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A Calderón–Zygmund estimate with applications to generalized Radon transforms and Fourier integral operators

by

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Abstract. We prove a Calderón–Zygmund type estimate which can be applied to sharpen known regularity results on spherical means, Fourier integral operators, generalized Radon transforms and singular oscillatory integrals.

1. Introduction. The main theme in this paper is to strengthen various sharp L^p-Sobolev regularity results for integral operators. To illustrate this we consider the example of spherical means.

Let σ denote surface measure on the unit sphere. Since

$$|\widehat{\sigma}(\xi)| \le C(1+|\xi|)^{-(d-1)/2}$$

the convolution operator $f \mapsto f * \sigma$ maps L^2 to the Sobolev space $L^2_{(d-1)/2}$. By complex interpolation with an L^∞-BMO estimate, Fefferman and Stein [4] proved that the operator maps L^p to $L^p_{(d-1)/p}$ for $2 < p < \infty$; here the regularity parameter $\alpha = (d-1)/p$ is optimal. It turns out, however, that the L^p-Sobolev result can be improved within the scale of Triebel–Lizorkin spaces [23] in two ways.

We recall the definition of the quasinorm

$$\|f\|_{F^p_{\alpha,q}} = \Big\| \Big(\sum_{k=0}^{\infty} 2^{k\alpha q} |\Pi_k f|^q \Big)^{1/q} \Big\|_{L^p}$$

which we will use for $1 < p < \infty$ and $0 < q < \infty$. Here the operators Π_k are defined by the standard smooth Littlewood–Paley cutoffs, so that $\widehat{\Pi_k f}$ is supported in $\{2^{k-1} \le |\xi| \le 2^{k+1}\}$ for $k \ge 1$ and in a neighborhood of the origin for $k = 0$; we assume that $\sum_{k=0}^{\infty} \Pi_k f = f$ for all Schwartz

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functions. It is well known, and immediate from Littlewood–Paley theory and embeddings for sequence spaces, that $L^p \subset F^p_{0,p} \equiv B^p_{0,p}$, $2 \le p < \infty$, and for all $p \in (1,\infty)$, $F^p_{\alpha,r} \subset F^p_{\alpha,s} \subset F^p_{\alpha,2} = L^p_\alpha$ if $0 < r \le s \le 2$. Thus the inequalities

$$\|f * \sigma\|_{F^p_{(d-1)/p,r}} \le C_{p,r}\|f\|_{F^p_{0,p}}, \qquad r > 0,\ 2 < p < \infty, \tag{1.1}$$

strengthen the standard regularity result. The case $r = 1$ also implies an $F^p_{0,\infty} \to F^p_{\alpha,p}$ estimate for $1 < p < 2$ and $\alpha = (d-1)/p'$, by duality and composition with Bessel derivatives $(I - \Delta)^{\alpha/2}$. Related phenomena have recently been observed in articles on space-time (or local smoothing) estimates for Schrödinger equations [17] and wave equations [6].

In §2 we formulate a general result which covers the spherical means and many other related applications. These are discussed in §3.

2. A Calderón–Zygmund estimate. For each $k \in \mathbb{N}$, we consider operators T_k defined on the Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ by

$$T_k f(x) = \int K_k(x,y) f(y)\,dy,$$

where each K_k is a continuous and bounded kernel (this qualitative assumption is made to avoid measurability questions). Let $\zeta \in \mathcal{S}(\mathbb{R}^d)$. Define $\zeta_k = 2^{kd}\zeta(2^k\cdot)$ and

$$P_k f = \zeta_k * f.$$

In applications the operators P_k often arise from dyadic frequency decompositions, however we emphasize that no cancellation condition on ζ is needed in the following result.

Theorem 2.1. *Let $0 < a < d$, $\varepsilon > 0$, and $1 < q < p < \infty$. Assume the operators T_k satisfy*

$$\sup_{k>0} 2^{ka/p}\|T_k\|_{L^p \to L^p} \le A, \tag{2.1}$$

$$\sup_{k>0} 2^{ka/q}\|T_k\|_{L^q \to L^q} \le B_0. \tag{2.2}$$

Furthermore let $\Gamma \ge 1$, and assume that for each cube Q there is a measurable set $\mathcal{E}_Q$ so that

$$|\mathcal{E}_Q| \le \Gamma \max\{|Q|^{1-a/d}, |Q|\}, \tag{2.3}$$

and for every $k \in \mathbb{N}$ and every cube Q with $2^k \operatorname{diam}(Q) \ge 1$,

$$\sup_{x \in Q} \int_{\mathbb{R}^d \setminus \mathcal{E}_Q} |K_k(x,y)|\,dy \le B_1 \max\{(2^k \operatorname{diam}(Q))^{-\varepsilon}, 2^{-k\varepsilon}\}. \tag{2.4}$$

Let

$$\mathcal{B} := B_0^{q/p}(A\Gamma^{1/p} + B_1)^{1-q/p}. \tag{2.5}$$

Then there is $C > 0$ (depending only on $d, \zeta, a, \varepsilon, p, q, r$) so that

(2.6)
$$\Big\|\Big(\sum_k 2^{kar/p}|P_kT_kf_k|^r\Big)^{1/r}\Big\|_p \le CA\Big[\log\Big(3+\frac{\mathcal{B}}{A}\Big)\Big]^{1/r-1/p}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}.$$

In some interesting applications $A \ll \mathcal{B}$ so that the logarithmic growth in (2.6) is helpful. The power of the logarithm is sharp (see [8], [22], [23] for a relevant counterexample and [18], [1] for positive results on families of translation invariant and pseudo-differential operators).

To prove Theorem 2.1 we begin with a standard L^∞-bound. In what follows the notation $\fint_Q f$ will be used for the average $|Q|^{-1}\int_Q f$.

