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Periodic solutions of degenerate differential equations in vector-valued function spaces

by

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Abstract. Let A and M be closed linear operators defined on a complex Banach space X. Using operator-valued Fourier multiplier theorems, we obtain necessary and sufficient conditions for the existence and uniqueness of periodic solutions to the equation $\frac{d}{dt}(Mu(t)) = Au(t) + f(t)$, in terms of either boundedness or R-boundedness of the modified resolvent operator determined by the equation. Our results are obtained in the scales of periodic Besov and periodic Lebesgue vector-valued spaces.

1. Introduction. We are concerned with the regularity of solutions to the equation

$$\frac{d}{dt}(Mu(t)) = Au(t) + f(t), \quad 0 \le t \le 2\pi, \tag{1.1}$$

where $(A, D(A))$ and $(M, D(M))$ are (unbounded) closed linear operators on a Banach space X, with $D(A) \subseteq D(M)$. The model (1.1), in case $A = \Delta$ is the Laplacian and $M = m$ is multiplication by a function $m(x)$, was first considered by Carroll and Showalter [CS] and has recently been studied by Marinoschi [M1]. This model describes, for example, the infiltration of water in unsaturated porous media, in which saturation might occur. The function m characterizes the porosity of the nonhomogeneous medium, while the fact that m is zero indicates the existence of impermeable intrusions in the soil. A study of solutions for this model, with $m(x) = 1$ and periodic initial conditions, was made in [M2] in the case of a nonlinear convection, in connection with some results given in [H]. An interesting analysis of periodic solutions to a nonlinear model involving a degenerate diffusion equation of the form (1.1) with homogeneous Dirichlet boundary conditions, where A is a multivalued linear operator, has recently been given in [FM].

A detailed study of linear abstract degenerate differential equations, using both the semigroups generated by multivalued (linear) operators and

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extensions of the operational method of Da Prato and Grisvard, has been described in the monograph [FY3].

Regularity of solutions in various vector-valued function spaces for the abstract equation (1.1) with periodic initial condition

$$Mu(0) = Mu(2\pi) \tag{1.2}$$

has been studied in [BF] using the sum method. The results obtained give sufficient conditions for periodicity, but leave it as an open problem to *characterize* the maximal regularity in terms of a hypothesis on the modified resolvent operator $(\lambda M - A)^{-1}$ of the operators M and A.

On the other hand, Arendt and Bu [AB1], using operator-valued Fourier multiplier theorems, have derived spectral characterizations of maximal regularity in Lebesgue spaces for the equation (1.1) with $M = I$, the identity operator, and periodic initial conditions. Similar characterizations were then obtained for the scale of Besov spaces [AB2], and subsequently the scale of Triebel–Lizorkin spaces [BK]. See also [KLP] and the references therein. This motivates the question whether it is possible to obtain a similar characterization for the problem (1.1)–(1.2). We note that, starting with [AB1], the problem of characterizing maximal regularity for evolution equations with periodic initial conditions has been intensively studied in the last years. See e.g. [BF1], [BF2], [BF3], [KLP], [L], [P] and the references therein. The main novelty of the present paper lies in the presence of two non-commuting operators A and M, which are only related by the domain. There are only a few papers dealing with this situation (see [LP]). Furthermore, our approach lends itself to immediate application to degenerate evolution equations, arising from applications. Some examples are given in the last section of this paper.

This work is organized as follows: After some preliminaries in the first section, and under a geometrical assumption on the Banach space X, we are able to characterize in Section 2 the uniqueness and existence of a strong L^p-solution for the problem (1.1)–(1.2) solely in terms of a boundedness property for the sequence of operators $ikM(ikM - A)^{-1}$. We remark that no additional assumption on the operator A is required. In Section 3, we prove a characterization in the context of Besov spaces. We notice that in this case the additional hypothesis on X is no longer required. In the particular case of Hölder spaces $C^s((0, 2\pi), X)$, $0 < s < 1$, we show that the following assertions are equivalent in general Banach spaces, provided $D(A) \subset D(M)$:

(1) $ikM - A$ is bijective for all $k \in \mathbb{Z}$ and $\sup_{k\in\mathbb{Z}} \|ikM(ikM - A)^{-1}\| < \infty$.
(2) For every $f \in C^s((0, 2\pi), X)$ there exists a unique function $u \in C^s((0, 2\pi), D(A))$ such that $Mu \in C^{s+1}((0, 2\pi), X)$ and (1.1)–(1.2) holds for a.e. $t \in [0, 2\pi]$.

We remark that this result extends and improves [BF, Theorem 2.1]. Finally, some concrete examples are examined.

2. Preliminaries. Let X, Y be Banach spaces. We denote by $\mathcal{B}(X,Y)$ the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$. For a linear operator A on X, we denote its domain by $D(A)$ and its resolvent set by $\rho(A)$. By $[D(A)]$ we denote the domain of A equipped with the graph norm.

