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Derivations mapping into the socle, III

by

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Abstract. Let A be a Banach algebra, and let $d : A \to A$ be a continuous derivation such that each element in the range of d has a finite spectrum. In a series of papers it has been proved that such a derivation is an inner derivation implemented by an element from the socle modulo the radical of A (a precise formulation of this statement can be found in the Introduction). The aim of this paper is twofold: we extend this result to the case where d is not necessarily continuous, and we give a complete description of such maps in the semisimple case.

1. Introduction and statement of the main results. There have been a lot of results considering derivations whose ranges are small in some sense. Let us mention the survey papers of Mathieu [9] and Murphy [10], where one can find conditions which imply that the image of a derivation is contained in the radical.

It was proved in [1, Theorem 5.2.1] that the range of an inner derivation such that the spectrum of each element in the range is a singleton is contained in the radical. This motivated Brešar and the second author [5] to initiate the study of derivations with the property that every element in the range has finite spectrum. They proved that for every inner derivation acting on a unital Banach algebra A this property yields the existence of a positive integer n such that the spectrum of every element in the range has at most n elements. Under the additional condition that A be semisimple the range of such an inner derivation is contained in $\mathrm{soc}(A)$, the socle of A, and this is true if and only if the range is contained in the set of algebraic elements of A.

Later Brešar [3] extended this result to all continuous (not necessarily inner) derivations. And finally, the first author and Mathieu [2] proved that for such a derivation d there exists $a \in A$ such that $a + \mathrm{rad}\, A \in \mathrm{soc}(A/\mathrm{rad}\, A)$ and $d(x) - [x, a] \in \mathrm{rad}\, A$ for all $x \in A$.

There are two natural questions that remain to be solved. Can we extend the last result to the case where d is not necessarily continuous? Can we get

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a complete description of such derivations on semisimple Banach algebras? It is the aim of this paper to answer both of these questions.

All algebras in this paper will be complex Banach algebras. In order to make our presentation simpler we shall always assume that they are unital. There is no loss of generality in adding this assumption. Namely, if A is without unit, then we denote by $A^\flat$ the unitization $A^\flat = \mathbb{C} \oplus A$. We are interested in derivations $d : A \to A$ with the property that there exists a positive integer n such that every $d(x)$, $x \in A$, has at most n elements in the spectrum. It is trivial to check that then the derivation $d^\flat : A^\flat \to A^\flat$ defined by $d^\flat(\lambda + a) = d(a)$, $\lambda \in \mathbb{C}$, $a \in A$, has the same property. Using this observation one can easily extend our main results to the nonunital case.

Let us now fix some notation and recall some definitions.

By $\sharp\sigma(a)$ we denote the cardinality of the spectrum of a. A linear mapping $d : A \to A$ is called a *derivation* if $d(xy) = xd(y) + d(x)y$, $x, y \in A$. Recall that if A is semisimple, then the sum of all the minimal left ideals of A coincides with the sum of all the minimal right ideals of A, and is called the *socle* of A. It will be denoted by $\operatorname{soc}(A)$. If A does not have minimal one-sided ideals, we define $\operatorname{soc}(A) = \{0\}$. According to the definition, for every nonzero element $a \in \operatorname{soc}(A)$ there exist finitely many minimal left ideals such that a belongs to their sum. The element $0 \in \operatorname{soc}(A)$ has rank zero. We define an element $a \in \operatorname{soc}(A)$ to be of *rank one* if it is nonzero and belongs to some minimal left ideal of A. And finally, an element $a \in \operatorname{soc}(A)$ is said to be of *rank* $n > 1$ if a belongs to a sum of n minimal left ideals, but does not belong to a sum of fewer than n minimal left ideals. For equivalent definitions of rank and structural results for finite rank elements we refer to [6]. In particular, in the case of $n \times n$ matrices or finite-rank operators the above definition of the rank coincides with the usual one.

We shall first extend the result of Boudi and Mathieu [2] to not necessarily continuous derivations.

THEOREM 1.1. *Let A be a complex Banach algebra, and let $d : A \to A$ be a derivation. Then the following conditions are equivalent:*

(i) $\sharp\sigma(d(x)) < \infty$ *for every* $x \in A$;
(ii) *there exists a positive integer n such that* $\sharp\sigma(d(x)) \le n$ *for every* $x \in A$;
(iii) $d(x) + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$ *for every* $x \in A$;
(iv) *there exists $a \in A$ and a closed ideal J of A such that* $a + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$, $\dim(J/\operatorname{rad} A) < \infty$, $d(J) \subset J$, *and* $d(x) - [x, a] \in J$ *for every* $x \in A$.

We need some preparation to formulate the main result of our paper. Let A be a finite-dimensional algebra. Set $n_A = \max\{\sharp\sigma(x) : x \in A\}$. We are interested in derivations $d : A \to A$ with the property that there exists

$x \in A$ such that $d(x)$ is invertible and $\sharp\sigma(d(x)) = n_A$. Every such derivation d will be called a *maximal spectral derivation.*

Let us give a complete description of maximal spectral derivations on finite-dimensional semisimple algebras. We first consider the case where $A = \mathbb{M}_n$ and recall that every derivation on $\mathbb{M}_n$ is inner and $n_{\mathbb{M}_n} = n$. Here, $\mathbb{M}_n$ denotes the algebra of all $n \times n$ complex matrices. Thus, $d(X) = XT - TX$, $X \in \mathbb{M}_n$, for some $T \in \mathbb{M}_n$. Note that if d is an inner derivation induced by T, then d is induced by $T - \lambda I$ for every $\lambda \in \mathbb{C}$. We have to characterize matrices $T \in \mathbb{M}_n$ with the property that $X \mapsto [X,T] = XT - TX$ is a maximal spectral derivation. By the previous remark, there is no loss of generality in assuming that $\operatorname{rank} T = \min\{\operatorname{rank}(T - \lambda I) : \lambda \in \mathbb{C}\}$. So, assume from now on that this condition is fulfilled. Obviously, we need to have $\operatorname{rank} T \geq n/2$ if we want the inner derivation induced by T to be a maximal spectral derivation since otherwise $\operatorname{rank}[X,T] \leq \operatorname{rank} XT + \operatorname{rank} TX < n$. As we shall see later (Lemma 2.2) this condition is not only necessary but also sufficient for T to induce a maximal spectral derivation.

