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Non-hyperreflexive reflexive spaces of operators

by

Roman V. Bessonov (St. Petersburg), Janko Bračič (Ljubljana) and Michal Zajac (Bratislava)

Abstract. We study operators whose commutant is reflexive but not hyperreflexive. We construct a C_0 contraction and a Jordan block operator S_B associated with a Blaschke product B which have the above mentioned property. A sufficient condition for hyperreflexivity of S_B is given. Some other results related to hyperreflexivity of spaces of operators that could be interesting in themselves are proved.

Introduction. Let $\mathcal{X}$ be a complex Banach space and let $\mathcal{L}(\mathcal{X})$ be the algebra of all bounded linear operators on $\mathcal{X}$. For a linear subspace $\mathcal{S} \subseteq \mathcal{L}(\mathcal{X})$, the *reflexive closure* of $\mathcal{S}$ is defined by $\operatorname{Ref}\mathcal{S} = \{T \in B(\mathcal{X}); Tx \in \overline{\mathcal{S}x}$ for every $x \in \mathcal{X}\}$, where $\overline{\mathcal{S}x}$ denotes the closure of the orbit $\mathcal{S}x$. It is easily seen that $\operatorname{Ref}\mathcal{S}$ is a strongly closed and hence also norm closed linear space that contains $\mathcal{S}$. The space $\mathcal{S}$ is said to be *reflexive* if $\mathcal{S} = \operatorname{Ref}\mathcal{S}$.

For an operator $T \in \mathcal{L}(\mathcal{X})$, the usual distance and the Arveson distance of T to a linear subspace $\mathcal{S} \subseteq \mathcal{L}(\mathcal{X})$ are given by

$$\mathrm{d}(T,\mathcal{S}) = \inf_{S\in\mathcal{S}} \sup_{\|x\|\le 1} \|Tx - Sx\| \quad \text{and} \quad \alpha(T,\mathcal{S}) = \sup_{\|x\|\le 1} \inf_{S\in\mathcal{S}} \|Tx - Sx\|,$$

respectively. It is obvious that $\mathrm{d}(\cdot,\mathcal{S})$ and $\alpha(\cdot,\mathcal{S})$ are seminorms on $\mathcal{L}(\mathcal{X})$ satisfying $\alpha(T,\mathcal{S}) \le \mathrm{d}(T,\mathcal{S})$ for every $T \in \mathcal{L}(\mathcal{X})$. The space $\mathcal{S}$ is said to be *hyperreflexive* if $\mathrm{d}(\cdot,\mathcal{S})$ and $\alpha(\cdot,\mathcal{S})$ are equivalent, i.e., if there exists a constant $c \ge 1$ such that

$$\mathrm{d}(T,\mathcal{S}) \le c\alpha(T,\mathcal{S}) \quad (T \in \mathcal{L}(\mathcal{X})). \tag{0.1}$$

Let $\kappa(\mathcal{S})$ be the infimum of all numbers $c \ge 1$ satisfying (0.1). Then $\kappa(\mathcal{S})$ satisfies (0.1) as well and it is the smallest number with this property. For a hyperreflexive linear space $\mathcal{S} \subseteq \mathcal{L}(\mathcal{X})$, the number $\kappa(\mathcal{S})$ is called the *hyperreflexivity constant* of $\mathcal{S}$.

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Every norm closed hyperreflexive linear space $\mathcal{S}$ is reflexive. Indeed, note that $T \in \operatorname{Ref} \mathcal{S}$ if and only if $\alpha(T, \mathcal{S}) = 0$, and $T \in \mathcal{S}$ if and only if $\mathrm{d}(T, \mathcal{S}) = 0$. On the other hand, there are reflexive linear spaces of operators which are not hyperreflexive [10]. However, if $\mathcal{S}$ is a finite-dimensional space, then $\mathcal{S}$ is hyperreflexive if and only if it is reflexive (see [11]).

Let $\mathcal{W}(A)$ denote the smallest strongly closed (i.e., closed in the strong operator topology) subalgebra of $\mathcal{L}(\mathcal{X})$ containing $A \in \mathcal{L}(\mathcal{X})$ and the identity operator I. Note that if $\mathcal{X}$ is a Hilbert space, then $\mathcal{W}(A)$ is also weakly closed. See, e.g. [3, p. 38] for the proof of the well-known fact that any strongly closed convex set of Hilbert space operators is closed in the weak operator topology. The operator A is said to be *reflexive*, respectively *hyperreflexive*, if so is $\mathcal{W}(A)$. For a hyperreflexive operator A, the hyperreflexivity constant of $\mathcal{W}(A)$ is denoted by $\kappa(A)$.

The hyperreflexivity concept for operator algebras was introduced by Arveson in the late 1970's and it turned out to be useful for some problems in perturbation theory. Until 1985 it was unknown whether hyperreflexivity and reflexivity are equivalent. The first example of a reflexive algebra of operators which is not hyperreflexive was given by Kraus and Larson [10].

This paper consists of two parts. Section 1 is devoted to the hyperreflexivity of spaces of operators on general Banach and Hilbert spaces. It is well known that similarity preserves hyperreflexivity in Hilbert spaces [7]. We extend this fact to Banach spaces. Next, we show that a direct sum of two spaces of Hilbert space operators is hyperreflexive if and only if the summands are hyperreflexive (see also [9]). Estimates of the hyperreflexivity constant of the direct sum are given in terms of the angle between the underlying Hilbert spaces.

An operator on a two-dimensional space is reflexive, and therefore hyperreflexive, if and only if it has two distinct eigenvalues. We express the hyperreflexivity constant of such an operator in terms of its matrix elements. This allows us to construct a non-hyperreflexive reflexive operator.

In Section 2 we consider the hyperreflexivity of C_0 contractions. An operator A acting on a Hilbert space $\mathcal{H}$ is called a C_0 *contraction* if $\|A\| \leq 1$ and there exists a bounded non-zero analytic function f on $\mathbb{D}$, the unit disk in the complex plane, such that $f(A) = 0$ (see [14]). It is well known that C_0 contractions share many properties with finite-dimensional operators. As mentioned above, each finite-dimensional reflexive operator is hyperreflexive. It is a natural question whether each reflexive C_0 contraction is hyperreflexive. We give a negative answer to this question.

We consider a so-called Jordan block, a functional model of a C_0 contraction A satisfying $\dim \operatorname{Ran}(I - AA^*) = \dim \operatorname{Ran}(I - A^*A) = 1$. First, we show that a Jordan block is hyperreflexive whenever its minimal function is a Blaschke product satisfying the Carleson condition. Next, we construct a Jordan block S and two-dimensional invariant subspaces $\mathcal{H}_n$ of it such that the orthogonal sum of the operators $S|\mathcal{H}_n$ is a non-hyperreflexive reflexive C_0 contraction T satisfying $\dim \operatorname{Ran}(I - TT^*) = \dim \operatorname{Ran}(I - T^*T) = \infty$. Finally, applying a deep result on Riesz bases due to Vasyunin, we show that this Jordan block S is itself a non-hyperreflexive reflexive operator.

1. Hyperreflexivity in general Banach and Hilbert spaces

1.1. Hyperreflexivity of similar spaces. It is well known that similarity preserves reflexivity and hyperreflexivity in Hilbert spaces. The following proposition extends [7, Proposition 1.3] to general Banach spaces.