A Banach space X is said to be *UMD* if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1,\infty)$. Here the Hilbert transform H of a function f in $\mathcal{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi}\,\mathrm{PV}\left(\frac{1}{t}\right) * f.$$

These spaces are also called $\mathcal{HT}$ spaces. It is well known that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain [Bo] and Burkholder [Bu].

Definition 2.1. Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{B}(X,Y)$ is called *R-bounded* if there is a constant $C > 0$ and $p \in [1,\infty)$ such that for all $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and all independent, symmetric, $\{-1,1\}$-valued random variables r_j on a probability space $(\Omega, \mathcal{M}, \mu)$,

$$\Big\|\sum_{j=1}^{N} r_j T_j x_j\Big\|_{L^p(\Omega,Y)} \le C \Big\|\sum_{j=1}^{N} r_j x_j\Big\|_{L^p(\Omega,X)}. \tag{2.1}$$

The smallest such C is called the *R-bound* of $\mathcal{T}$, denoted by $R_p(\mathcal{T})$.

We remark that large classes of classical operators are R-bounded (cf. [GW1] and the references therein). Hence, this assumption is not too restrictive for the applications that we consider in this article.

Remark 2.2. Several properties of R-bounded families can be found in the recent monograph of Denk–Hieber–Prüss [DHP]. For the reader's convenience, we here summarize some results from [DHP, Section 3].

(a) If $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded then it is uniformly bounded, with
$$\sup\{\|T\| : T \in \mathcal{T}\} \le R_p(\mathcal{T}).$$
(b) The definition of R-boundedness is independent of $p \in [1,\infty)$.
(c) When X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.
(d) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X,Y)$ be R-bounded. Then
$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T},\ S \in \mathcal{S}\}$$
is R-bounded as well, and $R_p(\mathcal{T} + \mathcal{S}) \le R_p(\mathcal{T}) + R_p(\mathcal{S})$.

(e) Let X, Y, Z be Banach spaces, and $\mathcal{T} \subset \mathcal{B}(X, Y)$ and $\mathcal{S} \subset \mathcal{B}(Y, Z)$ be R-bounded. Then

$$\mathcal{S}\mathcal{T} = \{ST : S \in \mathcal{S},\, T \in \mathcal{T}\}$$

is R-bounded, and $R_p(\mathcal{S}\mathcal{T}) \le R_p(\mathcal{S})R_p(\mathcal{T})$.

(g) Let X, Y be Banach spaces and $\mathcal{T} \subset \mathcal{B}(X, Y)$ be R-bounded. If $\{\alpha_k\}_{k\in\mathbb{Z}}$ is a bounded sequence, then $\{\alpha_k T : T \in \mathcal{T}\}$ is R-bounded.

Given $1 \le p < \infty$, we denote by $L^p_{2\pi}(\mathbb{R}, X)$ the space of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0, 2\pi]$ is p-integrable.

For a function $f \in L^1_{2\pi}(\mathbb{R}, X)$ we denote by $\hat{f}(k)$, $k \in \mathbb{Z}$, the kth Fourier coefficient of f:

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)\, dt.$$

We need the following lemma.

Lemma 2.3 ([AB1]). *Let $f, g \in L^p_{2\pi}(\mathbb{R}, X)$, where $1 \le p < \infty$, and let A be a closed linear operator on a Banach space X. Then the following assertions are equivalent:*

(i) *$f(t) \in D(A)$ and $Af(t) = g(t)$ a.e.*

(ii) *$\hat{f}(k) \in D(A)$ and $A\hat{f}(k) = \hat{g}(k)$, for all $k \in \mathbb{Z}$.*

The proof of the following lemma is analogous to that of [AB1, Lemma 2.1] and therefore omitted.

Lemma 2.4. *Let $1 \le p < \infty$, let M be a closed linear operator on X, $u \in L^p_{2\pi}(\mathbb{R}, [D(M)])$ and $u' \in L^p_{2\pi}(\mathbb{R}, X)$. Then the following assertions are equivalent:*

(i) *$\int_0^{2\pi} (Mu)'(t)\, dt = 0$ and there exists $x \in X$ such that $Mu(t) = x + \int_0^t (Mu)'(s)\, ds$ a.e. on $[0, 2\pi]$;*

(ii) *$\widehat{(Mu)}'(k) = ikM\hat{u}(k)$ for all $k \in \mathbb{Z}$.*

We also recall the following definition from [AB1].

Definition 2.5. For $1 \le p < \infty$, we say that a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an *L^p-multiplier* if, for each $f \in L^p_{2\pi}(\mathbb{R}, X)$, there exists $u \in L^p_{2\pi}(\mathbb{R}, Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

We finally recall the following results.