If we write $A = A_1 \oplus \cdots \oplus A_k$ we mean that the algebra A is a direct sum of ideals $A_1, \ldots, A_k$ and all operations are defined componentwise. Of course, all subalgebras $A_1, \ldots, A_k$ are unital. Indeed, as A is unital we have $1 = e_1 + \cdots + e_k$ with $e_j \in A_j$ for $j = 1, \ldots, k$. Clearly, each e_j is a unit of A_j and $\{e_1, \ldots, e_k\}$ is a set of pairwise orthogonal central idempotents. If $d : A \to A$ is a derivation, then each of the subalgebras $A_1, \ldots, A_k$ is invariant under d. Indeed, we have $d(1) = 0$, and therefore $d(e_j) = e_j d(e_j) + d(e_j)e_j \in A_j$ implies that $d(e_j) = 0$. For $x_j \in A_j$ we have $d(x_j) = d(x_j e_j) = d(x_j)e_j \in A_j$. Denote by d_j, $j = 1, \ldots, k$, the restriction of d to A_j.

We shall show that all d_j's are maximal spectral derivations if and only if d is a maximal spectral derivation. Suppose that each d_j is a maximal spectral derivation. Let $n = \max\{\sharp\sigma(x) : x \in A\}$, and let $n_j = \max\{\sharp\sigma(x_j, A_j) : x_j \in A_j\}$, $j = 1, \ldots, k$. Obviously we have $n \leq n_1 + \cdots + n_k$. For each $j = 1, \ldots, k$, we can find $x_j \in A_j$ such that $d(x_j)$ is invertible in A_j and $\sharp\sigma(d(x_j)) = n_j$. Replacing x_j by $\lambda_j x_j$ for suitable nonzero scalars $\lambda_1, \ldots, \lambda_k$, we may assume that the spectra of $d_1(x_1), \ldots, d_k(x_k)$ are pairwise disjoint. If $x = x_1 + \cdots + x_k$, then $d(x)$ is invertible and $n_1 + \cdots + n_k \leq \sharp\sigma(d(x)) \leq n \leq n_1 + \cdots + n_k$. We have thereby proved that d is maximal spectral. The converse is easy.

Now, if A is a finite-dimensional semisimple algebra, then by the Wedderburn theorem it is isomorphic to a direct sum of matrix algebras. Hence, a derivation on A (we identify A with the direct sum of matrix algebras) is a maximal spectral derivation if and only if it is a direct sum of inner derivations induced by $n_j \times n_j$ matrices T_j having the property that $\min\{\operatorname{rank}(T_j - \lambda I) : \lambda \in \mathbb{C}\} \geq n_j/2$.

Assume that a semisimple Banach algebra A can be decomposed into a direct sum of ideals as $A = A_1 \oplus A_2$ with A_1 finite-dimensional. The radical of an ideal of an algebra is equal to the intersection of this ideal and the radical of the algebra. It follows that both A_1 and A_2 are semisimple. Assume further that $d_1 : A_1 \to A_1$ is a maximal spectral derivation (by the above remark we completely understand the structure of such maps) and that $d_2 : A_2 \to A_2$ is an inner derivation induced by $a \in \operatorname{soc}(A_2)$ of rank k. Then each $d_2(x_2)$, $x_2 \in A_2$, is of rank at most $2k$, and consequently $\sharp\sigma(d_2(x_2)) \le 2k+1$ for every $x_2 \in A_2$ [6]. Set $n_1 = \max\{\sharp\sigma(x_1, A_1) : x_1 \in A_1\}$ and let $d : A \to A$ be a direct sum of d_1 and d_2. Then, clearly, $\sharp\sigma(d(x)) \le n_1 + 2k + 1$, $x \in A$.

Now we are ready to formulate our main result.

Theorem 1.2. *Let A be a complex semisimple Banach algebra, and let d be a derivation on A. Suppose that* $\max\{\sharp\sigma(d(x)) : x \in A\} = n$. *Then either*

- *A is finite-dimensional and d is maximal spectral, or*
- *n is odd, $n = 2k+1$ for some integer $k \ge 0$, and d is an inner derivation induced by an element $a \in \operatorname{soc}(A)$ of rank k, or*
- *A is a direct sum of ideals A_1 and A_2 with A_1 finite-dimensional and $n = n_1 + (2k+1)$, where $k \ge 0$ and $n_1 = \max\{\sharp\sigma(x_1, A_1) : x_1 \in A_1\} > 0$. Moreover, $d_{|A_1}$ is maximal spectral, and $d_{|A_2}$ is an inner derivation induced by an element $a \in \operatorname{soc}(A_2)$ of rank k.*

In the next section we shall prove some preliminary results. Then we shall deal with not necessarily continuous derivations with the property that every element in the range has a finite spectrum. The last section will be devoted to the proof of the main theorem.

2. Preliminary results. We shall start with some linear algebra results. We shall suppose throughout that X is a complex vector space and $T : X \to X$ a linear operator. Let r be a positive integer. We say that T has *property P_r* if there exist $x_1, \ldots, x_r \in X$ such that

$$\{x_1, \ldots, x_r, Tx_1, \ldots, Tx_r\}$$

is linearly independent. Let $\lambda \in \mathbb{C}$ be any scalar. Note that T has property P_r if and only if $T - \lambda I$ has property P_r. It is well-known (and easy to verify) that T has property P_1 if and only if T is not a scalar operator, that is, $T \notin \mathbb{C}I$. Our first goal is to prove the following lemma.

Lemma 2.1. *Let r be a positive integer and assume that* $\dim X \ge 2r$. *If*

$$\min\{\operatorname{rank}(T - \lambda I) : \lambda \in \mathbb{C}\} \ge r,$$

then T has property P_r.

Proof. We shall first consider the case where $\operatorname{rank} T = r$. Then we can choose a linearly independent set $\{u_1, \ldots, u_r\} \subset X$ such that

$$X = \operatorname{Ker} T \oplus \operatorname{span}\{u_1, \ldots, u_r\}.$$

The proof of this special case will be completed once we show inductively that for every $k = 0, \ldots, r$ there exist $v_1, \ldots, v_k \in X$ such that $Tv_j = Tu_j$, $j \leq k$, and

$$\{v_1, \ldots, v_k, Tv_1, \ldots, Tv_k, Tu_{k+1}, \ldots, Tu_r\}$$

is linearly independent.

