duggal and Müller [17]. Our approach also yields necessary and sufficient conditions for strict left n-inverses.

In Section 4, we make the following conjecture: for $S_1, T_1 \in B(X)$ and $S_2, T_2 \in B(Y)$, the tensor product $S_1 \otimes S_2$ is a strict left n-inverse of $T_1 \otimes T_2$ if and only if there exist m and l such that $m + l - 1 = n$ and S_1 is a strict left m-inverse of $(1/\lambda)T_1$ for some constant λ and S_2 is a strict left l-inverse of λT_2. The "if" part belongs to [17] and is also reproved here in Section 3. We verify this conjecture for $n = 1, 2, 3$ using a detailed algebraic approach as in [11], [12] for n-isometries. Furthermore we prove this conjecture under a general technical assumption, so the conjecture is very promising.

An actual theorem is obtained for the sum of tensor products of left n-inverses. Namely, we show that for $S, T \in B(X)$ and $Q \in B(Y)$, the tensor sum $S \otimes I + I \otimes Q$ is a strict left n-inverse of $T \otimes I$ if and only if there exist m and l such that $m + l - 1 = n$ and $S + \lambda I$ is a strict left m-inverse of

T for some constant λ and $Q - \lambda I$ is a nilpotent operator of order l. Again the "if" part has already been proved in Section 2.

In Section 5, we remark that the results from previous sections are also valid for essentially left n-invertible operators defined in [17]. We also apply our results to answer the question when elementary operators of length one or generalized derivations on $B(X)$ are left n-invertible operators. Some preliminary results on this question were obtained by Sid Ahmed [25] and more complete results were obtained by Duggal and Müller [17]. Our results on generalized derivations and elementary operators of length two are new (see Theorems 24 and 26).

Finally, we acknowledge that some ideas and techniques are borrowed without explicit mention from the author's paper [18] and from the author and Stankus's paper [19] where related questions and more for n-isometries and n-symmetric operators are studied. But this paper is self-contained and will focus on left n-invertible operators. Several informative examples are given to illustrate the results and to show they are sharp. Moreover, most results are valid for elements in Banach algebras with identity since our approach is purely algebraic.

2. Constructing left n-inverses. Recall that for two operators A, B in $B(X)$, the *commutator* $[A, B]$ is defined to be

$$[A, B] = AB - BA.$$

Two operators A and B are *commuting* if $[A, B] = 0$.

The following recursive formula can be proved by definition. We omit the proof since it is simpler than the proof of the following lemma.

$$\beta_n(S, T) = S\beta_{n-1}(S, T)T - \beta_{n-1}(S, T). \tag{2}$$

Lemma 1. *Assume $S, Q \in B(X)$ are commuting and $T \in B(X)$. Then*

$$\beta_n(S + Q, T) = \sum_{k=0}^{n} \binom{n}{k} Q^{n-k}\beta_k(S, T)T^{n-k}, \tag{3}$$

$$\beta_n(T, S + Q) = \sum_{k=0}^{n} \binom{n}{k} T^{n-k}\beta_k(T, S)Q^{n-k}.$$

Proof. We first give a heuristic argument. Note that

$$((y + z)x - 1)^n = (yx - 1 + zx)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k}(yx - 1)^k x^{n-k}$$

The lemma follows by substituting S for y, Q for z and T for x in the above formula using the functional calculus defined in (1).

For clarity, we now prove the lemma by induction. For $n = 1$, note that

$$\begin{aligned}\text{Left side of (3)} &= \beta_1(S+Q,T) = (S+Q)\,T - I\\ &= QT + ST - I.\end{aligned}$$

Since by definition $\beta_0(S,T) = I$, we have

$$\begin{aligned}\text{Right side of (3)} &= \sum_{k=0}^{1}\binom{1}{k}Q^{1-k}\beta_k(S,T)T^{1-k}\\ &= Q\beta_0(S,T)T + \beta_1(S,T)\\ &= QT + ST - I.\end{aligned}$$

Thus (3) holds for $n = 1$. Assume now (3) holds for n. By (2) and the induction hypothesis,

$$\begin{aligned}&\beta_{n+1}(S+Q,T) = (S+Q)\beta_n(S+Q,T)T - \beta_n(S+Q,T)\\ &= (S+Q)\left[\sum_{k=0}^{n}\binom{n}{k}Q^{n-k}\beta_k(S,T)T^{n-k}\right]T - \left[\sum_{k=0}^{n}\binom{n}{k}Q^{n-k}\beta_k(S,T)T^{n-k}\right]\\ &= \sum_{k=0}^{n}\binom{n}{k}Q^{n-k}S\beta_k(S,T)TT^{n-k} + \sum_{k=0}^{n}\binom{n}{k}Q^{n-k+1}\beta_k(S,T)T^{n-k+1}\\ &\quad - \sum_{k=0}^{n}\binom{n}{k}Q^{n-k}\beta_k(S,T)T^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}Q^{n-k}[S\beta_k(S,T)T - \beta_k(S,T)]T^{n-k}\\ &\quad + \sum_{k=0}^{n}\binom{n}{k}Q^{n-k+1}\beta_k(S,T)T^{n-k+1}\\ &= \sum_{k=0}^{n}\binom{n}{k}Q^{n-k}\beta_{k+1}(S,T)T^{n-k} + \sum_{k=0}^{n}\binom{n}{k}Q^{n+1-k}\beta_k(S,T)T^{n+1-k}\\ &= \sum_{k=1}^{n+1}\binom{n}{k-1}Q^{n+1-k}\beta_k(S,T)T^{n+1-k} + \sum_{k=0}^{n}\binom{n}{k}Q^{n+1-k}\beta_k(S,T)T^{n+1-k}\\ &= \sum_{k=0}^{n+1}\left[\binom{n}{k-1} + \binom{n}{k}\right]Q^{n+1-k}\beta_k(S,T)T^{n+1-k}\\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}Q^{n+1-k}\beta_k(S,T)T^{n+1-k},\end{aligned}$$

where in the third equality we use the assumption that S and Q are commuting, in the fifth equality we use (2) again, in the third to last equality we re-index the summation, and the last equality follows from the fact that $\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}$ and the convention that $\binom{n}{k-1}=0$ for $k=0$ and $\binom{n}{k}=0$ for $k=n+1$. This proves that (3) holds for $n+1$. ∎

THEOREM 2. *Assume $S, Q \in B(X)$ are commuting and Q is a nilpotent operator of order l.*

(a) *If S is a left m-inverse of T, then $S+Q$ is a left n-inverse of T where $n=m+l-1$. Furthermore $S+Q$ is a strict left n-inverse of T if and only if $Q^{l-1}\beta_{m-1}(S,T)T^{l-1} \neq 0$.*
(b) *If S is a right m-inverse of T, then $S+Q$ is a right n-inverse of T where $n=m+l-1$. Furthermore $S+Q$ is a strict right n-inverse of T if and only if $T^{l-1}\beta_{m-1}(T,S)Q^{l-1} \neq 0$.*
(c) *If S is an m-inverse of T, then $S+Q$ is an n-inverse of T where $n=m+l-1$. Furthermore $S+Q$ is a strict n-inverse of T if and only if either $T^{l-1}\beta_{m-1}(T,S)Q^{l-1} \neq 0$ or $Q^{l-1}\beta_{m-1}(S,T)T^{l-1} \neq 0$.*