PROPOSITION 1.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Banach spaces and let $\mathcal{S} \subseteq \mathcal{L}(\mathcal{X})$ be a hyperreflexive subspace. If $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ are invertible, then $A\mathcal{S}B \subseteq \mathcal{L}(\mathcal{Y})$ is a hyperreflexive subspace and*

$$\frac{1}{\|A\|\,\|B\|\,\|A^{-1}\|\,\|B^{-1}\|}\kappa(\mathcal{S}) \le \kappa(A\mathcal{S}B) \le \|A\|\,\|B\|\,\|A^{-1}\|\,\|B^{-1}\|\kappa(\mathcal{S}).$$

Proof. First we note that $\{x\in\mathcal{X};\ \|x\|\le 1\} \subseteq \{\|B^{-1}\|By;\ y\in\mathcal{Y},\ \|y\|\le 1\}$. Indeed, if $x \in \mathcal{X}$, $\|x\| \le 1$, then $x = \|B^{-1}\|B\frac{B^{-1}x}{\|B^{-1}\|}$ and $\big\|\frac{B^{-1}x}{\|B^{-1}\|}\big\| \le 1$. Let $T \in \mathcal{L}(\mathcal{X})$ be arbitrary. Then

$$\begin{aligned}
&\alpha(T, \mathcal{S}) = \sup\{\inf\{\|(T - S)x\|;\ S \in \mathcal{S}\};\ x \in \mathcal{X},\ \|x\| \le 1\}\\
&= \sup\{\inf\{\|A^{-1}(ATB - ASB)B^{-1}x\|;\ S \in \mathcal{S}\};\ x \in \mathcal{X},\ \|x\| \le 1\}\\
&\le \sup\{\inf\{\|A^{-1}(ATB - ASB)B^{-1}(\|B^{-1}\|By)\|;\ S \in \mathcal{S}\};\ y \in \mathcal{Y},\ \|y\| \le 1\}\\
&\le \|A^{-1}\|\,\|B^{-1}\| \sup\{\inf\{\|(ATB - ASB)y\|;\ S \in \mathcal{S}\};\ y \in \mathcal{Y},\ \|y\| \le 1\}\\
&= \|A^{-1}\|\,\|B^{-1}\|\alpha(ATB, A\mathcal{S}B).
\end{aligned}$$

A similar reasoning gives $\mathrm{d}(T, \mathcal{S}) \le \|A^{-1}\|\,\|B^{-1}\|\,\mathrm{d}(ATB, A\mathcal{S}B)$. It is obvious that $\mathrm{d}(ATB, A\mathcal{S}B) \le \|A\|\,\|B\|\,\mathrm{d}(T, \mathcal{S})$ as well. Thus, if $\mathcal{S}$ is hyperreflexive, then

$$\begin{aligned}
\mathrm{d}(ATB, A\mathcal{S}B) &\le \|A\|\,\|B\|\,\mathrm{d}(T, \mathcal{S}) \le \|A\|\,\|B\|\kappa(\mathcal{S})\alpha(T, \mathcal{S})\\
&\le \|A\|\,\|B\|\,\|A^{-1}\|\,\|B^{-1}\|\kappa(\mathcal{S})\alpha(ATB, A\mathcal{S}B)
\end{aligned}$$

for every $T \in \mathcal{L}(\mathcal{X})$. Hence $A\mathcal{S}B$ is a hyperreflexive subspace and $\kappa(A\mathcal{S}B) \le \|A\|\,\|B\|\,\|A^{-1}\|\,\|B^{-1}\|\kappa(\mathcal{S})$. The last inequality implies that $\kappa(\mathcal{S}) \le \|A\|\,\|B\| \cdot \|A^{-1}\|\,\|B^{-1}\|\kappa(A\mathcal{S}B)$ if we interchange the spaces $\mathcal{S}$ and $A\mathcal{S}B$. ∎

COROLLARY 1.2. *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$ be a hyperreflexive linear space. If U and V are unitary operators on $\mathcal{H}$, then the space $U\mathcal{S}V$ is hyperreflexive and $\kappa(U\mathcal{S}V) = \kappa(\mathcal{S})$.* ∎

COROLLARY 1.3. *Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces. Assume that $A \in \mathcal{L}(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{Y})$ are similar, i.e., there exists an invertible $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that $B = SAS^{-1}$. Then A is hyperreflexive if and only if B is hyperreflexive.* ∎

Operators $A \in \mathcal{L}(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{Y})$ are said to be *quasi-similar* if there exist injective operators with dense ranges, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, such that $SA = BS$ and $TB = AT$. Note that, by [16, Example 5.2], quasi-similarity does not preserve hyperreflexivity.

1.2. Hyperreflexivity of orthogonal sums of hyperreflexive spaces. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be their orthogonal sum. If $\mathcal{S}_i \subset \mathcal{L}(\mathcal{H}_i)$ $(i = 1, 2)$ are reflexive, then $\mathcal{S}_1 \oplus \mathcal{S}_2 \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is reflexive as well (see [3, Proposition 56.2]). For hyperreflexivity we have an analogous result.

PROPOSITION 1.4. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be their orthogonal sum. Assume that $\mathcal{S}_i \subset \mathcal{L}(\mathcal{H}_i)$ $(i = 1, 2)$ are closed linear subspaces and let $T_i \in \mathcal{L}(\mathcal{H}_i)$ $(i = 1, 2)$. Then*

(i) $\max\{\mathrm{d}(T_1, \mathcal{S}_1), \mathrm{d}(T_2, \mathcal{S}_2)\} \le \mathrm{d}(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2) \le \mathrm{d}(T_1, \mathcal{S}_1) + \mathrm{d}(T_2, \mathcal{S}_2)$,
(ii) $\max\{\alpha(T_1, \mathcal{S}_1), \alpha(T_2, \mathcal{S}_2)\} \le \alpha(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2) \le \alpha(T_1, \mathcal{S}_1) + \alpha(T_2, \mathcal{S}_2)$.
(iii) *$\mathcal{S}_1 \oplus \mathcal{S}_2$ is hyperreflexive if and only if both $\mathcal{S}_1$ and $\mathcal{S}_2$ are, and* $\max\{\kappa(\mathcal{S}_1), \kappa(\mathcal{S}_2)\} \le \kappa(\mathcal{S}_1 \oplus \mathcal{S}_2) \le 1 + 2\max\{\kappa(\mathcal{S}_1), \kappa(\mathcal{S}_2)\}$.

Proof. Most of these assertions are particular cases of those obtained in [9, Sec. 5]. For the reader's convenience, we give another proof here.

(i) Since

$$\begin{aligned}\mathrm{d}(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2)^2 &= \inf\{\sup\{\|(T_1 - S_1)x_1\|^2 + \|(T_2 - S_2)x_2\|^2; \\ &\qquad \|x_1\|^2 + \|x_2\|^2 \le 1\}; \ S_1 \in \mathcal{S}_1, \ S_2 \in \mathcal{S}_2\} \\ &\ge \inf\{\sup\{\|(T_i - S_i)x_i\|^2; \ \|x_i\| \le 1\}; \ S_i \in \mathcal{S}_i\} = \mathrm{d}(T_i, \mathcal{S}_i)^2 \quad \text{for } i = 1, 2,\end{aligned}$$

one has $\max\{\mathrm{d}(T_1, \mathcal{S}_1), \mathrm{d}(T_2, \mathcal{S}_2)\} \le \mathrm{d}(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2)$. On the other hand, $\mathrm{d}(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2) \le \mathrm{d}(T_1 \oplus 0, \mathcal{S}_1 \oplus \mathcal{S}_2) + \mathrm{d}(0 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2)$ since $\mathrm{d}(\cdot, \mathcal{S}_1 \oplus \mathcal{S}_2)$ is a seminorm. However,

$$\begin{aligned}\mathrm{d}(T_1 \oplus 0, \mathcal{S}_1 \oplus \mathcal{S}_2)^2 &= \inf\{\|T_1 - S_1\|^2 + \|S_2\|^2; \ S_1 \in \mathcal{S}_1, \ S_2 \in \mathcal{S}_2\} \\ &= \inf\{\|T_1 - S_1\|^2; \ S_1 \in \mathcal{S}_1\} = \mathrm{d}(T_1, \mathcal{S}_1)^2,\end{aligned}$$

and similarly $\mathrm{d}(0 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2) = \mathrm{d}(T_2, \mathcal{S}_2)$. Thus, $\mathrm{d}(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2) \le \mathrm{d}(T_1, \mathcal{S}_1) + \mathrm{d}(T_2, \mathcal{S}_2)$.