In the case $k = 0$ we have to show that $\{Tu_1, \ldots, Tu_r\}$ is linearly independent. This is true since these vectors form a basis of TX. So, assume that the assertion holds for some $k < r$ and we want to prove it for $k+1$. If

$$u_{k+1} \notin \operatorname{span}\{v_1, \ldots, v_k, Tv_1, \ldots, Tv_k, Tu_{k+1}, \ldots, Tu_r\}$$

set $v_{k+1} = u_{k+1}$ to complete the induction step. In the remaining case we use the induction hypothesis to conclude that

$$X = \operatorname{Ker} T \oplus \operatorname{span}\{v_1, \ldots, v_k, u_{k+1}, \ldots, u_r\}.$$

Let $P : X \to X$ be the idempotent operator whose range is $\operatorname{Ker} T$ and whose null space is $\operatorname{span}\{v_1, \ldots, v_k, u_{k+1}, \ldots, u_r\}$. If $\{PTu_1, \ldots, PTu_r\}$ is linearly independent, then from

$$u_{k+1} \in \operatorname{span}\{v_1, \ldots, v_k, Tv_1, \ldots, Tv_k, Tu_{k+1}, \ldots, Tu_r\}$$

and $Pu_{k+1} = Pv_1 = \cdots = Pv_k = 0$ we get $u_{k+1} \in \operatorname{span}\{v_1, \ldots, v_k\}$, a contradiction. Hence, $\{PTu_1, \ldots, PTu_r\}$ is linearly dependent, and since $\operatorname{Ker} T$ is at least r-dimensional we can find $z \in \operatorname{Ker} T$ such that z does not belong to the linear span of $PTu_1, \ldots, PTu_r$. Set $v_{k+1} = u_{k+1} + z$. In order to complete the induction step we have to prove that

$$\{v_1, \ldots, v_k, v_{k+1}, Tv_1, \ldots, Tv_k, Tv_{k+1} = Tu_{k+1}, Tu_{k+2}, \ldots, Tu_r\}$$

is linearly independent. So, let $\lambda_1, \ldots, \lambda_{k+1}, \mu_1, \ldots, \mu_r$ be scalars such that

$$\lambda_1 v_1 + \cdots + \lambda_{k+1} v_{k+1} + \mu_1 Tu_1 + \cdots + \mu_r Tu_r = 0.$$

Applying P to both sides of this equation we conclude that $\lambda_{k+1} = 0$. It follows that all the λ's and μ's are zero, as desired.

The next case we shall treat is that there exists $u \in X$ such that

$$\{u, Tu, T^2u, \ldots, T^{2r-1}u\}$$

is linearly independent. The choice $x_1 = u, x_2 = T^2u, \ldots, x_r = T^{2r-2}u$ completes the proof in this special case.

We shall now prove our statement by induction on r. We already know that the assertion is true when $r = 1$. So, assume that the conclusion of our theorem holds for $r - 1 \geq 1$ and we want to prove it for r. As T is not a scalar operator we can find $x \in X$ such that x and Tx are linearly independent. If $Y = \operatorname{span}\{x, Tx, \ldots, T^{2r-2}x\}$ is not invariant under T, then $\{x, Tx, \ldots, T^{2r-2}x, T^{2r-1}x\}$ is linearly independent, and we are done by the previous case. So, we may assume that Y is invariant under T and also $\min\{\operatorname{rank}(T - \lambda I) : \lambda \in \mathbb{C}\} > r$. Using a Jordan canonical form of the restriction of T to Y we can further find a two-dimensional subspace $X_1 \subset Y$ that is invariant under T and the restriction of T to X_1 is not a scalar operator. Let $X = X_1 \oplus X_2$. With respect to this direct sum decomposition T has a matrix representation

$$T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix}.$$

As T_1 is not a scalar operator it has property P_1. If we show that T_2 has property P_{r-1} then one can easily verify that T has the desired property P_r. We know that $\dim X_2 \geq 2(r-1)$. From

$$\operatorname{rank}(T - \lambda I) \leq \operatorname{rank}(T_2 - \lambda I) + 2, \qquad \lambda \in \mathbb{C},$$

we conclude that

$$\min\{\operatorname{rank}(T_2 - \lambda I) : \lambda \in \mathbb{C}\} \geq r - 1.$$

Thus the fact that T_2 has property P_{r-1} follows from the inductive hypothesis. ∎

LEMMA 2.2. *Let n be a positive integer, and let T be an $n \times n$ complex matrix. Then the inner derivation implemented by T, $(S \mapsto [T,S], S \in \mathbb{M}_n)$ is maximal spectral if and only if* $\min\{\operatorname{rank}(T - \lambda I) : \lambda \in \mathbb{C}\} \geq n/2$.

Proof. The "only if" part is obvious; we shall prove the "if" part.

If $n = 2k$, then $\min\{\operatorname{rank}(T - \lambda I) : \lambda \in \mathbb{C}\} \geq k$, and if $n = 2k+1$, then $\min\{\operatorname{rank}(T - \lambda I) : \lambda \in \mathbb{C}\} \geq k+1$. In both cases we use Lemma 2.1 to conclude that there exist k vectors $\zeta_1, \ldots, \zeta_k \in \mathbb{C}^n$ such that

$$\{\zeta_1, \ldots, \zeta_k, T\zeta_1, \ldots, T\zeta_k\}$$

is linearly independent. In the case where $n = 2k+1$ we also choose $\zeta \in \mathbb{C}^n$ such that

$$\{\zeta_1, \ldots, \zeta_k, T\zeta_1, \ldots, T\zeta_k, \zeta\}$$

is a basis in $\mathbb{C}^n$. Let $S \in \mathbb{M}_n$ be the matrix satisfying

$$S\zeta_j = 0, \qquad ST\zeta_j = j\zeta_j, \qquad 1 \leq j \leq k,$$

and let $S\zeta = 0$ when $n = 2k+1$. Then $(TS - ST)\zeta_j = -j\zeta_j$ and $(TS - ST)T\zeta_j = jT\zeta_j - ST^2\zeta_j$, $j = 1, \ldots, k$. Note that the image of S is contained in the linear span of $\zeta_1, \ldots, \zeta_k$. It is now easy to see that in the case where $n = 2k$ we have $\sigma(TS - ST) = \{-1, \ldots, -k, 1, \ldots, k\}$. Since $0 \notin \sigma(TS - ST)$, the matrix $TS - ST$ is invertible, and so this completes the proof of our lemma in the case where n is even.