Proof. We will prove (a); the proofs of (b) and (c) are similar. Let $n = m+l-1$. By the previous lemma,

$$\beta_n(S+Q,T) = \sum_{k=0}^{n} \binom{n}{k} Q^{n-k}\beta_k(S,T)T^{n-k}.$$

Note that if $k \geq m$, then $\beta_k(S,T)=0$, and if $k<m$, then $n-k>n-m=l-1$ and $Q^{n-k}=0$. Therefore $\beta_n(S+Q,T)=0$. Similarly

$$\begin{aligned}\beta_{n-1}(S+Q,T) &= \sum_{k=0}^{n-1} \binom{n-1}{k} Q^{n-1-k}\beta_k(S,T)T^{n-1-k}\\ &= \binom{n-1}{m-1} Q^{l-1}\beta_{m-1}(S,T)T^{l-1}.\end{aligned}$$

Thus $S+Q$ is a strict left n-inverse if and only if $Q^{l-1}\beta_{m-1}(S,T)T^{l-1} \neq 0$. ∎

COROLLARY 3. *If S is a left inverse (or a right inverse or an inverse) of T and Q is a nilpotent operator of order l such that $[S,Q]=0$, then $S+Q$ is a strict left l-inverse (or a strict right l-inverse or a strict l-inverse) of T.*

Proof. Note that in this case

$$\begin{aligned}\beta_{l-1}(S+Q,T) &= (l-1)Q^{l-1}\beta_0(S,T)T^{l-1}\\ &= (l-1)Q^{l-1}T^{l-1} \neq 0.\end{aligned}$$

Otherwise, $Q^{l-1}T^{l-1}=0$ implies that

$$S^{l-1}Q^{l-1}T^{l-1} = Q^{l-1}S^{l-1}T^{l-1} = Q^{l-1} = 0,$$

which contradicts $Q^{l-1} \neq 0$. ∎

Now we use Theorem 2 to construct examples of left n-inverses. We recall

$$\beta_n(S,T) = S\beta_{n-1}(S,T)T - \beta_{n-1}(S,T).$$

LEMMA 4. *If S is a left n-inverse of T, then*

$$S^i\beta_{n-1}(S,T)T^i = \beta_{n-1}(S,T) \quad \text{for } i \geq 0.$$

In particular, if S is a strict left n-inverse of T, then $\beta_{n-1}(S,T)T^i \neq 0$ for $i \geq 0$.

The first example seems to indicate an intimate relation between left n-inverses of T and the Fredholm index of T. We will construct our examples on a separable Hilbert space H.

EXAMPLE 5. Let U be the unilateral shift on H, that is, if $\{e_i : i \geq 0\}$ is an orthonormal basis of H, then $Ue_i = e_{i+1}$ for $i \geq 0$. Let $T = U^n$ for some fixed $n \geq 2$, and let S be a left inverse of T defined by

$$Se_{i+n} = e_i \quad \text{for } i \geq 0 \quad \text{and} \quad Se_i = 0 \quad \text{for } 0 \leq i \leq n-1.$$

Fix l such that $2 \leq l \leq n$. Let Q be the nilpotent operator of order l defined by

$$Qe_{0+kn} = 0, \quad Qe_{i+1+kn} = e_{i+kn} \quad \text{for } 0 \leq i \leq l-2,\ k \geq 0,$$
$$Qe_{i+kn} = 0 \quad \text{for } l \leq i \leq n-1,\ k \geq 0.$$

Then $S+Q$ is a strict left l-inverse of T and $S+Q$ is not invertible. We need to show that $QS = SQ$. For notational simplicity, we set $l = 2$ and $n = 3$. Then

$$Se_0 = Se_1 = Se_2 = 0, \quad Se_{i+3} = e_i \quad \text{for } i \geq 0,$$
$$Qe_{3k} = 0, \quad Qe_{3k+1} = e_{3k}, \quad Qe_{3k+2} = 0 \quad \text{for } k \geq 0.$$

Therefore

$$QSe_0 = Q(0) = 0, \quad QSe_1 = Q(0) = 0, \quad QSe_2 = Q(0) = 0,$$
$$SQe_0 = S(0) = 0, \quad SQe_1 = Se_0 = 0, \quad SQe_2 = S(0) = 0,$$

and for $k \geq 1$,

$$QSe_{3k} = Qe_{3(k-1)} = 0, \qquad SQe_{3k} = S(0) = 0,$$
$$QSe_{3k+1} = Qe_{3(k-1)+1} = e_{3(k-1)}, \quad SQe_{3k+1} = Se_{3k} = e_{3(k-1)},$$
$$QSe_{3k+2} = Qe_{3(k-1)+2} = 0, \qquad SQe_{3k+2} = S(0) = 0.$$

Let T be a left invertible operator with Fredholm index $-n$. The above example suggests the question: Is it always possible to construct a strict left k-inverse of T for each $k \leq n$? The second example is inspired by the structure of sub-Jordan operators [1].

EXAMPLE 6. Let S be a strict left m-inverse (or a strict right m-inverse or a strict m-inverse) of T. Let S_l and T_l be two operators on the direct sum of l copies of H,

$$S_l = \begin{bmatrix} S & 0 & \cdots & 0 \\ 0 & S & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & S \end{bmatrix}, \quad Q_l = \begin{bmatrix} 0 & cI & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & cI \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$T_l = \begin{bmatrix} T & 0 & \cdots & 0 \\ 0 & T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & T \end{bmatrix}$$

where $c \neq 0$ is a constant. Then $S_l + Q_l$ is a strict left n-inverse (or a strict right n-inverse or a strict n-inverse) of T_l where $n = m + l - 1$. Again the claim follows by a direct computation:

$$\begin{aligned} \beta_{n-1}(S_l + Q_l, T_l) &= \binom{n-1}{m-1} Q_l^{l-1} \beta_{m-1}(S_l, T_l) T_l^{l-1} \\ &= c^{l-1} \binom{n-1}{m-1} \begin{bmatrix} 0 & 0 & \cdots & \beta_{m-1}(S,T)T^{l-1} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \neq 0 \end{aligned}$$

since by Lemma 4, $\beta_{m-1}(S,T)T^{l-1} \neq 0$.

If S is a strict left m-inverse of T and Q is a nilpotent operator of order l, the next example shows that when $m > 1$, it is possible that $S + Q$ is not a strict left n-inverse with $n = m + l - 1$. So the results in Theorem 2 and Corollary 3 are sharp.

EXAMPLE 7. Let $S_1, T_1, Q_1 \in B(H)$ and

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & I \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & Q_1 \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & 0 \\ 0 & I \end{bmatrix}$$

where S_1 a strict left m-inverse of T_1 and Q_1 nilpotent operator of order l. It is easy to see that S a strict left m-inverse of T. Note also that $I + Q_1$ is a strict l-inverse of I. Since for any $k > 0$,

$$\beta_k(S + Q, T) = \begin{bmatrix} \beta_k(S_1, T_1) & 0 \\ 0 & \beta_k(I + Q_1, I) \end{bmatrix},$$

$S + Q$ is a strict left n-inverse of T with $n = \max\{m, l\}$. But $\max\{m, l\} < m + l - 1$ for $m > 1$.

We now attempt to study the converse of Theorem 2. That is, to what extent does a left n-inverse S of T arise as a sum of a left l-inverse of T and a nilpotent operator? We first observe that if S is a left n-inverse of T and if in addition $ST = TS$, then $\beta_n(T, S) = \beta_n(S, T) = 0$, thus S is also a right n-inverse of T. That is, S is an n-inverse of T. This leads to the following proposition which can be viewed as a partial converse of Theorem 2(c).