(ii) One has

$$\begin{aligned}\alpha(T_1,\mathcal{S}_1) &= \sup\{\inf\{\|(T_1-S_1)x_1\|;\ S_1\in\mathcal{S}_1\};\ \|x_1\|\le 1\}\\ &= \sup\{\inf\{\|(T_1\oplus T_2 - S_1\oplus S_2)(x_1\oplus 0)\|,\ S_1\in\mathcal{S}_1,\ S_2\in\mathcal{S}_2\};\ \|x_1\|\le 1\}\\ &\le \alpha(T_1\oplus T_2,\mathcal{S}_1\oplus\mathcal{S}_2)\le \alpha(T_1\oplus 0,\mathcal{S}_1\oplus\mathcal{S}_2)+\alpha(0\oplus T_2,\mathcal{S}_1\oplus\mathcal{S}_2),\end{aligned}$$

where the last inequality holds as $\alpha(\cdot,\mathcal{S}_1\oplus\mathcal{S}_2)$ is a seminorm. Again,

$$\begin{aligned}\alpha(T_1\oplus 0,\mathcal{S}_1\oplus\mathcal{S}_2)^2 &= \sup\{\inf\{\|(T_1-S_1)x_1\|^2+\|S_2x_2\|^2;\\ &\qquad S_1\in\mathcal{S}_1,\ S_2\in\mathcal{S}_2\};\ \|x_1\|^2+\|x_2\|^2\le 1\}\\ &= \sup\{\inf\{\|(T_1-S_1)x_1\|^2;\ S_1\in\mathcal{S}_1\};\ \|x_1\|\le 1\}=\alpha(T_1,\mathcal{S}_1)^2.\end{aligned}$$

Of course, the same inequalities hold if index 1 is replaced by index 2. We conclude that (ii) holds.

(iii) Assume first that $\mathcal{S}_1\oplus\mathcal{S}_2$ is hyperreflexive. Then $\mathrm{d}(T_1,\mathcal{S}_1) = \mathrm{d}(T_1\oplus 0,\mathcal{S}_1\oplus\mathcal{S}_2)\le\kappa(\mathcal{S}_1\oplus\mathcal{S}_2)\alpha(T_1\oplus 0,\mathcal{S}_1\oplus\mathcal{S}_2)=\kappa(\mathcal{S}_1\oplus\mathcal{S}_2)\alpha(T_1,\mathcal{S}_1)$, and similarly $\mathrm{d}(T_2,\mathcal{S}_2)\le\kappa(\mathcal{S}_1\oplus\mathcal{S}_2)\alpha(T_2,\mathcal{S}_2)$. This proves the inequality

$$\max\{\kappa(\mathcal{S}_1),\kappa(\mathcal{S}_2)\}\le\kappa(\mathcal{S}_1\oplus\mathcal{S}_2).$$

On the other hand, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are hyperreflexive, then $\kappa(\mathcal{S}_1\oplus\mathcal{S}_2)\le 1+2\max\{\kappa(\mathcal{S}_1),\kappa(\mathcal{S}_2)\}$, by [9, Corollary 5.4]. ∎

It is obvious that Proposition 1.4(iii) implies the following corollary.

Corollary 1.5. *Let $\mathcal{H}$ be the orthogonal sum of Hilbert spaces $\mathcal{H}_i$ ($i\in\mathbb{N}$) and let $\mathcal{S}_i\subseteq\mathcal{L}(\mathcal{H}_i)$ be weakly closed subspaces. If $\mathcal{S}=\bigoplus_{i\in\mathbb{N}}\mathcal{S}_i$ is hyperreflexive, then each $\mathcal{S}_i$ is hyperreflexive and $\kappa(\mathcal{S}_i)\le\kappa(\mathcal{S})$.* ∎

1.3. Hyperreflexivity of a direct sum of hyperreflexive spaces. Let $\mathcal{M}$ and $\mathcal{N}$ be closed non-trivial subspaces of a separable complex Hilbert space $\mathcal{H}$. Recall that the *angle* between $\mathcal{M}$ and $\mathcal{N}$ is the number $\varphi\in[0,\pi/2]$ which is given by

$$\cos\varphi = \sup\{|\langle x,y\rangle|;\ x\in\mathcal{M}\ominus(\mathcal{M}\cap\mathcal{N}),\ y\in\mathcal{N}\ominus(\mathcal{M}\cap\mathcal{N})\text{ and }\|x\|=\|y\|=1\}.\tag{1.1}$$

From now on we assume that $\mathcal{M}\cap\mathcal{N}=\{0\}$ and $\mathcal{M}+\mathcal{N}=\mathcal{H}$, i.e., $\mathcal{H}$ is the direct sum of $\mathcal{M}$ and $\mathcal{N}$, which we briefly write as $\mathcal{H}=\mathcal{M}\dotplus\mathcal{N}$. In this case (1.1) simplifies to

$$\cos\varphi=\sup\{|\langle x,y\rangle|;\ x\in\mathcal{M},\ y\in\mathcal{N}\text{ and }\|x\|=\|y\|=1\}.$$

Note that the sum $\mathcal{M}+\mathcal{N}$ is a closed subspace of $\mathcal{H}$ if and only if the angle between $\mathcal{M}$ and $\mathcal{N}$ is strictly greater than 0. Thus, from now on it is assumed that $\varphi>0$. We use the symbol $\oplus$ for the orthogonal direct sum, i.e., $\mathcal{M}\dotplus\mathcal{N}=\mathcal{M}\oplus\mathcal{N}$ if and only if $\varphi=\pi/2$.

Lemmas 1.6 and 1.7 below will be used to estimate the hyperreflexivity constant of a direct sum of hyperreflexive spaces.

LEMMA 1.6. *If $x \in \mathcal{M}$, $y \in \mathcal{N}$ and $\|x\|^2 + \|y\|^2 = 1$, then*

$$1 - \cos\varphi \le \|x+y\|^2 \le 1 + \cos\varphi. \tag{1.2}$$

On the other hand, if $\|x+y\| = 1$ for some $x \in \mathcal{M}$ and $y \in \mathcal{N}$, then

$$\frac{1}{1+\cos\varphi} \le \|x\|^2 + \|y\|^2 \le \frac{1}{1-\cos\varphi}. \tag{1.3}$$

Proof. If $x = 0$ or $y = 0$, then the statements hold trivially. Assume therefore that $x \neq 0$ and $y \neq 0$. Note that $2\|x\|\,\|y\| \le \|x\|^2 + \|y\|^2$ and $\operatorname{Re}\langle x, y\rangle \le |\langle x, y\rangle|$ for arbitrary vectors x and y.