For the remainder of the proof, we suppose that $n = 2k + 1$ is odd. We use the fact that

$$\operatorname{rank}(TS - ST) \leq 2 \operatorname{rank} S = 2k$$

to conclude that $\sigma(TS - ST) = \{-1, \ldots, -k, 1, \ldots, k, 0\}$.

In order to complete the proof we only need to show that there exists $R \in \mathbb{M}_{2k+1}$ such that $TR - RT$ is invertible. Indeed, assume for the moment that we have already proved this. Then $T(S+\lambda R) - (S+\lambda R)T$ is invertible for all but at most $2k+1$ complex numbers λ. Moreover, by the continuity of the spectrum, $\sharp\sigma(T(S+\lambda R) - (S+\lambda R)T) = n$ for all complex numbers λ that are close enough to 0. It follows that $X \mapsto TX - XT$ is maximal spectral, as desired.

Hence, in the remainder of the proof we suppose that $n = 2k+1, T \in \mathbb{M}_n$ and $\min\{\operatorname{rank}(T - \lambda I) : \lambda \in \mathbb{C}\} \geq k+1$. We must find $R \in \mathbb{M}_n$ such that $TR - RT$ is invertible. Set $Y = \operatorname{span}\{\zeta_1, \ldots, \zeta_k, T\zeta_1, \ldots, T\zeta_k\}$ and write

$$T\zeta = \mu\zeta + \sum_{j=1}^{k} (\mu'_j \zeta_j + h_j T\zeta_j), \qquad \mu, \mu'_j, h_j \in \mathbb{C}.$$

We claim that we can suppose with no loss of generality that $\mu'_j + h_j\mu \neq 0$ for some $j \in \{1, \ldots, k\}$. Suppose first that there exists $j \in \{1, \ldots, n\}$ such that $T^2\zeta_j \notin Y$. A straightforward computation shows that replacing ζ by $\zeta + \lambda T\zeta_j$ for a convenient $\lambda \in \mathbb{C}$, we may suppose that $\mu'_j + h_j\mu \neq 0$. Suppose now that $TY \subseteq Y$ and let us replace T by $T - \mu I$. Then $T\mathbb{C}^n \subset Y$. Since $\operatorname{rank} T \geq k+1$, we have $T\mathbb{C}^n \cap \operatorname{span}\{\zeta_1, \ldots, \zeta_k\} \neq \{0\}$. Replacing, if necessary, one of the ζ_j's by a suitable linear combination of $\zeta_1, \ldots, \zeta_k$, we may assume without loss of generality that there exists $j \in \{1, \ldots, k\}$ such that $\zeta_j \in T\mathbb{C}^n$. Write $\zeta_j = T\zeta'$ for some vector $\zeta' \in \mathbb{C}^n$. Observe that replacing ζ by $\zeta + \lambda\zeta'$, for a suitable $\lambda \in \mathbb{C}$, we can suppose that $\mu'_j \neq 0$. This proves the claim.

Hence, we may assume that $\mu'_k + h_k\mu \neq 0$. Replacing ζ by $\zeta + y$, where y is a suitable linear combination of $\{\zeta_1, \ldots, \zeta_{k-1}\}$, we may also suppose that

$$T\zeta = \mu\zeta + \mu'_k\zeta_k + h_k T\zeta_k + y', \qquad y' \in \operatorname{span}\{\zeta_1, \ldots, \zeta_{k-1}\}.$$

Write

$$T^2\zeta_k = \sum_{j=1}^{k} \nu_j\zeta_j + \sum_{j=1}^{k} \nu'_j T\zeta_j + \tau\zeta, \qquad \tau, \nu_j, \nu'_j \in \mathbb{C}.$$

Fix any $\alpha \in \mathbb{C}$ such that $-\alpha^2(\mu'_k + h_k\mu) + \alpha\tau \neq 0$. Let $R \in \mathbb{M}_n$ be the matrix satisfying

$$R\zeta = \zeta_k, \qquad R\zeta_j = 0, \qquad 1 \leq j \leq k,$$
$$RT\zeta_j = \zeta_j, \qquad 1 \leq j \leq k-1, \qquad RT\zeta_k = \zeta_k + \alpha\zeta.$$

Observe that $R\mathbb{C}^n \subset \mathrm{span}\{\zeta_1, \ldots, \zeta_k, \zeta\}$. Then we have

$$RT^2\zeta_k = (\tau + \nu_k')\zeta_k + \alpha\nu_k'\zeta + y'', \qquad y'' \in \mathrm{span}\{\zeta_1, \ldots, \zeta_{k-1}\}.$$

The corresponding matrix representation of $TR - RT$ with respect to the basis

$$\{\zeta_1, \ldots, \zeta_k, T\zeta_k, \zeta, T\zeta_1, \ldots, T\zeta_{k-1}\}$$

has the upper block triangular form

$$\begin{pmatrix} B_1 & * & * \\ 0 & M(\alpha) & * \\ 0 & 0 & B_2 \end{pmatrix},$$

where $B_1 = \mathrm{diag}(-1, \ldots, -1)$, $B_2 = \mathrm{diag}(1, \ldots, 1)$ and

$$M(\alpha) = \begin{pmatrix} -1 & \alpha\mu_k' - \tau - \nu_k' & -\mu - h_k \\ 0 & \alpha h_k + 1 & 1 \\ -\alpha & \alpha(\mu - \nu_k') & -\alpha h_k \end{pmatrix}.$$

Obviously,

$$\det M(\alpha) = -\alpha^2(\mu_k' + h_k\mu) + \alpha\tau,$$

and therefore $TR - RT$ is an invertible matrix, as desired. ■

For a vector space X and a linear operator T on X, we denote by $\sigma_{\mathrm{p}}(T)$ the set of eigenvalues of T. Recall that an algebra $\mathcal{A}$ of linear operators on X is called *dense* if for every positive integer n, every n-tuple of linearly independent vectors $(x_1, \ldots, x_n)$ of X and every n-tuple of vectors $(y_1, \ldots, y_n)$ of X there exists an operator $T \in \mathcal{A}$ such that $Tx_j = y_j$ for $j = 1, \ldots, n$.