Lemma 2.2. *Assuming* (2.1), (2.3) *and* (2.4), *the following statements hold true.*

(i) *If* $2^{-k} \le \operatorname{diam}(Q) \le 1$, *then*

(2.7)
$$\fint_Q |P_kT_kh|\,dy \le C(A\Gamma^{1/p}(2^k\operatorname{diam}(Q))^{-a/p} + B_1(2^k\operatorname{diam}(Q))^{-\varepsilon})\|h\|_\infty.$$

(ii) *If* $\operatorname{diam}(Q) \ge 1$, *then*

(2.8)
$$\fint_Q |P_kT_kh|\,dy \le C(A\Gamma^{1/p}2^{-ka/p} + B_12^{-k\varepsilon})\|h\|_\infty.$$

Proof. We split $h = h\chi_{\mathcal{E}_Q} + h\chi_{\mathbb{R}^d\setminus\mathcal{E}_Q}$. By Hölder's inequality, (2.1), and then (2.3) ([1]),

$$\begin{aligned}\fint_Q |T_k[h\chi_{\mathcal{E}_Q}]|\,dx &\le |Q|^{-1/p}\Big(\int |T_k[h\chi_{\mathcal{E}_Q}]|^p\,dx\Big)^{1/p}\\ &\lesssim |Q|^{-1/p}A2^{-ka/p}\|h\chi_{\mathcal{E}_Q}\|_p \lesssim A2^{-ka/p}|Q|^{-1/p}|\mathcal{E}_Q|^{1/p}\|h\|_\infty\\ &\lesssim A\Gamma^{1/p}2^{-ka/p}\max\{\operatorname{diam}(Q)^{-a/p},1\}\|h\|_\infty.\end{aligned}$$

On the other hand, by (2.4),

$$\begin{aligned}\fint_Q |T_k[h\chi_{\mathbb{R}^d\setminus\mathcal{E}_Q}]|\,dx &\le \sup_{x\in Q}\int_{\mathbb{R}^d\setminus\mathcal{E}_Q} |K_k(x,y)|h(y)\,dy\\ &\lesssim B_1\max\{(2^k\operatorname{diam}(Q))^{-\varepsilon}, 2^{-k\varepsilon}\}\|h\|_\infty.\end{aligned}$$

A combination of these two bounds shows that the stated estimates hold with P_kT_k replaced by T_k.

([1]) The expression $v \lesssim w$ denotes $v \le Cw$, where $C > 0$ is independent of A, B_0, B_1, Γ.

We now use straightforward estimates to incorporate the operators P_k. In view of the rapid decay of ζ we have

$$\fint_Q |P_k T_k h(x)|\,dx \le C_N \fint_Q \int \frac{2^{kd}}{(1+2^k|x-w|)^N} |T_k h(w)|\,dw\,dx.$$

Now for $m = 0, 1, 2, \ldots$ we let Q_m^* denote the cube parallel to Q with the same center, but with sidelength equal to 2^{m+1} times the sidelength of Q. Then the last estimate (with $N \gg d$) implies

$$\begin{aligned}&\fint_Q |P_k T_k h(x)|\,dx \\ &\qquad \le C_N' \fint_{Q_0^*} |T_k h(w)|\,dw + \sum_{m=1}^{\infty} (2^k \operatorname{diam}(Q_m^*))^{d-N} \fint_{Q_m^*} |T_k h(w)|\,dw.\end{aligned}$$

The term corresponding to $m = 0$ has already been estimated and, also by the bounds above applied to Q_m^*, the mth term is controlled by

$$2^{-m(N-d)} (2^k \operatorname{diam}(Q))^{d-N} \Big(\frac{A\Gamma^{1/p} 2^{-ma/p}}{(2^k \operatorname{diam}(Q))^{a/p}} + \frac{B 2^{-m\varepsilon}}{(2^k \operatorname{diam}(Q))^{\varepsilon}} \Big) \|h\|_\infty$$

if $2^m \operatorname{diam}(Q) \le 1$, and by

$$2^{-m(N-d)} (2^k \operatorname{diam}(Q))^{d-N} (A\Gamma^{1/p} 2^{-ka/p} + B 2^{-k\varepsilon}) \|h\|_\infty$$

if $2^m \operatorname{diam}(Q) > 1$. We sum in m to obtain the claimed result. ∎

Proof of Theorem 2.1. We first note that the asserted inequality for $r = p$ follows by assumption (2.1) and Fubini's theorem. We prove the theorem for $r \le 1$, and the intermediate cases $1 < r < p$ follow by interpolation.

By the monotone convergence theorem it suffices to prove (2.6) for all *finite* sequences $F = \{f_k\}_{k\in\mathbb{N}}$, i.e., we may assume that $f_k = 0$ for large k.

We use the Fefferman–Stein theorem [4] for the #-maximal operator. The left hand side of (2.6) is then rewritten and estimated as

$$\begin{aligned}&\Big\| \sum_k |2^{ka/p} P_k T_k f_k|^r \Big\|_{p/r}^{1/r} \\ &\quad \lesssim \Big\| \sup_{Q: x\in Q} \fint_Q \Big| \sum_k |2^{ka/p} P_k T_k f_k(y)|^r - \fint_Q \sum_k |2^{ka/p} P_k T_k f_k(z)|^r\,dz \Big|\,dy \Big\|_{L^{p/r}(dx)}^{1/r} \\ &\quad \lesssim \Big\| \sup_{Q: x\in Q} \sum_k 2^{kar/p} \fint_Q \fint_Q |P_k T_k f_k(y) - P_k T_k f_k(z)|^r\,dz\,dy \Big\|_{L^{p/r}(dx)}^{1/r}.\end{aligned}$$

In the last step we simply use $|u^r - v^r| \le |u-v|^r$ for nonnegative u, v and $0 < r \le 1$, combined with the triangle inequality.