Assume first that $x \in \mathcal{M}$, $y \in \mathcal{N}$ and $\|x\|^2 + \|y\|^2 = 1$. Then

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x, y\rangle = \|x\|^2 + \|y\|^2 \\ &\quad + \frac{2\|x\|\,\|y\|}{\|x\|^2+\|y\|^2}\operatorname{Re}\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle \le 1 + \left|\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle\right| \\ &\le 1 + \cos\varphi\end{aligned}$$

and

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x, y\rangle = \|x\|^2 + \|y\|^2 \\ &\quad - \frac{2\|x\|\,\|y\|}{\|x\|^2+\|y\|^2}\operatorname{Re}\left\langle -\frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle \ge 1 - \left|\left\langle -\frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle\right| \\ &\ge 1 - \cos\varphi.\end{aligned}$$

Let now $\|x+y\| = 1$. Then

$$\begin{aligned}\|x\|^2 + \|y\|^2 = \frac{\|x\|^2+\|y\|^2}{\|x+y\|^2} &= \left(1 - \operatorname{Re}\left(\left\langle -\frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle\right)\frac{2\|x\|\,\|y\|}{\|x\|^2+\|y\|^2}\right)^{-1} \\ &\le \left(1 - \left|\left\langle -\frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle\right|\right)^{-1} \le \frac{1}{1-\cos\varphi}\end{aligned}$$

and

$$\|x\|^2 + \|y\|^2 \ge \left(1 + \left|\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle\right|\right)^{-1} \ge \frac{1}{1+\cos\varphi}. \ \blacksquare$$

Define $\mathcal{K} = \mathcal{M} \oplus \mathcal{N}$, i.e., the underlying vector space for $\mathcal{K}$ is the same as for $\mathcal{H}$, but the inner product is $\langle x_1 + y_1, x_2 + y_2\rangle_{\mathcal{K}} = \langle x_1, x_2\rangle_{\mathcal{H}} + \langle y_1, y_2\rangle_{\mathcal{H}}$ for $x_1, x_2 \in \mathcal{M}$ and $y_1, y_2 \in \mathcal{N}$. From now on we omit the subscripts indicating where an inner product is computed. Let $\Phi : \mathcal{H} \to \mathcal{K}$ be given by $\Phi(x+y) = x \oplus y$ for $x \in \mathcal{M}$, $y \in \mathcal{N}$.

LEMMA 1.7. *Φ is an invertible linear operator with*

$$\|\Phi\| = \frac{1}{\sqrt{1-\cos\varphi}} \quad \textit{and} \quad \|\Phi^{-1}\| = \sqrt{1+\cos\varphi}.$$

Proof. It is obvious that Φ is an invertible linear operator. Let $x \in \mathcal{M}$ and $y \in \mathcal{N}$ be such that $x + y \in \mathcal{H}$ is a vector of norm 1. Then, by (1.3),

$$\|\Phi(x+y)\|^2 = \|x \oplus y\|^2 = \|x\|^2 + \|y\|^2 \le \frac{1}{1-\cos\varphi},$$

which shows that $\|\Phi\| \le 1/\sqrt{1-\cos\varphi}$. Now, let $\varepsilon > 0$. By the definition of the angle between subspaces, there exist $x \in \mathcal{M}$ and $y \in \mathcal{Y}$ of norm 1 such that $\cos\varphi \le |\langle x,y\rangle| + \varepsilon$. We may assume that $\langle x,y\rangle < 0$ (replace, for instance, x by $-\frac{|\langle x,y\rangle|}{\langle x,y\rangle}x$). Then

$$\begin{aligned}\left\|\Phi\left(\frac{x+y}{\|x+y\|}\right)\right\|^2 &= \frac{\|x\|^2+\|y\|^2}{\|x\|^2+\|y\|^2+2\operatorname{Re}\langle x,y\rangle} = \frac{2}{2-2|\langle x,y\rangle|}\\ &\ge \frac{1}{1-\cos\varphi+\varepsilon},\end{aligned}$$

so $\|\Phi\| \ge 1/\sqrt{1-\cos\varphi}$.

The norm of Φ^{-1} is computed in a similar way. The inequality $\|\Phi^{-1}\| \le \sqrt{1+\cos\varphi}$ follows from (1.2). On the other hand, for a given $\varepsilon > 0$, there exist $x \in \mathcal{M}$ and $y \in \mathcal{N}$ with $\|x\| = \|y\| = 1$ such that $\cos\varphi \le |\langle x,y\rangle| + \varepsilon$ and $\langle x,y\rangle > 0$. Then $\frac{1}{\sqrt{2}}(x \oplus y)$ is a vector of norm 1 in $\mathcal{K}$ such that

$$\begin{aligned}\left\|\Phi^{-1}\left(\frac{1}{\sqrt{2}}(x\oplus y)\right)\right\|^2 &= \frac{1}{2}(\|x\|^2+\|y\|^2+2\operatorname{Re}\langle x,y\rangle) = 1+|\langle x,y\rangle|\\ &\ge 1+\cos\varphi-\varepsilon.\end{aligned}$$

Thus, $\|\Phi^{-1}\| \ge \sqrt{1+\cos\varphi}$. ∎

Let $\mathcal{S} \subseteq \mathcal{L}(\mathcal{M})$ and $\mathcal{T} \subseteq \mathcal{L}(\mathcal{N})$ be weakly closed spaces of operators. We denote by $\mathcal{S} \dot{+} \mathcal{T}$ the set of all operators in $\mathcal{L}(\mathcal{H})$ which have a block matrix representation $\left(\begin{smallmatrix} S & 0\\ 0 & T\end{smallmatrix}\right)$ ($S \in \mathcal{S}$, $T \in \mathcal{T}$) with respect to the decomposition $\mathcal{H} = \mathcal{M} \dot{+} \mathcal{N}$. It is obvious that $\mathcal{S} \dot{+} \mathcal{T}$ is a weakly closed subspace and it is not hard to see that it is reflexive if and only if $\mathcal{S}$ and $\mathcal{T}$ are reflexive.

Proposition 1.8. *The space $\mathcal{S} \dot{+} \mathcal{T} \subseteq \mathcal{L}(\mathcal{H})$ is hyperreflexive if and only if $\mathcal{S}$ and $\mathcal{T}$ are hyperreflexive. Moreover,*

$$\frac{1}{3}\,\frac{1-\cos\varphi}{1+\cos\varphi}\kappa(\mathcal{S}\dot{+}\mathcal{T}) \le \max\{\kappa(\mathcal{S}),\kappa(\mathcal{T})\} \le \frac{1-\cos\varphi}{1+\cos\varphi}\kappa(\mathcal{S}\dot{+}\mathcal{T}). \tag{1.4}$$

Proof. As before, let $\mathcal{K} = \mathcal{M} \oplus \mathcal{N}$. It is obvious that $\mathcal{S} \oplus \mathcal{T} := \{S \oplus T;\ S \in \mathcal{S},\ T \in \mathcal{T}\}$ is a linear subspace of $\mathcal{L}(\mathcal{K})$. Let $\Phi : \mathcal{H} \to \mathcal{K}$ be the linear operator defined above. Then $\mathcal{S} \oplus \mathcal{T} = \Phi(\mathcal{S} \dot{+} \mathcal{T})\Phi^{-1}$. Thus, by Proposition 1.1,

$$\frac{1-\cos\varphi}{1+\cos\varphi}\kappa(\mathcal{S}\dot{+}\mathcal{T}) \le \kappa(\mathcal{S}\oplus\mathcal{T}) = \kappa(\Phi(\mathcal{S}\dot{+}\mathcal{T})\Phi^{-1}) \le \frac{1+\cos\varphi}{1-\cos\varphi}\kappa(\mathcal{S}\dot{+}\mathcal{T})\,.$$

Since every hyperreflexivity constant is greater than or equal to 1, we have (1.4), by Proposition 1.4(iii). ∎

1.4. Hyperreflexivity of 2×2 matrices. Let $\mathcal{X}$ be a two-dimensional complex vector space. It is an easy exercise to show that $\mathcal{W}(A) = \{A\}'$ for every operator $A \in B(\mathcal{X})$ which is not a scalar multiple of I. Thus, such an operator is reflexive if and only if its commutant is reflexive, which happens precisely when A has two distinct eigenvalues (see [5]). Let us identify $\mathcal{X}$ with $\mathbb{C}^2$. Then every operator is represented by a 2×2 matrix. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{M}_{2\times 2}(\mathbb{C})$ have distinct eigenvalues. Our aim is to express $\kappa(A)$ as a function of a, b, c, d. This will be used to construct an operator with reflexive but non-hyperreflexive commutant.

