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Structure of Rademacher subspaces in Cesàro type spaces

by

Sergey V. Astashkin (Samara) and Lech Maligranda (Luleå)

Dedicated to the memory of Aleksander Pełczyński
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Abstract. The structure of the closed linear span $\mathcal{R}$ of the Rademacher functions in the Cesàro space Ces_∞ is investigated. It is shown that every infinite-dimensional subspace of $\mathcal{R}$ either is isomorphic to l_2 and uncomplemented in Ces_∞, or contains a subspace isomorphic to c_0 and complemented in $\mathcal{R}$. The situation is rather different in the p-convexification of Ces_∞ if $1 < p < \infty$.

1. Introduction. The behaviour of the Rademacher functions in the spaces $L_p = L_p[0,1]$ is well known. By the classical *Khintchine inequality*, there exists a constant $A_p > 0$ such that for all real numbers a_k, $k = 1, 2, \ldots,$ we have

$$A_p^{-1}\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2} \le \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{L_p} \le A_p\Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2}, \quad 0 < p < \infty,$$

that is, $\{r_n\}_{n=1}^{\infty}$ spans an isomorphic copy of l_2 in L_p for every $0 < p < \infty$. Moreover, the subspace $[r_n]$ is complemented in L_p for $1 < p < \infty$, and not complemented in L_1 since no complemented infinite-dimensional subspace of L_1 can be reflexive. Moreover, $\|\sum_{k=1}^{n} a_k r_k\|_{L_\infty[0,1]} = \sum_{k=1}^{n} |a_k|$, and so the Rademacher functions span in L_∞ an isometric copy of l_1, which is known to be uncomplemented (see [13, Theorem 2.b.4(ii)], [15, Theorem 1] and [2, Theorem 3.4]).

Investigations of Rademacher sums in the Cesàro function spaces $\mathrm{Ces}_p := \mathrm{Ces}_p[0,1]$ were initiated in [6]. The Cesàro spaces consist of all Lebesgue

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measurable real-valued functions f on $[0,1]$ such that

$$\|f\|_{\mathrm{Ces}_p} = \left[\int_0^1 \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx\right]^{1/p} < \infty \quad \text{for } 1 \le p < \infty$$

and

$$\|f\|_{\mathrm{Ces}_\infty} = \sup_{0<x\le 1} \frac{1}{x}\int_0^x |f(t)|\,dt < \infty \quad \text{for } p = \infty.$$

The latter space Ces_∞ appeared already in 1948 [10] (see also [14], [17, p. 469] and [16, p. 26]) and is known as the *Korenblyum–Kreĭn–Levin space* K.

Further, we will also consider the p-convexification of the space $K = \mathrm{Ces}_\infty$, $1 < p < \infty$, which will be denoted by K_p, consisting of all Lebesgue measurable real-valued functions f on $[0,1]$ such that the norm

$$\|f\|_{K_p} = \sup_{0<x\le 1} \left(\frac{1}{x}\int_0^x |f(t)|^p\,dt\right)^{1/p}$$

is finite.

The Cesàro function spaces Ces_p, $1 \le p \le \infty$, are not rearrangement invariant, and are not isomorphic to L_q-spaces for any $1 \le q \le \infty$ (see [5], [7] and [8], where also other properties are investigated). However, similarly to L_p-spaces, there is also an essential difference in the behaviour of Rademacher sums in Ces_p for $1 \le p < \infty$ and $p = \infty$. Namely, as proved in [6], for any $1 \le p < \infty$, the sequence $\{r_n\}_{n=1}^\infty$ is equivalent in Ces_p to the unit vector basis of l_2, i.e.,

$$B_p^{-1}\Big(\sum_{k=1}^\infty a_k^2\Big)^{1/2} \le \Big\|\sum_{k=1}^\infty a_k\,r_k\Big\|_{\mathrm{Ces}_p} \le B_p\Big(\sum_{k=1}^\infty a_k^2\Big)^{1/2}$$

for a suitable constant $B_p > 0$ and for all real a_k, $k = 1, 2, \ldots$. On the other hand, we have

$$(1.1)\qquad C_p^{-1}\Big[\Big(\sum_{k=1}^\infty a_k^2\Big)^{1/2} + \sup_{m\in\mathbb{N}}\Big|\sum_{k=1}^m a_k\Big|\Big] \le \Big\|\sum_{k=1}^\infty a_k\,r_k\Big\|_{K_p} \le C_p\Big[\Big(\sum_{k=1}^\infty a_k^2\Big)^{1/2} + \sup_{m\in\mathbb{N}}\Big|\sum_{k=1}^m a_k\Big|\Big],$$

with some constant $C_p > 0$ which depends only on $p \in [1,\infty)$. In particular, $\sum_{k=1}^\infty a_k r_k$ converges in K_p if and only if both $\sum_{k=1}^\infty a_k^2$ and $\sum_{k=1}^\infty a_k$ are convergent. Moreover, (1.1) shows that the Rademacher functions form a conditional basis in the subspace

$$\mathcal{R}_p := [r_k] \text{ spanned by } r_k,\ k = 1, 2, \ldots, \text{ in } K_p,\ 1 \le p < \infty.$$

The aim of this paper is to describe the geometrical structure of the space $\mathcal{R}_p$ for $1 \le p < \infty$. The following main results, which can be treated as a Kadec–Pełczyński type alternative for the Rademacher subspaces of K_p, indicate that their structures in the cases $p = 1$ and $1 < p < \infty$ are different.

THEOREM 1. *Every infinite-dimensional subspace of $\mathcal{R}$ either is isomorphic to l_2 and uncomplemented in $K = \mathrm{Ces}_\infty$, or contains a subspace isomorphic to c_0 and complemented in $\mathcal{R}$.*

THEOREM 2. *Every infinite-dimensional subspace of $\mathcal{R}_p$, $1 < p < \infty$, either is isomorphic to l_2 and complemented in K_p, or contains a subspace isomorphic to c_0 and complemented in $\mathcal{R}_p$.*

It is worth noting that comparing Theorem 2 with Leibov's results relating to the space of functions of bounded mean oscillation (see [11]) shows that the structures of the Rademacher subspaces in K_p, $1 < p < \infty$, and in BMO are similar. Generally speaking, this is not surprising, because Rademacher sums satisfy in BMO inequalities completely analogous to (1.1) (see also [4], where it is proved, among other results, that the subspace $[r_k]$ spanned by the Rademacher functions in BMO is not complemented in BMO). At the same time, it is instructive to emphasize the following point. In [11], Leibov uses the fact that the continuous embedding BMO $\hookrightarrow L_1$ factorizes through L_p for any $p \in (1, \infty)$, which allows him to apply the well-known Kadec–Pełczyński result about complementability of any subspace of L_p, $p \ge 2$, isomorphic to l_2. However, by contrast, the continuous embedding $\mathrm{Ces}_\infty \hookrightarrow L_p$ holds if and only if $p = 1$; hence we cannot now use the Kadec–Pełczyński argument, and the result obtained (Theorem 1) essentially differs from the one proved by Leibov [11].

In what follows, given two positive functions (quasi-norms) f and g we write $f \asymp g$ if there exists a constant $C > 0$ independent of all or of a part of parameters such that $C^{-1} f \le g \le Cf$. As usual, we denote by $[x_n]$ the closed linear span of a sequence $\{x_n\}_{n=1}^\infty$ in a Banach space X, and set $\|f\|_d := \left(\sum_{n=1}^\infty a_n^2\right)^{1/2}$ for a Rademacher sum $f = \sum_{n=1}^\infty a_n r_n$ converging a.e. on $[0, 1]$. Moreover, we write $K_1 = K$ and $\mathcal{R}_1 = \mathcal{R}$.