Note that the application of the Fefferman–Stein inequality is valid because of our *a priori* assumption involving finite sums.

Given a sequence f_k we can choose cubes $Q(x)$ depending measurably on x so that the supremum in Q can be up to a factor of two realized by the choice of $Q(x)$. This means that it suffices to prove the inequality

$$\Big\| \sum_k 2^{kar/p} \fint_{Q(x)} \fint_{Q(x)} |P_k T_k f_k(y) - P_k T_k f_k(z)|^r \, dz \, dy \Big\|_{L^{p/r}(dx)}^{1/r} \leq CA \Big[\log\Big(3 + \frac{\mathcal{B}}{A}\Big)\Big]^{1/r-1/p} \Big(\sum_k \|f_k\|_p^p \Big)^{1/p} \tag{2.9}$$

where C does not depend on the choice of $x \mapsto Q(x)$. We define $L(x)$ to be the integer L for which the sidelength of $Q(x)$ belongs to $[2^L, 2^{L+1})$.

Let $X = \{x : L(x) \leq 0\}$. We shall first estimate the $L^{p/r}$ norm over X (the main and more interesting part) and then provide the bound on $L^{p/r}(\mathbb{R}^d \setminus X)$ separately.

Define

$$\mathcal{G}_k h(x) = \Big(\fint_{Q(x)} \fint_{Q(x)} |P_k T_k h(y) - P_k T_k h(z)|^r \, dz \, dy \Big)^{1/r}$$

so that the left hand side of (2.9) is equal to $\| \sum_k 2^{kar/p} |\mathcal{G}_k f_k|^r \|_{p/r}^{1/r}$. Let $\mathcal{N}$ be a positive integer (it will later be chosen as $C \log(3 + \mathcal{B}/A)$ with a large C). For $x \in X$ we split the k-sum into three pieces acting on $F = \{f_k\}$:

$$\sum_k 2^{kar/p} |\mathcal{G}_k f_k(x)|^r = |\mathfrak{S}^{\mathrm{low}}[F](x)|^r + |\mathfrak{S}^{\mathrm{mid}}[F](x)|^r + |\mathfrak{S}^{\mathrm{high}}[F](x)|^r,$$

where

$$\mathfrak{S}^{\mathrm{low}}[F](x) = \Big(\sum_{k+L(x)<0} 2^{kar/p} |\mathcal{G}_k f_k(x)|^r \Big)^{1/r}$$

$$\mathfrak{S}^{\mathrm{mid}}[F](x) = \Big(\sum_{0 \leq k+L(x) \leq \mathcal{N}} 2^{kar/p} |\mathcal{G}_k f_k(x)|^r \Big)^{1/r}$$

$$\mathfrak{S}^{\mathrm{high}}[F](x) = \Big(\sum_{k+L(x)>\mathcal{N}} 2^{kar/p} |\mathcal{G}_k f_k(x)|^r \Big)^{1/r}.$$

We need to bound the L^p norms of the three terms by the right hand side of (2.9). The terms $\mathfrak{S}^{\mathrm{low}}[F]$ and $\mathfrak{S}^{\mathrm{mid}}[F]$ will be estimated by using just hypothesis (2.1).

To bound $\mathfrak{S}^{\mathrm{low}}[F]$ we first consider the expression

$$P_k T_k f_k(y) - P_k T_k f_k(z) = \int_0^1 \!\! \int \langle 2^k (y-z), 2^{kd} \nabla \zeta(2^k(z - w + s(y-z))) \rangle T_k f_k(w) \, dw \, ds.$$

For $y,z\in Q(x)$ we have $2^k|y-z|\lesssim 2^{k+L(x)}$, and by Hölder's inequality and the rapid decay of ζ,

$$|\mathcal{G}_k f_k(x)| \le \Big(\fint_{Q(x)} \fint_{Q(x)} |P_k T_k f_k(y) - P_k T_k f_k(z)|^r \, dz\, dy\Big)^{1/r}$$
$$\lesssim 2^{k+L(x)} M_{\mathrm{HL}}[T_k f_k](x).$$

Here M_{HL} denotes the standard Hardy–Littlewood maximal operator. Now, by Hölder's inequality with respect to the k-summation,

$$\Big(\sum_{k+L(x)\le 0} |2^{ka/p}\mathcal{G}_k f_k(x)|^r\Big)^{1/r} \lesssim \Big(\sum_k |2^{ka/p} M_{\mathrm{HL}}[T_k f_k](x)|^p\Big)^{1/p}.$$

Thus

$$\|\mathfrak{S}^{\mathrm{low}}[F]\|_p \le \Big(\sum_k 2^{ka}\|M_{\mathrm{HL}}[T_k f_k]\|_p^p\Big)^{1/p} \tag{2.10}$$
$$\lesssim \Big(\sum_k 2^{ka}\|T_k f_k\|_p^p\Big)^{1/p} \lesssim A\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}.$$

Next we take care of $\mathfrak{S}^{\mathrm{mid}}[F](x)$, which may often be considered the main term but is also estimated using just (2.1). Now

$$|\mathcal{G}_k f_k(x)|^r \le 2 \fint_{Q(x)} |P_k T_k f_k(y)|^r\, dy$$

and therefore

$$\sum_{0\le k+L(x)\le \mathcal{N}} 2^{kar/p}|\mathcal{G}_k f_k(x)|^r$$
$$\lesssim \fint_{Q(x)} \mathcal{N}^{1-r/p}\Big(\sum_{0\le k+L(x)\le\mathcal{N}} |2^{ka/p}P_k T_k f_k(y)|^p\Big)^{r/p} dy.$$