LEMMA 2.3. *Let X be a complex vector space, and let $\mathcal{A}$ be a dense algebra of linear operators on X. Let T be a linear operator on X, and let n be a positive integer. Suppose that* $\max\{\sharp\sigma_{\mathrm{p}}(TS - ST) : S \in \mathcal{A}\} = n$. *If* $\dim X \geq n + 1$, *then n is odd, $n = 2k + 1$. Moreover,* $\min \mathrm{rank}(T - \lambda I) = k$.

Proof. We first show that for every $x_1, \ldots, x_{n+1} \in X$ the set

$$\{x_1, \ldots, x_{n+1}, Tx_1, \ldots, Tx_{n+1}\}$$

is linearly dependent. Indeed, if this were not true, we would be able to find $S \in \mathcal{A}$ such that $Sx_j = 0$ and $STx_j = jx_j$, $j = 1, \ldots, n + 1$. Then $(TS - ST)x_j = -jx_j$, contradicting our assumption on the cardinality of the point spectrum of $TS - ST$.

Let r be the largest positive integer such that there exist $x_1, \ldots, x_r \in X$ with the property that

$$\{x_1, \ldots, x_r, Tx_1, \ldots, Tx_r\}$$

is linearly independent. Set

$$Y = \operatorname{span}\{x_1, \dots, x_r, Tx_1, \dots, Tx_r\}.$$

Choose $S \in \mathcal{A}$ such that $Sx_j = 0$, $STx_j = jx_j$, $j = 1, \dots, r$. If $T^2x_j \in Y$ for some j, we have $ST^2x_j \in \operatorname{span}\{x_1, \dots, x_r\}$. Suppose that there exists j such that $T^2x_j \notin Y$. With no loss of generality, we may suppose that the set

$$\{x_1, \dots, x_r, Tx_1, \dots, Tx_r, T^2x_1, \dots, T^2x_s\}$$

is linearly independent ($s \le r$) and that for every $j = s+1, \dots, r$, we have $T^2x_j \in Y + \operatorname{span}\{T^2x_1, \dots, T^2x_s\}$. Then S can be chosen so that $ST^2x_j = 0, 1 \le j \le s$. For each j, $1 \le j \le r$, we have

$$(TS - ST)x_j = -jx_j, \ \ (TS - ST)Tx_j = jTx_j + u_j, \ u_j \in \operatorname{span}\{x_1, \dots, x_r\}.$$

Thus, Y is invariant under $TS - ST$. If we consider the matrix representation of $TS - ST$ with respect to $\{x_1, \dots, x_r, Tx_1, \dots, Tx_r\}$, we see that $\{1, \dots, r, -1, \dots, -r\} \subset \sigma_{\mathrm{p}}(TS - ST)$. Consequently, $2r \le n$.

We are now ready to prove our statement. Suppose first that n is odd, that is, $n = 2k+1$ for some nonnegative integer k. If $\min\operatorname{rank}(T - \lambda I) \ge k+1$, Lemma 2.1 implies that there exist $k+1$ vectors $x_1, \dots, x_{k+1}$ such that the set

$$\{x_1, \dots, x_{k+1}, Tx_1, \dots, Tx_{k+1}\}$$

is linearly independent. The above argument shows that there exists $S \in \mathcal{A}$ such that $\sharp\sigma_p(TS - ST) \ge 2k+2$, a contradiction. Thus $\min\operatorname{rank}(T - \lambda I) \le k$. Of course we must have $\min\operatorname{rank}(T - \lambda I) = k$.

In order to complete the proof we have to show that n cannot be even. Assume on the contrary that $n = 2k$, where $k \in \mathbb{N}$. If $\min\operatorname{rank}(T - \lambda I) \ge k$, Lemma 2.1 implies that there exist k vectors $x_1, \dots, x_k$ such that the set $\{x_1, \dots, x_k, Tx_1, \dots, Tx_k\}$ is linearly independent. Since $\dim X \ge 2k+1$, the above argument shows that there exists $S \in \mathcal{A}$ such that $\sharp\sigma_p(TS - ST) \ge 2k+1$. This contradiction implies $\min\operatorname{rank}(T - \lambda I) \le k-1$. It follows that $\sharp\sigma_p(TS - ST) \le 2k-1$. Consequently, this case cannot occur. ∎

By combining Lemmas 2.2 and 2.3, we obtain the following

OBSERVATION 2.4. *Let n be a positive integer, and let A be an $n \times n$ complex matrix. Suppose that the inner derivation $X \mapsto [X, A]$, $X \in \mathbb{M}_n$, is not maximal spectral. Then there exists an integer $k \ge 0$ such that $\max\{\sharp\sigma([X, A]) : X \in \mathbb{M}_n\} = 2k+1$ and $\min\operatorname{rank}\{A - \lambda I : \lambda \in \mathbb{C}\} = k$. Furthermore, 0 belongs to $\sigma([X, A])$ for every $X \in \mathbb{M}_n$.*

3. The case where d is not necessarily continuous. Let A be a Banach algebra, and let $d : A \to A$ be a derivation. According to [13], the set $\varDelta$ of primitive ideals which are not invariant under d is at most finite, and each primitive ideal in $\varDelta$ has finite codimension. Recall that the noncommutative Singer–Wermer conjecture states that $\varDelta$ is actually empty [13, 14]. If d is continuous, then every primitive ideal is invariant under d [12], and derivations on semisimple Banach algebras are continuous [8]. Using the result of [13], we shall show that if $\sharp\sigma(d(x))$ is finite for every $x \in A$, then d maps A into the socle modulo the radical.