The paper is organized as follows. In Section 2, based on some constructions from [11] and [4], we study properties of block bases of the Rademacher system in Cesàro type spaces. We show that, depending on whether $\liminf_{n\to\infty} \|u_n\|_d > 0$ or $\liminf_{n\to\infty} \|u_n\|_d = 0$, a block basis $\{u_n\}_{n=1}^\infty$ of the Rademacher functions weakly converging to zero in K_p, $1 \le p < \infty$, and such that $C^{-1} \le \|u_n\|_{K_p} \le C$ for some $C > 0$ and all $n \in \mathbb{N}$ contains a subsequence equivalent to the unit vector basis of l_2 or c_0 (Theorem 3). This allows us to prove, in Theorem 4, that for each $1 \le p < \infty$ every infinite-dimensional subspace of $\mathcal{R}_p$ either is isomorphic to l_2, or contains a subspace isomorphic to c_0 and complemented in $\mathcal{R}_p$.

In Sections 3 and 4, we complete the proof of Theorems 1 and 2 by exhibiting an essential difference in the geometrical structure of subspaces of $\mathcal{R}_p$ in the cases $p = 1$ and $1 < p < \infty$. Finally, in Section 5, we finish with some remarks relating to a more general weighted version of the Cesàro space.

2. Block bases of the Rademacher system in K_p, $1 \le p < \infty$. Let $\{u_n\}_{n=1}^{\infty}$ be a block basis of the Rademacher system $\{r_k\}_{k=1}^{\infty}$, that is,

$$u_n = \sum_{k=m_n+1}^{m_{n+1}} a_k r_k, \quad 0 < m_1 < m_2 < \cdots, \ n \in \mathbb{N}. \tag{2.1}$$

THEOREM 3. *Let $1 \le p < \infty$, and $\{u_n\}_{n=1}^{\infty}$ be a block basis of $\{r_k\}_{k=1}^{\infty}$ weakly converging to zero in K_p with $1/c_0 \le \|u_n\|_{K_p} \le c_0$ for some constant $c_0 > 0$ and for all $n \in \mathbb{N}$.*

(a) *If there is $\varepsilon > 0$ such that $\|u_n\|_d \ge \varepsilon$ $(n \in \mathbb{N})$, then $\{u_n\}_{n=1}^{\infty}$ contains a subsequence equivalent to the unit vector basis of l_2.*
(b) *If $\|u_n\|_d \to 0$ as $n \to \infty$, then $\{u_n\}_{n=1}^{\infty}$ contains a subsequence equivalent to the unit vector basis of c_0.*

To prove Theorem 3 we will need some auxiliary facts. First, we observe that $\{r_k\}_{k=1}^{\infty}$ is not weakly convergent to zero in K_p. In fact, let φ_0 be a linear functional defined on the linear span of r_k, $k = 1, 2, \ldots$, by

$$\varphi_0\Big(\sum_{k=1}^{n} a_k r_k\Big) = \sum_{k=1}^{n} a_k,$$

where $n \in \mathbb{N}$, $a_k \in \mathbb{R}$, $k = 1, \ldots, n$. By (1.1), we have

$$\Big|\varphi_0\Big(\sum_{k=1}^{n} a_k r_k\Big)\Big| = \Big|\sum_{k=1}^{n} a_k\Big| \le C \Big\|\sum_{k=1}^{n} a_k r_k\Big\|_{K_p}.$$

Therefore, φ_0 can be extended to a functional $\tilde{\varphi_0} \in (K_p)^*$. Since $\tilde{\varphi_0}(r_n) = \varphi_0(r_n) = 1$ for all $n \in \mathbb{N}$, it follows that $r_n \nrightarrow 0$ weakly in K_p.

Now, we show that the sequence

$$s_n := r_n - r_{n-1}, \quad n = 1, 2, \ldots, \quad \text{where } r_0 = 0,$$

converges weakly to zero in $\mathcal{R}_p$, and it even forms a shrinking basis.

PROPOSITION 1. *The sequence $\{s_n\}_{n=1}^{\infty}$ is a shrinking basis in the space $\mathcal{R}_p$ for every $1 \le p < \infty$.*

Proof. First, we show that $\{s_n\}_{n=1}^\infty$ is a basis in $\mathcal{R}_p$. To this end, we consider a function $f = \sum_{n=1}^\infty \beta_n s_n \in \mathcal{R}_p$. Since

$$f = \sum_{n=1}^\infty \beta_n (r_n - r_{n-1}) = \sum_{n=0}^\infty (\beta_n - \beta_{n-1}) r_n, \qquad \text{where } \beta_0 = 0,$$

from (1.1) it follows that $f \in \mathcal{R}_p$ if and only if $\{\beta_n\}_{n=1}^\infty$ converges and $\sum_{n=1}^\infty (\beta_n - \beta_{n-1})^2 < \infty$. Moreover, $\{s_n\}_{n=1}^\infty$ is complete in $\mathcal{R}_p$ and

$$\|f\|_{K_p} \asymp \sup_{n=1,2,\dots} |\beta_n| + \Big(\sum_{n=1}^\infty (\beta_n - \beta_{n-1})^2\Big)^{1/2}, \tag{2.2}$$

which implies that $\{s_n\}_{n=1}^\infty$ forms a basis in $\mathcal{R}_p$.

To prove the shrinking property of $\{s_n\}_{n=1}^\infty$ we need to show that for any $\varphi \in (K_p)^*$,

$$\|\varphi|_{[s_n]_{n=m}^\infty}\|_{K_p} \to 0 \qquad \text{as } m \to \infty. \tag{2.3}$$

Assume (2.3) does not hold. Then there exist $\varepsilon \in (0,1)$, a functional $\varphi \in (K_p)^*$ with $\|\varphi\|_{(K_p)^*} = 1$, and a sequence of functions $f_n = \sum_{k=m_n}^\infty a_k^{m_n} s_k$, where $m_1 < m_2 < \cdots$, such that $\|f_n\|_{K_p} = 1$, $n = 1,2,\dots$, and

$$\varphi(f_n) \ge \varepsilon \qquad \text{for all } n = 1,2,\dots. \tag{2.4}$$

It is not hard to construct two sequences of positive integers, $\{q_i\}_{i=1}^\infty$ and $\{p_i\}_{i=1}^\infty$, such that $q_i = m_{n_i}$, $i = 1,2,\dots$, $1 < q_1 < p_1 < q_2 < p_2 < \cdots$ and

$$\Big\| \sum_{n=p_i+1}^\infty a_k^{q_i} s_k \Big\|_{K_p} \le \frac{\varepsilon}{2}. \tag{2.5}$$

In fact, setting $q_1 = m_1$, we can find $p_1 > q_1$ such that $\|\sum_{n=p_1+1}^\infty a_k^{q_1} s_k\|_{K_p} \le \varepsilon/2$. Then, taking for q_2 the smallest m_n which is larger than p_1, we find $p_2 > q_2$ satisfying $\|\sum_{n=p_2+1}^\infty a_k^{q_2} s_k\|_{K_p} \le \varepsilon/2$. Continuing in the same way, we come to the required sequences $\{q_i\}_{i=1}^\infty$ and $\{p_i\}_{i=1}^\infty$.

Since $\|f_n\|_{K_p} = 1$, $n = 1,2,\dots$, by (2.5) the sequence

$$u_i := \sum_{k=q_i}^{p_i} a_k^{q_i} s_k, \qquad i = 1,2,\dots,$$

is bounded in K_p. Moreover, from (2.4) and (2.5) it follows that

$$\begin{aligned} \varphi(u_i) &= \varphi(f_i) - \varphi\Big(\sum_{k=p_i+1}^\infty a_k^{q_i} s_k\Big) \\ &\ge \varphi(f_i) - \Big\| \sum_{k=p_i+1}^\infty a_k^{q_i} s_k \Big\|_{K_p} \ge \frac{\varepsilon}{2}, \qquad i = 1,2,\dots. \end{aligned} \tag{2.6}$$

Setting $\alpha_k^i = a_k^{q_i}$ if $q_i \le k \le p_i$ and $\alpha_k^i = 0$ if $p_i < k < q_{i+1}$, $i = 1, 2, \dots$, we have

$$u_i := \sum_{k=q_i}^{q_{i+1}-1} \alpha_k^i s_k, \quad i = 1, 2, \dots,$$

that is, $\{u_i\}_{i=1}^\infty$ is a bounded block basis of $\{s_k\}_{k=1}^\infty$.