Proposition 2.6 ([AB1]). *Let X, Y be Banach spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ be an L^p-multiplier for some $1 \le p < \infty$. Then the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

THEOREM 2.7 ([AB1]). *Let X, Y be UMD spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for all $1 < p < \infty$.*

3. A characterization on vector-valued Lebesgue spaces. In this section we consider the problem

$$\begin{cases} \dfrac{d}{dt}(Mu(t)) = Au(t) + f(t), & 0 \le t \le 2\pi, \\ Mu(0) = Mu(2\pi), \end{cases} \tag{3.1}$$

where $A : D(A) \subseteq X \to X$ and $M : D(M) \subseteq X \to X$ are closed linear operators, $D(A) \subseteq D(M)$ and $f \in L^p_{2\pi}(\mathbb{R}, X)$, $p \ge 1$. For a given closed operator M, and $1 \le p < \infty$, we define the set

$$H^{1,p}_{\mathrm{per},M}(\mathbb{R},[D(M)]) \\ = \{u \in L^p_{2\pi}(\mathbb{R},[D(M)]) : \exists v \in L^p_{2\pi}(\mathbb{R},X),\ \hat{v}(k) = ikM\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

If $M = I$, we write $H^{1,p}_{\mathrm{per}}(\mathbb{R}, X)$ (see [AB1]). Next, we introduce the following definition.

DEFINITION 3.1. We say that a function $u \in H^{1,p}_{\mathrm{per},M}(\mathbb{R},[D(M)])$ is a *strong L^p-solution* of (3.1) if $u(t) \in D(A)$ and (3.1) holds for a.e. $t \in [0, 2\pi]$.

Define the *M-resolvent set* of A by

$$\rho_M(A) \\ = \{\lambda \in \mathbb{C} : \lambda M - A : D(A) \to X \text{ is bijective and } (\lambda M - A)^{-1} \in \mathcal{B}(X)\}.$$

We begin with the following result.

PROPOSITION 3.2. *Let $A : D(A) \subseteq X \to X$ and $M : D(M) \subseteq X \to X$ be closed linear operators defined on a UMD space X. Suppose that $D(A) \subseteq D(M)$. Then, for all $1 < p < \infty$, the following assertions are equivalent:*

(i) *$\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{ikM(ikM - A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.*
(ii) *$\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{ikM(ikM - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. Define $M_k = ikM(ikM - A)^{-1}$. Since A is closed, by the identity $M_k = A(ikM - A)^{-1} + I$ and the Closed Graph Theorem we conclude that M_k is a bounded operator for each $k \in \mathbb{Z}$. By Proposition 2.6 it follows that (i) implies (ii). Conversely, by Theorem 2.7 it is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. In fact,

$$(3.2)\quad \begin{aligned} k(M_{k+1} - M_k) \\ &= k[i(k+1)M(i(k+1)M - A)^{-1} - ikM(ikM - A)^{-1}] \\ &= kM[i(k+1)(i(k+1)M - A)^{-1} - ik(ikM - A)^{-1}] \\ &= kM(i(k+1)M - A)^{-1} \\ &\quad \cdot [i(k+1)(ikM - A) - ik(i(k+1)M - A)] \\ &\quad \cdot (ikM - A)^{-1} \\ &= kM(i(k+1)M - A)^{-1}(-iA)(ikM - A)^{-1} \\ &= -ikM(i(k+1)M - A)^{-1}(ikM(ikM - A)^{-1} - I), \end{aligned}$$

where in the last equality we use $A(ikM - A)^{-1} = ikM(ikM - A)^{-1} - I$. Therefore, since the products and sums of R-bounded sequences are R-bounded, by (d) and (g) of Remark 2.2, the proof is finished. ∎

The following is one of the main results in this paper. It is an extension of [AB1, Theorem 2.3], which corresponded to $M = I$.

THEOREM 3.3. *Let X be a UMD space, and $A : D(A) \subseteq X \to X$ and $M : D(M) \subseteq X \to X$ be closed linear operators. Suppose that $D(A) \subseteq D(M)$ and $ikM - A$ is a closed operator for all $k \in \mathbb{Z}$. Then, for all $1 < p < \infty$, the following assertions are equivalent:*

(i) *For every $f \in L^p_{2\pi}(\mathbb{R}, X)$, there exists a unique strong L^p-solution of* (3.1).
(ii) *$\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{ikM(ikM - A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.*
(iii) *$\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{ikM(ikM - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. (i)⇒(ii). We follow the same lines of [AB1, Theorem 2.3]. Let $k \in \mathbb{Z}$ and $y \in X$ be given. Define $f(t) = e^{ikt}y$. By hypothesis, there exists $u \in H^{1,p}_{\mathrm{per},M}(\mathbb{R}, [D(M)])$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + f(t)$. Taking the Fourier transform on both sides, we have $\hat{u}(k) \in D(A)$ and

$$ikM\hat{u}(k) = A\hat{u}(k) + \hat{f}(k) = A\hat{u}(k) + y.$$

Thus, $(ikM - A)\hat{u}(k) = y$ for all $k \in \mathbb{Z}$ and therefore $ikM - A$ is surjective. Let $x \in D(A)$. If $(ikM - A)x = 0$, then $u(t) = e^{ikt}x$ defines a periodic solution of (3.1). In fact, $(Mu)'(t) - Au(t) = ike^{ikt}Mx - e^{ikt}Ax = e^{ikt}(ikM - A)x = 0$. Hence $u \equiv 0$ by uniqueness, and thus $x = 0$. Therefore, $ikM - A$ is bijective. We conclude, from the Closed Graph Theorem, that $ik \in \rho_M(A)$ for all $k \in \mathbb{Z}$. We will see that $\{ikM(ikM - A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.