By Hölder's inequality, this implies

$$|\mathfrak{S}^{\mathrm{mid}}[F](x)| \lesssim \mathcal{N}^{1/r-1/p} M_{\mathrm{HL}}\Big[\Big(\sum_k |2^{ka/p}P_k T_k f_k|^p\Big)^{1/p}\Big](x),$$

so that

$$\|\mathfrak{S}^{\mathrm{mid}}[F]\|_p \lesssim \mathcal{N}^{1/r-1/p}\Big\| M_{\mathrm{HL}}\Big[\Big(\sum_k |2^{ka/p}P_k T_k f_k|^p\Big)^{1/p}\Big]\Big\|_p \tag{2.11}$$
$$\lesssim \mathcal{N}^{1/r-1/p}\Big(\sum_k 2^{ka}\|P_k T_k f_k\|_p^p\Big)^{1/p}$$
$$\lesssim A\mathcal{N}^{1/r-1/p}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}.$$

We now turn to the expression $\mathfrak{S}^{\mathrm{high}}$ which we estimate for $L(x) \le 0$. Again by Hölder's inequality,

$$\mathfrak{S}^{\mathrm{high}}[F](x) \le \Big(2 \sum_{k>\mathcal{N}-L(x)} 2^{kar/p} \fint_{Q(x)} |P_k T_k f_k(y)|^r \, dy\Big)^{1/r}$$

$$\le \Big(2 \sum_{k>\mathcal{N}-L(x)} 2^{kar/p} \Big(\fint_{Q(x)} |P_k T_k f_k(y)| \, dy\Big)^r\Big)^{1/r}.$$

If $r < 1$ then we choose a small $\delta > 0$ and use Hölder's inequality with respect to the k-summation to get

$$\mathfrak{S}^{\mathrm{high}}[F](x) \le C(r,\delta) \sum_{k>\mathcal{N}-L(x)} 2^{ka/p} 2^{(k+L(x))\delta} \fint_{Q(x)} |P_k T_k f_k(y)| \, dy, \tag{2.12}$$

where

$$C(r,\delta) = 2^{1/r}\Big(\sum_{k>\mathcal{N}-L(x)} 2^{-(k+L(x))\delta r/(1-r)}\Big)^{1-r} \lesssim 2^{-\mathcal{N}\delta r}(r\delta)^{r-1},$$

so that $C(r,\delta) \lesssim (r\delta)^{r-1}$.

In order to estimate the expression (2.12) it suffices to bound the L^p norm of

$$\mathcal{T}^{\mathrm{lin}}[F](x) = \sum_{k>\mathcal{N}-L(x)} 2^{ka/p} 2^{(k+L(x))\delta} \fint_{Q(x)} \omega_k(x,y) P_k T_k f_k(y) \, dy,$$

where $\omega_k(x,y)$ are measurable functions satisfying $\sup_{x,y,k} |\omega_k(x,y)| \le 1$, with the constants in the estimates independent of the particular choice of the ω_k. We now fix one such choice.

Write $n = k + L(x)$, so that $n > \mathcal{N}$, and define, for $0 \le \mathrm{Re}(z) \le 1$,

$$S_n^z F(x) = 2^{(n-L(x))a(1-z)/q} \fint_{Q(x)} \omega_{n-L(x)}(x,y) P_{n-L(x)} T_{n-L(x)} f_{n-L(x)}(y) \, dy. \tag{2.13}$$

Observe that

$$\mathcal{T}^{\mathrm{lin}}[F](x) = \sum_{n>\mathcal{N}} 2^{n\delta} S_n^{\theta} F(x) \quad \text{for } \theta = 1 - \frac{q}{p}. \tag{2.14}$$

We estimate the L^p norm of $S_n^z F$ for $z = \theta$ by interpolating between an L^q bound for $\mathrm{Re}(z) = 0$ and an L^∞ bound for $\mathrm{Re}(z) = 1$.

For $z = i\tau$, $\tau \in \mathbb{R}$ we obtain

$$|S_n^{i\tau} F(x)| \le \fint_{Q(x)} \sup_k 2^{ka/q} |P_k T_k f_k(y)| \, dy$$

$$\le M_{\mathrm{HL}}\Big[\Big(\sum_k |2^{ka/q} P_k T_k f_k|^q\Big)^{1/q}\Big](x)$$

and therefore, by the L^q estimate for M_{HL}, Fubini, and assumption (2.2),

$$\|S_n^{i\tau}F\|_q \lesssim \Big(\sum_k 2^{ka}\|P_kT_kf_k\|_q^q\Big)^{1/q} \lesssim B_0\Big(\sum_k \|f_k\|_q^q\Big)^{1/q}.$$

The L^∞ estimate for $\mathrm{Re}(z)=1$ follows from Lemma 2.2; for $L(x)\le 0$, we get

$$\begin{aligned}|S_n^{1+i\tau}F(x)| &\le \fint_{Q(x)} |P_{n-L(x)}T_{n-L(x)}f_{n-L(x)}(y)|\,dy \\ &\lesssim (A\Gamma^{1/p}2^{-na/p}+B_12^{-n\varepsilon})\|f_{n-L(x)}\|_\infty\end{aligned}$$

and of course $\|f_{n-L(x)}\|_\infty \le \sup_k \|f_k\|_\infty$. Interpolating the two bounds yields

$$\|S_n^\theta F\|_{L^p(X)} \lesssim 2^{-\varepsilon_0 n(1-q/p)}\mathcal{B}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p} \tag{2.15}$$

with $\varepsilon_0 := \min\{a/p,\varepsilon\}$ and $\mathcal{B}$ as in (2.5). Choosing $\delta=(1-q/p)\varepsilon_0/2$, this yields

$$\begin{aligned}\|\mathcal{T}^{\mathrm{lin}}[F]\|_{L^p(X)} &\lesssim \sum_{n>\mathcal{N}} 2^{n\delta}\|S_n^\theta F\|_{L^p(X)} \\ &\lesssim \varepsilon_0^{-1}(1-q/p)^{-1}\mathcal{B}2^{-\mathcal{N}(1-q/p)\varepsilon_0/2}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}\end{aligned}$$

and then, by suitably choosing ω_k,

$$\|\mathfrak{S}^{\mathrm{high}}F\|_{L^p(X)} \lesssim \varepsilon_0^{-2}(1-q/p)^{-2}\mathcal{B}2^{-\mathcal{N}(1-q/p)\varepsilon_0/2}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}.$$

