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M-ideals of homogeneous polynomials

by

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Dedicated to the memory of Nigel Kalton

Abstract. We study the problem of whether $\mathcal{P}_w(^nE)$, the space of n-homogeneous polynomials which are weakly continuous on bounded sets, is an M-ideal in the space $\mathcal{P}(^nE)$ of continuous n-homogeneous polynomials. We obtain conditions that ensure this fact and present some examples. We prove that if $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$, then $\mathcal{P}_w(^nE)$ coincides with $\mathcal{P}_{w0}(^nE)$ (n-homogeneous polynomials that are weakly continuous on bounded sets at 0). We introduce a polynomial version of property (M) and derive that if $\mathcal{P}_w(^nE) = \mathcal{P}_{w0}(^nE)$ and $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$, then $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$. We also show that if $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$, then the set of n-homogeneous polynomials whose Aron–Berner extension does not attain its norm is nowhere dense in $\mathcal{P}(^nE)$. Finally, we discuss an analogous M-ideal problem for block diagonal polynomials.

Introduction. In the theory of Banach spaces, the concept of M-ideal, since its introduction by Alfsen and Effros [5], has proved to be an important tool in the study of geometric and isometric properties of the spaces. As stated in the exhaustive book on the topic written by Harmand, Werner and Werner [23],

> "The fact that Y is an M-ideal in X has a strong impact on both Y and X since there are a number of important properties shared by M-ideals, but not by arbitrary subspaces."

Since $\mathcal{K}(X, Y)$, the space of compact operators between Banach spaces X and Y, is a distinguished subspace of the space $\mathcal{L}(X, Y)$ of all bounded linear operators, many specialists have been interested in characterizing when $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$ and deriving consequences of this fact (see, for example, [23, 24, 26–29]).

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When changing to the polynomial setting, the role of compact operators is usually played by the homogeneous polynomials which are weakly continuous on bounded sets (that is, send bounded weak convergent nets into convergent nets). Recall that "compact linear operators" are the same as "linear operators weakly continuous on bounded sets". For polynomials, these two concepts do not coincide, because continuous polynomials are not necessarily weak-to-weak continuous. Moreover, every scalar valued continuous polynomial is compact but not every scalar valued continuous polynomial is weakly continuous on bounded sets. Many authors have studied for which values of n, for a fixed Banach space E, all the continuous n-homogeneous polynomials on E are weakly continuous on bounded sets (see, for instance, [4, 10, 13, 14, 18, 20, 21, 30, 31]). Here, assuming this is not the case, we are interested in determining whether there is an M-structure.

Let us recall the definition and some facts about M-ideals that we use. Naturally, our guide in this topic is the book [23].

Definition 0.1. A closed subspace J of a Banach space X is an *M-ideal* in X if

$$X^* = J^\perp \oplus_1 J^\sharp,$$

where $J^\perp$ is the annihilator of J, and $J^\sharp$ is a closed subspace of X^*.

When J is an M-ideal in X, then

$$J^\sharp = \{x^* \in X^* : \|x^*\| = \|x^*|_J\|\},$$

and so $J^\sharp$ can be (isometrically) identified with J^*. Thus, we usually write $X^* = J^\perp \oplus_1 J^*$. Another relevant result is that if J is an M-ideal in X, we have the following equality for the sets of extreme points of unit balls:

$$\operatorname{Ext} B_{X^*} = \operatorname{Ext} B_{J^\perp} \cup \operatorname{Ext} B_{J^*}.$$

Also, recall the 3-*ball property* [23, Theorem I.2.2(iv)] that we will use repeatedly: a closed subspace J is an M-ideal in X if and only if for every $y_1, y_2, y_3 \in B_J$, $x \in B_X$ and $\varepsilon > 0$, there exists $y \in J$ satisfying

$$\|x + y_i - y\| \leq 1 + \varepsilon, \quad i = 1, 2, 3.$$

Throughout this paper, E will denote a Banach space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The closed unit ball of E will be denoted by B_E and the unit sphere by S_E. As usual, $\mathcal{L}(E)$ and $\mathcal{K}(E)$ will stand for the spaces of continuous linear operators and compact operators from E to E. $\mathcal{P}(^n E)$ will denote the space of all continuous n-homogeneous polynomials from E to $\mathbb{K}$, which is a Banach space with the norm

$$\|P\| = \sup\{|P(x)| : x \in B_E\}.$$

If $P \in \mathcal{P}(^nE)$, there exists a unique symmetric n-linear mapping

$$P^\vee \colon \underbrace{E \times \cdots \times E}_{n} \to \mathbb{K}$$

such that

$$P(x) = P^\vee(x, \ldots, x).$$

For a polynomial $P \in \mathcal{P}(^nE)$, its Aron–Berner extension to E^{**} will be denoted by $\overline{P}$. Also, for each $z \in E^{**}$, e_z will refer to the map given by $e_z(P) = \overline{P}(z)$.

We will denote by $\mathcal{P}_w(^nE)$ the space of n-homogeneous polynomials on E that are weakly continuous on bounded sets (equivalently, send bounded weak convergent nets into convergent nets) and by $\mathcal{P}_{w0}(^nE)$ the space of n-homogeneous polynomials on E that are weakly continuous on bounded sets at 0 (equivalently, send bounded weakly null nets into null nets). Polynomials of the form $P = \sum_{j=1}^N \pm\gamma_j^n$ with $\gamma_j \in E^*$ are said to be *of finite type.* The space of finite type n-homogeneous polynomials on E will be denoted by $\mathcal{P}_f(^nE)$, and its closure (in the polynomial norm), which is the space of *approximable* n-homogeneous polynomials on E, will be denoted by $\mathcal{P}_A(^nE)$.

Recall that if E does not contain a subspace isomorphic to ℓ_1, then $\mathcal{P}_w(^nE)$ coincides with the space of weakly sequentially continuous n-homogeneous polynomials on E [9].

We refer to [16] for the necessary background on polynomials.