LEMMA 1.9. *If $A = \left(\begin{smallmatrix} \lambda & \omega \\ 0 & \mu \end{smallmatrix}\right) \in \mathbb{M}_{2\times 2}(\mathbb{C})$ with $\lambda \neq \mu$, then*

$$\kappa(A) = \frac{\sqrt{|\omega|^2 + |\mu - \lambda|^2}}{|\mu - \lambda|}.$$

Proof. Since A has two distinct eigenvalues there are precisely two nontrivial proper A-invariant subspaces. These are complex lines spanned by the eigenvectors $e = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ with eigenvalue λ and $f = (|\omega|^2 + |\mu - \lambda|^2)^{-1/2} \left(\begin{smallmatrix} \omega \\ \mu-\lambda \end{smallmatrix}\right)$ with eigenvalue μ. The angle φ between these subspaces is given by $\cos\varphi = |\langle e, f\rangle| = |\omega|/\sqrt{|\omega|^2 + |\mu - \lambda|^2}$. It follows, by [15], that

$$\kappa(A) = \sin^{-1}\varphi = \frac{\sqrt{|\omega|^2 + |\mu - \lambda|^2}}{|\mu - \lambda|}. \qquad \blacksquare$$

If, in Lemma 1.9, A is a diagonal operator, i.e., $\omega = 0$, then $\mathcal{W}(A) = \{A\}' = \mathcal{D}_2$, the algebra of all diagonal 2×2 matrices. Thus, $\kappa(\mathcal{D}_2) = \kappa(A) = 1$. Note, however, that the hyperreflexivity constant of $\mathcal{D}_3$, the algebra of all 3×3 diagonal matrices, is $\sqrt{3/2}$ [4, Theorem 2.3].

Let $\mathcal{H}$ be a Hilbert space of dimension $N \in \{4, 5, \ldots\} \cup \{\infty\}$ and let $\{e_n\}_{n=1}^N$ be an orthonormal basis for $\mathcal{H}$. Then the algebra $\mathcal{D}_N \subset \mathcal{L}(\mathcal{H})$ of all operators which are diagonal with respect to $\{e_n\}_{n=1}^N$ can be identified with $\mathcal{D}_3 \oplus \mathcal{D}_{N-3}$, where $\mathcal{D}_{N-3} = \mathcal{D}_\infty$ if $N = \infty$. It follows from Proposition 1.4(iii) that $\kappa(\mathcal{D}_N) \geq \sqrt{3/2}$. On the other hand, by [13, Theorem 3.5], $\kappa(D_N) \leq 2$.

LEMMA 1.10. *Let $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{M}_{2\times 2}(\mathbb{C})$, $\gamma \neq 0$, be a matrix with eigenvalues 0 and 1. Then $\delta = 1 - \alpha$, $\beta = \alpha(1 - \alpha)\gamma^{-1}$, and*

$$\kappa(A) = \frac{\sqrt{|\gamma|^2 + |\overline{\alpha} - |\alpha|^2 - |\gamma|^2|^2}}{|\gamma|}. \tag{1.5}$$

Proof. Since A has eigenvalues 0 and 1 one has $\alpha + \delta = \operatorname{trace} A = 1$ and $\alpha\delta - \beta\gamma = \det A = 0$, which gives $\delta = 1 - \alpha$ and $\beta = \alpha(1 - \alpha)\gamma^{-1}$. Let $U = (|\alpha|^2 + |\gamma|^2)^{-1/2}\left(\begin{smallmatrix} \alpha & -\overline{\gamma} \\ \gamma & \overline{\alpha} \end{smallmatrix}\right)$. It is easily seen (by direct computation) that U is a unitary matrix such that $A = UTU^*$ (Schur decomposition), where $T = \left(\begin{smallmatrix} 1 & (\overline{\alpha}-|\alpha|^2-|\gamma|^2)\gamma^{-1} \\ 0 & 0 \end{smallmatrix}\right)$. Note that $\{T\}' = U^*\{A\}'U$, which means, by Corollary 1.2, that $\kappa(A) = \kappa(T)$. By Lemma 1.9, one has (1.5). ∎

PROPOSITION 1.11. *Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{M}_{2\times 2}(\mathbb{C})$ be a matrix with two distinct eigenvalues $\lambda_{1,2} = \frac{1}{2}(a + d \pm \sqrt{(a-d)^2 + 4bc})$.*

(i) *If $c = 0$, then*

$$\kappa(A) = \frac{\sqrt{|b|^2 + |d-a|^2}}{|d-a|}.$$

(ii) *If $c \neq 0$, then*

$$\kappa(A) = \frac{\sqrt{|c|^2|\lambda_2-\lambda_1|^2 + \left|(\overline{a}-\overline{\lambda}_1)(\lambda_2-\lambda_1) - |a-\lambda_1|^2 - |c|^2\right|^2}}{|c|\,|\lambda_2-\lambda_1|}. \tag{1.6}$$

Proof. It is obvious that (i) is just Lemma 1.9. Assume therefore that $c \neq 0$. The matrix $A - \lambda_1 I$ has eigenvalues 0 and $\lambda_2 - \lambda_1$. Dividing by $\lambda_2 - \lambda_1$ we obtain

$$B = \frac{1}{\lambda_2 - \lambda_1}(A - \lambda_1 I) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\alpha = \frac{a-\lambda_1}{\lambda_2-\lambda_1}, \quad \beta = \frac{b}{\lambda_2-\lambda_1}, \quad \gamma = \frac{c}{\lambda_2-\lambda_1}, \delta = \frac{d-\lambda_1}{\lambda_2-\lambda_1}.$$

The matrix B has eigenvalues 0 and 1. The commutants of A and B coincide, which means $\kappa(A) = \kappa(B)$. Thus, by Lemma 1.10, we have (1.6). ∎

Now we use the above results to construct an example of a non-hyperreflexive reflexive operator.

PROPOSITION 1.12. *There exists a sequence $\{A_n\}_{n=1}^\infty$ of 2×2 complex matrices having the following properties:*

(i) *there exists $m \geq \sqrt{2}$ such that $\|A_n\| \leq m$ for all $n \in \mathbb{N}$;*
(ii) *each A_n has two eigenvalues $\mu_n \neq \nu_n$, and if $n \neq k$, then $\{\mu_n, \nu_n\} \cap \{\mu_k, \nu_k\} = \emptyset$;*
(iii) *the operator $T = \bigoplus_{n=1}^\infty A_n$ is reflexive but not hyperreflexive.*

Proof. Put

$$A_n = \begin{pmatrix} 1 - 1/n\sqrt{n} & 1/n \\ 0 & 1 - 1/n\sqrt{n} + 1/n^2 \end{pmatrix}.$$

It is obvious that $\lim_{n\to\infty} \|I - A_n\| = 0$, which gives $\lim_{n\to\infty} \|A_n\| = 1$ and therefore $\{\|A_n\|\}_{n=1}^\infty$ is a bounded sequence. Since $\|A_1\| = \sqrt{2}$ this proves assertion (i).

To prove (ii) it suffices to show that there are no $n, k \in \mathbb{N}$ for which

$$\frac{1}{n\sqrt{n}} - \frac{1}{n^2} = \frac{1}{k\sqrt{k}}.$$

The latter can be seen by simple number-theoretical arguments.