Proposition 8. *If S is an n-inverse of T and $ST = TS$, then T is invertible and $S = T^{-1} + Q$ where $Q^n = 0$ and $QT = TQ$.*

Proof. It is clear that if T has an n-inverse S, then T is invertible. To prove the proposition, define Q as

$$Q = S - T^{-1} \quad \text{or} \quad S = T^{-1} + Q.$$

Since by assumption $ST = TS$ and $TT^{-1} = T^{-1}T$, we have $QT = TQ$. Now by Lemma 1,

$$\begin{aligned} 0 = \beta_n(T^{-1} + Q, T) &= \sum_{k=0}^{n} \binom{n}{k} Q^{n-k} \beta_k(T^{-1}, T) T^{n-k} \\ &= \binom{n}{0} Q^{n-0} \beta_0(T^{-1}, T) T^{n-0} = Q^n T^n \end{aligned}$$

since $\beta_k(T^{-1}, T) = 0$ for $k \geq 1$. Thus $Q^n = 0$ since T is invertible. ∎

We next show that the condition $ST = TS$ is not needed when $n = 2$.

Proposition 9. *S is a 2-inverse of T if and only if $ST = TS$ and either S is a left 2-inverse of T, or S is a right 2-inverse of T.*

Proof. By definition, S being a 2-inverse of T implies that

$$S^2T^2 - 2ST + I = 0, \qquad T^2S^2 - 2TS + I = 0. \tag{4}$$

Therefore

$$S(2T - ST^2) = I, \qquad (2T - T^2S)S = I. \tag{5}$$

So both $2T - ST^2$ and $2T - T^2S$ are the inverses of S. Thus $ST^2 = T^2S$ and $S^2T^2 = T^2S^2$. Now substituting $S^2T^2 = T^2S^2$ into (4), we get $ST = TS$. ∎

The following example shows that the above result cannot extend to $n \geq 3$.

Example 10. For any two constants $a \neq 0$ and $b \neq 0$, let

$$S = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

A direct computation shows that S is a strict 3-inverse of T. But

$$ST = \begin{bmatrix} 1 & b \\ a & ab + 1 \end{bmatrix} \neq TS = \begin{bmatrix} 1 + ab & b \\ a & 1 \end{bmatrix}.$$

So $S \neq T^{-1} + Q$ for any nilpotent Q such that $Q^3 = 0$ and $QT^{-1} = T^{-1}Q$. Nevertheless

$$S = I + Q_1 \quad \text{where} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q_1 = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}.$$

Furthermore I is a strict 2-inverse of T, $Q_1^2 = 0$ and $Q_1 \cdot I = I \cdot Q_1$.

By combining the previous two results, we get the following corollary which gives a complete characterization of a 2-inverse S of T.

Corollary 11. *If $T \in B(X)$ has a 2-inverse S, then $S = T^{-1} + Q$ where $Q^2 = 0$ and $QT^{-1} = T^{-1}Q$.*

By Corollary 11, the matrix

$$T = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \qquad \lambda \neq 0,\ \mu \neq 0,\ \lambda \neq \mu,$$

does not have a strict 2-inverse since the only Q such that $QT = TQ$ is a multiple of the identity, which is not nilpotent. A direct calculation shows that T has neither a strict left 2-inverse nor a strict right 2-inverse.

When X is a finite-dimensional vector space, if S is a left inverse of T, then in fact S is an inverse of T. Thus we ask: if S is a left n-inverse of T on a finite-dimensional vector space, is S automatically an n-inverse of T?

3. Products and tensor products of left n-invertible operators. In this section we first discuss the product of left n-inverses.

Lemma 12. *Assume $S_1, S_2, T_1, T_2 \in B(X)$ and $[S_1, S_2] = [T_1, T_2] = [T_1, S_2] = 0$. Then*

$$\beta_n(S_1S_2, T_1T_2) = \sum_{k=0}^{n} \binom{n}{k} S_1^{n-k}\beta_k(S_1, T_1)T_1^{n-k}\beta_{n-k}(S_2, T_2) \tag{6}$$
$$= \sum_{k=0}^{n} \binom{n}{k} \beta_k(S_1, T_1)S_2^k\beta_{n-k}(S_2, T_2)T_2^k.$$

Proof. We first give a heuristic argument. Note that

$$(y_1y_2x_1x_2 - 1)^n = ([y_1x_1 - 1] + y_1[y_2x_2 - 1]x_1)^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} y_1^{n-k}(y_1x_1 - 1)^k x_1^{n-k}(y_2x_2 - 1)^{n-k}.$$

The lemma follows by substituting S_1 for y_1, T_1 for x_1, S_2 for y_2 and T_2 for x_2 in the above formula using the functional calculus defined in (1).

For clarity, we now prove the lemma by induction. For $n = 1$, note that

$$\text{Left side of (6)} = \beta_1(S_1S_2, T_1T_2) = S_1S_2T_1T_2 - I.$$

Since by definition $\beta_0(S_1,T_1)=\beta_0(S_2,T_2)=I$,

$$\begin{aligned}
\text{Right side of (6)} &= \sum_{k=0}^{1}\binom{1}{k} S_1^{1-k}\beta_k(S_1,T_1)T_1^{1-k}\beta_{1-k}(S_2,T_2)\\
&= S_1\beta_0(S_1,T_1)T_1\beta_1(S_2,T_2)+\beta_1(S_1,T_1)\beta_0(S_2,T_2)\\
&= S_1T_1(S_2T_2-I)+(S_1T_1-I)\\
&= S_1T_1S_2T_2-S_1T_1+S_1T_1-I=S_1T_1S_2T_2-I\\
&= S_1S_2T_1T_2-I,
\end{aligned}$$

where in the last equality we use the assumption that T_1 and S_2 are commuting. Thus (6) holds for $n=1$. Assume now (6) holds for n. By (2) and the induction hypothesis,

$$\begin{aligned}
&\beta_{n+1}(S_1S_2,T_1T_2) = S_1S_2\beta_n(S_1S_2,T_1T_2)T_1T_2-\beta_n(S_1S_2,T_1T_2)\\
&\quad= S_1S_2\beta_n(S_1S_2,T_1T_2)T_1T_2-S_1\beta_n(S_1S_2,T_1T_2)T_1\\
&\qquad+S_1\beta_n(S_1S_2,T_1T_2)T_1-\beta_n(S_1S_2,T_1T_2)\\
&\quad= S_1\left[\sum_{k=0}^{n}\binom{n}{k}S_1^{n-k}\beta_k(S_1,T_1)T_1^{n-k}[S_2\beta_{n-k}(S_2,T_2)T_2-\beta_{n-k}(S_2,T_2)]\right]T_1\\
&\qquad+\sum_{k=0}^{n}\binom{n}{k}S_1^{n-k}[S_1\beta_k(S_1,T_1)T_1-\beta_k(S_1,T_1)]T_1^{n-k}\beta_{n-k}(S_2,T_2)\\
&\quad= S_1\left[\sum_{k=0}^{n}\binom{n}{k}S_1^{n-k}\beta_k(S_1,T_1)T_1^{n-k}\beta_{n+1-k}(S_2,T_2)\right]T_1\\
&\qquad+\sum_{k=0}^{n}\binom{n}{k}S_1^{n-k}\beta_{k+1}(S_1,T_1)T_1^{n-k}\beta_{n-k}(S_2,T_2)\\
&\quad= \sum_{k=0}^{n}\binom{n}{k}S_1^{n-k+1}\beta_k(S_1,T_1)T_1^{n-k+1}\beta_{n+1-k}(S_2,T_2)\\
&\qquad+\sum_{k=1}^{n+1}\binom{n}{k-1}S_1^{n+1-k}\beta_k(S_1,T_1)T_1^{n+1-k}\beta_{n+1-k}(S_2,T_2)\\
&\quad= \sum_{k=0}^{n+1}\left[\binom{n}{k}+\binom{n}{k-1}\right]S_1^{n+1-k}\beta_k(S_1,T_1)T_1^{n+1-k}\beta_{n+1-k}(S_2,T_2)\\
&\quad= \sum_{k=0}^{n+1}\binom{n+1}{k}S_1^{n+1-k}\beta_k(S_1,T_1)T_1^{n+1-k}\beta_{n+1-k}(S_2,T_2),
\end{aligned}$$

where in the third and the fifth equalities we use the assumption that $[S_1,S_2]=[T_1,T_2]=[T_1,S_2]=0$, in the fourth equality we use (2), and the remaining equalities follow from rewriting the summation and the combinatorial fact that $\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}$ and the convention that $\binom{n}{k-1}=0$ for $k=0$ and $\binom{n}{k}=0$ for $k=n+1$. This proves that (6) holds for $n+1$. ■