In the proof we shall need the following well-known facts. If A is a complex Banach algebra and $x \in A$, then $\sigma(x + \operatorname{rad} A) = \sigma(x)$. Assume that A is semisimple. If for an $x \in A$ we have $\dim xAx < \infty$, then $x \in \operatorname{soc}(A)$. If $J \subset A$ is a finite-dimensional ideal, then there exists a closed ideal $L \subset A$ such that $A = J \oplus L$. Indeed, denote the unit of J by e. We shall verify that e belongs to the centre of A. As $ex, xe \in J$ for every $x \in A$, we have $ex = (ex)e = e(xe) = xe$. Set $L = \{x - ex : x \in A\}$. Clearly, L is a closed ideal and $A = J \oplus L$.

Lemma 3.1. *Let A be a complex Banach algebra, and let $d : A \to A$ be a derivation. Then there exists a closed ideal J of A such that* $\operatorname{rad} A \subset J$, $\dim J/\operatorname{rad} A < \infty$ *and* $d(J) \subset J$.

Proof. Suppose first that for every primitive ideal P of A, $d(P) \subset P$. In this case set $J = \operatorname{rad} A$. Clearly, $d(J) \subset J$ and we are done.

So, assume now that there exists a primitive ideal P of A which is not invariant under d. It follows from [13] that there exist a finite number of primitive ideals which are not invariant under d. Let $P_1, \ldots, P_r$ be those exceptional primitive ideals. Set

$$\varGamma = \{P \in \operatorname{Prim}(A) : P \neq P_1, \ldots, P_r\} \quad \text{and} \quad J = \bigcap_{P \in \varGamma} P.$$

Here, $\operatorname{Prim}(A)$ denotes the set of all primitive ideals of A. Of course, if $\varGamma = \emptyset$ we have $J = A$. Clearly, J is a closed ideal of A and $d(J) \subset J$. Let us consider the linear map

$$\varphi : J \to A/P_1 \times \cdots \times A/P_r, \qquad a \mapsto (a + P_1, \ldots, a + P_r).$$

Then $\operatorname{Ker} \varphi = \operatorname{rad} A$. Thus the induced map $\varphi : J/\operatorname{rad} A \to A/P_1 \times \cdots \times A/P_r$ is injective. Since the algebras A/P_i are finite-dimensional [13], so is the algebra $J/\operatorname{rad} A$. ∎

Using this, we can now prove our first theorem.

Proof of Theorem 1.1. Suppose that (i) is true and we want to show (iv). It follows from Lemma 3.1 that there exists a closed ideal J of A such that $\operatorname{rad} A \subset J$, $\dim J/\operatorname{rad} A < \infty$ and $d(J) \subset J$. Observe that the algebra A/J is semisimple. Indeed, as $J/\operatorname{rad} A$ is finite-dimensional, there exists a closed ideal $L \subset A/\operatorname{rad} A$ such that $(J/\operatorname{rad} A) \oplus L = A/\operatorname{rad} A$. We know that A/J is isomorphic to $(A/\operatorname{rad} A)/(J/\operatorname{rad} A)$, and thus A/J is isomorphic to L, which is semisimple because each ideal of a semisimple algebra is semisimple. Set $A/J = \overline{A}$ and let $\overline{d}$ be the induced derivation on $\overline{A}$. Then $\overline{d}$ is continuous [8]. For every $x \in A$, we have $\sigma(\overline{d}(x+J)) \subset \sigma(d(x))$, thus $\sigma(\overline{d}(x))$ is finite for all $x \in \overline{A}$. It follows from [2, Theorem 2.4] that there exists $a \in A$ such that $a+J \in \operatorname{soc} \overline{A}$ and $\overline{d}(x) = [a+J, x]$ for every $x \in \overline{A}$. Thus $d(x) - [a, x] \in J$ for every $x \in A$. On the other hand, since $a + J \in \operatorname{soc} \overline{A}$ and $\dim J/\operatorname{rad} A < \infty$, we have $a + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$.

Next assume (iv). Since $J/\operatorname{rad} A \subset \operatorname{soc}(A/\operatorname{rad} A)$ we have (iv)⇒(iii)⇒(i). Clearly, (ii)⇒(i). Let us show that (iv)⇒(ii). Suppose (iv). We use the fact that the ideal $J/\operatorname{rad} A$ is finite-dimensional once more to get a closed ideal L of $A/\operatorname{rad} A$ such that

$$(J/\operatorname{rad} A) \oplus L = A/\operatorname{rad} A.$$

Set $a = a_1 + a_2$, where $a_1 + \operatorname{rad} A \in J/\operatorname{rad} A$ and $a_2 + \operatorname{rad} A \in L$. Observe that $a_2 + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$. For every $x \in A$, $d(x) + \operatorname{rad} A = y + [a_2, x] + \operatorname{rad} A$ for some $y \in J$. Thus, $\sigma(d(x) + \operatorname{rad} A) \subset \sigma([a_2, x]) \cup \sigma(y + \operatorname{rad} A)$. On the other hand, using [5] once again, we see that there exists $m \in \mathbb{N}$ such that $\sharp\sigma([a_2, x]) \le m$ for every $x \in A$. Further, we know that $\sharp\sigma(y + \operatorname{rad} A) \le \dim(J/\operatorname{rad} A) + 1$ for every $y \in J$. Thus, (ii) holds. ■

4. Proof of the main result. Let A be a complex Banach algebra. Recall that every primitive ideal is prime. In the case that $\operatorname{Prim}(A)$ is nonempty, we shall often use the following result [11, Theorem 2.2.9]: $\sigma(x) = \bigcup_{P \in \operatorname{Prim}(A)} \sigma(x + P)$, $x \in A$.

Proof of Theorem 1.2. It follows from [2, Theorem 2.4] that there exists $b \in \operatorname{soc}(A)$ such that d is the inner derivation implemented by b. If $d(A) \not\subset P$ for some primitive ideal P, then $b \notin P$. The fact that $b \in \operatorname{soc}(A)$ implies that this can happen for at most finitely many primitive ideals P [5, Proposition 2.2].

Let $\{P_1, \ldots, P_r\}$ be the set of primitive ideals of A such that $d(A) \not\subset P_j$, $j = 1, \ldots, r$. Observe that $P_k \not\subset P_j$ for $k \neq j$, $1 \le k, j \le r$. Indeed, otherwise we have $\operatorname{soc}(A/P_k) \subset P_j/P_k$. Here we have used the fact that $x + P_k \in \operatorname{soc}(A/P_k)$ if and only if $\pi_k(x)$ is a finite rank operator in $\pi_k(A)$, where π_k is an irreducible representation of A whose kernel is P_k, together with the fact that every nonzero ideal of $\pi_k(A)$ contains all finite rank operators from $\pi_k(A)$. It follows that $b \in P_j$. This contradicts our assumption on b.