Let $\{\gamma_n\}_{n=1}^\infty$ be an arbitrary sequence of positive numbers such that

$$\sum_{n=1}^\infty \gamma_n^2 < \infty \quad \text{and} \quad \sum_{n=1}^\infty \gamma_n = \infty. \tag{2.7}$$

We want to show that the series $\sum_{n=1}^\infty \gamma_n u_n$ converges in K_p. To this end, we set

$$b_k := \alpha_k^i \gamma_i \quad \text{if } q_i \le k < q_{i+1},\ i = 1, 2, \dots.$$

Then, by (2.2), if $k \ge q_j$ with some $j = 1, 2, \dots$, we have

$$\begin{aligned} |b_k| &\le \sup_{i\ge j} \max_{q_i \le k < q_{i+1}} |a_k^{q_i}\gamma_i| \le C \sup_{i\in\mathbb{N}} \|u_i\|_{K_p} \sup_{i\ge j} \gamma_i \\ &\le C \sup_{i\in\mathbb{N}} \|u_i\|_{K_p} \sup_{i\ge j} \gamma_i. \end{aligned}$$

Hence, thanks to (2.7), we obtain $\lim_{k\to\infty} b_k = 0$. Moreover,

$$\begin{aligned} \sum_{k=1}^\infty (b_k - b_{k+1})^2 &= \sum_{i=1}^\infty \sum_{k=q_i}^{q_{i+1}-2} (\alpha_k^i \gamma_i - \alpha_{k+1}^i \gamma_i)^2 \\ &\quad + \sum_{i=1}^\infty (\alpha_{q_{i+1}-1}^i \gamma_i - \alpha_{q_{i+1}}^{i+1} \gamma_{i+1})^2 = A_1 + A_2. \end{aligned}$$

Let us estimate A_1 and A_2 separately. In view of (2.2) and (2.7) we have

$$A_1 = \sum_{i=1}^\infty \gamma_i^2 \sum_{k=q_i}^{q_{i+1}-2} (\alpha_k^i - \alpha_{k+1}^i)^2 \le C \sup_{i\in\mathbb{N}} \|u_i\|_{K_p} \sum_{i=1}^\infty \gamma_i^2 < \infty,$$

and similarly

$$A_2 \le 2 \sum_{i=1}^\infty \big[(\alpha_{q_{i+1}-1}^i)^2 \gamma_i^2 + (\alpha_{q_{i+1}}^{i+1})^2 \gamma_{i+1})^2\big] \le C \sum_{i=1}^\infty \gamma_i^2 \sup_{i\in\mathbb{N}} \|u_i\|_{K_p} < \infty.$$

The above observations combined with (2.2) show that the series

$$\sum_{n=1}^\infty \gamma_n u_n = \sum_{k=1}^\infty b_k s_k$$

converges in K_p. At the same time, since $\varphi \in (K_p)^*$, by (2.6) and (2.7) we have

$$\varphi\Big(\sum_{n=1}^\infty \gamma_n u_n\Big) = \sum_{n=1}^\infty \gamma_n \varphi(u_n) \ge \frac{\varepsilon}{2} \sum_{n=1}^\infty \gamma_n = \infty,$$

and therefore (2.3) is proved. ∎

COROLLARY 1. *Let $\{u_n\}_{n=1}^\infty$ be a block basis defined as in* (2.1), *suppose* $\|u_n\|_{K_p} \le C$, $n = 1, 2, \ldots$, *for some* $C > 0$, *and let*

$$\gamma_n = \gamma_n(\{u_n\}) := \sum_{k=m_n+1}^{m_{n+1}} a_k, \quad n = 1, 2, \ldots .$$

Then $u_n \to 0$ weakly in K_p if and only if $\gamma_n \to 0$ as $n \to \infty$.

Proof. Denote by $\{s_n^*\}_{n=1}^\infty$ the system biorthogonal to the above basis $\{s_n\}_{n=1}^\infty$. From Proposition 1 and [12, Proposition 1.b.1] it follows that $\{s_n^*\}_{n=1}^\infty$ is a basis in the dual space $(\mathcal{R}_p)^*$. By definition, we have

$$s_n^*(s_m) = 0 \text{ if } n \ne m \quad \text{and} \quad s_n^*(s_n) = 1,\ n = 1, 2, \ldots .$$

Since $s_1 = r_1$, this implies that

$$s_n^*(r_m) = 1 \text{ if } n \le m \quad \text{and} \quad s_n^*(r_m) = 0 \text{ if } n > m. \tag{2.8}$$

Now, define the sequence $\{r_n^*\}_{n=0}^\infty$ from $(\mathcal{R}_p)^*$ by setting

$$r_0^* = s_1^* \quad \text{and} \quad r_n^* = s_n^* - s_{n+1}^*, \quad n = 1, 2, \ldots .$$

Clearly, $\{r_n^*\}_{n=0}^\infty$ is complete in $(\mathcal{R}_p)^*$ together with $\{s_n^*\}_{n=1}^\infty$, and from (2.8) it follows that

$$r_0^*(r_m) = 1\ (m = 1, 2, \ldots), \quad r_n^*(r_m) = 0\ (n \ne m), \quad r_n^*(r_n) = 1\ (n = 1, 2, \ldots).$$

Since $r_0^*(u_n) = \gamma_n$, $n = 1, 2, \ldots$, we have

$$r_k^*(u_n) \to 0 \quad \text{for every } k = 0, 1, 2, \ldots \tag{2.9}$$

if and only if $\gamma_n \to 0$ as $n \to \infty$. On the other hand, in view of the boundedness of $\{u_n\}_{n=1}^\infty$ in K_p, condition (2.9) is equivalent to the weak convergence of $\{u_n\}$ to zero in K_p. Therefore, the result follows. ∎

Proof of Theorem 3. Let $\{u_n\}_{n=1}^\infty$ be a block basis of the Rademacher functions defined in (2.1). First, by assumption and Corollary 1, we have

$$\gamma_n := \sum_{k=m_n+1}^{m_{n+1}} a_k \to 0 \quad \text{as } n \to \infty,$$

and passing to a subsequence of $\{u_n\}_{n=1}^\infty$ if necessary, we can assume that

$$|\gamma_n| \le 2^{-n}, \quad n = 1, 2, \ldots . \tag{2.10}$$

(a) Let $f = \sum_{n=1}^\infty b_n u_n \in \mathcal{R}_p$. Then

$$f = \sum_{n=1}^\infty \sum_{k=m_n+1}^{m_{n+1}} b_n a_k r_k = \sum_{k=1}^\infty \beta_k r_k. \tag{2.11}$$

First, we estimate $\sum_{k=p}^q \beta_k$ for $p \le q$. Let $m_{n-1} \le p < m_n < m_{n+l} < q \le$

m_{n+l+1} for some n and l. Then

$$\Big|\sum_{k=p}^{q}\beta_k\Big| = \Big|\sum_{k=p}^{m_n}\beta_k + \sum_{k=m_n+1}^{m_{n+l}}\beta_k + \sum_{k=m_{n+l}+1}^{q}\beta_k\Big|$$
$$= \Big|\sum_{k=p}^{m_n} b_{n-1}a_k + \sum_{i=n}^{n+l-1}\sum_{k=m_i+1}^{m_{i+1}} b_i a_k + \sum_{k=m_{n+l}+1}^{q} b_{n+l}a_k\Big|$$
$$\le |b_{n-1}|\Big|\sum_{k=p}^{m_n} a_k\Big| + \sum_{i=n}^{n+l-1}|b_i|\Big|\sum_{k=m_i+1}^{m_{i+1}} a_k\Big| + |b_{n+l}|\Big|\sum_{k=m_{n+l}+1}^{q} a_k\Big|$$
$$\le \sup_{n\in\mathbb{N}}|b_n|\Big(\Big|\sum_{k=p}^{m_n} a_k\Big| + \sum_{i=n}^{n+l-1}\Big|\sum_{k=m_i+1}^{m_{i+1}} a_k\Big| + \Big|\sum_{k=m_{n+l}+1}^{q} a_k\Big|\Big).$$