We combine the three bounds for $\mathfrak{S}^{\mathrm{high}}$, $\mathfrak{S}^{\mathrm{mid}}$ and $\mathfrak{S}^{\mathrm{low}}$ and get

$$\begin{aligned}&\Big\|\sum_k 2^{kar/p}|\mathcal{G}_kf_k|^r\Big\|_{L^{p/r}(X)}^{1/r} \\ &\le C_r(A\mathcal{N}^{1/r-1/p}+\varepsilon_0^{-2}(1-q/p)^{-2}\mathcal{B}2^{-\mathcal{N}(1-q/p)\varepsilon_0/2})\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}\end{aligned}$$

and choosing $\mathcal{N}=C_{\mathrm{large}}\log(3+\mathcal{B}/A)$ (with C_{large} depending on p, q and ε_0), we obtain the bound

$$\Big\|\sum_k 2^{kar/p}|\mathcal{G}_kf_k|^r\Big\|_{L^{p/r}(X)}^{1/r} \le CA\Big[\log\Big(3+\frac{\mathcal{B}}{A}\Big)\Big]^{1/r-1/p}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}. \tag{2.16}$$

It remains to give the estimation on $\mathbb{R}^d\setminus X$ (the set where $L(x)>0$), which is similar in spirit, but more straightforward. We first single out the terms for $k\le\mathcal{N}$ and by an estimate similar to the one for $\mathfrak{S}^{\mathrm{mid}}$ above we get

$$\Big\| \sum_{k\le\mathcal{N}} 2^{kar/p}|\mathcal{G}_k f_k|^r\Big\|_{L^{p/r}}^{1/r} \lesssim A\mathcal{N}^{1/r-1/p}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}. \tag{2.17}$$

On the other hand, by assumption (2.2),

$$2^{ka/q}\|\mathcal{G}_k f_k\|_q \lesssim B_0\|f_k\|_q$$

and by (2.8).

$$\|\mathcal{G}_k f_k\|_{L^\infty(\mathbb{R}^d\setminus X)} \lesssim (A\Gamma^{1/p}2^{-ka/p} + B_1 2^{-k\varepsilon})\|f_k\|_\infty.$$

Thus, with $\varepsilon_0 = \min\{a/p, \varepsilon\}$, by interpolation we get

$$2^{ka/p}\|\mathcal{G}_k f_k\|_{L^p(\mathbb{R}^d\setminus X)} \lesssim 2^{-k\varepsilon_0(1-q/p)}\mathcal{B}\|f_k\|_p.$$

By a straightforward application of Hölder's inequality,

$$\Big\| \sum_{k>\mathcal{N}} 2^{kar/p}|\mathcal{G}_k f_k|^r\Big\|_{L^{p/r}}^{1/r} \lesssim \varepsilon_0^{-1/r}(1-q/p)^{-1/r}2^{-\mathcal{N}\varepsilon_0(1-q/p)/2}\mathcal{B}\sup_k \|f_k\|_p, \tag{2.18}$$

which is slightly better than the $\ell^p(L^p)$ bound that we are aiming for. Combining (2.17) and (2.18), and choosing $\mathcal{N}$ as before, yields

$$\Big\| \sum_{k} 2^{kar/p}|\mathcal{G}_k f_k|^r\Big\|_{L^{p/r}(\mathbb{R}^d\setminus X)}^{1/r} \le CA\Big[\log\Big(3+\frac{\mathcal{B}}{A}\Big)\Big]^{1/r-1/p}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p},$$

which concludes the proof. ∎

3. Applications

Integrals over hypersurfaces. Consider the example of spherical means. For $k\in\mathbb{N}$, let P_k be a Littlewood–Paley cutoff operator $\widetilde{\Pi}_k$ (localizing to frequencies of size $\approx 2^k$ as in the introduction) such that $\widetilde{\Pi}_k\Pi_k = \Pi_k$. Take $T_k f = \sigma * \widetilde{\Pi}_k f$ and $f_k = \Pi_k f$. If Q is a cube satisfying $2^{-k} \le \operatorname{diam}(Q) \le 1$, with center x_Q, then the exceptional set $\mathcal{E}_Q$ is the tubular neighborhood of the unit sphere centered at x_Q, with width $C\operatorname{diam}(Q)$; if $\operatorname{diam}(Q) > 1$ we can simply choose the double cube. Then the hypotheses of Theorem 2.1 are easily verified with $a = d-1$, $q = 2$, any $p > 2$, and with A, B_0, B_1, Γ all comparable. Then (1.1) is implied by Theorem 2.1.