The paper is organized as follows. In Section 1 we present some general results and consequences of the fact that $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$. This enables us to conclude that, for each Banach space E, there is at most one value of n where the stated problem makes sense. We also obtain a Bishop–Phelps type result for polynomials. In Section 2 we give a set of conditions that ensure that $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$. This enables us to produce some nice examples where this happens. Section 3 is devoted to a polynomial version of property (M). With this property we can make a link with the linear theory by showing that if $\mathcal{P}_w(^nE) = \mathcal{P}_{w0}(^nE)$ and $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$, then $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$. In the last section we consider the space of block diagonal polynomials (with respect to a fixed sequence J of integers) that are defined on a Banach space E with an unconditional finite-dimensional decomposition. We study in this context the same question as before, that is, whether the space of block diagonal polynomials weakly continuous on bounded sets is an M-ideal in the space of all block diagonal polynomials.

1. General results. We begin by stating some easy results which are polynomial analogues to [23, Propositions VI.4.2 and VI.4.3]. Their proofs are straightforward.

First, recall that J is an *M-summand* in X if

$$X = J \oplus_\infty \widehat{J},$$

where $\widehat{J}$ is a closed subspace of X. Clearly M-summands are M-ideals.

Proposition 1.1. *If $\mathcal{P}_w({}^nE)$ is an M-summand in $\mathcal{P}({}^nE)$, then $\mathcal{P}_w({}^nE) = \mathcal{P}({}^nE)$.*

Proposition 1.2.

(a) *If $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$ and $F \subset E$ is a 1-complemented subspace, then $\mathcal{P}_w({}^nF)$ is an M-ideal in $\mathcal{P}({}^nF)$.*

(b) *The class of Banach spaces E for which $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$ is closed with respect to the Banach–Mazur distance.*

One useful tool when computing norms in Banach spaces is the characterization of the extreme points of dual unit balls. Let us state some facts about extreme points when the space considered is a space of polynomials.

First, note that if J is a subspace of $\mathcal{P}({}^nE)$ that contains $\mathcal{P}_f({}^nE)$, then, for every $x \in S_E$, the map e_x belongs to S_{J^*}. Indeed, it is clear that $e_x(P) = P(x)$ is a linear functional on J and that $\|e_x\| = \sup\{|e_x(P)| : P \in B_J\} \le 1$. Also, since J contains all finite type n-homogeneous polynomials, it contains γ^n for every $\gamma \in E^*$, and so $\|e_x\| = 1$.

Proposition 1.3.

(a) *If J is a subspace of $\mathcal{P}({}^nE)$ that contains all finite type n-homogeneous polynomials, then*

$$\operatorname{Ext} B_{J^*} \subset \overline{\{\pm e_x : x \in S_E\}}^{w^*},$$

where the $\pm$ is needed only in the real case and w^* is the topology $\sigma(J^*, J)$.

(b) *For the particular case $J = \mathcal{P}_w({}^nE)$ we can be more precise:*

$$\operatorname{Ext} B_{\mathcal{P}_w({}^nE)^*} \subset \{\pm e_z : z \in S_{E^{**}}\},$$

where the $\pm$ is needed only in the real case.

Proof. (a) We have

$$B_{J^*} = \overline{\Gamma\{\pm e_x : x \in S_E\}}^{w^*},$$

where Γ stands for the closed convex hull. Indeed, one inclusion follows from the comment before the proposition and the other is easily obtained through the Hahn–Banach theorem.

Now, Milman's theorem [17, Theorem 3.41] yields

$$\operatorname{Ext} B_{J^*} \subset \overline{\{\pm e_x : x \in S_E\}}^{w^*}.$$

(b) When $J = \mathcal{P}_w({}^nE)$, we will show that $\overline{\{\pm e_x : x \in S_E\}}^{w^*} \subset \{\pm e_z : z \in S_{E^{**}}\}$. Indeed, if $\Phi \in \overline{\{\pm e_x : x \in S_E\}}^{w^*}$, then there exists a net $\{x_\alpha\}_\alpha$ in S_E such that $e_{x_\alpha} \xrightarrow{w^*} \Phi$ (or $-e_{x_\alpha} \xrightarrow{w^*} \Phi$). Passing to appropriate subnets, we can suppose that $\{x_\alpha\}_\alpha$ is w^*-convergent to an element z in $S_{E^{**}}$ (here, w^* means $\sigma(E^{**}, E^*)$). Since the Aron–Berner extension of a polynomial weakly continuous on bounded sets is w^*-continuous on bounded sets, we derive that $\overline{P}(x_\alpha) \to \overline{P}(z)$ for every $P \in \mathcal{P}_w({}^nE)$. Thus, $e_{x_\alpha} \xrightarrow{w^*} e_z$. Therefore, $\Phi = e_z$. ∎

Some observations are in order.

Remark 1.4.

(1) For the particular case $J = \mathcal{P}({}^nE)$, the previous result can also be proved using the representation of $\mathcal{P}({}^nE)$ as the dual of $\widehat{\otimes}^n_{\pi_s} E$ (the symmetric projective n-tensor product of E), the description of the unit ball of this space given in [19] and Goldstine's theorem.
(2) When E^* has the approximation property, item (b) of the proposition above can also be obtained by another argument. Indeed, in this case $\mathcal{P}_w({}^nE) = \mathcal{P}_A({}^nE)$, and the dual of this space is the space of integral n-homogeneous polynomials on E^*: $\mathcal{P}_w({}^nE)^* = \mathcal{P}_I({}^nE^*)$. By the description of the set of extreme points of the ball of the space of integral polynomials in [11, Proposition 1] (see also [12, proof of Theorem 1.5]), the result follows.

The essential norm of a linear operator is the distance to the subspace of compact operators. Analogously, we define:

Definition 1.5. For an n-homogeneous polynomial $P \in \mathcal{P}({}^nE)$, the *essential norm* of P is

$$\|P\|_{\mathrm{es}} = d(P, \mathcal{P}_w({}^nE)) = \inf\{\|P - Q\| : Q \in \mathcal{P}_w({}^nE)\}.$$

The following result, which is the polynomial version of [23, Proposition VI.4.7], has an important consequence stated in the corollary below.