Part of the following theorem belongs to Duggal and Müller [17] (see the remark after this theorem).

THEOREM 13. *Assume $S_1, S_2, T_1, T_2 \in B(X)$ and $[S_1, S_2]=[T_1, T_2]=[T_1, S_2]=0$.*

(a) *If S_1 is a left m-inverse of T_1 and S_2 is a left l-inverse of T_2, then S_1S_2 is a left n-inverse of T_1T_2 with $n=m+l-1$. Furthermore S_1S_2 is a strict left n-inverse of T_1T_2 if and only if*
$$\beta_{m-1}(S_1,T_1)\beta_{l-1}(S_2,T_2)\neq 0.$$
(b) *If S_1 is a right m-inverse of T_1 and S_2 is a right l-inverse of T_2, then S_1S_2 is a right n-inverse of T_1T_2 with $n=m+l-1$. Furthermore S_1S_2 is a strict right n-inverse of T_1T_2 if and only if $\beta_{l-1}(T_2,S_2)\beta_{m-1}(T_1,S_1)\neq 0$.*
(c) *If S_1 is a m-inverse of T_1 and S_2 is a l-inverse of T_2, then S_1S_2 is a n-inverse of T_1T_2 with $n=m+l-1$. Furthermore S_1S_2 is a strict n-inverse of T_1T_2 if and only if either $\beta_{m-1}(S_1,T_1)\beta_{l-1}(S_2,T_2)\neq 0$ or $\beta_{l-1}(T_2,S_2)\beta_{m-1}(T_1,S_1)\neq 0$.*

Proof. The proof is straightforward by using Lemma 12 and similar to the proof of Theorem 2. We will only prove (a). Let $n=m+l-1$. Since $\beta_k(S_1,T_1)=0$ for $k\geq m$ and $\beta_{n-k}(S_2,T_2)=0$ for $k<m$, we have
$$\beta_n(S_1S_2,T_1T_2)=\sum_{k=0}^{n}\binom{n}{k}S_1^{n-k}\beta_k(S_1,T_1)T_1^k\beta_{n-k}(S_2,T_2)=0.$$
Furthermore, by Lemma 4, $S_1^{l-1}\beta_{m-1}(S_1,T_1)T_1^{l-1}=\beta_{m-1}(S_1,T_1)$, so we have
$$\begin{aligned}\beta_{n-1}(S_1S_2,T_1T_2)&=\sum_{k=0}^{n-1}\binom{n-1}{k}S_1^{n-1-k}\beta_k(S_1,T_1)T_1^{n-1-k}\beta_{n-1-k}(S_2,T_2)\\&=\binom{n-1}{m-1}S_1^{l-1}\beta_{m-1}(S_1,T_1)T_1^{l-1}\beta_{l-1}(S_2,T_2)\\&=\binom{n-1}{m-1}\beta_{m-1}(S_1,T_1)\beta_{l-1}(S_2,T_2).\end{aligned}$$
This completes the proof of the theorem. ■

REMARK 14. In fact the condition $[T_1,S_2]=0$ is not needed in the first half of the above theorem as shown by [17, Corollary 2.6].

The following result can also be easily seen by definition; we still state it for completeness.

Corollary 15. *Assume* $S_1, S_2, T_1, T_2 \in B(X)$ *and* $[S_1, S_2] = [T_1, T_2] = 0$. *If* S_1 *is a left inverse* (*or a right inverse or an inverse*) *of* T_1 *and* S_2 *is a strict left* l*-inverse* (*or a right inverse or an inverse*) *of* T_2, *then* S_1S_2 *is a strict left* l*-inverse* (*or a right inverse or an inverse*) *of* T_1T_2.

Again if $m > 1$ and $l > 1$, then $\beta_{m-1}(S_1, T_1)\beta_{l-1}(S_2, T_2)$ could be zero. An example is given in [17, p. 121]. Here we give a more transparent example by using the direct sum. Let

$$S_1 = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}, \quad S_2 = \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix}, \quad T_1 = \begin{bmatrix} B_1 & 0 \\ 0 & I \end{bmatrix}, \quad T_2 = \begin{bmatrix} I & 0 \\ 0 & B_2 \end{bmatrix},$$

where $A_1, B_1 \in B(H)$, A_1 is a strict left m-inverse of B_1 and A_2 is a strict left l-inverse of B_2. Since for any $k > 0$,

$$\beta_k(S_1S_2, T_1T_2) = \begin{bmatrix} \beta_k(A_1, B_1) & 0 \\ 0 & \beta_k(A_2, B_2) \end{bmatrix},$$

S_1S_2 is a strict left n-inverse of T_1T_2 with $n = \max\{m, l\}$. But $n = \max\{m, l\} < m + l - 1$ if $m > 1$ and $l > 1$.

Let X and Y be two Banach spaces, and let $X \otimes Y$ denote the tensor product Banach space with an appropriate norm. Again for our approach which is mostly algebraic, the norm seems irrelevant as long as it has the property that if $A \in B(X)$ and $B \in B(Y)$, then $A \otimes B \in B(X \otimes Y)$. More precise results for tensor products of operators are obtained by applying Theorems 2 and 13 to tensor products of operators. We will first prove a lemma which is similar to one of the several equivalent conditions for a strict left n-inverse in [17, Theorem 2.10]. See also a similar result for an m-isometry in [18, Proposition 3].

Lemma 16. *If* S *is a strict left* m*-inverse of* T, *then for any* $n \geq m$, *the list of operators* $\{S^{n-k}\beta_k(S, T)T^{n-k} : k = 0, 1, \ldots, m-1\}$ *or the list of operators* $\{\beta_k(S, T)T^{n-k} : k = 0, 1, \ldots, m-1\}$ *is linearly independent. If* Q *is a nilpotent operator of order* l, *then the list of operators* $\{I, Q, Q^2, \ldots, Q^{l-1}\}$ *is linearly independent.*

Proof. We will prove $\{\beta_k(S, T)T^{n-k} : k = 0, 1, \ldots, m-1\}$ is linearly independent. Assume for some constants a_k,

$$\sum_{k=0}^{m-1} a_k\beta_k(S, T)T^{n-k} = 0.$$

Then multiplying the above equation on the left by S and on the right by

T and subtracting the resulting two equations, we obtain

$$\begin{aligned}&\sum_{k=0}^{m-1} a_k S\beta_k(S,T)TT^{n-k} - \sum_{k=0}^{m-1} a_k\beta_k(S,T)T^{n-k}\\ &= \sum_{k=0}^{m-1} a_k[S\beta_k(S,T)T - \beta_k(S,T)]T^{n-k} = \sum_{k=0}^{m-1} a_k[\beta_{k+1}(S,T)]T^{n-k} = 0.\end{aligned}$$

Applying the procedure $m-2$ more times, we get

$$\sum_{k=0}^{m-1} a_k\beta_{k+j}(S,T)T^{n-k} = 0, \quad j = 1,\dots,m-1.$$

Set $j = m-1$; then $a_0 = 0$ since by Lemma 4, $\beta_{m-1}(S,T)T^n \neq 0$. Now setting $j = m-2$, we find $a_1 = 0$ and so on. Therefore all a_k are zero. ∎

We also state the following simple fact as a lemma. For $f \in X$ and $g^* \in X^*$ (the dual space of X), $\langle f, g^*\rangle = g^*(f)$.