Let $\pi_1, \ldots, \pi_r$ be irreducible representations of A on Banach spaces $X_1, \ldots, X_r$, respectively, such that $\operatorname{Ker} \pi_j = P_j$ for all j. For each j, set $n_j = \max\{\sharp\sigma(d(x) + P_j) : x \in A\}$ and let d_j denote the induced derivation on A/P_j. For $1 \le j \le r$, choose $x_j \in A$ such that $\sharp\sigma(d(x_j) + P_j) = n_j$. In the case that $\dim(A/P_j) < \infty$ and d_j is maximal spectral, we further suppose that $0 \notin \sigma(d(x_j) + P_j)$. Replacing each x_j by $\alpha_j x_j$ for a suitable

$\alpha_j \in \mathbb{C}$, we can assume that $\sigma(d(x_k)+P_k)\cap\sigma(d(x_j)+P_j) \subset \{0\}$, $k \neq j$. Now observe that each element of $\sigma(\pi_j d(x_j))$ is an eigenvalue, since $\pi_j d(x_j)$ is a finite rank operator. Moreover, $\sigma(d(x_j)+P_j) = \sigma(\pi_j d(x_j))$. For each j, let $\lambda_1^j, \dots, \lambda_{n_j}^j$ be the eigenvalues of $\pi_j d(x_j)$, and let $\zeta_1^j, \dots, \zeta_{n_j}^j$ be associated eigenvectors. Using the extended Jacobson density theorem [7, p. 283], we can find $x \in A$ such that $\pi_j(x)u_j = \pi_j(x_j)u_j$ for each $1 \le j \le r$ and each $u_j \in \operatorname{span}\{\zeta_t^j, \pi_j(b)\zeta_t^j : t = 1, \dots, n_j\}$. Then we have

$$\pi_j(d(x))\zeta_t^j = \pi_j(bx - xb)\zeta_t^j = \pi_j(bx_j - x_j b)\zeta_t^j = \lambda_t^j \zeta_t^j.$$

Hence, $\sigma(d(x_j)+P_j) \subset \sigma(d(x)+P_j)$ for each $j \in \{1, \dots, r\}$. In fact, since $n_j = \max\{\sharp\sigma(d(z)+P_j) : z \in A\}$, we have $\sigma(d(x_j)+P_j) = \sigma(d(x)+P_j)$ for each $j \in \{1, \dots, r\}$. We may suppose that for $1 \le j \le t$, $\dim(A/P_j) < \infty$ and d_j is maximal spectral, and for $t+1 \le j \le r$ either $\dim(A/P_j) < \infty$ and d_j is not maximal spectral, or A/P_j is infinite-dimensional. Of course, we can have $t = 0$. Using Observation 2.4 we see that $0 \in \sigma(d(y)+P)$ for every primitive ideal P different from $P_1, \dots, P_t$. We have

$$\sigma(d(y)) = \bigcup_{P\in \operatorname{Prim}(A)} \sigma(d(y)+P)$$

for every $y \in A$. Recall that $n = \max\{\sharp\sigma(d(y)) : y \in A\}$. Observe that $\sharp\sigma(d(x)) = n$.

We distinguish three cases.

(a) If $r \neq t$, then $\sharp\sigma(d(x)) = n_1 + \dots + n_t + n_{t+1} + \sum_{j=t+2}^{r} n_j - 1$.
(b) If $r = t$ and there exists $P \in \operatorname{Prim}(A)$ such that $d(A) \subset P$, then $\sharp\sigma(d(x)) = n_1 + \dots + n_t + 1$.
(c) If $r = t$ and $\operatorname{Prim}(A) = \{P_1, \dots, P_t\}$, then $\sharp\sigma(d(x)) = n_1 + \dots + n_t$.

In the case where $t < r$ we fix j such that $t+1 \le j \le r$. According to Lemma 2.3 and Observation 2.4, $n_j = 2k_j + 1$, where $k_j = \min\{\operatorname{rank}(\pi_j(b) - \lambda I) : \lambda \in \mathbb{C}\}$.

Suppose first that $\dim(A/P_j) < \infty$. Then the algebra A/P_j is simple and unital. The intersection $\bigcap_{P\neq P_j} P + P_j$ is a nonzero ideal of A/P_j. Indeed, each $P + P_j$, $P \neq P_j$, is an ideal of A/P_j, and therefore it is either equal to A/P_j, or it is zero. In the latter case we would have $P \subset P_j$. But this is impossible because $P_k \not\subset P_j$, $k = 1, \dots, r$, $k \neq j$, and $d(A) \subset P$ for all $P \in \operatorname{Prim}(A) \setminus \{P_1, \dots, P_r\}$. It follows that $\bigcap_{P\neq P_j} P + P_j = A$. Choose $e_j \in \bigcap_{P\neq P_j} P$ such that $e_j + P_j$ is the unit element of A/P_j. Moreover, choose $\lambda_j \in \mathbb{C}$ such that $\operatorname{rank} \pi_j(b - \lambda_j e_j) = k_j$. Put $b_j = (b - \lambda_j)e_j$.

Next suppose that $\dim(A/P_j) = \infty$. Then d_j is the inner derivation implemented by $b + P_j$ and $b + P_j \in \operatorname{soc}(A/P_j)$. Since $\operatorname{soc}(A/P_j) \subset \bigcap_{P\neq P_j} P + P_j$, there exists $b_j \in \bigcap_{P\neq P_j} P$ such that $b_j + P_j = b + P_j$.

Now put $a = \sum_{k=t+1}^{r} b_k$. Then for each $t+1 \le j \le r$, $\pi_j(a) = \pi_j(b_j)$. Let

$$\Gamma = \{P \in \mathrm{Prim}(A) : P \neq P_{t+1}, \ldots, P_r\}.$$

Then $a \in \bigcap_{P\in\Gamma} P$. Using [6], we infer that $\mathrm{rank}(a) = \sum_{j=t+1}^{r} k_j$.