Proposition 1.6. *For any $P \in \mathcal{P}({}^nE)$, set*

$$w(P) = \sup\{\limsup |P(x_\alpha)| : \|x_\alpha\| = 1,\ x_\alpha \xrightarrow{w} 0\}.$$

If $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$, then $\|P\|_{\mathrm{es}} = w(P)$.

Proof. Let $Q \in \mathcal{P}_w({}^nE)$ and let $\{x_\alpha\}_\alpha$ be a weakly null net with $\|x_\alpha\| = 1$ for all α. Then

$$\|P - Q\| \ge |(P - Q)(x_\alpha)| \ge |P(x_\alpha)| - |Q(x_\alpha)|.$$

Since $Q(x_\alpha) \to 0$ it follows that $\|P - Q\| \ge \limsup |P(x_\alpha)|$ and thus $\|P\|_{\mathrm{es}} \ge w(P)$.

Now suppose that $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$. Then

$$\mathcal{P}({}^nE)^* = \mathcal{P}_w({}^nE)^\perp \oplus_1 \mathcal{P}_w({}^nE)^*,$$
$$\operatorname{Ext} B_{\mathcal{P}({}^nE)^*} = \operatorname{Ext} B_{\mathcal{P}_w({}^nE)^\perp} \cup \operatorname{Ext} B_{\mathcal{P}_w({}^nE)^*}.$$

The essential norm of P, $\|P\|_{\mathrm{es}}$, is the norm of the class of P in the quotient space $\mathcal{P}({}^nE)/\mathcal{P}_w({}^nE)$, and the dual of this quotient can be isometrically identified with $\mathcal{P}_w({}^nE)^\perp$. Thus, there exists $\Phi \in \operatorname{Ext} B_{\mathcal{P}_w({}^nE)^\perp}$ such that $\Phi(P) = \|P\|_{\mathrm{es}}$. So, $\Phi \in \operatorname{Ext} B_{\mathcal{P}({}^nE)^*}$ and, by Proposition 1.3(a), $\Phi \in \overline{\{\pm e_x : x \in S_E\}}^{w^*}$.

Consequently, there exists a net $\{x_\alpha\}_\alpha$ in S_E such that $e_{x_\alpha} \xrightarrow{w^*} \Phi$ (or $-e_{x_\alpha} \xrightarrow{w^*} \Phi$), where w^* means the topology $\sigma(\mathcal{P}({}^nE)^*, \mathcal{P}({}^nE))$. Passing to appropriate subnets, we can suppose that $\{x_\alpha\}_\alpha$ is w^*-convergent to an element z in $S_{E^{**}}$ (here, w^* means $\sigma(E^{**}, E^*)$).

If $\gamma \in E^*$, since $\gamma^n \in \mathcal{P}_w({}^nE)$ we obtain

$$0 = \Phi(\gamma^n) = \lim_\alpha \gamma(x_\alpha)^n = z(\gamma)^n.$$

Thus we have proved that $z(\gamma) = 0$ for all $\gamma \in E^*$, which implies that $z = 0$ and that $\{x_\alpha\}_\alpha$ is weakly null. As a consequence,

$$\|P\|_{\mathrm{es}} = \Phi(P) = \lim_\alpha P(x_\alpha) \le w(P),$$

which completes the proof. ∎

Since for every polynomial P which is weakly continuous on bounded sets at 0 we have $w(P) = 0$, and the equality $\|P\|_{\mathrm{es}} = 0$ implies that P is weakly continuous on bounded sets, we obtain:

Corollary 1.7. *If $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$, then $\mathcal{P}_w({}^nE) = \mathcal{P}_{w0}({}^nE)$.*

This corollary tells us that we have at most one value of n (which we call the "critical degree") where $\mathcal{P}_w({}^nE)$ could be a nontrivial M-ideal in $\mathcal{P}({}^nE)$. Indeed, recall the following simple facts, whose proofs appeared in (or can be derived from) [10, 7]:

- If an n-homogeneous polynomial P is weakly continuous on bounded sets at any point $x \in E$, then it is weakly continuous on bounded sets at 0.
- If $\mathcal{P}_w({}^kE) = \mathcal{P}({}^kE)$ for all $1 \le k < n$, then $\mathcal{P}_w({}^nE) = \mathcal{P}_{w0}({}^nE)$.
- If there exists an n-homogeneous polynomial P which is not weakly continuous on bounded sets at a point $x \in E$, $x \ne 0$, then for every $\gamma \in E^*$ such that $\gamma(x) \ne 0$, the $(n+k)$-homogenous polynomial $Q = \gamma^k P$ belongs to $\mathcal{P}_{w0}({}^{n+k}E) \setminus \mathcal{P}_w({}^{n+k}E)$.

So, the situation can be summarized as follows:

REMARK 1.8. For a Banach space E, either $\mathcal{P}_w({}^kE) = \mathcal{P}_{w0}({}^kE) = \mathcal{P}({}^kE)$ for all k, or there exists $n \in \mathbb{N}$ such that:

- $\mathcal{P}_w({}^kE) = \mathcal{P}_{w0}({}^kE) = \mathcal{P}({}^kE)$ for all $k < n$.
- $\mathcal{P}_w({}^nE) = \mathcal{P}_{w0}({}^nE) \subsetneq \mathcal{P}({}^nE)$.
- $\mathcal{P}_w({}^kE) \subsetneq \mathcal{P}_{w0}({}^kE) \subset \mathcal{P}({}^kE)$ for all $k > n$.

When this value of n does exist, we call it the *critical degree* of E and write $n = \mathrm{cd}(E)$.

Therefore, if there exists a polynomial on E which is not weakly continuous on bounded sets, the critical degree is the minimum k such that $\mathcal{P}_w({}^kE) \neq \mathcal{P}({}^kE)$.

REMARK 1.9. If E is a reflexive Banach space with the approximation property, then $\mathcal{P}_w({}^kE)^{**} = \mathcal{P}({}^kE)$ for every k. When this is the case, the problem that we are studying here is whether $\mathcal{P}_w({}^nE)$ is an M-ideal in its bidual.