Lemma 17. *Let $A_i \in B(X)$ and $B_i \in B(Y)$ for $i = 1,\dots,n$. If $A_1\otimes B_1 + \dots + A_n \otimes B_n = 0$ and $\{A_1,\dots,A_n\}$ is linearly independent, then $B_i = 0$ all i.*

Proof. Without loss of generality, assume $B_1 \neq 0$. Then there exist $y \in Y$ and $y \in Y^*$ such that $y^*(B_1y) \neq 0$. Let $x \in X$ and $x^* \in X^*$. Then

$$\langle [A_1 \otimes B_1 + \dots + A_n \otimes B_n](x \otimes y), x^* \otimes y^*\rangle = x^*\Big(\sum_{i=1}^{n} y^*(B_iy)A_ix\Big) = 0.$$

Since x and x^* are arbitrary, $\sum_{i=1}^{n} y^*(B_iy)A_i = 0$, contradicting the linear independence of $\{A_1,\dots,A_n\}$. ∎

Theorem 18. *Assume $S,T \in B(X)$ and $Q \in B(Y)$. Then any two of statements* (a)–(c) *imply the third, where:*

(a) *S is a strict left m-inverse of T.*
(b) *Q is a nilpotent operator of order l.*
(c) *$S\otimes I + I \otimes Q$ is a strict left n-inverse of $T \otimes I$ where $n = m+l-1$.*

Proof. We first show (a) and (b) imply (c). Indeed, by Theorem 2, we only need to note that

$$\beta_{n-1}(S\otimes I + I\otimes Q, T\otimes I) = \beta_{m-1}(S,T)T^{l-1} \otimes Q^{l-1} \neq 0,$$

since by Lemma 4, $\beta_{m-1}(S,T)T^{l-1} \neq 0$.

We next show (a) and (c) imply (b). By Lemma 1 and (a),

$$\begin{aligned}\beta_n(S\otimes I+I\otimes Q,T\otimes I)&=\sum_{k=0}^{n}\binom{n}{k}\beta_k(S,T)T^{n-k}\otimes Q^{n-k}\\&=\sum_{k=0}^{m-1}\binom{n}{k}\beta_k(S,T)T^{n-k}\otimes Q^{n-k}=0.\end{aligned}$$

Now by Lemma 16, $\{\beta_k(S,T)T^{n-k}:k=0,1,\dots,m-1\}$ is linearly independent, so $Q^{n-k}=0$ for $0\le k\le m-1$. Therefore $Q^l=0$ for $l=n-(m-1)$. Also $Q^{l-1}\neq 0$, since otherwise $S\otimes I+I\otimes Q$ will not be a strict left n-inverse of $T\otimes I$.

Finally, we prove that (b) and (c) imply (a). Again by Lemma 1 and (b),

$$\begin{aligned}\beta_n(S\otimes I+I\otimes Q,T\otimes I)&=\sum_{k=0}^{n}\binom{n}{k}\beta_k(S,T)T^{n-k}\otimes Q^{n-k}\\&=\sum_{k=n-(l-1)}^{n}\binom{n}{k}\beta_k(S,T)T^{n-k}\otimes Q^{n-k}=0.\end{aligned}$$

By the linear independence of $\{Q^{n-k}:n-(l-1)\le k\le n\}$ and Lemma 17,

$$\beta_k(S,T)T^{n-k}=0\quad\text{ for } n-(l-1)\le k\le n.$$

In particular, letting $k=n,n-1$, we see that $\beta_n(S,T)=0$ and $\beta_{n-1}(S,T)T=0$. But by Lemma 4,

$$\beta_{n-1}(S,T)=S\beta_{n-1}(S,T)T.$$

Thus $\beta_{n-1}(S,T)=0$. By Lemma 4 again and the fact $\beta_{n-2}(S,T)T^2=0$ ($k=n-2$), we have

$$\beta_{n-2}(S,T)=S\beta_{n-2}(S,T)T=S^2\beta_{n-2}(S,T)T^2=0.$$

Continuing this process until $k=n-(l-1)$, we obtain $\beta_m(S,T)=0$ for $m=n-(l-1)$. Also $\beta_{m-1}(S,T)\neq 0$, since otherwise $S\otimes I+I\otimes Q$ will not be a strict left n-inverse of $T\otimes I$. ∎

Now we state a theorem for tensor products of left n-inverses which essentially puts together Theorems 2.11, 2.12 and 2.13 from Duggal and Müller [17]. The proof is similar to the proof of the above theorem by using Lemma 12 instead of Lemma 1, but it is short, so we include it for completeness.

THEOREM 19 ([17]). *Assume $S_1,T_1\in B(X)$ and $S_2,T_2\in B(Y)$. Then any two of statements* (a)–(c) *imply the third, where:*

(a) *S_1 is a strict left m-inverse of T_1.*
(b) *S_2 is a strict left l-inverse of T_2.*
(c) *$S_1\otimes S_2$ is a strict left n-inverse of $T_1\otimes T_2$ where $n=m+l-1$.*

Proof. We first show (a) and (b) imply (c). Indeed, by Theorem 13, we only need to note that

$$\beta_{n-1}(S_1 \otimes S_2, T_1 \otimes T_2) = \beta_{m-1}(S_1, T_1) \otimes \beta_{l-1}(S_2, T_2) \neq 0.$$

We next show (a) and (c) imply (b). By Lemma 12 and (a),

$$\begin{aligned} \beta_n(S_1 \otimes S_2, T_1 \otimes T_2) &= \sum_{k=0}^{n} \binom{n}{k} S_1^{n-k} \beta_k(S_1, T_1) T_1^{n-k} \otimes \beta_{n-k}(S_2, T_2) \\ &= \sum_{k=0}^{m-1} \binom{n}{k} S_1^{n-k} \beta_k(S_1, T_1) T_1^{n-k} \otimes \beta_{n-k}(S_2, T_2) = 0. \end{aligned}$$

Now by Lemma 16, $\{S_1^{n-k} \beta_k(S_1, T_1) T_1^{n-k} : k = 0, 1, \ldots, m-1\}$ is linearly independent, so $\beta_{n-k}(S_2, T_2) = 0$ for $0 \leq k \leq m-1$. Therefore $\beta_l(S_2, T_2) = 0$ for $l = n - (m-1)$. Also $\beta_{l-1}(S_2, T_2) \neq 0$, since otherwise $S_1 \otimes S_2$ will not be a strict left n-inverse of $T_1 \otimes T_2$.

By symmetry, (b) and (c) also imply (a). ∎

We remark that there are also right n-inverses and n-inverses versions of the above two theorems.