Using again the Closed Graph Theorem, we find that there exists a constant $C > 0$ independent of $f \in L^p_{2\pi}(\mathbb{R}, X)$ such that

$$\|(Mu)'\|_{L^p} + \|Au\|_{L^p} \le C\|f\|_{L^p}.$$

Note that for $f(t) = e^{itk}y$, $y \in X$, the solution u of (3.1) is given by $u(t) = (ikM - A)^{-1}e^{ikt}y$. Hence,

$$\|ikM(ikM - A)^{-1}y\| \le C\|y\|.$$

So, we see that $ikM(ikM - A)^{-1}$ is a bounded operator for all $k \in \mathbb{Z}$. Let $f \in L^p_{2\pi}(\mathbb{R}, X)$. By hypothesis, there exists $u \in H^{1,p}_{\mathrm{per},M}(\mathbb{R}, [D(M)])$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + f(t)$. Taking the Fourier transform on both sides, and using the bijectivity of $ikM - A$, we have $\hat{u}(k) \in D(A)$ and $\hat{u}(k) = (ikM - A)^{-1}\hat{f}(k)$. Now, since $u \in H^{1,p}_{\mathrm{per},M}(\mathbb{R}, [D(M)])$ and by definition of $H^{1,p}_{\mathrm{per},M}(\mathbb{R}, [D(M)])$, there exists $v \in L^p_{2\pi}(\mathbb{R}, X)$ such that $\hat{v}(k) = ikM\hat{u}(k)$ for all $k \in \mathbb{Z}$. Therefore, $\hat{v}(k) = ikM(ikM - A)^{-1}\hat{f}(k)$.

(ii)⇒(i). Define $M_k = ikM(ikM-A)^{-1}$. Suppose that $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. Let $f \in L^p_{2\pi}(\mathbb{R}, X)$. Then there exists $u \in L^p_{2\pi}(\mathbb{R}, X)$ such that $\hat{u}(k) = ikM(ikM - A)^{-1}\hat{f}(k)$ for all $k \in \mathbb{Z}$. Now by the identity $I = ikM(ikM - A)^{-1} - A(ikM - A)^{-1}$ it follows that

$$\hat{u}(k) = ikM(ikM - A)^{-1}\hat{f}(k) = (I + A(ikM - A)^{-1})\hat{f}(k).$$

So, we obtain $\widehat{(u - f)}(k) = A(ikM - A)^{-1}\hat{f}(k)$. Putting $v := u - f$, we have $v \in L^p_{2\pi}(\mathbb{R}, X)$, and $\hat{v}(k) = A(ikM - A)^{-1}\hat{f}(k)$. Observe that A^{-1} is an isomorphism of X onto $D(A)$ (seen as a Banach space with the graph norm). Therefore, $A^{-1}\hat{v}(k) = (ikM - A)^{-1}\hat{f}(k)$. Let $w := A^{-1}v$. Since A^{-1} is a bounded operator, we deduce that $w \in L^p_{2\pi}(\mathbb{R}, X)$, $\hat{w}(k) \in D(A)$ and $\hat{w}(k) = (ikM - A)^{-1}\hat{f}(k)$. So,

$$\begin{aligned} ikM\hat{w}(k) - A\hat{w}(k) &= ikM(ikM - A)^{-1}\hat{f}(k) - A(ikM - A)^{-1}\hat{f}(k) \\ &= (ikM - A)(ikM - A)^{-1}\hat{f}(k) = \hat{f}(k). \end{aligned}$$

Now, observe that

$$\hat{u}(k) = ikM(ikM - A)^{-1}\hat{f}(k) = ikM\hat{w}(k)$$

for all $k \in \mathbb{Z}$. Therefore, $w \in H^{1,p}_{\mathrm{per},M}(\mathbb{R}, [D(M)])$. Moreover $Mw(0) = Mw(2\pi)$, since $w(0) = w(2\pi)$ and $w(t) \in D(A)$. Since A and M are closed operators and $\widehat{(Mw)'}(k) = ikM\hat{w}(k) = A\hat{w}(k) + \hat{f}(k)$ for all $k \in \mathbb{Z}$, one has $(Mw)'(t) = Aw(t) + f(t)$ a.e. by Lemmas 2.3 and 2.4. So w is a strong L^p-solution of (3.1).