We finish this section by a polynomial version of [23, Proposition VI.4.8]. This result can be related to extensions of the Bishop–Phelps theorem to the polynomial setting. There is no polynomial (or multilinear) Bishop–Phelps theorem [2, 1], but there are some variations that are valid. Aron, García and Maestre [8] proved that the set of 2-homogeneous polynomials whose Aron–Berner extension attains its norm is dense in the set of 2-homogeneous polynomials. It is an open problem whether this result can be generalized to n-homogeneous polynomials. Here, with the very strong hypothesis of $\mathcal{P}_w({}^nE)$ being an M-ideal in $\mathcal{P}({}^nE)$, we obtain a stronger conclusion.

PROPOSITION 1.10. *Let E be a Banach space and suppose that $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$.*

(a) *If $P \in \mathcal{P}({}^nE)$ is such that its Aron–Berner extension $\overline{P}$ does not attain its norm in $B_{E^{**}}$, then $\|P\| = \|P\|_{\mathrm{es}}$.*
(b) *The set of polynomials in $\mathcal{P}({}^nE)$ whose Aron–Berner extension does not attain its norm is nowhere dense in $\mathcal{P}({}^nE)$.*

Proof. (a) Let $\Phi \in \operatorname{Ext} B_{\mathcal{P}({}^nE)^*}$ be such that $\|P\| = \Phi(P)$. Since $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$, it follows that $\Phi \in \operatorname{Ext} B_{\mathcal{P}_w({}^nE)^*}$ or $\Phi \in \operatorname{Ext} B_{\mathcal{P}_w({}^nE)^\perp}$. If $\Phi \in \operatorname{Ext} B_{\mathcal{P}_w({}^nE)^*}$ then, by Proposition 1.3(b), $\Phi = \pm e_z$ for some $z \in E^{**}$ with $\|z\| = 1$. Then

$$\|\overline{P}\| = \|P\| = \Phi(P) = |\overline{P}(z)|,$$

which is a contradiction. Thus, $\Phi \in \operatorname{Ext} B_{\mathcal{P}_w({}^nE)^\perp}$. Consequently,

$$\|P\| = \Phi(P) = \sup\{|\Psi(P)| : \Psi \in \operatorname{Ext} B_{\mathcal{P}_w({}^nE)^\perp}\} = \|P\|_{\mathrm{es}}.$$

(b) By (a), the set of polynomials in $\mathcal{P}(^nE)$ whose Aron–Berner extension does not attain its norm is contained in the metric complement

$$\mathcal{P}_w(^nE)^{\Theta} = \{P \in \mathcal{P}(^nE) : \|P\| = \|P\|_{\mathrm{es}}\}.$$

Since this set is closed, we have to prove that it has empty interior. By [23, Proposition II.1.11 and Corollary II.1.7], $\mathcal{P}_w(^nE)^{\Theta}$ has empty interior if and only if

$$\inf\Big\{\sup_{\Phi \in B_{\mathcal{P}_w(^nE)^*}} |\langle \Phi, P\rangle| : \|P\|_{\mathrm{es}} = 1\Big\} = 1.$$

But

$$\begin{aligned}\sup_{\Phi \in B_{\mathcal{P}_w(^nE)^*}} |\langle \Phi, P\rangle| &\le \sup_{\Phi \in B_{\mathcal{P}(^nE)^*}} |\langle \Phi, P\rangle| = \|P\| = \sup_{x \in B_E} |P(x)| \\ &= \sup_{x \in B_E} |\langle e_x, P\rangle| \le \sup_{\Phi \in B_{\mathcal{P}_w(^nE)^*}} |\langle \Phi, P\rangle|.\end{aligned}$$

Thus,

$$\inf\Big\{\sup_{\Phi \in B_{\mathcal{P}_w(^nE)^*}} |\langle \Phi, P\rangle| : \|P\|_{\mathrm{es}} = 1\Big\} = \inf\{\|P\| : \|P\|_{\mathrm{es}} = 1\} = 1,$$

since $\|P\|_{\mathrm{es}} \le \|P\|$ for all P, with equality for $P \in \mathcal{P}_w(^nE)^{\Theta}$. ∎

2. Compact approximations. Several criteria for $\mathcal{K}(X,Y)$ to be an M-ideal in $\mathcal{L}(X,Y)$ were related to the so-called "shrinking compact approximations of the identity" satisfying certain properties. In order to obtain examples of Banach spaces E such that $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$, we present a sufficient condition for this to happen, also involving nets of compact operators. Due to Corollary 1.7 and Remark 1.8, we have to add the hypothesis of $n = \mathrm{cd}(E)$. To prove our result we need the following simple lemma.

LEMMA 2.1. *Let E be a Banach space and suppose that there exists a bounded net $\{S_\alpha\}_\alpha$ of linear operators from E to E satisfying $S_\alpha^*\gamma \to \gamma$ for all $\gamma \in E^*$. Then $\|P - P \circ S_\alpha\| \to 0$ for all $P \in \mathcal{P}_w(^nE)$.*

Proof. Since $\{S_\alpha^*\}_\alpha$ is a bounded net it follows that $S_\alpha^*\gamma \to \gamma$ uniformly for γ in a relatively compact set. Thus, for every Banach space F and every compact operator $K \in \mathcal{K}(E,F)$, it is clear that

$$\|K - K \circ S_\alpha\| \to 0.$$

It is known [9] that for $P \in \mathcal{P}_w(^nE)$, the associated linear operator T_P belongs to $\mathcal{K}(E, \mathcal{L}_s(^{n-1}E))$ (where $\mathcal{L}_s(^{n-1}E)$ denotes the space of symmetric $(n-1)$-linear forms on E). If C is a bound for $\|S_\alpha\|$, we obtain

$$\begin{aligned}|P(x)-P\circ S_\alpha(x)| &= \Big|\sum_{j=1}^{n}\binom{n}{j}P^\vee\big((x-S_\alpha(x))^j, S_\alpha(x)^{n-j}\big)\Big|\\ &\le \sum_{j=1}^{n}\binom{n}{j}\Big|T_P(x-S_\alpha(x))\big((x-S_\alpha(x))^{j-1}, S_\alpha(x)^{n-j}\big)\Big|\\ &\le \sum_{j=1}^{n}\binom{n}{j}\|T_P-T_P\circ S_\alpha\|\,\|I-S_\alpha\|^{j-1}\|S_\alpha\|^{n-j}\|x\|^n\\ &\le \sum_{j=1}^{n}\binom{n}{j}\|T_P-T_P\circ S_\alpha\|(1+C)^{j-1}C^{n-j}\|x\|^n.\end{aligned}$$