Set

$$\Gamma' = \{P \in \mathrm{Prim}(A) : P \neq P_1, \ldots, P_t\} \text{ and } A_1 = \bigcap_{P\in\Gamma'} P.$$

Then A_1 is a finite-dimensional ideal of A isomorphic to $A/P_1 \oplus \cdots \oplus A/P_t$. Let e be the unit element of A_1. Then e is a central idempotent of A. Set $A_2 = A(1-e) = \bigcap_{j=1}^{t} P_j$. We denote by d_j, $j = 1, 2$, the restriction of d to A_j. Then $d(x) + P_j = d_1(xe) + P_j$ for all $1 \le j \le t$, and therefore, $d_1(xe)$ is an invertible element of A_1. Moreover, $\sharp\sigma(d_1(xe)) = \sum_{j=1}^{t} n_j$. Thus, d_1 is maximal spectral. In the case (c) we are done.

So, from now on we may suppose that $A_2 \neq \{0\}$. We fix $y \in A_2$. Observe that $d(y) + P = [b, y] + P = [a, y] + P$ for every $P \in \Gamma'$. Since A_2 is semisimple and $[b-a, y] \in P$ for every $P \in \Gamma'$, we infer that $[b-a, y] = 0$ and so $d(y) = [a, y]$.

Next we claim that $\sigma(d(y), A_2) = \sigma(d(y), A)$. Note that $\sigma(d(y), A)\setminus\{0\} = \sigma(d(y), A_2) \setminus \{0\}$. Furthermore, $0 \in \sigma(d(y), A)$. It remains to show that $0 \in \sigma(d(y), A_2)$. Let P be a primitive ideal of A different from $P_1, \ldots, P_t$. Notice that $(\bigcap_{j=1}^{t} P_j + P)/P$ is a nonzero ideal of A/P and is isomorphic to $\bigcap_{j=1}^{t} P_j/(\bigcap_{j=1}^{t} P_j \cap P)$. Moreover, $\bigcap_{j=1}^{t} P_j \cap P$ is a primitive ideal of $\bigcap_{j=1}^{t} P_j = A_2$. Since $0 \in \sigma(d(y) + P, A/P)$, we have $0 \in \sigma(d(y) + P, \bigcap_{j=1}^{t} P_j + P)$ and so $0 \in \sigma(d(y) + \bigcap_{j=1}^{t} P_j \cap P, \bigcap_{j=1}^{t} P_j)$. Thus 0 belongs to $\sigma(d(y), A_2)$ and the claim is proved.

Thus, in the case (b) we have the third possibility from the conclusion of our theorem with $k = 0$.

It remains to consider the case (a). Set $n' = \max\{\sharp\sigma(d(y)) : y \in A_2\}$. Since $\sigma(d(y)) = \bigcup_{P\in\Gamma'} \sigma(d(y) + P)$, $y \in A_2$, and $x - xe + P = x + P$ for every $P \in \Gamma'$, we have $n' = \sharp\sigma(d(x - xe))$. Hence, $n' = 2(\sum_{j=t+1}^{r} k_j) + 1$ $= 2\,\mathrm{rank}(a) + 1$. This completes the proof. ■

Let A be an associative algebra. The Lie product of A is defined by $[x, y] = xy - yx$. Let J be an ideal of A. According to [4], a Lie ideal L of A is said to be *embraced* by J if

$$[J, A] \subseteq L \subseteq \{x \in A : [x, A] \subseteq [J, A]\}.$$

In the case that A is a semisimple Banach algebra, recall that the socle of A is the largest ideal with finite spectrum. Using [2] and [5] we can show that there exists a largest Lie ideal L of A such that every element of L has finite spectrum. In particular, we shall show that the Lie ideal L is embraced by the socle of A. Set

$$Z'(A) = \{x \in Z(A) : \sharp\sigma(x) < \infty\}.$$

It follows from the fact that primitive complex Banach algebras are central that $Z'(A)$ is a vector subspace of $Z(A)$.

COROLLARY 4.1. *Let A be a complex unital semisimple Banach algebra. Then* $\mathrm{soc}(A)+Z'(A)$ *coincides with the largest spectrum finite Lie ideal of A.*

Proof. It is obvious that $\mathrm{soc}(A) + Z'(A)$ is a Lie ideal of A. We first show that every element in $\mathrm{soc}(A) + Z'(A)$ has finite spectrum. Let $0 \neq x \in \mathrm{soc}(A)$, and let $u \in Z'(A)$. According to [5, Proposition 2.2], there exist a finite number of primitive ideals $P_1, \ldots, P_r$ such that $x \in P$ for every $P \notin \{P_1, \ldots, P_r\}$. Set $\Gamma = \{P \in \mathrm{Prim}(A) : P \neq P_1, \ldots, P_r\}$. Then we have

$$\sigma(x+u) = \Big(\bigcup_{j=1}^{r} \sigma(x+u+P_j)\Big) \cup \Big(\bigcup_{P\in\Gamma} \sigma(u+P)\Big).$$

But $\bigcup_{P\in\Gamma} \sigma(u+P) \subseteq \sigma(u)$ and for each $j \in \{1, \ldots, r\}$, there exists $\lambda \in \mathbb{C}$ such that $\sigma(x+u+P_j) = \sigma(x+\lambda+P_j)$. Thus $\sharp\sigma(x+u) < \infty$. Now let L be a Lie ideal of A such that $\sharp\sigma(a) < \infty$ for every $a \in L$. Fix an element $a \in L$. Then $\sharp\sigma([a,x]) < \infty$ for every $x \in A$. It follows from [2, 5] that there exists $u \in Z(A)$ such that $a + u \in \mathrm{soc}(A)$. In particular, $\sharp\sigma(a+u) < \infty$. For every $P \in \mathrm{Prim}(A)$, set $u + P = \lambda_P + P, \lambda_P \in \mathbb{C}$. Then $\sigma(a+u) = \bigcup_{P\in\mathrm{Prim}(A)} \sigma(a+\lambda_P+P)$. But $\bigcup_{P\in\mathrm{Prim}(A)} \sigma(a+P) = \sigma(a)$ and $\sharp\sigma(a) < \infty$, thus the set $\{\lambda_P : P \in \mathrm{Prim}(A)\} = \sigma(u)$ is finite. This completes the proof. ∎

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