Now, to see the uniqueness, let $u \in H^{1,p}_{\mathrm{per},M}(\mathbb{R}, [D(M)])$ be such that $(Mu)'(t) = Au(t)$. Then $\hat{u}(k) \in D(A)$, and $(ikM - A)\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$. Since $ikM - A$ is bijective for all $k \in \mathbb{Z}$, we obtain $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$, and thus $u \equiv 0$.

(ii)⇔(iii). Proposition 3.2. ■

COROLLARY 3.4. *Let H be a Hilbert space, let $A : D(A) \subset H \to H$ and $M : D(M) \subset H \to H$ be closed linear operators satisfying $D(A) \subseteq D(M)$ and suppose that $ikM - A$ is a closed operator for all $k \in \mathbb{Z}$. Then, for all $1 < p < \infty$, the following assertions are equivalent:*

(i) *For every $f \in L^p_{2\pi}(\mathbb{R}, H)$, there exists a unique strong L^p-solution of* (3.1).
(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ *and* $\sup_k \|ikM(ikM - A)^{-1}\| < \infty$.

Proof. Follows from Theorem 3.3, and the fact that in Hilbert spaces the concepts of R-boundedness and boundedness are equivalent [DHP]. ∎

The solution $u(\cdot)$ given in Theorem 3.3 actually has the following maximal regularity property.

COROLLARY 3.5. *In the context of Theorem* 3.3, *if condition* (iii) *is fulfilled, we have $(Mu)', Au \in L^p_{2\pi}(\mathbb{R}, X)$. Moreover, there exists a constant $C > 0$ independent of $f \in L^p_{2\pi}(\mathbb{R}, X)$ such that*

$$\|(Mu)'\|_{L^p} + \|Au\|_{L^p} \le C\|f\|_{L^p}. \tag{3.3}$$

REMARK 3.6. From the inequality (3.3) we deduce that the operator L defined by

$$(Lu)(t) = (Mu)'(t) - Au(t)$$

with domain

$$D(L) = H^{1,p}_{\mathrm{per},M}(\mathbb{R}, [D(M)]) \cap L^p_{2\pi}(\mathbb{R}, [D(A)])$$

is an isomorphism. Indeed, since A and M are closed, the space

$$H^{1,p}_{\mathrm{per},M}(\mathbb{R}, [D(M)]) \cap L^p_{2\pi}(\mathbb{R}, [D(A)])$$

becomes a Banach space under the norm

$$|||u||| := \|u\|_p + \|(Mu)'\|_p + \|Au\|_p.$$

We remark that such isomorphisms are crucial for nonlinear evolution equations (see [A2]).

4. Maximal regularity on the scale of vector-valued Besov spaces. In this section we consider solutions in $B^s_{p,q}((0, 2\pi), X)$, the vector-valued periodic Besov space for $1 \le p \le \infty$, $s > 0$. For the definition and main properties of these spaces we refer to [AB2] or [KL2]. For the scalar case, see [BB], [ST]. In contrast to the L^p case, the multiplier theorems established for vector-valued Besov space are valid for arbitrary Banach spaces X; see [A1], [AB2] and [GW2]. Special cases here allow one to treat Hölder–Zygmund spaces. Specifically, we have $B^s_{\infty,\infty} = \mathcal{C}^s$ for $s > 0$. Moreover, if $0 < s < 1$ then $B^s_{\infty,\infty}$ is just the usual Hölder space C^s.

We summarize some useful properties of $B^s_{p,q}((0, 2\pi), X)$ (see [AB2, Section 2] for a proof):

(i) $B^s_{p,q}((0,2\pi),X)$ is a Banach space.
(ii) If $s>0$, then the natural injection $B^s_{p,q}((0,2\pi),X)\hookrightarrow L^p((0,2\pi),X)$ is a continuous linear operator.
(iii) Let $s>0$. Then $f\in B^{s+1}_{p,q}((0,2\pi),X)$ if and only if f is differentiable a.e. and $f'\in B^s_{p,q}((0,2\pi),X)$.

We begin with the definition of operator-valued Fourier multipliers in the context of periodic Besov spaces.

DEFINITION 4.1. Let $1\le p\le\infty$. A sequence $\{M_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(X,Y)$ is a $B^s_{p,q}$*-multiplier* if for each $f\in B^s_{p,q}((0,2\pi),X)$ there exists a function $g\in B^s_{p,q}((0,2\pi),Y)$ such that

$$M_k\hat{f}(k)=\hat{g}(k),\quad k\in\mathbb{Z}.$$

The following concept was studied in [KLP].

DEFINITION 4.2. We say that $\{M_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(X,Y)$ is *M-bounded* if

$$\sup_k\|M_k\|<\infty,\qquad \sup_k\|k(M_{k+1}-M_k)\|<\infty, \tag{4.1}$$

$$\sup_k\|k^2(M_{k+1}-2M_k+M_{k-1})\|<\infty. \tag{4.2}$$

We recall the following operator-valued Fourier multiplier theorem on Besov spaces.