Consequently, $\|P-P\circ S_\alpha\|\to 0$. ∎

PROPOSITION 2.2. *Let E be a Banach space and let $n=\mathrm{cd}(E)$. Suppose that there exists a bounded net $\{K_\alpha\}_\alpha$ of compact operators from E to E satisfying the following two conditions:*

- *$K_\alpha^*\gamma\to\gamma$ for all $\gamma\in E^*$.*
- *For all $\varepsilon>0$ and all α_0 there exists $\alpha>\alpha_0$ such that for every $x\in E$,*

$$\|K_\alpha x\|^n+\|x-K_\alpha x\|^n\le(1+\varepsilon)\|x\|^n.$$

Then $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$.

Proof. Let $P_1,P_2,P_3\in B_{\mathcal{P}_w({}^nE)}$, $Q\in B_{\mathcal{P}({}^nE)}$ and $\varepsilon>0$. In order to verify the 3-ball property [23, Theorem I.2.2], we have to show that there exists a polynomial $P\in\mathcal{P}_w({}^nE)$ such that

$$\|Q+P_i-P\|\le 1+\varepsilon\qquad\text{for } i=1,2,3. \tag{2.1}$$

Since, by the previous lemma, $\|P_i-P_i\circ K_\alpha\|\to 0$ for $i=1,2,3$, let us fix a value of α such that

$$\|P_i-P_i\circ K_\alpha\|\le\varepsilon/2\qquad\text{for } i=1,2,3$$

and also

$$\|K_\alpha x\|^n+\|x-K_\alpha x\|^n\le(1+\varepsilon/2)\|x\|^n.$$

Consider the polynomial $P\in\mathcal{P}({}^nE)$ given by

$$P(x)=Q(x)-Q(x-K_\alpha x).$$

We have to prove that P is weakly continuous on bounded sets and that it satisfies inequality (2.1).

Let $\{x_\beta\}_\beta$ be a bounded weakly null net. If we show that $P(x_\beta)\to 0$, then P will be weakly continuous on bounded sets (since $\mathcal{P}_w({}^nE)=\mathcal{P}_{w0}({}^nE)$). By the compactness of K_α, we know that $K_\alpha x_\beta\to 0$ as $\beta\to\infty$, and so

$$
\begin{aligned}
|P(x_\beta)| &= |Q(x_\beta) - Q(x_\beta - K_\alpha x_\beta)| \\
&\le \sum_{j=1}^{n} \binom{n}{j} \big|Q^\vee\big((K_\alpha x_\beta)^j, (x_\beta - K_\alpha x_\beta)^{n-j}\big)\big| \\
&\le \sum_{j=1}^{n} \binom{n}{j} \|Q^\vee\|\, \|K_\alpha x_\beta\|^j \|x_\beta - K_\alpha x_\beta\|^{n-j} \\
&\le \sum_{j=1}^{n} \binom{n}{j} \|Q^\vee\|\, \|K_\alpha x_\beta\|^j ((1+C_1)C_2)^{n-j} \underset{\beta}{\to} 0,
\end{aligned}
$$

where C_1 and C_2 are bounds for the nets $\{K_\alpha\}_\alpha$ and $\{x_\beta\}_\beta$, respectively.

Now (2.1) follows from the inequalities

$$
\begin{aligned}
\|Q + P_i - P\| &\le \|Q + P_i \circ K_\alpha - P\| + \|P_i - P_i \circ K_\alpha\| \\
&\le \|Q + P_i \circ K_\alpha - P\| + \varepsilon/2
\end{aligned}
$$

and

$$
\begin{aligned}
\|Q + P_i \circ K_\alpha - P\| &= \sup_{x \in B_E} |Q(x - K_\alpha x) + P_i(K_\alpha x)| \\
&\le \sup_{x \in B_E} (\|K_\alpha x\|^n + \|x - K_\alpha x\|^n) \\
&\le \sup_{x \in B_E} (1 + \varepsilon/2)\|x\|^n = 1 + \varepsilon/2. \ \blacksquare
\end{aligned}
$$

Remark 2.3. In [29], Oja and Werner introduced the concept of (M_p)-space as a space X such that $\mathcal{K}(X \oplus_p X)$ is an M-ideal in $\mathcal{L}(X \oplus_p X)$. By [23, Theorem VI.5.3], for $p \le n$, every (M_p)-space fulfils the conditions (on the net of compact operators) of Proposition 2.2.

For spaces with a shrinking finite-dimensional decomposition we have the following simpler version of Proposition 2.2.

Corollary 2.4. *Let E be a Banach space and let $n = \mathrm{cd}(E)$. Suppose that E has a shrinking finite-dimensional decomposition with associated projections $\{\pi_m\}_m$ such that:*

- *For all $\varepsilon > 0$ and $m_0 \in \mathbb{N}$ there exists $m > m_0$ such that for every $x \in E$,*

$$\|\pi_m x\|^n + \|x - \pi_m x\|^n \le (1+\varepsilon)\|x\|^n.$$

Then $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$.

From Proposition 2.2, Corollary 2.4 and Remark 1.8, we can derive the following examples.

Example 2.5. If H is a Hilbert space, then $\mathcal{P}(^2H) \ne \mathcal{P}_w(^2H)$, because if $\{e_\alpha\}_\alpha$ is an orthonormal basis then the polynomial

$$P(x) = \sum_{\alpha} \langle x, e_\alpha \rangle^2$$

is not weakly continuous on bounded sets. So the critical degree is $n = 2$ and since it is clear that the hypotheses of Proposition 2.2 are valid, we conclude that $\mathcal{P}_w(^2H)$ is an M-ideal in $\mathcal{P}(^2H)$.