One can extend this observation to more general averaging operators over hypersurfaces which are not necessarily translation invariant. Let $\chi \in C_c^\infty(\mathbb{R}^d\setminus\{0\})$ and let $(x,y)\mapsto \Phi(x,y)$ be a smooth function defined in a neighborhood of $\operatorname{supp}\chi$ and assume that $\nabla_x\Phi(x,y)\ne 0$ and $\nabla_y\Phi(x,y)\ne 0$. Let δ be the Dirac measure on the real line and define the generalized Radon transform $\mathcal{R}$ as the integral operator with Schwartz kernel

$$K_{\mathcal{R}}(x,y) = \chi(x,y)\delta(\Phi(x,y)).$$

As shown in [21] (cf. also [7]), regularity properties of $\mathcal{R}$ are determined by the rotational curvature

$$\kappa(x,y) = \det\begin{pmatrix} \Phi_{xy} & \Phi_x \\ \Phi_y & 0 \end{pmatrix}.$$

Strengthening the results in [21] slightly, we obtain

Corollary 3.1. (i) *Let $d \geq 2$, $2 < p < \infty$, $r > 0$, and suppose that $\kappa(x,y) \neq 0$ on* $\operatorname{supp}\chi$. *Then $\mathcal{R} : F^p_{0,p}(\mathbb{R}^d) \to F^p_{(d-1)/p,r}(\mathbb{R}^d)$.*

(ii) *Let $d \geq 2$, $r > 0$, and suppose that $\kappa(x,y) \neq 0$ vanishes only of finite order on* $\operatorname{supp}\chi$, *i.e. there is n such that $\sum_{|\gamma|\leq n} |\partial_y^\gamma \kappa(x,y)| \neq 0$. Then there is a $p_0(n,d) < \infty$ so that $\mathcal{R} : F^p_{0,p}(\mathbb{R}^d) \to F^p_{(d-1)/p,r}(\mathbb{R}^d)$ for $p_0(n,d) < p < \infty$.*

The proof of (i) is essentially the same as for the spherical means. One decomposes $\mathcal{R} = \sum_{k=0}^\infty \mathcal{R}_k$ where for $k > 0$ the Schwartz kernel of $\mathcal{R}_k$ is given by

$$R_k(x,y) = \int \eta(2^{-k}|\tau|)\chi(x,y)\, e^{i\tau\Phi(x,y)}\, d\tau \tag{3.1}$$

with a suitable η supported in $(1/2, 2)$. One may then write

$$\mathcal{R} = \sum_{k=0}^\infty \Pi_k \mathcal{R}_k \Pi_k + \sum_{k=0}^\infty E_k,$$

where E_k is negligible, i.e. mapping L^p to any Sobolev space L^p_N with norm $\leq C_N 2^{-kN}$; this decomposition follows by an integration by parts argument in [7], and uses only the assumptions $\Phi_x \neq 0$ and $\Phi_y \neq 0$ (see also §2 in [19] for an exposition of this kind of argument). To estimate the main operator $\sum_{k=1}^\infty \Pi_k \mathcal{R}_k \Pi_k$ we use Theorem 2.1, setting $P_k = \Pi_k$, $f_k = \Pi_k f$, $T_k = \mathcal{R}_k$, and choose all parameters as in the example for the spherical means. For the exceptional sets $\mathcal{E}_Q$ we choose a tubular neighborhood of width $C\operatorname{diam}(Q)$ of the surface $\{y : \Phi(x_Q, y) = 0\}$.

For part (ii) one decomposes the operators $\mathcal{R}$ according to the size of κ, using a suitable cutoff function of the form $\beta_1(2^\ell|\kappa(x,y)|)$ where β_1 is supported in $(1/2, 2)$. Let R_k^ℓ be defined as in (3.1) but with $\chi(x,y)$ replaced by $\chi(x,y)\beta_1(2^\ell|\kappa(x,y)|)$. Then the proof of Proposition 2.2 in [21] shows that the operators $\mathcal{R}_k^\ell$ are bounded on L^2 with operator norm $\lesssim 2^{\ell M} 2^{-k(d-1)/2}$ (in fact with $M = 5d/2 + (d-1)/2$). By the finite type assumption on κ (and a standard sublevel set estimate related to van der Corput's lemma) the operator $\mathcal{R}_k^\ell$ is bounded on L^∞ with operator norm $\lesssim 2^{-\ell/n}$. Hence for $p > q > (2Mn+1)$ hypotheses (2.1) and (2.2) are satisfied with $A = 2^{-\ell\varepsilon(p)}$, $B_0 = 2^{-\ell\varepsilon(q)}$ for some $\varepsilon(p) > 0$, $\varepsilon(q) > 0$. We choose the exceptional set as in part (i), and (2.3), (2.4) hold as well with some B_1, Γ independent of ℓ.

Fourier integral operators. Another application concerns general Fourier integral operators associated to a canonical graph. Let $\chi \in C_0^\infty(\mathbb{R}^d)$, let a be a standard smooth symbol supported in $\{\xi : |\xi| \geq 1\}$. Let

$$Sf(x) = \chi(x) \int a(x,\xi) \widehat{f}(\xi)\, e^{i\phi(x,\xi)}\, d\xi$$

where ϕ is smooth in $\mathbb{R}^d \setminus \{0\}$ and $\xi \mapsto \phi(x,\xi)$ is homogeneous of degree 1. We assume that $\det \phi''_{x\xi} \neq 0$ on the support of the symbol. The following statement sharpens the L^p estimates of [11], [9] for the wave equation and of [20] for more general Fourier integral operators. One can use general facts about Fourier integral operators [7] to see that it implies part (i) of Corollary 3.1.

Corollary 3.2. *Let $d \geq 2$, $2 < p < \infty$, $r > 0$, and suppose that a is a standard symbol of order $-(d-1)(1/2-1/p)$. Then $S : F^p_{0,p}(\mathbb{R}^d) \to F^p_{0,r}(\mathbb{R}^d)$.*

The statement is equivalent to the $F^p_{0,p} \to F^p_{(d-1)/p,r}$ boundedness of a similar Fourier integral operator T of order $-(d-1)/2$. We use the dyadic decomposition in ξ to split $T = T_0 + \sum_{k=1}^\infty T_k$ where T_0 is smoothing to arbitrary order. Exceptional sets are also constructed as in [20]. Given a cube Q with center x_Q and diameter $d_Q \leq 1$ one chooses a maximal $\sqrt{d_Q}$-separated set of unit vectors ξ_ν, thus this set has cardinality $O(d_Q^{-(d-1)/2})$. For each ν let π_ν be the orthogonal projection to the hyperplane perpendicular to ξ_ν. Form for large C the rectangle $\rho_\nu(Q)$ consisting of y for which $|\langle y - \phi_\xi(x_Q,\xi_\nu), \xi_\nu\rangle| \leq C d_Q$ and $|\pi_\nu(y - \phi_\xi(x_Q,\xi_\nu))| \leq C d_Q^{1/2}$. The exceptional set $\mathcal{E}_Q$ for $|Q| < 1$ is then defined to be the union of the $\rho_\nu(Q)$ and has measure $O(|Q|^{-1/d})$. We refer to [20] for the arguments proving $\|T_k\|_{L^p \to L^p} \lesssim 2^{-k(d-1)/p}$, $2 < p < \infty$, and the integration by parts arguments leading to (2.4).