By (1.1), we have

$$\max\Big(\Big|\sum_{k=p}^{m_n} a_k\Big|, \Big|\sum_{k=m_{n+l}+1}^{q} a_k\Big|\Big) \le C\|u_n\|_{K_p} \le C_1.$$

Moreover, from (2.10) it follows that

$$\sum_{i=n}^{n+l-1}\Big|\sum_{k=m_i+1}^{m_{i+1}} a_k\Big| = \sum_{i=n}^{n+l-1}|\gamma_i| \le \sum_{i=n}^{n+l-1} 2^{-i} < 1.$$

Therefore, from the preceding estimates we infer that

$$\Big|\sum_{k=p}^{q}\beta_k\Big| \le (2C_1+1)\sup_{n\in\mathbb{N}}|b_n| \le (2C_1+1)\Big(\sum_{n=1}^{\infty} b_n^2\Big)^{1/2} \quad \text{for all } 1\le p\le q<\infty. \tag{2.12}$$

By assumption and (1.1), there is a constant $C_2>0$ such that for all $n\in\mathbb{N}$,

$$\sum_{k=m_n+1}^{m_{n+1}} a_k^2 \le C\|u_n\|_{K_p}^2 \le C_2^2,$$

and so

$$\Big(\sum_{k=1}^{\infty}\beta_k^2\Big)^{1/2} \le \Big(\sum_{n=1}^{\infty} b_n^2\cdot\sum_{k=m_n+1}^{m_{n+1}} a_k^2\Big)^{1/2} \le C_2\Big(\sum_{n=1}^{\infty} b_n^2\Big)^{1/2}. \tag{2.13}$$

On the other hand, since $\|u_n\|_d = \big(\sum_{k=m_n+1}^{m_{n+1}} a_k^2\big)^{1/2} \ge \varepsilon$, we have

$$\Big(\sum_{k=1}^{\infty}\beta_k^2\Big)^{1/2} = \Big(\sum_{n=1}^{\infty} b_n^2\|u_n\|_d^2\Big)^{1/2} \ge \varepsilon\Big(\sum_{n=1}^{\infty} b_n^2\Big)^{1/2}. \tag{2.14}$$

From (2.12)–(2.14) and (1.1), it follows that $\{u_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis in l_2, so the proof of part (a) is complete.

(b) Since $\|u_n\|_d \to 0$ as $n \to \infty$, we can assume that

$$\|u_n\|_d \le \eta\, 2^{-n}, \qquad n = 1, 2, \ldots, \tag{2.15}$$

where $\eta > 0$ will be chosen later on. Suppose that $f = \sum_{n=1}^{\infty} b_n u_n \in \mathcal{R}_p$. Then, representing f by formula (2.11), and applying (1.1), (2.15) and the first inequality of (2.12) (which is still valid), we obtain

$$\begin{aligned}\|f\|_{K_p} &\le C\Big(\|f\|_d + \sup_{1\le p\le q<\infty}\Big|\sum_{k=p}^{q}\beta_k\Big|\Big)\\ &\le C\Big(\sum_{n=1}^{\infty}|b_n|\,\|u_n\|_d + (2C_1+1)\sup_{n\in\mathbb{N}}|b_n|\Big) \le C_3\sup_{n\in\mathbb{N}}|b_n|.\end{aligned}$$

On the other hand, by (1.1), we have

$$\|f\|_{K_p} \ge c\sup_{1\le p\le q<\infty}\Big|\sum_{k=p}^{q}\beta_k\Big| \ge c\sup_{n\in\mathbb{N}}|b_n|\cdot\max_{m_n<j\le m_{n+1}}\Big|\sum_{k=m_n+1}^{j}a_k\Big|.$$

Since $\|u_n\|_{K_p} \ge 1/c_0$, $n = 1, 2, \ldots$, choosing $\eta > 0$ in (2.15) sufficiently small, for every $n \in \mathbb{N}$ we can find l_n with $m_n < l_n \le m_{n+1}$ such that for some $\delta > 0$,

$$\Big|\sum_{k=m_n+1}^{l_n}a_k\Big| \ge \delta.$$

Combining this observation with the preceding estimate we obtain

$$\|f\|_{K_p} \ge c\delta\sup_{n\in\mathbb{N}}|b_n|,$$

and therefore (b) is proved. ∎

Now, we are ready to prove the main result of Section 2.

THEOREM 4. *Let $1 \le p < \infty$. Every infinite-dimensional subspace X of $\mathcal{R}_p$ either is isomorphic to l_2, or contains a subspace isomorphic to c_0 and complemented in $\mathcal{R}_p$.*

Proof. Suppose that for every $f = \sum_{k=1}^{\infty} b_k r_k \in X$ we have

$$\|f\|_{K_p} \asymp \|f\|_d = \Big(\sum_{k=1}^{\infty}b_k^2\Big)^{1/2}.$$

This means that X is isomorphic to some subspace of l_2 and so to l_2 itself.

Otherwise, by (1.1), there is a sequence $\{f_n\}_{n=1}^{\infty} \subset X$ with $\|f_n\|_{K_p} = 1$ for which $\|f_n\|_d \to 0$ as $n \to \infty$. Observe that $\{f_n\}_{n=1}^{\infty}$ has no subsequence converging in K_p-norm. In fact, if $\|f_{n_k} - f\|_{K_p} \to 0$ for some $\{f_{n_k}\} \subset \{f_n\}$ and $f \in X$, then $\|f_{n_k} - f\|_d \to 0$ and hence $\|f\|_d = 0$, i.e., $f = 0$. On the other hand, $\|f\|_{K_p}$ should be equal to 1, and we come to a contradiction. Thus, passing to a subsequence if necessary, we can assume that

$$(2.16)\qquad \|f_n - f_m\|_{K_p} \geq \varepsilon > 0 \quad \text{for all } n \neq m.$$

Recall that $\{s_n^*\}_{n=1}^\infty$ is a basis in $(\mathcal{R}_p)^*$. Applying the diagonal process, we can construct a sequence $\{n_k\}_{k=1}^\infty$, $n_1 < n_2 < \cdots$, such that $\lim_{k\to\infty} s_i^*(f_{n_k})$ exists for every $i = 1, 2, \ldots$. Then

$$\lim_{k\to\infty} s_i^*(f_{n_{2k+1}} - f_{n_{2k}}) = 0 \quad \text{for all } i = 1, 2, \ldots,$$

and since $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^\infty$ is bounded in $\mathcal{R}_p$, we infer that $f_{n_{2k+1}} - f_{n_{2k}} \to 0$ weakly in K_p. Now, taking into account (2.16) and applying the well-known Bessaga–Pełczyński Selection Principle (see [1, Proposition 1.3.10] or [12, Proposition 1.a.12]), we can find a subsequence of $\{f_{n_{2k+1}} - f_{n_{2k}}\}$ (not relabelled) and a block basis $\{u_k\}_{k=1}^\infty$ of the Rademacher functions such that

$$(2.17)\qquad \|u_k - (f_{n_{2k+1}} - f_{n_{2k}})\|_{K_p} \leq B_0^{-1} 2^{-k-1}, \quad k = 1, 2, \ldots,$$

where B_0 is the basic constant of $\{r_k\}$ in $\mathcal{R}_p$. Then the sequences $\{u_k\}_{k=1}^\infty$ and $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^\infty$ are equivalent in K_p (cf. [12, Proposition 1.a.9]). Moreover, from (2.17) it follows that $u_k \to 0$ weakly in K_p and $\|u_k\|_d \to 0$. Therefore, by Theorem 3(b), the sequence $\{u_k\}_{k=1}^\infty$ (and so $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^\infty$) contains a subsequence equivalent to the unit vector basis of c_0. Complementability in $\mathcal{R}_p$ of the subspace spanned by the latter subsequence is an immediate consequence of Sobczyk's theorem [1, Corollary 2.5.9]. ∎

Arguing as in the proof of Theorem 4, we also obtain the following result.