EXAMPLE 2.6. Let $E = \bigoplus_{\ell_p} X_m$, where each X_m is a finite-dimensional space and $1 < p < \infty$. Then $\mathcal{P}(^kE) = \mathcal{P}_w(^kE)$ if and only if $k < p$. This means that the critical degree is the number n that satisfies $p \le n < p+1$. It is obvious that E satisfies the hypotheses of Corollary 2.4, and thus $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$.

In particular, we have the result for ℓ_p spaces: $\mathcal{P}_w(^n\ell_p)$ is an M-ideal in $\mathcal{P}(^n\ell_p)$ for $p \le n < p+1$.

EXAMPLE 2.7. Let us consider the dual of a Lorentz sequence space, $E = d^*(w,p)$ with $1 < p < \infty$. Our result works for certain sequences w. If $n-1$ is the greatest integer strictly smaller than p^* (the conjugate of p), suppose that $w \notin \ell_s$, where $s = ((n-1)^*/p)^*$. Then, by [25, Proposition 2.4], $n = \operatorname{cd}(E)$. We deduce that $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$. Indeed, $d^*(w,p)$ has a shrinking Schauder basis $\{e_j\}_j$ and if π_m is the projection $\pi_m(x) = \sum_{j=1}^m x_j e_j$, then

$$\|\pi_m x\|^n + \|x - \pi_m x\|^n \le (\|\pi_m x\|^{p^*} + \|x - \pi_m x\|^{p^*})^{n/p^*} \le \|x\|^n.$$

The last inequality follows by duality, since if y and z are disjointly supported vectors in $d(w,p)$, then

$$\|y + z\| \le (\|x\|^p + \|y\|^p)^{1/p}.$$

Observe that, in particular, we have proved that for $p \ge 2$, $\mathcal{P}_w(^2d^*(w,p))$ is an M-ideal in $\mathcal{P}(^2d^*(w,p))$ for any sequence w (the above condition in this case is $w \notin \ell_1$, which is implied by the definition of $d^*(w,p)$).

EXAMPLE 2.8. Let $1 < p < 2$ and consider the space $L_p[0,1]$. Since $L_p[0,1]$ contains a complemented subspace isomorphic to ℓ_2, it follows that $\mathcal{P}(^2L_p[0,1]) \ne \mathcal{P}_w(^2L_p[0,1])$ and $n = 2$ is the critical degree. Even though we will see in Example 3.8 that $\mathcal{P}_w(^2L_p[0,1])$ is not an M-ideal in $\mathcal{P}(^2L_p[0,1])$, the space $L_p[0,1]$ can be renormed to a Banach space E such that $\mathcal{P}_w(^2E)$ is an M-ideal in $\mathcal{P}(^2E)$. Indeed, the renorming considered in [23, Proposition 6.8] satisfies all the conditions of Corollary 2.4.

REMARK 2.9. The spaces E of the previous examples are all reflexive with the approximation property. So, by Remark 1.9, the corresponding spaces $\mathcal{P}_w(^nE)$ are M-embedded.

3. Polynomial property (M)

LEMMA 3.1. *If $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$, then for each $P \in \mathcal{P}({}^nE)$ there exists a bounded net $\{P_\alpha\}_\alpha \subset \mathcal{P}_w({}^nE)$ such that*

$$\overline{P}_\alpha(z) \to \overline{P}(z) \quad \textit{for all } z \in E^{**}.$$

Proof. By [23, Remark I.1.13], if $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$ then $B_{\mathcal{P}_w({}^nE)}$ is $\sigma(\mathcal{P}({}^nE), \mathcal{P}_w({}^nE)^*)$-dense in $B_{\mathcal{P}({}^nE)}$. So, for each $P \in B_{\mathcal{P}({}^nE)}$ there exists a net $\{P_\alpha\}_\alpha \subset B_{\mathcal{P}_w({}^nE)}$ such that $P_\alpha \to P$ in the topology $\sigma(\mathcal{P}({}^nE), \mathcal{P}_w({}^nE)^*)$.

Note that $\mathcal{P}_w({}^nE)^*$ can be viewed as a subset of $\mathcal{P}({}^nE)^*$ by identification with the set

$$\mathcal{P}_w({}^nE)^\sharp = \{\Phi \in \mathcal{P}({}^nE)^* : \|\Phi\| = \|\Phi|_{\mathcal{P}_w({}^nE)}\|\}.$$

Since $\|e_z\| = \|e_z|_{\mathcal{P}_w({}^nE)}\|$, this implies that $\overline{P}_\alpha(z) \to \overline{P}(z)$ for all $z \in E^{**}$. ■

As a consequence of [32, Proposition 2.3] and the previous lemma, we have the following result which can be proved analogously to [32, Theorem 3.1]:

THEOREM 3.2. *Let E be a Banach space. The following are equivalent:*

(i) *$\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$.*
(ii) *For all $P \in \mathcal{P}({}^nE)$ there exists a net $\{P_\alpha\}_\alpha \subset \mathcal{P}_w({}^nE)$ such that $\overline{P}_\alpha(z) \to \overline{P}(z)$ for all $z \in E^{**}$ and*

$$\limsup_\alpha \|Q + P - P_\alpha\| \le \max\{\|Q\|, \|Q\|_{\mathrm{es}} + \|P\|\} \quad \textit{for all } Q \in \mathcal{P}({}^nE).$$

(iii) *For all $P \in \mathcal{P}({}^nE)$ there exists a net $\{P_\alpha\}_\alpha \subset \mathcal{P}_w({}^nE)$ such that $\overline{P}_\alpha(z) \to \overline{P}(z)$ for all $z \in E^{**}$ and*

$$\limsup_\alpha \|Q + P - P_\alpha\| \le \max\{\|Q\|, \|P\|\} \quad \textit{for all } Q \in \mathcal{P}_w({}^nE).$$

Property (M), introduced by Kalton in [26], proves to be useful to characterize the spaces X such that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$. Recall one of its equivalent definitions [23, Lemma VI.4.13]:

DEFINITION 3.3. A Banach space E has *property* (M) if whenever $u, v \in E$ with $\|u\| \le \|v\|$ and $\{x_\alpha\}_\alpha \subset E$ is a bounded weakly null net, then

$$\limsup_\alpha \|u + x_\alpha\| \le \limsup_\alpha \|v + x_\alpha\|.$$

In [27] an operator version of this property is introduced to study when $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$. Here we propose a polynomial version of property (M).