THEOREM 4.3 ([AB2]). *Let X,Y be Banach spaces and let $\{M_k\}_{k\in\mathbb{Z}}\subseteq\mathcal{B}(X,Y)$ be an M-bounded sequence. Then for all $1\le p,q\le\infty$ and $s\in\mathbb{R}$, $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier.*

We next prove the following result, which is an analogue to Proposition 3.2.

PROPOSITION 4.4. *Let $A:D(A)\subseteq X\to X$ and $M:D(M)\subseteq X\to X$ be closed linear operators. Suppose that $D(A)\subseteq D(M)$. Then, for all $1\le p,q\le\infty$ and $s\in\mathbb{R}$ the following assertions are equivalent:*

(i) $\{ik\}_{k\in\mathbb{Z}}\subset\rho_M(A)$ *and* $\{ikM(ikM-A)^{-1}\}_{k\in\mathbb{Z}}$ *is a* $B^s_{p,q}$*-multiplier.*
(ii) $\{ik\}_{k\in\mathbb{Z}}\subset\rho_M(A)$ *and* $\sup_{k\in\mathbb{Z}}\|ikM(ikM-A)^{-1}\|<\infty$.

Proof. (i)⇒(ii). Follow the proof in [KL1, Proposition 3.4].

(ii)⇒(i). For $k\in\mathbb{Z}$, define $M_k=ikM(ikM-A)^{-1}$. From the identity (3.2) we obtain

$$\sup_k\|k(M_{k+1}-M_k)\|<\infty, \tag{4.3}$$

proving (4.1). To verify (4.2), we notice

$$
\begin{aligned}
&k^2(M_{k+1} - 2M_k + M_{k-1}) \\
&\quad = k^2\big[i(k+1)M[i(k+1)M - A]^{-1} - 2ikM(ikM - A)^{-1} \\
&\qquad + i(k-1)M(i(k-1)M - A)^{-1}\big] \\
&\quad = k^2 M[i(k+1)M - A]^{-1}[i(k+1)(ikM - A) - 2ik[i(k+1)M - A] \\
&\qquad + i(k-1)[i(k+1)M - A][i(k-1)M - A]^{-1}(ikM - A)](ikM - A)^{-1} \\
&\quad = k^2 M[i(k+1)M - A]^{-1}\big[i(k+1)(ikM - A) - 2ik(ikM - A) - 2ikiM \\
&\qquad + i(k-1)[i(k-1)M - A][i(k-1)M - A]^{-1}(ikM - A) \\
&\qquad + 2i \cdot i(k-1)M[i(k-1)M - A]^{-1}(ikM - A)\big](ikM - A)^{-1} \\
&\quad = k^2 M[i(k+1)M - A]^{-1} \\
&\qquad \cdot [(i(k+1) - 2ik + i(k-1) + 2iM_{k-1})[ikM - A] - 2ikiM] \\
&\qquad \cdot (ikM - A)^{-1} \\
&\quad = k^2 M[i(k+1)M - A]^{-1}\big[2iM_{k-1}(ikM - A) - 2ikiM\big](ikM - A)^{-1} \\
&\quad = kM[i(k+1)M - A]^{-1}\big[2ikM_{k-1}(ikM - A) - 2ikikM\big](ikM - A)^{-1} \\
&\quad = kM[i(k+1)M - A]^{-1}\big[2ikM_{k-1} \cdot I - 2ikikM(ikM - A)^{-1}\big] \\
&\quad = kM[i(k+1)M - A]^{-1}(2ikM_{k-1} - 2ikM_k) \\
&\quad = kM[i(k+1)M - A]^{-1}[-2ik(M_k - M_{k-1})] \\
&\quad = kM[i(k+1)M - A]^{-1}[-2i(k-1)(M_k - M_{k-1}) - 2i(M_k - M_{k-1})].
\end{aligned}
$$

Since we know that $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is bounded, and $\{M_k\}_{k\in\mathbb{Z}}$ is bounded by hypothesis, we conclude from the above identity that

$$\sup_k \|k^2(M_{k+1} - 2M_k + M_{k-1})\| < \infty. \tag{4.4}$$

So, $\{M_k\}_{k\in\mathbb{Z}}$ is M-bounded and therefore, by Theorem 4.3, it is a $B^s_{p,q}$-multiplier. ∎

Definition 4.5. Let $1 \le p, q \le \infty$ and $s > 0$ be given. A function $u \in B^s_{p,q}((0, 2\pi), [D(A)])$ is said to be a *strong* $B^s_{p,q}$*-solution* of (3.1) if $Mu \in B^{s+1}_{p,q}((0, 2\pi), X)$ and equation (3.1) holds for a.e. $t \in (0, 2\pi)$.

The next theorem is the main result of this section. It extends [AB2, Theorem 5.1] which corresponds to $M = I$.