THEOREM 5. *Let $1 \leq p < \infty$, and let $\{f_n\}_{n=1}^\infty$ be a basic sequence in $\mathcal{R}_p$ weakly converging to zero in K_p with $1/c_0 \leq \|f_n\|_{K_p} \leq c_0$ for some constant $c_0 > 0$ and for all $n \in \mathbb{N}$. Then $\{f_n\}_{n=1}^\infty$ contains a subsequence equivalent to the unit vector basis of l_2 or c_0.*

3. Structure of Rademacher subspaces in K. In this section we complete the proof of Theorem 1. In view of Theorem 3 all we need is the following result.

THEOREM 6. *Let X be a subspace of $K = \mathrm{Ces}_\infty$ which is isomorphic to l_2 and such that $X \subset \mathcal{R}$. Then X is uncomplemented in K.*

Proof. On the contrary, assume that an X as above is complemented in K. Let $\{x_n\}_{n=1}^\infty \subset X$ be equivalent to the unit vector basis $\{e_n\}_{n=1}^\infty$ of l_2. Since $e_n \xrightarrow{w} 0$ in l_2, it follows that $x_n \xrightarrow{w} 0$ in K. Noting that $x_n \in \mathcal{R}$ and $\|x_n\|_K \asymp 1$, $n = 1, 2, \ldots$, by applying the Bessaga–Pełczyński theorem once more (see [1, Proposition 1.3.10] or [12, Proposition 1.a.12]), we select a subsequence $\{x_{n_i}\} \subset \{x_n\}$ equivalent to a suitable block basis $\{u_n\}$ of the Rademacher functions,

$$u_n = \sum_{k=m_n+1}^{m_{n+1}} a_k r_k, \qquad 0 < m_1 < m_2 < \cdots,\ n \in \mathbb{N},$$

such that $\|x_{n_i} - u_i\|_K \to 0$ as $i \to \infty$. Hence, $\{u_n\}$ is equivalent in K to the unit vector basis in l_2, i.e.,

$$\Big\| \sum_{n=1}^{\infty} b_n u_n \Big\|_K \asymp \Big(\sum_{n=1}^{\infty} b_n^2 \Big)^{1/2}, \qquad b_n \in \mathbb{R}. \tag{3.1}$$

Taking into account the principle of small perturbations [1, Proposition 1.3.9], we can also assume that the closed linear span $[u_n]$ is also complemented in K. Moreover,

$$\|u_n\|_K \asymp \Big(\sum_{k=m_n+1}^{m_{n+1}} a_k^2 \Big)^{1/2} \asymp 1, \qquad n = 1, 2, \ldots. \tag{3.2}$$

In fact, otherwise we have

$$\liminf_{n\to\infty} \sum_{k=m_n+1}^{m_{n+1}} a_k^2 = 0.$$

Also if $\varphi \in K^*$, then

$$|\varphi(u_i)| \le |\varphi(u_i - x_{n_i})| + |\varphi(x_{n_i})| \le \|\varphi\|\, \|u_i - x_{n_i}\| + |\varphi(x_{n_i})|.$$

Since $x_n \overset{w}{\to} 0$ in K and $\|u_i - x_{n_i}\|_K \to 0$, we find that $u_n \overset{w}{\to} 0$ in K as well. Therefore, by Theorem 3(b), $\{u_n\}_{n=1}^{\infty}$ contains a subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ equivalent in K to the unit vector basis in c_0. Since this contradicts $\{u_n\}$ being equivalent in K to the unit vector basis in l_2, (3.2) is proved.

Let P be a bounded projection from K onto $[u_n]$. Since $\{u_n\}$ is a basic sequence in K, we can find functionals $\varphi_n \in K^*$, $n = 1, 2, \ldots$, such that

$$Pf(x) = \sum_{n=1}^{\infty} \varphi_n(f)\, u_n(x), \qquad f \in K.$$

Since $K \subset L_1$, the Köthe dual K' contains the space L_∞, and therefore K' is a total set on K. Thus, by [9, Chapter 10, Theorem 3.6], we have

$$K^* = K^c \oplus K^s, \tag{3.3}$$

where K^c (respectively, K^s) is the set of all order continuous linear functionals on K generated by the space K' (respectively, singular bounded linear functionals on K). Hence, if $\theta \in K^s$, then

$$\theta(f) = 0 \quad \text{for every } f \in K_0, \tag{3.4}$$

where K_0 is the separable part of K (the set of all elements in K having absolutely continuous norm). In particular, from (3.3) it follows that

$$\varphi_n = \psi_n + \theta_n \quad \text{with } \psi_n \in K^c \text{ and } \theta_n \in K^s,\ n = 1, 2, \ldots.$$

Moreover, since P is a projection onto $[u_n]$, we have

$$\psi_n(u_n) + \theta_n(u_n) = 1\ (n = 1, 2, \ldots), \qquad \psi_n(u_i) + \theta_n(u_i) = 0 \text{ for } i \neq n. \tag{3.5}$$

Since from (3.1) it follows that

$$\|Pf\|_K \asymp \Big(\sum_{n=1}^{\infty} \varphi_n(f)^2\Big)^{1/2} < \infty \quad \text{for every } f \in K,$$

$\{\varphi_n\}$ is a weakly* null sequence in K^*. Therefore,

$$\|\varphi_n\|_{K^*} \leq A \quad (n = 1, 2, \ldots) \text{ for some } A > 0. \tag{3.6}$$

On the other hand, taking into account (3.4), we see that the operator

$$Qf(x) = \sum_{n=1}^{\infty} \psi_n(f)\, u_n(x), \qquad f \in K_0,$$

coincides with P on K_0 and hence $Q : K_0 \to K$ is bounded. Let us show that Q acts boundedly in K.

Since $\psi_n \in K^c$, we have

$$\psi_n(f) = \int_0^1 g_n(t) f(t)\, dt \quad \text{for some } g_n \in K'. \tag{3.7}$$

For every $f \in K$ we have $|f|\chi_{[1/m,1]} \cdot \operatorname{sign} g_n \in K_0$, $m, n = 1, 2, \ldots,$ and therefore in view of (3.6) and (3.4),

$$\begin{aligned}\int_0^1 |g_n(t) f(t) \chi_{[1/m,1]}(t)|\, dt &= \psi_n(|f|\chi_{[1/m,1]} \cdot \operatorname{sign} g_n) \\ &= \varphi_n\big(|f|\chi_{[1/m,1]} \operatorname{sign} g_n\big) \\ &\leq A\, \|f\|_K, \qquad m, n = 1, 2, \ldots.\end{aligned}$$

Letting $m \to \infty$, by the Fatou lemma we have

$$\int_0^1 |g_n(t) f(t)|\, dt \leq A \|f\|_K \quad \text{for all } f \in K \text{ and } n \in \mathbb{N},$$

whence $\|\psi_n\|_{K^*} \leq A$. Combining this inequality with (3.6), we infer that

$$\|\theta_n\|_{K^*} \leq 2A, \qquad n = 1, 2, \ldots. \tag{3.8}$$

Moreover, by (3.1),

$$\sum_{n=1}^{\infty}\Big(\int_0^1 g_n(t)f(t)\chi_{[1/m,1]}(t)\,dt\Big)^2 \asymp \|Q(f\chi_{[1/m,1]})\|_K^2$$
$$\le C\|f\chi_{[1/m,1]}\|_K^2 \le C\|f\|_K^2.$$

Since $g_n f \in L_1$ for all $n = 1, 2, \ldots$, as above we obtain

$$\|Qf\|_K^2 \asymp \sum_{n=1}^{\infty}\Big(\int_0^1 g_n(t)f(t)\,dt\Big)^2 \le C\|f\|_K^2, \tag{3.9}$$

and the assertion is proved.