DEFINITION 3.4. Let $P \in \mathcal{P}(^nE)$ with $\|P\| \le 1$. We say that P has *property* (M) if for all $\lambda \in \mathbb{K}$ and $v \in E$ with $|\lambda| \le \|v\|^n$, and for every bounded weakly null net $\{x_\alpha\}_\alpha \subset E$,

$$\limsup_\alpha |\lambda + P(x_\alpha)| \le \limsup_\alpha \|v + x_\alpha\|^n.$$

Analogously to [27, Lemma 6.2], we can prove:

LEMMA 3.5. *Let $P \in \mathcal{P}(^nE)$ with $\|P\| \le 1$. If P has property (M) then for every net $\{v_\alpha\}_\alpha$ contained in a compact subset of E, for every net $\{\lambda_\alpha\}_\alpha \subset \mathbb{K}$ with $|\lambda_\alpha| \le \|v_\alpha\|^n$, and for every bounded weakly null net $\{x_\alpha\}_\alpha \subset E$,*

$$\limsup_\alpha |\lambda_\alpha + P(x_\alpha)| \le \limsup_\alpha \|v_\alpha + x_\alpha\|^n.$$

DEFINITION 3.6. We say that a Banach space E has the *n-polynomial property* (M) if every $P \in \mathcal{P}(^nE)$ with $\|P\| \le 1$ has property (M).

The following proposition and theorem are the polynomial versions of [27, Theorem 6.3] and their proofs follow the ideas of the proof of that theorem, with the necessary changes.

PROPOSITION 3.7. *If $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$ then E has the n-polynomial property (M).*

Proof. Let $P \in \mathcal{P}(^nE)$ with $\|P\| \le 1$, and pick $\lambda \in \mathbb{K}$ and $v \in E$ with $|\lambda| \le \|v\|^n$. Consider a bounded weakly null net $\{x_\alpha\}_\alpha \subset E$. Take $Q \in \mathcal{P}_w(^nE)$ such that $\|Q\| \le 1$ and $Q(v) = \lambda$. Given $\varepsilon > 0$, by Theorem 3.2(iii), there exists a polynomial $R \in \mathcal{P}_w(^nE)$ such that

$$|P(v) - R(v)| < \varepsilon \quad \text{and} \quad \|Q + P - R\| \le 1 + \varepsilon.$$

Since $Q(v + x_\alpha) \to Q(v)$ and $R(x_\alpha) \to 0$ we obtain

$$\begin{aligned}\limsup_\alpha |\lambda + P(x_\alpha)| &= \limsup_\alpha |Q(v) + P(x_\alpha)| \\ &= \limsup_\alpha |Q(v + x_\alpha) + (P - R)(x_\alpha)| \\ &\le \limsup_\alpha |Q(v + x_\alpha) + (P - R)(x_\alpha) + (P - R)(v)| + \varepsilon.\end{aligned}$$

Recall that $\mathcal{P}_w(^nE)$ being an M-ideal in $\mathcal{P}(^nE)$ implies that for every $1 \le k \le n-1$, all the polynomials in $\mathcal{P}(^kE)$ are weakly continuous on bounded sets. So we derive that

$$\begin{aligned}&|(P - R)(v + x_\alpha) - [(P - R)(v) + (P - R)(x_\alpha)]| \\ &\qquad = \Big|\sum_{j=1}^{n-1} \binom{n}{j} (P - R)^\vee(v^j, x_\alpha^{n-j})\Big| \underset{\alpha}{\to} 0.\end{aligned}$$

This implies

$$\begin{aligned}\limsup_\alpha |\lambda + P(x_\alpha)| &\le \limsup_\alpha |Q(v + x_\alpha) + (P - R)(v + x_\alpha)| + \varepsilon\\ &\le (1 + \varepsilon) \limsup_\alpha \|v + x_\alpha\|^n + \varepsilon.\end{aligned}$$

As ε is arbitrary, the proof is complete. ∎

Example 3.8. From the previous proposition we infer that $\mathcal{P}_w(^2L_p[0,1])$ is not an M-ideal in $\mathcal{P}(^2L_p[0,1])$ for $1 < p < 2$, because $L_p[0,1]$ does not have the 2-polynomial property (M). Indeed, let $\{r_n\}_n$ be the sequence of Rademacher functions and consider the polynomial $P \in \mathcal{P}(^2L_p[0,1])$ given by

$$P(f) = \sum_n \Big(\int f r_n \, d\mu\Big)^2.$$

The norm of this polynomial is bounded by $B_{p^*}^2$, where B_{p^*} is the upper bound in Khinchin's inequality (this can be derived from the norm of the usual projection from $L_p[0,1]$ to ℓ_2 [3, Proposition 6.4.2]). So, the polynomial $P/B_{p^*}^2$ has norm smaller than 1 and it does not have property (M). The inequality of Definition 3.4 fails when we consider the sequence $\{r_n\}_n$, which is weakly null, and the function $v \equiv 1$. Since $P(r_n) = 1$ for all n, and $\|v\| = 1$, we see that

$$\left|1 + \frac{P}{B_{p^*}^2}(r_n)\right| = 1 + \frac{1}{B_{p^*}^2}, \quad \text{while} \quad \|v + r_n\|^2 = 2^{2/p^*}, \quad \text{for all } n.$$