4. A promising conjecture. Can we improve Theorems 18 and 19? We make the following conjecture which, if confirmed, completely characterizes when a tensor product of two operators is a left n-inverse. A related conjecture for tensor products of n-isometries on Hilbert spaces has been proved by the author in [18, Theorem 7]; in fact, the proof of Proposition 23 below is adapted from the n-isometries case.

CONJECTURE 20. *The tensor product $S_1 \otimes S_2$ is a strict left n-inverse of $T_1 \otimes T_2$ if and only if there exist m and l such that $m + l - 1 = n$ and S_1 is a strict left m-inverse of $(1/\lambda)T_1$ for some constant λ and S_2 is a strict left l-inverse of λT_2.*

We now prove the conjecture for $n = 1, 2, 3$. The proof is by a careful algebra argument and is similar in nature to the proof for n-isometries for $n = 1, 2, 3$ [11, Theorems 3.2 and 4.2]. It seems that the only fact used is Lemma 17. Note we use I to denote the identity on both X and Y.

PROPOSITION 21. *The above conjecture is true for $n = 1, 2, 3$.*

Proof. We first deal with the case $n = 2$. Assume

$$S_1^2 T_1^2 \otimes S_2^2 T_2^2 - 2 S_1 T_1 \otimes S_2 T_2 + I \otimes I = 0. \tag{7}$$

CASE 1: $S_1 T_1 = aI$ for some constant a. That is, S_1 is a strict left 1-inverse of $(1/a)T_1$. Then

$$S_1^2 T_1^2 = S_1(S_1 T_1) T_1 = S_1(aI) T_1 = a S_1 T_1 = a^2 I. \tag{8}$$

Therefore equation (7) becomes

$$a^2 I \otimes S_2^2 T_2^2 - 2aI \otimes S_2 T_2 + I \otimes I = I \otimes (a^2 S_2^2 T_2^2 - 2aS_2 T_2 + I) = 0.$$

So $a^2 S_2^2 T_2^2 - 2aS_2 T_2 + I = 0$ and S_2 is a left 2-inverse of aT_2. Strictness follows from Theorem 19.

CASE 2: $S_1^2 T_1^2 = aI + bS_1 T_1$. Then

$$\begin{aligned}(aI + bS_1T_1) \otimes S_2^2T_2^2 - 2S_1T_1 \otimes S_2T_2 + I \otimes I \\ &= I \otimes aS_2^2T_2^2 + S_1T_1 \otimes bS_2^2T_2^2 - 2S_1T_1 \otimes S_2T_2 + I \otimes I \\ &= I \otimes (aS_2^2T_2^2 + I) + S_1T_1 \otimes (bS_2^2T_2^2 - 2S_2T_2) = 0.\end{aligned}$$

Therefore

$$aS_2^2T_2^2 + I = 0 \quad \text{and} \quad bS_2^2T_2^2 - 2S_2T_2 = 0$$

Clearly $a \neq 0$, $b \neq 0$. The two equations above yield $S_2T_2 = -\frac{b}{2a}I$. By symmetry, this reduces to Case 1.

Now we deal with the case $n = 3$. To ease notation we let

$$A_i = S_1^i T_1^i, \qquad B_i = S_2^i T_2^i, \qquad i = 1, 2, 3.$$

Then $S_1 \otimes S_2$ being a strict left 3-inverse of $T_1 \otimes T_2$ means

$$A_3 \otimes B_3 - 3A_2 \otimes B_2 + 3A_1 \otimes B_1 - I \otimes I = 0. \tag{9}$$

CASE 1: $A_1 = aI$. That is, S_1 is a strict left 1-inverse of $(1/a)T_1$. Furthermore, as in (8), $A_2 = a^2 I$ and $A_3 = a^3 I$. Thus equation (9) becomes

$$I \otimes (a^3 B_3 - 3a^2 B_2 + 3aB_1 - I) = 0.$$

Hence S_2 is a left 3-inverse of aT_2.

CASE 2: $A_2 = aI + bA_1$, and I and A_1 are linearly independent. We also assume I and B_1 are linearly independent. Otherwise by symmetry, this reduces to Case 1. As in (8),

$$A_3 = aA_1 + bA_2 = aA_1 + b(aI + bA_1) = abI + (a + b^2)A_1.$$

Plugging the formulas for A_2 and A_3 into (9) and rearranging the terms, we obtain

$$I \otimes (abB_3 - 3aB_2 - I) + A_1 \otimes [(a + b^2)B_3 - 3bB_2 + 3B_1] = 0.$$

Therefore

$$abB_3 - 3aB_2 - I = 0, \tag{10}$$

$$(a + b^2)B_3 - 3bB_2 + 3B_1 = 0. \tag{11}$$

Write (10) as $a(bB_3 - 3B_2) = I$ and plug this into (11) to obtain

$$\begin{aligned}(a + b^2)B_3 - 3bB_2 + 3B_1 &= aB_3 + b(bB_3 - 3B_2) + 3B_1 \\ &= aB_3 + \frac{b}{a}I + 3B_1 = 0.\end{aligned} \tag{12}$$

Multiplying (12) by b and subtracting (10) gives

$$3aB_2 + 3bB_1 + \left(\frac{b^2}{a} + 1\right)I = 0. \tag{13}$$

Multiply (13) on the left by S_2 and on the right by T_2 to get

$$3aB_3 + 3bB_2 + \left(\frac{b^2}{a} + 1\right)B_1 = 0. \tag{14}$$

Now (14) minus 3 times (12) gives

$$3bB_2 + \left(\frac{b^2}{a} + 1 - 9\right)B_1 - 3\frac{b}{a}I = 0. \tag{15}$$

Finally, multiplying (13) by b/a and subtracting (15) yields

$$\left(3\frac{b^2}{a} - \frac{b^2}{a} + 8\right)B_1 + \left[\frac{b}{a}\left(\frac{b^2}{a} + 1\right) + 3\frac{b}{a}\right] = 0.$$

Since I and B_1 are linearly independent, we have

$$2\frac{b^2}{a} + 8 = 0 \quad \text{and} \quad \frac{b}{a}\left(\frac{b^2}{a} + 1\right) + 3\frac{b}{a} = 0.$$

The two equations above reduce to $b^2 = -4a$. Set $\lambda = 2/b$. Then $b = 2/\lambda$ and $a = -b^2/4 = -1/\lambda^2$. Therefore

$$\begin{aligned} A_2 - aI - bA_1 &= A_2 - \frac{2}{\lambda}A_1 + \frac{1}{\lambda^2}I \\ &= \frac{1}{\lambda^2}(\lambda^2 A_2 - 2\lambda A_1 + I) = 0. \end{aligned}$$

That is, S_1 is a left 2-inverse of λT_1. Now by Theorem 19, S_2 is a left 2-inverse of $(1/\lambda)T_2$. The proof is complete. ∎

Is there an analogous conjecture related to Theorem 18? In fact in this case we have a theorem. The proof uses the approximate point spectrum $\sigma_{\mathrm{ap}}(Q)$ of an operator Q instead of the algebraic approach as in the previous proposition.