Note that $r_i - \chi_{[0,1]} \in K_0$ for all $i \in \mathbb{N}$. Therefore, by (3.5), $\theta_n(r_i) = \theta_n(\chi_{[0,1]}) := c_n$ for all $n, i \in \mathbb{N}$, and

$$\theta_n(u_i) = c_n \sum_{k=m_i+1}^{m_{i+1}} a_k, \qquad n, i \in \mathbb{N}.$$

Moreover, since $u_i \xrightarrow{w} 0$ in K, by Corollary 1 we obtain

$$\sum_{k=m_i+1}^{m_{i+1}} a_k \to 0 \quad \text{ as } i \to \infty.$$

Therefore, by (3.8), for all positive integers n and i,

$$|c_n| = |\theta_n(\chi_{[0,1]})| \le \|\theta_n\|_{K^*}\|\chi_{[0,1]}\|_K \le 2A,$$

whence

$$\lim_{i\to\infty}\sup_{n\in\mathbb{N}}|\theta_n(u_i)| = 0.$$

On the other hand, by (3.5), we have

$$\psi_n(u_i) = -\theta_n(u_i)\,(i \ne n) \quad \text{and} \quad \psi_n(u_n) = 1 - \theta_n(u_n), \qquad n \in \mathbb{N},$$

whence

$$\lim_{n\to\infty}\psi_n(u_n) = 1 \quad \text{and} \quad \lim_{i\to\infty}\sup_{n\ne i}|\psi_n(u_i)| = 0.$$

Thus, passing to subsequences of $\{u_n\}$ and $\{\psi_n\}$, and keeping the same notation, we deduce that the operator

$$Rf(x) := \sum_{n=1}^{\infty}\psi_n(f)u_n(x), \qquad f \in K,$$

where

$$\psi_n(u_n) \ge 1 - 2^{-n}\ (n = 1, 2, \ldots) \quad \text{and} \quad |\psi_n(u_i)| \le 2^{-i},\ n \ne i, \tag{3.10}$$

acts boundedly in K. As above, the functionals ψ_n are defined by (3.7).

Since $K \subset L_1[0,1]$, the operator R is also bounded from K into L_1. Let us show that $R : K \to L_1$ is weakly compact. By the Dunford–Pettis theorem (see, for example, [1, Theorem 5.2.9]), it is sufficient to check that the set $\{Rf : \|f\|_K \le 1\}$ is uniformly integrable on $[0,1]$. In fact, by (3.2) and (3.9), for every $f \in K$ with $\|f\|_K \le 1$ and any set $E \subset [0,1]$, we obtain

$$\|Rf \cdot \chi_E\|_{L_1} \le m(E)^{1/2}\, \|Rf\|_{L_2} = m(E)^{1/2} \Big(\sum_{n=1}^{\infty} \psi_n(f)^2 \cdot \sum_{k=m_n+1}^{m_{n+1}} a_k^2 \Big)^{1/2}$$
$$\asymp m(E)^{1/2} \Big(\sum_{n=1}^{\infty} \psi_n(f)^2 \Big)^{1/2} \le C\|R\| m(E)^{1/2},$$

whence

$$\lim_{m(E)\to 0} \sup\{\|Rf \cdot \chi_E\|_{L_1} : \|f\|_K \le 1\} = 0$$

(here, $m(E)$ is the Lebesgue measure of a set E). Thus, R is a weakly compact operator from K into L_1.

Now, we consider separately two cases. Firstly, assume that there are $\delta \in (0,1)$ and a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that

$$(3.11) \qquad |\psi_{n_k}(u_{n_k} \cdot \chi_{[\delta,1]})| = \Big| \int_{\delta}^{1} g_{n_k}(t) u_{n_k}(t)\, dt \Big| \ge \frac{1}{2}, \qquad k = 1, 2, \ldots.$$

Note that for every measurable function f on $[0,1]$ with $\operatorname{supp} f \subset [\delta, 1]$ we have

$$\|f\|_{L_1[\delta,1]} \le \|f\|_K = \sup_{\delta \le x \le 1} \frac{1}{x} \int_{\delta}^{1} |f(t)|\, dt \le \frac{1}{\delta} \|f\|_{L_1[\delta,1]}.$$

Therefore, $R_\delta f := R(f\chi_{[\delta,1]})$ is a weakly compact operator in $L_1[0,1]$. Since $L_1[0,1]$ has the Dunford–Pettis property (see [1, Theorem 5.4.5]), we conclude that R_δ is weak-to-norm sequentially continuous. Clearly, from $u_n \overset{w}{\to} 0$ in K it follows that $u_n\chi_{[\delta,1]} \overset{w}{\to} 0$ in $L_1[0,1]$. Thus, $\|R(u_n\chi_{[\delta,1]})\|_{L_1} \to 0$ as $n \to \infty$. On the other hand, by the Khintchine inequality in L_1, (3.2) and (3.11), we have

$$\|R(u_{n_k} \cdot \chi_{[\delta,1]})\|_{L_1} \asymp \Big\| \sum_{i=1}^{\infty} \psi_i(u_{n_k} \cdot \chi_{[\delta,1]}) \sum_{j=m_i+1}^{m_{i+1}} a_j r_j \Big\|_{L_1}$$
$$\asymp \Big[\sum_{i=1}^{\infty} \psi_i(u_{n_k} \cdot \chi_{[\delta,1]})^2 \sum_{j=m_i+1}^{m_{i+1}} a_j^2 \Big]^{1/2}$$
$$\asymp \Big[\sum_{i=1}^{\infty} \psi_i(u_{n_k} \cdot \chi_{[\delta,1]})^2 \Big]^{1/2} \ge |\psi_{n_k}(u_{n_k} \cdot \chi_{[\delta,1]})| \ge \frac{1}{2}$$

for all $k = 1, 2, \ldots$. This contradiction concludes the proof in the case when (3.11) holds.

Suppose now that (3.11) does not hold. Then, by (3.10) and (3.7), for any $\delta \in (0,1)$ and all sufficiently large $n \in \mathbb{N}$ we have

$$\Big|\int_0^\delta g_n(t)u_n(t)\,dt\Big| > \frac{1}{4}. \tag{3.12}$$

Setting $\delta_0 = 1/2$, we find $n_1 \in \mathbb{N}$ and $\delta_1 \in (0, \delta_0)$ such that

$$\Big|\int_{\delta_1}^{\delta_0} g_{n_1}(t)u_{n_1}(t)\,dt\Big| > \frac{1}{4}.$$

Denote $v_{n_1} := u_{n_1}\chi_{(\delta_1,\delta_0)}$. From (3.9) it follows that $g_n \overset{w^*}{\to} 0$ in K^*. Moreover, we know that $u_n \overset{w}{\to} 0$ in K. Therefore, by (3.12), there is $n_2 > n_1$ for which

$$\Big|\int_0^{\delta_1} g_{n_1}(t)u_{n_2}(t)\,dt\Big| < \frac{1}{2^3},$$

$$\Big|\int_0^1 g_{n_2}(t)v_{n_1}(t)\,dt\Big| < \frac{1}{2^3},$$

$$\Big|\int_0^{\delta_1} g_{n_2}(t)u_{n_2}(t)\,dt\Big| > \frac{1}{4}.$$

Furthermore, we can find $\delta_2 \in (0, \delta_1)$ such that the functions g_{n_i} and v_{n_i} $(i = 1, 2)$, where $v_{n_2} := u_{n_2}\chi_{[\delta_2,\delta_1]}$, satisfy

$$\Big|\int_0^1 g_{n_j}(t)v_{n_i}(t)\,dt\Big| < \frac{1}{2^3} \quad \text{for } 1 \le i \ne j \le 2,$$

$$\Big|\int_0^1 g_{n_2}(t)v_{n_2}(t)\,dt\Big| > \frac{1}{4}.$$

Suppose that for some $k \in \mathbb{N}$ we have chosen

$$n_1 < \cdots < n_k \quad \text{and} \quad 1/2 = \delta_0 > \delta_1 > \cdots > \delta_k > 0$$

so that the functions g_{n_i} and $v_{n_i} := u_{n_i}\chi_{[\delta_i,\delta_{i-1}]}$, $i = 1, \ldots, k$, satisfy

$$\Big|\int_0^1 g_{n_j}(t)v_{n_i}(t)\,dt\Big| < \frac{1}{2^{i+j}}, \quad 1 \le i \ne j \le k, \tag{3.13}$$

$$\Big|\int_0^1 g_{n_i}(t)v_{n_i}(t)\,dt\Big| > \frac{1}{4}, \qquad i = 1, \ldots, k. \tag{3.14}$$