Theorem 4.6. *Let $1 \le p, q \le \infty$ and $s > 0$. Let X be a Banach space and let $A : D(A) \subseteq X \to X$ and $M : D(M) \subseteq X \to X$ be closed linear operators. Suppose that $D(A) \subseteq D(M)$ and $ikM - A$ is a closed operator for all $k \in \mathbb{Z}$. Then the following assertions are equivalent:*

(i) *For every* $f \in B^s_{p,q}((0,2\pi),X)$ *there exists a unique strong* $B^s_{p,q}$-*solution of* (3.1).
(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ *and* $\{ikM(ikM-A)^{-1}\}_{k\in\mathbb{Z}}$ *is a* $B^s_{p,q}$-*multiplier.*
(iii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ *and* $\sup_{k\in\mathbb{Z}} \|ikM(ikM-A)^{-1}\| < \infty$.

Proof. (ii)⇔(iii). Follows from Proposition 4.4.

(i)⇒(iii). Suppose that for every $f \in B^s_{p,q}((0,2\pi),X)$ there exists a unique strong $B^s_{p,q}$-solution of (3.1). Fix $x \in X$ and $k \in \mathbb{Z}$. Define $f(t) = e^{itk}x$. Then $f \in B^s_{p,q}((0,2\pi),X)$. By hypothesis there exist $u \in B^s_{p,q}((0,2\pi),[D(A)])$ with $Mu \in B^{s+1}_{p,q}((0,2\pi),X)$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t)+f(t)$ for a.e. $t \in (0,2\pi)$. By Lemma 2.4 we have $ikM\hat{u}(k) = A\hat{u}(k)+x$. Following the proof of Theorem 3.3 we find that $ikM-A$ is bijective for all $k \in \mathbb{Z}$. Let $M_k := ikM(ikM-A)^{-1}$. We will see that $\{M_k\}_{k\in\mathbb{Z}}$ is bounded. By the Closed Graph Theorem, there exists a constant C independent of f such that

$$\|Mu\|_{B^{s+1}_{p,q}((0,2\pi),X)} + \|Au\|_{B^s_{p,q}((0,2\pi),[D(A)])} \le C\|f\|_{B^s_{p,q}((0,2\pi),X)}.$$

Note that for $f(t) = e^{itk}x$, the solution u of (3.1) is given by $u(t) = (ikM-A)^{-1}e^{ikt}x$. Hence,

$$\sup_k \|ikM(ikM-A)^{-1}x\| \le C\|x\|.$$

(iii)⇒(i). Suppose that $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\sup_{k\in\mathbb{Z}} \|ikM(ikM-A)^{-1}\| < \infty$. Define $M_k := ikM(ikM-A)^{-1}$ and $N_k := (ikM-A)^{-1}$ for $k \in \mathbb{Z}$. Since $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$, Proposition 4.4 shows that $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Now, we will see that $\{N_k\}_{k\in\mathbb{Z}}$ is an M-bounded sequence. First note that, since $0 \in \rho_M(A)$, A is an invertible operator, and hence the identity $ikM(ikM-A)^{-1} = A(ikM-A)^{-1}+I$ implies $N_k = A^{-1}(M_k-I)$. So, $\sup_{k\in\mathbb{Z}} \|N_k\| < \infty$. Now, observe that

$$k(N_{k+1}-N_k) = k[(i(k+1)M-A)^{-1}-(ikM-A)^{-1}] = A^{-1}k(M_{k+1}-M_k).$$

Hence, by (4.3) we get $\sup_{k\in\mathbb{Z}} \|k(N_{k+1}-N_k)\| < \infty$. In the same way, we have

$$\begin{aligned} &k^2(N_{k+1}-2N_k+N_{k-1}) \\ &\qquad = k^2[A^{-1}M_{k+1}-A^{-1}-2[A^{-1}(M_k-I)]+A^{-1}M_{k-1}-A^{-1}] \\ &\qquad = A^{-1}k^2(M_{k+1}-2M_k+M_{k-1}). \end{aligned}$$

Therefore, using (4.4), we obtain

$$\sup_k \|k^2(N_{k+1}-2N_k+N_{k-1})\| < \infty.$$

So, $\{N_k\}_{k\in\mathbb{Z}}$ is an M-bounded sequence and, by Theorem 4.3, $\{N_k\}_{k\in\mathbb{Z}}$

is a $B^s_{p,q}$-multiplier. We conclude that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{N_k\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-multipliers. Let $f \in B^s_{p,q}((0,2\pi),X)$. There exist $u,v \in B^s_{p,q}((0,2\pi),X)$ such that $\hat{u}(k) = ikM(ikM-A)^{-1}\hat{f}(k)$ and $\hat{v}(k) = (ikM-A)^{-1}\hat{f}(k)$ for all $k \in \mathbb{Z}$. So, we have $ikM\hat{v}(k) = \hat{u}(k)$ for all $k \in \mathbb{Z}$. By Lemma 2.3 we obtain $(Mv)' = u$ a.e. Since $u \in B^s_{p,q}((0,2\pi),X)$ we have $(Mv)' \in B^s_{p,q}((0,2\pi),X)$, and so $Mv \in B^{s+1}_{p,q}((0,2\pi),X)$. Also, since $ikM-A$ is bijective for all $k \in \mathbb{Z}$ and $\hat{v}(k) = (ikM-A)^{-1}\hat{f}(k)$, we have $v(t) \in D(A)$ and $ikM\hat{v}(k) - \hat{f}(k) = A\hat{v}(k)$ for all $k \in \mathbb{Z}$. So, $(Mv)'(t) = Av(t) + f(t)$ a.e. $t \in (0,2\pi)$ by Lemma 2.3. The uniqueness follows as in the proof of Theorem 3.3. ∎