By (ii), we have $\{T\}' = \bigoplus_{n=1}^{\infty}\{A_n\}'$, and since $\mathcal{W}(T) \subset \{T\}'$, also $\mathcal{W}(T) = \bigoplus_{n=1}^{\infty} \mathcal{W}(A_n)$. Therefore both $\{T\}'$ and $\mathcal{W}(T)$, being direct sums of reflexive algebras, are reflexive [3, Proposition 56.2]. By Proposition 1.11(i), the hyperreflexivity constant of A_n is $\kappa(A_n) = \kappa(\{A_n\}') = \sqrt{1+n^2}$, which means, by Corollary 1.5, that neither T nor $\{T\}'$ is hyperreflexive. ∎

The operator T constructed above has norm $\|T\| \geq \sqrt{2}$. Dividing T by its norm we obtain a contraction, which, however, is not in the class C_0. At the end of the paper we shall improve this construction and obtain a C_0 contraction which is reflexive but not hyperreflexive.

2. Hyperreflexivity of C_0 contractions

2.1. Preliminaries. The content of this subsection is standard and can be found in several classical monographs, e.g. in [1, 6, 12, 14]. Let H^2 and H^∞ be the usual Hardy spaces of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$. Recall that a function $\theta \in H^\infty$ is *inner* if $|\theta(e^{it})| = 1$ almost everywhere. For an inner function $\theta \in H^\infty$, let $\mathcal{H}_\theta = H^2 \ominus \theta H^2$ and let P_θ be the orthogonal projection from H^2 onto $\mathcal{H}_\theta$. Of course, if θ is a constant (of modulus 1), then $\mathcal{H}_\theta = \{0\}$. The Jordan block S_θ associated with θ is an operator on $\mathcal{H}_\theta$ given by

$$(S_\theta f)(z) = P_\theta[zf(z)] \quad (f \in \mathcal{H}_\theta).$$

It is well known that each C_0 contraction T satisfying $\dim \operatorname{Ran}(1 - T^*T) = \dim \operatorname{Ran}(1-T^*T) = 1$ is unitarily equivalent to a Jordan block. By Sarason's theorem [1, Proposition 3.1.21], $\mathcal{W}(S_\theta) = \{S_\theta\}'$. It follows that the commutant $\{S_\theta\}'$ is reflexive, respectively hyperreflexive, if and only if S_θ is reflexive, respectively hyperreflexive. The equality $\mathcal{W}(S_\theta) = \{S_\theta\}'$ implies the equality of their invariant subspace lattices. By [1, Proposition 3.1.10(ii)], every invariant subspace $\mathcal{M}$ of S_θ has the form $\mathcal{M} = \theta_1 H^2 \ominus \theta H^2 = \theta_1 \mathcal{H}_{\theta/\theta_1}$ for an inner divisor θ_1 of θ.

For $\lambda \in \mathbb{D}$, let

$$b_\lambda(z) = \frac{|\lambda|}{\lambda}\,\frac{\lambda - z}{1 - \bar{\lambda} z}$$

be the corresponding Blaschke factor. Assume that $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{D}$ is a sequence of pairwise distinct numbers satisfying the Blaschke condition, i.e., $\sum_{n=1}^{\infty}(1 - |\lambda_n|) < \infty$. The Blaschke product $B(z) = \prod_{n=1}^{\infty} b_{\lambda_n}(z)$ is an inner function with simple zeroes. Let

$$B_{\lambda_n}(z) = \frac{B(z)}{b_{\lambda_n}(z)} \quad \text{and} \quad r_n(z) = \sqrt{1-|\lambda_n|^2}\,\frac{B(z)}{z-\lambda_n} \qquad (n \in \mathbb{N}). \tag{2.1}$$

Then $\{\lambda_n\}_{n=1}^{\infty}$ is the point spectrum of the Jordan block S_B, and r_n are the corresponding eigenvectors of norm one. Each invariant subspace $\mathcal{M}$ of S_B is the closed linear span of all eigenvectors $r_n \in \mathcal{M}$.

A Blaschke product $B(z)$ is said to satisfy the *Carleson condition* if

$$\inf_n |B_{\lambda_n}(\lambda_n)| > 0.$$

For a detailed discussion of Blaschke products satisfying this condition see [12]. The following lemma is a simple consequence of the Kabaila–Newman Lemma [12, p. 206 (159 in English transl.)].

Lemma 2.1. *For any $q \in (0,1)$ there exists a positive number $\delta(q)$ such that if $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{D}$ is a sequence of pairwise distinct numbers satisfying $1-|\lambda_{n+1}| \le q(1-|\lambda_n|)$ for $n = 1,2,\ldots,$ then the corresponding Blaschke product $B(z)$ satisfies the Carleson condition $\inf_n |B_{\lambda_n}(\lambda_n)| \ge \delta(q) > 0$.*

Recall that a family $\{u_n\}_{n=1}^{\infty}$ of vectors in an infinite-dimensional separable Hilbert space $\mathcal{H}$ is a *Riesz basis* of $\mathcal{H}$ if there exists an invertible operator $T \in \mathcal{L}(\mathcal{H})$ such that $\{Tu_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. It is known (see [12, p. 175 (135 in English transl.)]) that $B(z)$ satisfies the Carleson condition if and only if the family $\{r_n\}_{n=1}^{\infty}$ from (2.1) is a Riesz basis of $\mathcal{H}_B$.

Similarly, a family $\{\mathcal{M}_k\}_{k=1}^{\infty}$ of subspaces of a Hilbert space $\mathcal{H}$ is called a *Riesz basis* if

(i) $\mathcal{M}_k \cap \mathcal{M}_l = \{0\}$ if $k \ne l$,
(ii) $\mathcal{H} = \mathcal{M}_1 \dotplus \mathcal{M}_2 \dotplus \cdots$,
(iii) the operator $T : f_1 \dotplus f_2 \dotplus \cdots \mapsto f_1 \oplus f_2 \oplus \cdots$ from $\mathcal{H}$ to the orthogonal sum $\bigoplus_{k=1}^{\infty} \mathcal{M}_k$ and its inverse are bounded. Here, $f_k \in \mathcal{M}_k$ for every $k \in \mathbb{N}$.

We shall use the following result due to Vasyunin (for the proof see [12, pp. 279–287 (217–222 in English transl.)]).

Theorem 2.2. *Let θ and $\{\vartheta_k\}_{k=1}^{\infty}$ be inner functions such that $\theta = \prod_{k=1}^{\infty} \vartheta_k$ and denote $\theta_k = \theta/\vartheta_k$. Then the family $\mathcal{H}_k = \mathcal{H}_{\vartheta_k}$ is a Riesz basis in $\mathcal{H}_\theta$ if and only if*

$$\inf_k \inf_{z\in\mathbb{D}} (|\theta_k(z)| + |\vartheta_k(z)|) > 0.$$

2.2. Hyperreflexivity of a Jordan block: a sufficient condition. For any Blaschke product B with simple zeroes there exists a C_0 contraction T whose commutant $\mathcal{W}(T) = \{T\}'$ is hyperreflexive and whose minimal function is B. Namely, let T be the diagonal operator with the zeroes of

$B(z)$ along the diagonal. Then $\{T\}'$ is the algebra $\mathcal{D}$ of all diagonal operators and therefore, by [13, Theorem 3.5], $\kappa(T) \le 2$. The following theorem says that under a stronger condition on $B(z)$ the associated Jordan block is hyperreflexive.