THEOREM 22. *Assume $S, T \in B(X)$ and $Q \in B(Y)$. The tensor sum $S \otimes I + I \otimes Q$ is a strict left n-inverse of $T \otimes I$ if and only if there exist m and l such that $m + l - 1 = n$ and $S + \lambda I$ is a strict left m-inverse of T for some constant λ and $Q - \lambda I$ is a nilpotent operator of order l.*

Proof. The "if" part is proved in Theorem 18. We will prove the "only if" part. Let λ be any number in $\sigma_{\mathrm{ap}}(Q)$. That is, there is $y_i \in Y$ of unit norm such that $(Q - \lambda I)y_i \to 0$ as $i \to \infty$. Let $y_i^* \in Y^*$ be such that $\langle y_i, y_i^* \rangle = y_i^*(y_i) = 1$. It is clear that for any $j \geq 0$,

$$\langle Q^j y_i, y_i^* \rangle \to \lambda^j \quad \text{as } i \to \infty. \tag{16}$$

Then for any $x \in X$ and $x^* \in X^*$, by Lemma 1,

$$\begin{aligned} 0 &= \langle \beta_n(S \otimes I + I \otimes Q, T \otimes I)(x \otimes y_i), x^* \otimes y_i^* \rangle \\ &= \left\langle \left[\sum_{k=0}^{n} \binom{n}{k} \beta_k(S,T)T^{n-k} \otimes Q^{n-k} \right] (x \otimes y_i), x^* \otimes y_i^* \right\rangle \\ &= \sum_{k=0}^{n} \binom{n}{k} \langle \beta_k(S,T)T^{n-k}x, x^* \rangle \langle Q^{n-k} y_i, y_i^* \rangle. \end{aligned}$$

Taking the limit by using (16), we obtain

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} \langle \beta_k(S,T)T^{n-k}x, x^* \rangle \lambda^{n-k} &= \left\langle \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} \beta_k(S,T)T^{n-k}x, x^* \right\rangle \\ &= \langle \beta_n(S + \lambda I, T)x, x^* \rangle = 0, \end{aligned}$$

where we have used Lemma 1 in a reverse way. Therefore $S + \lambda I$ is a left n-inverse of T. Let $m \le n$ be such that $S + \lambda I$ is a strict left m-inverse of T. Note that

$$(S + \lambda I) \otimes I + I \otimes (Q - \lambda I) = S \otimes I + I \otimes Q.$$

Set $l = n - (m-1)$; by Theorem 18, $Q - \lambda I$ a nilpotent operator of order l. ■

We remark that the n-isometries version of the above result for tensor products of operators on Hilbert spaces is more involved and is proved to be almost true by the author in [18, Theorem 12]. The above approach leads to the confirmation of Conjecture 20 in a very general case. We need to make the following technical definition. For $S, T \in B(X)$, we say S and T *are not orthogonal* if there exist $\lambda \in \sigma_{\mathrm{ap}}(T)$, $\mu \in \sigma_{\mathrm{ap}}(S^*)$, $x_i \in X$, $x_i^* \in X^*$ such that $\|x_i\| = \|x_i^*\| = 1$, $(T - \lambda I)x_i \to 0$ and $(S^* - \mu I)x_i^* \to 0$ but $\langle x_i, x_i^* \rangle = x_i^*(x_i) \nrightarrow 0$ as $i \to \infty$. By passing to a subsequence we can assume $x_i^*(x_i) \to \alpha \neq 0$.

Proposition 23. *Let $S_1, T_1 \in B(X)$ and $S_2, T_2 \in B(Y)$. Assume either S_1 and T_1 are not orthogonal, or S_2 and T_2 are not orthogonal. Then $S_1 \otimes S_2$ is a strict left n-inverse of $T_1 \otimes T_2$ if and only if there exist m and l such that $m + l - 1 = n$ and S_1 is a strict left m-inverse of $(1/\alpha)T_1$ for some constant α and S_2 is a strict left l-inverse of αT_2.*

Proof. Assume $S_1 \otimes S_2$ is a strict left n-inverse of $T_1 \otimes T_2$, and S_1 and T_1 are not orthogonal. Let $\lambda \in \sigma_{\mathrm{ap}}(T_1)$, $\mu \in \sigma_{\mathrm{ap}}(S_1^*)$, $x_i \in X$, $x_i^* \in X^*$ be such that $\|x_i\| = \|x_i^*\| = 1$, $(T_1 - \lambda I)x_i \to 0$, $(S_1^* - \mu I)x_i^* \to 0$ and $\langle x_i, x_i^* \rangle = x_i^*(x_i) \to \alpha \neq 0$ as $i \to \infty$. Note that $\lambda \neq 0$ since if $S_1 \otimes S_2$ is a strict left n-inverse of $T_1 \otimes T_2$, then $T_1 \otimes T_2$ is left invertible, so both T_1 and T_2 are left invertible. Similarly $\mu \neq 0$ since both S_1 and S_2 are right invertible. Then for any $y \in Y$ and $y^* \in Y^*$,

$$\begin{aligned}0 &= \langle \beta_n(S_1 \otimes S_2, T_1 \otimes T_2)x_i \otimes y, x_i^* \otimes y^*\rangle \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\langle [S_1^k T_1^k \otimes S_2^k T_2^k]x_i \otimes y, x_i^* \otimes y^*\rangle \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\langle T_1^k x_i, S_1^{*k}x_i^*\rangle\langle S_2^k T_2^k y, y^*\rangle.\end{aligned}$$

By letting $i \to \infty$ and noting that $(T_1^k - \lambda^k I)x_i \to 0$ and $(S_1^{*k} - \mu^k I)x_i^* \to 0$, we obtain

$$\lim_{i\to\infty}\langle T_1^k x_i, S_1^{*k}x_i^*\rangle = \lim_{i\to\infty}\langle \lambda^k x_i, \mu^k x_i^*\rangle = \lambda^k\mu^k \lim_{i\to\infty} x_i^*(x_i) = \lambda^k\mu^k\alpha.$$

Therefore

$$\begin{aligned}0 &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\lambda^k\mu^k\alpha\langle S_2^k T_2^k y, y^*\rangle \\ &= \alpha\Big\langle \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} S_2^k(\lambda\mu T_2)^k y, y^*\Big\rangle \\ &= \alpha\langle \beta_n(S_2, \lambda\mu T_2)y, y^*\rangle.\end{aligned}$$

Hence $\beta_n(S_2, \lambda\mu T_2) = 0$ and S_2 is a left n-inverse of $\lambda\mu T_2$. Let $l \le n$ be such that S_2 is a strict left l-inverse of $\lambda\mu T_2$. Set $m = n - (l-1)$; by Theorem 19, S_1 is a strict left m-inverse of $(1/\lambda\mu)T_1$. ∎

One can replace the condition that S_1 and T_1 are not orthogonal by the condition $[S_1, T_1] = 0$. In this case, $S_1^k T_1^k = (S_1T_1)^k$. Let $\lambda \in \sigma_{\rm ap}(S_1T_1)$, $x_i \in X$, $x_i^* \in X^*$ be such that $\|x_i\| = \|x_i^*\| = 1$, $(S_1T_1 - \lambda I)x_i \to 0$ and $\langle x_i, x_i^*\rangle = 1$. Noting that $\lambda \neq 0$, the rest of the proof is similar.

5. Essential left n-inverses and left n-invertible elementary operators. This last section really consists of a few remarks. In Duggal and Müller [17], S is said to be an *essential left n-inverse* of T if

$$\beta_n(S,T) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}S^kT^k = K$$

for some compact operator $K \in K(X)$. Furthermore it is shown in [17] that by using a construction known in the literature as the Sadovskiĭ/Buoni, Harte, Wickstead construction [23, p. 159], one can represent the Calkin algebra $B(X)/K(X)$ as an algebra of operators on a suitable Banach space and thus all the previous results on left n-invertible operators transfer to corresponding results on essentially left n-invertible operators.