Remark 4.7. Note that the Besov spaces $B^s_{\infty,\infty}((0,2\pi),X)$ correspond to the familiar Hölder spaces C^s if $0 < s < 1$. Hence, Theorem 4.6 extends and improves Theorem 2.1 in [BF] where X was assumed to be a reflexive Banach space.

Example 4.8. Let us consider the periodic boundary value problem

$$\frac{\partial(m(x)u)}{\partial t} - \Delta u = f(t,x) \quad \text{in } [0,2\pi]\times\Omega, \tag{4.5}$$

$$u = 0 \quad \text{in } [0,2\pi]\times\partial\Omega, \tag{4.6}$$

$$m(x)u(0,x) = m(x)u(2\pi,x) \quad \text{in } \Omega, \tag{4.7}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $m(x) \geq 0$ is a given measurable bounded function on Ω, and f is a function on $[0,2\pi] \times \Omega$. The initial value problem (4.5)–(4.6) with $m(x)u(0,x) = v_0$ has been studied in [FY1], [FY2] both in the spaces $H^{-1}(\Omega)$, $L^2(\Omega)$ and in $L^p(\Omega)$, $p > 1$. The periodic problem (4.5)–(4.7) has been studied in [BF] in the spaces $H^{-1}(\Omega)$ and $L^2(\Omega)$.

Let M be the operator of multiplication by m. If we take $X = H^{-1}(\Omega)$ then by [BF, p. 38] (see also references therein), there exists a constant $c > 0$ such that

$$\|M(zM-\Delta)^{-1}\| \leq \frac{c}{1+|z|}$$

whenever $\operatorname{Re} z \geq -c(1+|\operatorname{Im} z|)$. In particular, on the imaginary axis we have $\|M(ikM-\Delta)^{-1}\| \leq c/(1+|k|)$ for all $k \in \mathbb{Z}$. Therefore, Theorem 4.6 applies immediately, yielding the existence and uniqueness of solutions of (4.5)–(4.7) in periodic Besov spaces, complementing the results in [BF]. On the other hand, and because $H^{-1}(\Omega)$ is a Hilbert space, Corollary 3.4 also applies, showing that for all $f \in L^p_{2\pi}(\mathbb{R}, H^{-1}(\Omega))$ the periodic problem (4.5)–(4.7) has precisely one strong solution u with maximal regularity.

Example 4.9. Consider, for $t \in [0, 2\pi]$ and $x \in [0, \pi]$, the problem

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + 1\right)u(t,x) = -a\frac{\partial^2}{\partial x^2}u(t,x) - ku(t,x) + f(t,x), \tag{4.8}$$

$$u(t,0) = u(t,\pi) = \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,\pi) = 0, \tag{4.9}$$

$$\left(\frac{\partial^2}{\partial x^2} + 1\right)u(0,x) = \left(\frac{\partial^2}{\partial x^2} + 1\right)u(2\pi,x), \tag{4.10}$$

where a is a positive constant and $-2a < k < 4a$. Set $X = C_0([0,\pi]) = \{u \in C([0,\pi]) : u(0) = u(\pi)\}$ and let K be the realization of $\partial^2/\partial x^2$ with domain

$$D(K) = \left\{u \in C^2([0,\pi]) : u(0) = u(\pi) = \frac{\partial^2}{\partial x^2}u(0) = \frac{\partial^2}{\partial x^2}u(\pi) = 0\right\}.$$

Then we take $M = K + I$ and $A = aM + (k-a)I$. By [BF, p. 39, Ex. 1.2] we have, as in the above example,

$$\|M(ikM - \Delta)^{-1}\| \leq \frac{c}{1+|k|}$$

for all $k \in \mathbb{Z}$. Therefore, Theorem 4.6 applies, and hence for all $f \in B^s_{p,q}((0,2\pi), C_0([0,\pi]))$, $s > 0$, $1 \leq p, q \leq \infty$ the problem (4.8)–(4.10) has a unique strong solution u with $\partial^2 u/\partial x^2 \in B^s_{p,q}((0,2\pi), C_0([0,\pi]))$. In particular, because the class of Besov spaces contains the class of Hölder spaces, our result recovers and extends Example 1.2 in [BF].

Remark 4.10. Following a similar method of proof, and using the operator-valued Fourier multiplier theorem stated in [BK, Theorem 3.2], one can prove a result analogous to Theorem 4.6 for the scale of Triebel–Lizorkin spaces.

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