Strongly singular integrals. Define the convolution operator $S^{b,\gamma}$ by

$$\widehat{S^{b,\gamma} f}(\xi) = \frac{\exp(i|\xi|^\gamma)}{(1+|\xi|^2)^{b/2}} \widehat{f}(\xi).$$

We assume $0 < \gamma < 1$ and $1 < p < \infty$. The classical result [4] states that $S^{b,\gamma}$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $b \geq \gamma d |1/2 - 1/p|$. Theorem 2.1 can be used to upgrade the endpoint version to

Corollary 3.3. *Let $d \geq 1$, $2 < p < \infty$, $r > 0$, $b = b(\gamma) = \gamma d(1/2 - 1/p)$. Then $S^{b,\gamma} : F^p_{0,p} \to F^p_{0,r}$.*

To prove it we define $\widehat{T^\gamma f}(\xi) = (1+|\xi|^2)^{-\gamma d/2p} \widehat{S^{b(\gamma),\gamma} f}(\xi)$. For $\operatorname{diam}(Q) < 1$ we choose for the exceptional set $\mathcal{E}_Q$ the cube with the same center but diameter $C(\operatorname{diam}(Q))^{1-\gamma}$, for large C. Then the verification of the hypotheses with $a = \gamma d$ is done using the arguments in [4] or [10].

Remarks. (i) For the range $2 \le p < s$, it is known that the operator $S^{b(\gamma),\gamma}$ is not bounded on $F^p_{0,s}$ (see [2]).

(ii) There are also corresponding results for the range $\gamma > 1$ which improve on the results in [10], but they do not fit precisely our setup of Theorem 2.1 (cf. [17] for the corresponding smoothing space-time estimate).

Integrals over curves. We consider the generalized Radon transform associated to curves given by the equations $\Phi_i(x,y) = 0$, $i = 1,\dots,d-1$, where the $\nabla_x \Phi_i$ are linearly independent and the $\nabla_y \Phi_i$ are linearly independent, for (x,y) in a neighborhood $\mathcal{U} = X \times Y$ of the support of a C_c^∞ function χ. For simplicity (and without loss of generality) we assume that $\Phi_i(x,y) := S^i(x,y_d) - y_i$ for $i = 1,\dots,d-1$, and $\nabla_x S^i$ are linearly independent.

An important model case arises when $S^i(x,y_d) = x_i + (x_d - y_d)^{d+1-i}$ (i.e. for convolution with arclength measure on the curve $(t^d, t^{d-1},\dots,t)$, for a compact t-interval). The complete sharp L^p-Sobolev estimates for $2<p<\infty$ are unknown in dimension $d \ge 3$. However in three dimensions the sharp estimates are known for some range of large p (see [14]), and this result is strongly related to deep questions on Wolff's inequality for decompositions of cone multipliers [24]. A variable coefficient generalization of the result in [14] is in [16]. To discuss and apply the latter result we now let δ be the Dirac measure on $\mathbb{R}^{d-1}$ and define the generalized Radon transform $\mathcal{R}$ as the operator with Schwartz kernel

$$\mathcal{K}(x,y) = \chi(x,y)\delta(\vec{\Phi}(x,y)).$$

Again we shall also consider the dyadic pieces $\mathcal{R}_k$ with Schwartz kernel

$$R_k(x,y) = \int \beta(2^{-k}|\tau|)\chi(x,y) e^{i\tau\cdot\vec{\Phi}(x,y)}\, d\tau. \tag{3.2}$$

The analogue of the rotational curvature now depends on τ; we define it as a homogeneous of degree zero function and, for $|\tau| = 1$, set

$$\kappa(x,y,\tau) = \det\begin{pmatrix} \tau\cdot\vec{\Phi}_{xy} & \vec{\Phi}_x \\ \vec{\Phi}_y & 0 \end{pmatrix} = \sum_{i=1}^{d-1} \tau_i \det(S^i_{xy_d}\ S^1_x\ \cdots\ S^{d-1}_x).$$

Note that for $d \ge 3$ there are always directions where $\kappa(x,y,\tau)$ vanishes.

In [16] the case $d = 3$ is considered; we refer to this paper for further discussion. Let $\mathcal{M} = \{(x,y) \in \mathcal{U} : \vec{\Phi}(x,y) = 0\}$ and let $N^*\mathcal{M}$ be the conormal bundle. We assume that $(N^*\mathcal{M})'$ is a folding canonical relation and satisfies an additional curvature condition. To describe the latter one consider the fold surface

$$\mathcal{L} = \{(x, \tau\cdot\vec{\Phi}_x(x,y), y, -\tau\cdot\vec{\Phi}_y(x,y)) : \vec{\Phi}(x,y) = 0,\ \kappa(x,y,\tau) = 0\},$$

and assume that the projection $\mathcal{L} \to X$ has surjective differential. Thus for any fixed x the set $\Sigma_x = \{\xi \in \mathbb{R}^3 : (x,\xi,y,\eta) \in \mathcal{L} \text{ for some } (y,\eta)\}$ is a twodimensional conic hypersurface, and the additional curvature assumption is that Σ_x has one nonvanishing principal curvature everywhere (see [5], [16] for further discussion). For $d = 3$ this covers perturbation of the translation invariant model case.