As mentioned in the introduction, an alternative approach is to work on a Banach algebra B with identity. Let s and t be two elements in B. We say s is a *left n-inverse* of t if

$$\beta_n(s,t) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} s^k t^k = 0.$$

All results, except possibly Theorem 22 and Proposition 23 where the approximate point spectrum and adjoint operators are used for the proofs, seem to hold because of the purely algebraic approach. For the small cases $n = 1, 2, 3$ in Theorem 22, an algebraic proof without using adjoint operators can also be given as in Proposition 21.

Now we introduce elementary operators. Let $S_1 \in B(X)$, $S_2 \in B(Y)$. The left multiplication operator L_{S_1} and the right multiplication R_{S_2} are defined by

$$L_{S_1}(W) = S_1 W, \quad R_{S_2}(W) = W S_2, \quad W \in B(Y,X).$$

The *elementary operator* $\tau_{S_1S_2}$ of length one and the *generalized derivation* $\delta_{S_1S_2}$ on $B(Y,X)$ are defined by

$$\tau_{S_1S_2}(W) = S_1 W S_2, \quad \delta_{S_1S_2}(W) = S_1 W - W S_2, \quad W \in B(Y,X).$$

Note that $\tau_{S_1S_2} = L_{S_1} R_{S_2}$, $\delta_{S_1S_2} = L_{S_1} - R_{S_2}$ and $[L_{S_1}, R_{S_2}] = 0$. Elementary operators such as $\tau_{S_1S_2}$ and $\delta_{S_1S_2}$ have been studied extensively in the past several decades: see for example the recent book [14] and also [11], [12], [15], [17], [18], [25] for works related to our paper. We will study left n-invertible operator $\tau_{S_1S_2}$ on $B(Y,X)$ and refer to Duggal and Müller [17] for the study of left n-invertible operators $\tau_{S_1S_2}$ on an operator ideal J of $B(Y,X)$ where by using the approach of [20], one can represent J as a tensor product Banach space. For $S_1, T_1 \in B(X)$ and $S_2, T_2 \in B(Y)$, note that

$$\beta_n(L_{S_1}, L_{T_1}) = L_{\beta_n(S_1,T_1)} \quad \text{and} \quad \beta_n(R_{S_2}, R_{T_2}) = R_{\beta_n(T_2,S_2)}.$$

Therefore L_{S_1} is a left n-inverse of L_{T_1} on $B(Y,X)$ if and only if S_1 is a left n-inverse of T_1 on X, while R_{S_2} is a left n-inverse of R_{T_2} on $B(Y,X)$ if and only if S_2 is a right n-inverse of T_2 on Y. Here are a couple of sample results.

THEOREM 24. *Let $S, T \in B(X)$ and $Q \in B(Y)$. Then δ_{SQ} is a strict left n-inverse of L_T on $B(Y,X)$ if and only if there exist m and l such that $m + l - 1 = n$ and $S + \lambda I_X$ is a strict left m-inverse of T on X for some constant λ and $Q - \lambda I_Y$ is a nilpotent operator on Y of order l.*

PROPOSITION 25. *Let $S_1, T_1 \in B(X)$ and $S_2, T_2 \in B(Y)$. For $n = 1, 2, 3$, $\tau_{S_1S_2}$ is a strict left n-inverse of $\tau_{T_1T_2}$ on $B(Y,X)$ if and only if there exist m and l such that $m + l - 1 = n$ and S_1 is a strict left m-inverse of $(1/\alpha)T_1$ on X for some constant α and S_2 is a strict right l-inverse of αT_2 on Y.*

Finally we state the following theorem for elementary operators of length two which in a way combines Theorems 2 and 13. We state the theorem in a slightly different way to avoid too many notations; we give a short and direct proof for clarity.

THEOREM 26. *Let $A_1, A_2 \in B(X)$ and $B_1, B_2 \in B(Y)$. Assume $[A_1, A_2] = [B_1, B_2] = 0$. Assume also A_1 is a left m-invertible operator, B_1 is a right j-invertible operator and either A_2 or B_2 is a nilpotent operator of order l. Then the operator Δ on $B(Y, X)$ defined by*

$$\Delta(W) = A_1 W B_1 + A_2 W B_2, \qquad W \in B(Y, X),$$

is a left n-invertible operator with $n = m + j + l - 2$.

Proof. Let $n_1 = m + j - 1$. We first prove $\tau_{A_1 B_1}$ is a left n_1-invertible operator. By the assumption, let C_1 be a left m-inverse of A_1 and D_1 be a right j-inverse of B_1. We show that $\tau_{C_1 D_1}$ is a left n_1-inverse of $\tau_{A_1 B_1}$. By Lemma 12 with $S_1 = L_{C_1}$, $S_2 = R_{D_1}$, $T_1 = L_{A_1}$ and $T_2 = R_{B_1}$, for any $W \in B(Y, X)$,

$$\begin{aligned}
(17) \qquad \beta_{n_1}(\tau_{C_1 D_1}, \tau_{A_1 B_1})(W) &= \beta_{n_1}(L_{C_1} R_{D_1}, L_{A_1} R_{B_1})(W) \\
&= \sum_{k=0}^{n_1} \binom{n_1}{k} L_{C_1}^{n_1-k} \beta_k(L_{C_1}, L_{A_1}) L_{A_1}^{n_1-k} \beta_{n_1-k}(R_{D_1}, R_{B_1})(W) \\
&= \sum_{k=0}^{n_1} \binom{n_1}{k} C_1^{n_1-k} \beta_k(C_1, A_1) A_1^{n_1-k} W \beta_{n_1-k}(D_1, B_1) = 0,
\end{aligned}$$

since $\beta_k(C_1, A_1) = 0$ if $k \geq m$ and $\beta_{n_1-k}(D_1, B_1) = 0$ if $k < m = n_1 - (j-1)$ (or $n_1 - k > j - 1$).

Now we show that $\tau_{C_1 D_1}$ is in fact a left n-inverse of Δ on $B(Y, X)$. By Lemma 1 with $T = \tau_{C_1 D_1}$, $S = \tau_{A_1 B_1}$ and $Q = \tau_{A_2 B_2}$, for any $W \in B(Y, X)$,

$$\begin{aligned}
\beta_n(\tau_{C_1 D_1}, \Delta)(W) &= \beta_n(\tau_{C_1 D_1}, \tau_{A_1 B_1} + \tau_{A_2 B_2})(W) \\
&= \Big(\sum_{k=0}^{n} \binom{n}{k} \tau_{C_1 D_1}^{n-k} \beta_k(\tau_{C_1 D_1}, \tau_{A_1 B_1}) \tau_{A_2 B_2}^{n-k} \Big)(W) \\
&= \sum_{k=0}^{n} \binom{n}{k} \tau_{C_1 D_1}^{n-k} \beta_k(\tau_{C_1 D_1}, \tau_{A_1 B_1})(A_2^{n-k} W B_2^{n-k}) = 0,
\end{aligned}$$

since if $k \geq n_1$, then $\beta_k(\tau_{C_1 D_1}, \tau_{A_1 B_1})(A_2^{n-k} W B_2^{n-k}) = 0$ by (17) (with W being $A_2^{n-k} W B_2^{n-k}$), and if $k < n_1 = n - (l-1)$ (or $n - k > l - 1$) then $A_2^{n-k} = 0$ or $B_2^{n-k} = 0$ by the assumption that either A_2 or B_2 is a nilpotent operator of order l and thus

$$\beta_k(\tau_{C_1 D_1}, \tau_{A_1 B_1})(A_2^{n-k} W B_2^{n-k}) = \beta_k(\tau_{C_1 D_1}, \tau_{A_1 B_1})(0) = 0.$$

The proof is complete. ∎

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