THEOREM 2.3. *If $B(z)$ satisfies the Carleson condition, then $\mathcal{W}(S_B) = \{S_B\}'$ is hyperreflexive.*

Proof. We use the notation introduced in (2.1). As mentioned above, the family $\{r_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}_B$ because $B(z)$ satisfies the Carleson condition. Hence, there exists an invertible $T \in \mathcal{L}(\mathcal{H}_B)$ such that $e_n = Tr_n$ ($n \in \mathbb{N}$) is an orthonormal basis for $\mathcal{H}_B$. With respect to the basis $\{e_n\}_{n=1}^{\infty}$ the operator TS_BT^{-1} is diagonal with $\{\lambda_n\}_{n=1}^{\infty}$ along the diagonal. Since these numbers are distinct we have $\mathcal{W}(TS_BT^{-1}) = \{TS_BT^{-1}\}' = \mathcal{D}$, the algebra of all operators which are diagonal with respect to the basis $\{e_n\}_{n=1}^{\infty}$. By [13, Theorem 3.5], $\mathcal{D}$ is hyperreflexive with $\kappa(\mathcal{D}) \le 2$. By Proposition 1.1, $\kappa(\{S_B\}') = \kappa(S_B) \le 2\|T\|^2\|T^{-1}\|^2$. ∎

2.3. Hyperreflexivity of a Jordan block: a counterexample. Recall that the operator T constructed in Proposition 1.12 is not a C_0 contraction and satisfies $\dim \operatorname{Ran}(1 - T^*T) = \dim \operatorname{Ran}(1 - T^*T) = \infty$. On the other hand, $\dim \operatorname{Ran}(1 - S_\theta S_\theta^*) = \dim \operatorname{Ran}(1 - S_\theta^* S_\theta) = 1$ for any Jordan block S_θ. To improve the result of Proposition 1.12 we are going to show that there exists a Blaschke product $B(z)$ such that the Jordan block S_B is reflexive but not hyperreflexive. We will need several lemmas. The first one is proved in [12, p. 280 (218 in English transl.)].

LEMMA 2.4. *Let $B = \prod_{k=1}^{\infty} b_{\lambda_k}$ be a Blaschke product with simple zeroes satisfying the Carleson condition $\inf_k |B_{\lambda_k}(\lambda_k)| \ge \delta > 0$. If $|b_{\lambda_n}(\mu)| \le \delta/3$ for some $n \in \mathbb{N}$ and $\mu \in \mathbb{D}$, then $|B_{\lambda_n}(\mu)| \ge \delta/2$.*

Now, for Blaschke factors $b_\mu(z)$ and $b_\nu(z)$ ($\mu, \nu \in \mathbb{D}$), we estimate the norm $\|b_\mu - b_\nu\|_\infty$.

LEMMA 2.5. *Assume that $\mu = re^{i\varphi} \in \mathbb{D}$ and $\nu = (r+\varepsilon)e^{i\psi} \in \mathbb{D}$, where $0 < r \le r + \varepsilon < 1$ and $0 \le \psi \le \varphi$. Then*

$$\sup_{z \in \mathbb{D}} |b_\mu(z) - b_\nu(z)| \le \frac{(\varphi - \psi)(1 - r(r+\varepsilon)) + 2\varepsilon}{(1-r)(1-r-\varepsilon)}. \tag{2.2}$$

Proof. According to the maximum modulus principle,

$$\sup_{z \in \mathbb{D}} |b_\mu(z) - b_\nu(z)| = \max_{|z|=1} |b_\mu(z) - b_\nu(z)|.$$

Since

$$
\begin{aligned}
|b_\mu(z) - b_\nu(z)| &= \left| \frac{|\mu|}{\mu} \frac{z-\mu}{1-\overline{\mu}z} - \frac{|\nu|}{\nu} \frac{z-\nu}{1-\overline{\nu}z} \right| \\
&= \left| \frac{r}{re^{i\varphi}} \frac{z - re^{i\varphi}}{1 - re^{-i\varphi}z} - \frac{r+\varepsilon}{(r+\varepsilon)e^{i\psi}} \frac{z-(r+\varepsilon)e^{i\psi}}{1-(r+\varepsilon)e^{-i\psi}z} \right| \\
&= \frac{|z(e^{i\psi} - e^{i\varphi})(1 - r(r+\varepsilon)) + \varepsilon(e^{i(\varphi+\psi)} - z^2)|}{|1 - re^{-i\varphi}z|\,|1-(r+\varepsilon)e^{-i\psi}z|}
\end{aligned}
$$

and $|e^{i(\varphi+\psi)} - z^2| \le 2$, $|1 - re^{-i\varphi}z| \ge 1 - r$, $|1-(r+\varepsilon)e^{-i\psi}z| \ge 1 - r - \varepsilon$ for every z of modulus 1, we have

$$
\begin{aligned}
\sup_{z\in\mathbb{D}} |b_\mu(z) - b_\nu(z)| &= \max_{|z|=1} \frac{|z(e^{i\psi} - e^{i\varphi})(1 - r(r+\varepsilon)) + \varepsilon(e^{i(\varphi+\psi)} - z^2)|}{|1 - re^{-i\varphi}z|\,|1-(r+\varepsilon)e^{-i\psi}z|} \\
&\le \frac{|e^{i\varphi} - e^{i\psi}|(1 - r(r+\varepsilon)) + 2\varepsilon}{(1-r)(1-r-\varepsilon)}.
\end{aligned}
$$

As $|e^{i\varphi} - e^{i\psi}| = 2\sin\frac{\varphi-\psi}{2} \le \varphi - \psi$ we conclude that (2.2) holds. ∎

The following lemma plays a central role in our considerations.

LEMMA 2.6. *There exist two Blaschke sequences* $\{\mu_n\}_{n=1}^\infty, \{\nu_n\}_{n=1}^\infty \subset \mathbb{D}$ *and a number* $\delta > 0$ *such that*

(i) *the Blaschke products* $C(z) = \prod_{n=1}^\infty b_{\mu_n}$ *and* $D(z) = \prod_{n=1}^\infty b_{\nu_n}$ *satisfy the Carleson condition:* $\inf_n |C_{\mu_n}(\mu_n)| > \delta$ *and* $\inf_n |D_{\nu_n}(\nu_n)| > \delta$, *where* $C_{\mu_n} = C/b_{\mu_n}$ *and* $D_{\nu_n} = D/b_{\nu_n}$, *for every* $n \in \mathbb{N}$;

(ii) $\sup_{z\in\mathbb{D}} |b_{\mu_n}(z) - b_{\nu_n}(z)| \le \delta/4$ *for every* $n \in \mathbb{N}$;

(iii) $\lim_{n\to\infty} \|p_n - q_n\|_\infty = 0$, *where*

$$
(2.3)\qquad p_n(z) = \sqrt{1-|\mu_n|^2}\,\frac{C(z)D(z)}{z-\mu_n}, \qquad q_n(z) = \sqrt{1-|\nu_n|^2}\,\frac{C(z)D(z)}{z-\nu_n}.
$$

If $\{\mu_n\}_{n=1}^\infty, \{\nu_n\}_{n=1}^\infty \subset \mathbb{D}$ *are sequences which satisfy* (i)–(iii), *then*

(iv) $\inf_n \inf_{z\in\mathbb{D}}(|C_{\mu_n}(z)D_{\nu_n}(z)| + |b_{\mu_n}(z)b_{\nu_n}(z)|) > (\delta/6)^4$.

Proof. Consider a sequence $\{\mu_n\}_{n=1}^\infty$ such that $1 - |\mu_{n+1}| < q(1-|\mu_n|)$ for some $0 < q < 1$. By Lemma 2.1 there exists $\delta = \delta(q) > 0$ such that $\inf_k |C_{\mu_k}(\mu_k)| > \delta$. For every $n \in \mathbb{N}$, we denote $r_n = |\mu_n|$ and $\varphi_n = \arg(\mu_n) \in [0, 2\pi)$. Choose

$$
(2.4)\qquad \psi_n \in \left(0, \frac{\delta}{2^{n+3}}(1-|\mu_n|)^2\right) \quad \text{and set} \quad \nu_n = \mu_n e^{i\psi_n} = r_n e^{i(\varphi_n+\psi_n)}
$$

for $n \in \mathbb{N}$. It is obvious that $\{\nu_n\}_{n=1}^\infty$ is a Blaschke sequence satisfying $|\nu_n| = |\mu_n| = r_n$. Again by Lemma 2.1, $\inf_k |D_{\nu_k}(\nu_k)| > \delta$ and (i) is fulfilled.