Using the facts that $u_n \overset{w}{\to} 0$ in K and $g_n \overset{w^*}{\to} 0$ in K^*, and (3.12) once more, we can find $n_{k+1} > n_k$ such that

$$\Big|\int_0^{\delta_k} g_{n_i}(t)u_{n_{k+1}}(t)\,dt\Big| < \frac{1}{2^{i+k+1}}, \qquad i = 1,\ldots,k,$$

$$\Big|\int_0^1 g_{n_{k+1}}(t)v_{n_i}(t)\,dt\Big| < \frac{1}{2^{i+k+1}}, \qquad i = 1,\ldots,k,$$

$$\Big|\int_0^{\delta_k} g_{n_{k+1}}(t)u_{n_{k+1}}(t)\,dt\Big| > \frac{1}{4}.$$

Clearly, there is $\delta_{k+1} \in (0,\delta_k)$ such that for the functions g_{n_i} and v_{n_i}, where $v_{n_{k+1}} := u_{n_{k+1}}\chi_{[\delta_{k+1},\delta_k]}$, inequalities (3.13) (respectively, (3.14)) hold for all $1 \le i \ne j \le k+1$ (respectively, $i = 1,\ldots,k+1$).

Thus, we can select sequences

$$n_1 < n_2 < \cdots \quad \text{and} \quad 1/2 = \delta_0 > \delta_1 > \cdots > 0$$

such that the functions g_{n_i} and $v_{n_i} := u_{n_i}\chi_{[\delta_i,\delta_{i-1}]}$, $i = 1,2,\ldots$, satisfy

$$\Big|\int_0^1 g_{n_j}(t)v_{n_i}(t)\,dt\Big| \le \frac{1}{2^{i+j}}, \quad 1 \le i \ne j < \infty, \tag{3.15}$$

$$\Big|\int_0^1 g_{n_i}(t)v_{n_i}(t)\,dt\Big| > \frac{1}{4}, \qquad i = 1,2,\ldots. \tag{3.16}$$

By [3, Proposition 1], every sequence $\{f_n\} \subset K$ such that $\operatorname{supp} f_n \subset [a_n,b_n]$ with $b_1 > a_1 > b_2 > a_2 > \cdots > 0$ and $b_n \to 0^+$ contains a subsequence $\{f_{n_k}\}$ which is equivalent in K to the unit vector basis of c_0. Therefore, we can assume that

$$\sup_{m=1,2,\ldots}\Big\|\sum_{i=1}^m v_{n_i}\Big\|_K \le C\sup_i \|v_{n_i}\|_K \le C\sup_{n=1,2,\ldots}\|u_n\|_K < \infty.$$

Moreover, it is clear that the operator

$$R'f(x) := \sum_{i=1}^\infty \int_0^1 g_{n_i}(t)f(t)\,dt\cdot u_{n_i}(x)$$

is bounded in K together with the operator R. Hence, on the one hand,

$$\Big\|R'\Big(\sum_{i=1}^m v_{n_i}\Big)\Big\|_K \le \|R'\|\Big\|\sum_{i=1}^m v_{n_i}\Big\|_K \le C\|R'\| \quad \text{for all } m \in \mathbb{N}.$$

On the other hand, by (3.1), (3.15) and (3.16), for every $m = 1,2,\ldots$, we have

$$\Big\|R'\Big(\sum_{i=1}^{m} v_{n_i}\Big)\Big\|_K = \Big\|\sum_{j=1}^{\infty}\sum_{i=1}^{m}\int_0^1 g_{n_j}(s)v_{n_i}(s)\,ds\cdot u_{n_j}\Big\|_K$$
$$\geq \Big\|\sum_{i=1}^{m}\int_0^1 g_{n_i}(s)v_{n_i}(s)\,ds\cdot u_{n_i}\Big\|_K - \sum_{i,j=1,i\neq j}^{\infty}\Big|\int_0^1 g_{n_j}(s)v_{n_i}(s)\,ds\Big|$$
$$\geq \Big\|\sum_{i=1}^{m}\int_0^1 g_{n_i}(s)v_{n_i}(s)\,ds\cdot u_{n_i}\Big\|_K - 1$$
$$\geq c\Big[\sum_{i=1}^{m}\Big|\int_0^1 g_{n_i}(s)v_{n_i}(s)\,ds\Big|^2\Big]^{1/2} - 1 \geq \frac{cm^{1/2}}{4} - 1,$$

where $c > 0$. This contradiction finishes the proof. ■

From Theorem 6, it follows that every subspace of $\mathcal{R}$ isomorphic to l_2 is uncomplemented in K. However, the following result holds.

Proposition 2. *Every subspace X of $\mathcal{R}$ isomorphic to l_2 contains a subspace complemented in $\mathcal{R}$.*

Proof. Let $\{x_n\}_{n=1}^{\infty} \subset X$ be a sequence equivalent to the unit vector basis in l_2. Arguing as in the proof of Theorem 6, we can find a subsequence $\{x_{n_i}\} \subset \{x_n\}$ which is equivalent in K to a suitable block basis $\{u_i\}_{i=1}^{\infty}$ of the Rademacher functions such that $\|u_i\|_K \geq \varepsilon$ and $u_i \to 0$ weakly in K. Moreover, we can assume that $[x_{n_i}]$ is complemented in $\mathcal{R}$ if and only if $[u_i]$ is complemented in $\mathcal{R}$. Since $\{u_i\}$ is equivalent to the unit vector basis in l_2, equivalence (3.1) holds. Let

$$u_n = \sum_{k=m_n+1}^{m_{n+1}} a_k r_k, \qquad 1 = m_1 < m_2 < \cdots,\ n \in \mathbb{N}.$$

For any $f = \sum_{n=1}^{\infty} c_k r_k \in \mathcal{R}$ we set

$$Pf = \sum_{n=1}^{\infty} b_n(f) u_n, \quad \text{where} \quad b_n(f) = \frac{1}{m_{n+1} - m_n}\sum_{k=m_n+1}^{m_{n+1}} c_k.$$

Then $Pg = g$ if $g \in [u_n]$. Moreover, by (3.1) and (1.1),

$$\|Pf\|_K \asymp \Big(\sum_{n=1}^{\infty} b_n(f)^2\Big)^{1/2} = \Big(\sum_{n=1}^{\infty}\frac{1}{(m_{n+1}-m_n)^2}\Big(\sum_{k=m_n+1}^{m_{n+1}} c_k\Big)^2\Big)^{1/2}$$
$$\leq \Big(\sum_{n=1}^{\infty}\frac{1}{m_{n+1}-m_n}\sum_{k=m_n+1}^{m_{n+1}} c_k^2\Big)^{1/2} \leq \Big(\sum_{k=1}^{\infty} c_k^2\Big)^{1/2} \leq C\|f\|_K.$$

Thus, P is a bounded projection from $\mathcal{R}$ onto $[u_n]$. By the above observation, this implies that the subspace $[x_{n_i}]$ of X is complemented in $\mathcal{R}$. ∎

4. Structure of Rademacher subspaces in K_p, $1 < p < \infty$. Here, we prove Theorem 2. Clearly, it is an immediate consequence of Theorem 3 and the following result.