Fix ℓ and, for $k > 3\ell$, define

$$R_k^\ell(x,y) = \int \eta(2^{-k}|\tau|)\chi(x,y)\widetilde{\beta}_1(2^\ell \kappa(x,y,\tau/|\tau|))\, e^{i\tau\cdot\vec{\Phi}(x,y)}\, d\tau,$$

where $\widetilde{\beta}_1$ is supported in $\{\xi : C^{-1} \le |\xi| \le C\}$ for large C, and, for $k = 3\ell$, define $R_k^\ell(x,y)$ in the same way but with β_1 replaced by β_0, a smooth cutoff function which is equal to 1 in a C-neighborhood of the origin. Let $\mathcal{R}_k^\ell$ be the operator with Schwartz kernel R_k^ℓ. We then have to estimate the L^p operator norm for

$$\mathcal{R}^\ell := \sum_{k\ge 3\ell} \mathcal{R}_k^\ell,$$

for each $l > 0$.

In [16] it is shown, based on the previously mentioned Wolff inequality, that under the above assumptions

$$\|\mathcal{R}_k^\ell\|_{L^p\to L^p} \lesssim C(\epsilon_\circ, p) 2^{-k/p} 2^{-\ell(1-\epsilon_\circ)/p}, \qquad p > p_W.$$

Here (p_W, ∞) is the range of Wolff's inequality (in [24], $p_W = 74$, but this has been improved since). Standard L^2 estimates (see [13], [12]) show that for $k \ge 3\ell$ the operators $\mathcal{R}_k^\ell$ are bounded on L^2 with norm $O(2^{(\ell-k)/2})$. By interpolation,

$$\|\mathcal{R}_k^\ell\|_{L^p\to L^p} \lesssim 2^{-k/p} 2^{-\ell\epsilon(p)} \qquad \text{with } \epsilon(p) > 0 \text{ for } p > (p_W+2)/2.$$

We claim that this yields the boundedness result

Corollary 3.4. *Let $(p_W + 2)/2 < p < \infty$, $r > 0$. Then $\mathcal{R} : F^p_{0,p}(\mathbb{R}^3) \to F^p_{1/p,r}(\mathbb{R}^3)$.*

To see this we use the assumption that $\nabla_x S^i$ are linearly independent and thus by integration by parts one can find a constant C_0 depending on $\vec{S}$ so that

$$\|\Pi_k \mathcal{R}_{k'}^\ell \Pi_{k''}\|_{L^p\to L^p} \le C_N \min\{2^{-kN}, 2^{-k'N}, 2^{-k''N}\}$$
$$\text{provided that} \max\{|k-k'|, |k'-k''|\} \ge C_0,\ k' \ge 3l.$$

Straightforward arguments (such as those used for the error terms in the proof of Corollary 3.1) reduce matters to the inequality

$$\Big\|\Big(\sum_{k>0} |2^{k/p}\Pi_{k+s_1}\mathcal{R}_k^\ell \Pi_{k+s_2} f|^r\Big)^{1/r}\Big\|_p \lesssim 2^{-\ell\epsilon'(p)} \Big\|\Big(\sum_{k>0} |\Pi_{k+s_2} f|^p\Big)^{1/p}\Big\|_p \tag{3.3}$$

with $\epsilon'(p) > 0$ for $p > (p_W + 2)/2$. Here $|s_1| \le C_0$ and $|s_2| \le C_0$. Indeed we apply, for fixed ℓ, Theorem 2.1 with $P_k = \Pi_{k+s_1}$, $f_k = \Pi_{k+s_2} f$, and $T_k = \mathcal{R}_k^\ell$ if $k \ge 3\ell$ (and $T_k = 0$ otherwise). For $p > q > (p_W + 2)/2$ assumption (2.1) holds with $A \lesssim 2^{-\ell\epsilon(p)}$ and assumption (2.2) holds with $B_1 \lesssim 2^{-\ell\epsilon(q)}$. We check assumption (2.4). By an integration by parts argument we derive the crude bound

$$|R_k^\ell(x,y)| \le C_N \frac{2^{2k}}{(1 + 2^{k-\ell}|y' - \vec{S}(x_Q, y_3)|)^N}.$$

Now for a given cube Q with center x_Q we let

$$\mathcal{E}_Q := \{y : |y' - \vec{S}(x_Q, y_3)| \le C2^\ell \operatorname{diam}(Q)\}$$

if $\operatorname{diam}(Q) \le 1$. If $\operatorname{diam}(Q) \ge 1$ then we let $\mathcal{E}_Q$ be a ball of diameter $C2^\ell \operatorname{diam}(Q)$ centered at x_Q. Clearly assumptions (2.3) and (2.4) are satisfied with $\Gamma \lesssim 2^{3\ell}$ and $B_1 \lesssim 2^{2\ell}$. By Theorem 2.1,

$$\Big\| \Big(\sum_{k \ge 3\ell} |2^{k/p} P_k \mathcal{R}_k^\ell f_k|^r \Big)^{1/r} \Big\|_p \lesssim (1+\ell) 2^{-\epsilon'(p)\ell} \Big(\sum_k \|f_k\|_p^p \Big)^{1/p}, \quad p > \frac{p_W + 2}{2},$$

which concludes the proof of (3.3) and yields

$$\|\mathcal{R}^\ell f\|_{F^p_{1/p,r}} \lesssim (1+\ell) 2^{-\varepsilon(p)\ell} \|f\|_{F^p_{0,p}}.$$

Corollary 3.4 follows by summation in $\ell \ge 0$.

Remark. A similar strengthening, with a similar argument, applies to the restricted X-ray transform model in [15].

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