Since $|\mu_n - \nu_n| = |\nu_n|\,|e^{i\psi_n} - 1| = 2r_n \sin(\psi_n/2)$ we have

$$0 < |\mu_n - \nu_n| < \psi_n < \frac{\delta}{2^{n+3}}(1 - |\mu_n|)^2.$$

By Lemma 2.5,

$$\sup_{z\in\mathbb{D}} |b_{\mu_n}(z) - b_{\nu_n}(z)| \le \psi_n \,\frac{1+r_n}{1-r_n}$$

and therefore

$$\sup_{z\in\mathbb{D}} |b_{\mu_n}(z) - b_{\nu_n}(z)| < \frac{\delta}{2^{n+3}}(1-r_n)^2\,\frac{1+r_n}{1-r_n} < \frac{\delta}{2^{n+2}}(1-r_n) \le \frac{\delta}{4},$$

which means (ii) is also fulfilled.

For (iii), note first that

$$p_n(z) = \frac{\sqrt{1-r_n^2}}{e^{i(2\varphi_n+\psi)}}\,\frac{(z-\nu_n)C_{\mu_n}(z)D_{\nu_n}(z)}{(1-\overline{\mu}_n z)(1-\overline{\nu}_n z)},$$
$$q_n(z) = \frac{\sqrt{1-r_n^2}}{e^{i(2\varphi_n+\psi)}}\,\frac{(z-\mu_n)C_{\mu_n}(z)D_{\nu_n}(z)}{(1-\overline{\mu}_n z)(1-\overline{\nu}_n z)}.$$

Hence

$$\begin{aligned}\|p_n - q_n\|_\infty &= \sqrt{1-r_n^2}\sup_{z\in\mathbb{D}}\left|(\mu_n-\nu_n)\frac{C_{\mu_n}(z)D_{\nu_n}(z)}{(1-\overline{\mu}_n z)(1-\overline{\nu}_n z)}\right| \\ &\le \frac{\sqrt{1-r_n^2}}{(1-r_n)^2}\,|\mu_n-\nu_n|\sup_{z\in\mathbb{D}}|C_{\mu_n}(z)D_{\nu_n}(z)| \\ &< \frac{\sqrt{1-r_n^2}}{(1-r_n)^2}\,\frac{\delta}{2^{n+3}}(1-r_n)^2\|C_{\mu_n}(z)D_{\nu_n}(z)\|_\infty < \frac{\delta}{2^{n+3}},\end{aligned}$$

which gives (iii).

Now take any sequences $\{\mu_n\}_{n=1}^\infty, \{\nu_n\}_{n=1}^\infty \subset \mathbb{D}$ which satisfy (i)–(iii). For every $n \in \mathbb{N}$, let $U_n = \{z \in \mathbb{D};\ |b_{\mu_n}(z)| \le \delta/3\}$. Since $|C_{\mu_n}(\mu_n)| > \delta$ we have, by Lemma 2.4, $|C_{\mu_n}(z)| \ge \delta/2$ for every $z \in U_n$. Condition (ii) gives

$$|b_{\nu_n}(z)| \le |b_{\mu_n}(z)| + |b_{\mu_n}(z) - b_{\nu_n}(z)| < \frac{\delta}{3} + \frac{\delta}{4} = \frac{1}{3}\,\frac{7\delta}{4}$$

for every $z \in U_n$. It follows, by Lemma 2.4 again, that $|D_{\nu_n}(z)| \ge 7\delta/8$ for every $z \in U_n$. Thus we have

$$|C_{\mu_n}(z)D_{\nu_n}(z)|^2 \ge \frac{49}{256}\delta^4 \qquad (z \in U_n). \tag{2.5}$$

Assume now that $z \in \mathbb{D}\setminus U_n$. Then $|b_{\mu_n}(z)| > \delta/3$. Consequently, using inequality (ii), we obtain $|b_{\nu_n}(z)| \ge |b_{\mu_n}(z)| - |b_{\mu_n}(z) - b_{\nu_n}(z)| > \delta/3 - \delta/4 = \delta/12$. Hence

$$|b_{\mu_n}(z)b_{\nu_n}(z)|^2 > (\delta/6)^4 \qquad (z \in \mathbb{D}\setminus U_n). \tag{2.6}$$

Inequalities (2.5) and (2.6) together give

$$\inf_{z\in\mathbb{D}}(|b_{\mu_n}(z)b_{\nu_n}(z)|^2+|C_{\mu_n}(z)D_{\nu_n}(z)|^2)>(\delta/6)^4,$$

as required. ■

In the remainder of this paper we consider the Blaschke products from Lemma 2.6 and denote

$$\theta=CD,\ \vartheta_n=b_{\mu_n}b_{\nu_n},\ \theta_n=\theta/\vartheta_n,\ \mathcal{H}_n=\theta_n\mathcal{H}_{\vartheta_n},\ M_n=S_\theta|\mathcal{H}_n. \tag{2.7}$$

Each operator M_n acts on the two-dimensional space $\mathcal{H}_n=\bigvee\{p_n,q_n\}$ and it has two eigenvalues $\mu_n\neq\nu_n$. The corresponding unital eigenvectors are p_n and q_n given by (2.3). Hence $\mathcal{W}(M_n)=\{M_n\}'$ and, by [15], $\kappa(M_n)=1/\sin\varphi_n$, where φ_n is the angle between p_n and q_n. It is easy to show that $\sin\varphi_n\le\|p_n-q_n\|_\infty$. By Lemma 2.6(iii), $\lim_{n\to\infty}\|p_n-q_n\|_\infty=0$, and consequently

$$\kappa(M_n)\xrightarrow{n\to\infty}\infty. \tag{2.8}$$

Now we can improve Proposition 1.12.

Proposition 2.7. *There exists a C_0 contraction T which is reflexive but not hyperreflexive.*

Proof. Let $\mathcal{H}=\bigoplus_{n=1}^\infty\mathcal{H}_n$ and $T=\bigoplus_{n=1}^\infty M_n$. Then T is a C_0 contraction and its minimal function is θ, i.e., a Blaschke product having only simple zeroes. It is well known that then T and $\{T\}'$ are reflexive (see, e.g., [1, 8]). According to (2.8) and Corollary 1.5 the operator T is not hyperreflexive. ■

The following theorem is the main result of this section.

Theorem 2.8. *There exists a Blaschke product such that the associated Jordan block is reflexive but not hyperreflexive.*

Proof. The function θ defined in (2.7) is a Blaschke product with simple zeroes, which means that the operator S_θ is reflexive. Observe that, for each $n\in\mathbb{N}$, multiplication by the inner function θ_n is a unitary operator from $\mathcal{H}_{\vartheta_n}$ onto $\mathcal{H}_n=\theta\mathcal{H}_{\vartheta_n}$. Therefore, by Theorem 2.2 and Lemma 2.6(iv), the family $\{\mathcal{H}_n\}_{n=1}^\infty$ forms a Riesz basis for $\mathcal{H}_\theta$. Hence S_θ is similar to

$$T=\bigoplus_{n=1}^\infty M_n,$$

which, by (2.8) and Corollary 1.5 is not hyperreflexive. By Proposition 1.1, S_θ is not hyperreflexive. ■

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Roman V. Bessonov
Steklov Institute of Mathematics
of Russian Academy of Sciences
St. Petersburg Department
E-mail: bessonov@pdmi.ras.ru

Janko Bračič
IMFM
University of Ljubljana
Jadranska 19
1000 Ljubljana, Slovenia
E-mail: janko.bracic@fmf.uni-lj.si

Michal Zajac
Department of Mathematics
Slovak University of Technology
SK-812 19 Bratislava, Slovakia
E-mail: michal.zajac@stuba.sk

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