THEOREM 7. *Let $1 < p < \infty$. Every subspace X of $\mathcal{R}_p$ isomorphic to l_2 is complemented in K_p.*

Proof. Let us prove that for every $x = \sum_{k=1}^{\infty} a_k r_k \in X$ we have

$$\|x\|_{K_p} \asymp \|x\|_d = \Big(\sum_{k=1}^{\infty} a_k^2\Big)^{1/2}, \tag{4.1}$$

with constants independent of $x \in X$. In view of (1.1), $\|x\|_{K_p} \ge c\|x\|_d$ for all $x \in \mathcal{R}_p$. Hence, assuming the contrary, we find a sequence $\{x_n\} \subset X$ such that $\|x_n\|_{K_p} = 1$ $(n = 1, 2, \ldots)$ and $\|x_n\|_d \to 0$ as $n \to \infty$.

Since X is isomorphic to l_2, we may assume that $x_n \to x$ weakly in K_p for some $x \in X$. Then, setting $x_n = \sum_{k=1}^{\infty} a_k^n r_k, n = 1, 2, \ldots,$ and $x = \sum_{k=1}^{\infty} a_k r_k$, we see that $\lim_{n\to\infty} a_k^n = a_k$ for each $k = 1, 2, \ldots$. On the other hand, $\lim_{n\to\infty} a_k^n = 0$, $k = 1, 2, \ldots$ because $\|x_n\|_d \to 0$, and so $x_n \to 0$ weakly in K_p. Therefore, applying the Bessaga–Pełczyński Selection Principle, we can find a subsequence $\{x_{n_i}\} \subset \{x_n\}$ and a block basis $\{u_i\}$ of the Rademacher functions such that $\|x_{n_i} - u_i\|_{K_p} \to 0$ as $i \to \infty$, and $\{x_{n_i}\}$ and $\{u_i\}$ are equivalent in K_p. It is obvious that $\|u_i\|_{K_p} \asymp 1$ $(i = 1, 2, \ldots)$ and $u_i \overset{w}{\to} 0$ as $i \to \infty$ in K_p. Moreover, by (1.1),

$$\|u_i\|_d \le \|x_{n_i} - u_i\|_d + \|x_{n_i}\|_d \le C\|x_{n_i} - u_i\|_{K_p} + \|x_{n_i}\|_d,$$

whence $\|u_i\|_d \to 0$ as $i \to \infty$. But then, by Theorem 3(b), $\{u_i\}_{i=1}^{\infty}$ contains a subsequence equivalent to the unit vector basis of c_0. Clearly, this contradicts the assumption, and relation (4.1) is proved.

Recall that the orthogonal projection $\mathcal{P}$ acts boundedly from L_p with $1 < p < \infty$ onto the closed linear span $[r_n]$ in L_p. Since the Rademacher functions are equivalent in $L_p, 1 \le p < \infty$, to the unit vector basis of l_2, from (4.1) it follows that X is a complemented subspace in $[r_n]$. Denote by R a projection from $[r_n]$ onto X. Then $S = R\mathcal{P}$ is a bounded projection from L_p onto X. Moreover, since

$$\|x\|_{L_p} \le \sup_{0<x\le 1} \Big(\frac{1}{x}\int_0^x |x(t)|^p\,dt\Big)^{1/p} = \|x\|_{K_p} \quad \text{for all } x \in K_p,$$

we have

$$\|Sx\|_{K_p} \asymp \|Sx\|_{L_p} \le C\|S\|_{L_p\to L_p}\|x\|_{L_p} \le C_1\|\mathcal{P}\|_{L_p\to L_p}\|x\|_{K_p}$$

for all $x \in K_p$. Thus, X is complemented in K_p, and the theorem is proved. ∎

From the Fatou lemma it follows that K_p, $1 \le p < \infty$, has the *Fatou property*, i.e., the conditions $f_n \in K_p$, $\|f_n\|_{K_p} \le C$, $n = 1, 2, \ldots$, $f_n \to f$ a.e. on $[0,1]$ imply that $f \in K_p$ and $\|f\|_{K_p} \le C$. Therefore, if X is a subspace of the Rademacher space $\mathcal{R}_p, 1 \le p < \infty$, isomorphic to c_0, then by the Bessaga–Pełczyński theorem, we can select a block basis $\{u_n\}_{n=1}^{\infty}$ of the Rademacher functions which is equivalent to the unit vector basis in c_0 and then K_p contains the subspace $\tilde{X}_p$ consisting of all functions

$$f = \sum_{n=1}^{\infty} a_n u_n, \quad \text{where} \quad (a_n) \in l_\infty \quad \text{(the series converges a.e. on } [0,1]).$$

Clearly, $\tilde{X}_p$ is isomorphic to l_∞. Note that the existence of a bounded projection from K_p onto $[u_n] \approx c_0$ would imply immediately that we have a bounded projection from $\tilde{X}_p \approx l_\infty$ onto $[u_n]$, which contradicts the well-known Phillips–Sobczyk theorem (see [1, Theorem 2.5.5]. Thus, we obtain

COROLLARY 2. *Every subspace of $\mathcal{R}_p$, $1 \le p < \infty$, isomorphic to c_0 is uncomplemented in K_p.*

5. Rademacher functions in weighted Cesàro spaces. In [6], we have also considered a more general weighted version of Cesàro type spaces, the space $K_{p,w} = K_{p,w}[0,1]$ with the norm

$$\|f\|_{K_{p,w}} = \sup_{0<x\le 1} \left(\frac{1}{w(x)} \int_0^x |f(t)|^p \, dt \right)^{1/p},$$

where $1 \le p < \infty$ and w is a quasi-concave function on $[0,1]$, that is, $w(0) = 0$, w is non-decreasing and $w(x)/x$ is non-increasing on $(0,1]$. Using the equivalence (cf. [6, Theorem 2])

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{K_{p,w}} \asymp \Big(\sum_{k=1}^{\infty} a_k^2 \Big)^{1/2} + \sup_{m\in\mathbb{N}} \left(\frac{2^{-m}}{w(2^{-m})} \right)^{1/p} \Big| \sum_{k=1}^{m} a_k \Big| \tag{5.1}$$

and the fact that the restriction of $K_{1,w}$ to any interval $[\delta,1]$, where $0 < \delta < 1$, coincides with $L_1[\delta,1]$ (with equivalence of norms), and applying the Dunford–Pettis property of the latter space, we proved in [6, Theorem 5] that the closed linear span $[r_n]$ of the Rademacher functions in $K_{1,w}$ is uncomplemented. The situation is different in the case when $1 < p < \infty$. If there is a constant $c > 0$ such that

$$w(t) \ge ct \log_2^{p/2}(2/t) \quad \text{for all } 0 < t \le 1, \tag{5.2}$$

then $\{r_n\}_{n=1}^{\infty}$ is equivalent in $K_{p,w}$ to the unit vector basis of l_2, and $[r_n]$ is complemented in $K_{p,w}$ (see [6, Theorems 3 and 5]).

Now, the techniques based on using block bases from this paper allow us to fill the gap in [6] related to the case when condition (5.2) does not hold.

THEOREM 8. *If* $1 < p < \infty$ *and condition* (5.2) *does not hold, then the subspace* $[r_n]$ *of the Rademacher functions is not complemented in* $K_{p,w}$.

Indeed, arguing as in the proof of Theorem 4, we can construct a block basis $\{u_n\}$ of the Rademacher functions equivalent to the unit vector basis of c_0 such that the closed linear span $[u_n]$ in $K_{p,w}$ is complemented in the subspace $[r_n]$ and not complemented in $K_{p,w}$ (see Corollary 2). Clearly, these facts imply that $[r_n]$ is not complemented in $K_{p,w}$. We omit the details.

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Sergey V. Astashkin
Department of Mathematics and Mechanics
Samara State University
Acad. Pavlova 1
443011 Samara, Russia
E-mail: astash@samsu.ru

Lech Maligranda
Department of Engineering Sciences
and Mathematics
Luleå University of Technology
SE-971 87 Luleå, Sweden
E-mail: lech.maligranda@ltu.se

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