We conclude our argument by proving the inequality

$$2^{2/p^*} < 1 + \frac{1}{B_{p^*}^2}.$$

From now on, we denote $q = p^*$. Since $q > 2$, we know from [22] that

$$B_q = \sqrt{2}\bigg(\frac{\Gamma(q + 1/2)}{\sqrt{\pi}}\bigg)^{1/q}.$$

This means that the inequality we want to prove is

$$2^{2/q} < 1 + \frac{1}{2} \cdot \bigg(\frac{\sqrt{\pi}}{\Gamma((q + 1)/2)}\bigg)^{2/q} \quad \text{for every } q > 2.$$

Our first step is to prove an appropriate bound for the gamma function:

$$\Gamma\Big(\frac{q+1}{2}\Big) \le \Big(\frac{q}{4}\Big)^{q/2} \sqrt{\pi} \quad \text{for every } q \ge 2. \tag{3.1}$$

This is equivalent to the negativity of the function

$$h(q) = \log\Big(\Gamma\Big(\frac{q+1}{2}\Big)\Big) - \frac{q}{2}\log\Big(\frac{q}{4}\Big) - \log(\sqrt{\pi})$$

in the interval $[2,\infty)$. The derivative of h is

$$h'(q) = \frac{1}{2}\left[\psi\left(\frac{q+1}{2}\right) - \log\left(\frac{q}{4}\right) - 1\right],$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, that is, the derivative of the logarithm of the gamma function. The function ψ satisfies the following inequality, for all $x > 0$ (see, for instance [6, (2.2)]):

$$\psi(x) < \log(x) - \frac{1}{2x}.$$

Thus, we obtain

$$h'(q) < \frac{1}{2}\left[\log\left(\frac{q+1}{2}\right) - \frac{1}{q+1} - \log\left(\frac{q}{4}\right) - 1\right] \le 0.$$

Consequently, h is decreasing and $h(q) < h(2) = 0$.

This proves (3.1), which implies that

$$1 + \frac{1}{2}\cdot\left(\frac{\sqrt{\pi}}{\Gamma((q+1)/2)}\right)^{2/q} \ge 1 + \frac{1}{2}\cdot\frac{4}{q} = 1 + \frac{2}{q}.$$

The desired inequality follows since $1 + 2/q > 2^{2/q}$ for all $q > 2$, because $f(q) = (1+2/q)^{q/2}$ is a strictly increasing function and $f(2) = 2$.

THEOREM 3.9. *Let E be a Banach space and let $n = \mathrm{cd}(E)$. Suppose that there exists a net $\{K_\alpha\}_\alpha$ of compact operators from E to E satisfying the following two conditions:*

- *$K_\alpha x \to x$ for all $x \in E$ and $K_\alpha^*\gamma \to \gamma$ for all $\gamma \in E^*$.*
- *$\|\mathrm{Id} - 2K_\alpha\| \underset{\alpha}{\to} 1$.*

Then $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$ if and only if E has the n-polynomial property (M).

Proof. One direction follows from the previous proposition. For the other, we will verify the 3-ball property. Let $P_1, P_2, P_3 \in B_{\mathcal{P}_w({}^nE)}$, $Q \in B_{\mathcal{P}({}^nE)}$ and $\varepsilon > 0$. We will prove that, for α large enough, the polynomial $P(x) = Q(x) - Q(x - K_\alpha x)$ satisfies $\|Q + P_i - P\| \le 1 + \varepsilon$. As in the proof of Proposition 2.2, it can be seen that P is weakly continuous on bounded sets.

Let β be such that

$$\|\mathrm{Id} - 2K_\beta\|^n \le 1 + \varepsilon/2 \quad \text{and} \quad \|P_i - P_i \circ K_\beta\| \le \varepsilon/2 \quad \text{for all } i = 1,2,3.$$

(Recall from Lemma 2.1 that $\|P_i - P_i \circ K_\alpha\| \to 0$.) We have

$$\begin{aligned}\|Q + P_i - P\| &\le \|Q + P_i \circ K_\beta - P\| + \|P_i - P_i \circ K_\beta\|\\ &\le \|P_i \circ K_\beta + Q \circ (\mathrm{Id} - K_\alpha)\| + \varepsilon/2.\end{aligned}$$

Let $\{x_\alpha\}_\alpha \subset B_E$ be such that

$$\limsup_\alpha \|P_1 \circ K_\beta + Q \circ (\mathrm{Id} - K_\alpha)\| = \limsup_\alpha |P_1(K_\beta x_\alpha) + Q(x_\alpha - K_\alpha x_\alpha)|.$$

Since $|P_1(K_\beta x_\alpha)| \le \|K_\beta x_\alpha\|^n$ and $\{K_\beta x_\alpha\}_\alpha$ is contained in a compact subset of E, and since $\{x_\alpha - K_\alpha x_\alpha\}_\alpha$ is a bounded weakly null net and E has the n-polynomial property (M), from Lemma 3.5 we get

$$\begin{aligned}\limsup_\alpha |P_1(K_\beta x_\alpha) + Q(x_\alpha - K_\alpha x_\alpha)| &\le \limsup_\alpha \|K_\beta x_\alpha + x_\alpha - K_\alpha x_\alpha\|^n\\ &\le \limsup_\alpha \|K_\beta + \mathrm{Id} - K_\alpha\|^n\\ &\le \|\mathrm{Id} - 2K_\beta\|^n \le 1 + \varepsilon/2,\end{aligned}$$

where the inequality of the last line is proved in [23, p. 300].

Therefore,

$$\limsup_\alpha \|Q + P_1 - (Q - Q \circ (\mathrm{Id} - K_\alpha))\| \le 1 + \varepsilon.$$

So, there exists a subnet $\{K_\gamma\}_\gamma$ of $\{K_\alpha\}_\alpha$ such that

$$\lim_\gamma \|Q + P_1 - (Q - Q \circ (\mathrm{Id} - K_\gamma))\| \le 1 + \varepsilon.$$

With the same argument, taking further subnets, we obtain the inequality for P_2 and P_3. Thus, the 3-ball property is proved. ■

The following proposition is the polynomial analogue of [23, Lemma VI.4.14] (again we borrow some ideas from that proof) and enables us to obtain a link with the linear theory.

Proposition 3.10. *Let E be a Banach space and let $n = \mathrm{cd}(E)$. If E has property (M), then E has the n-polynomial property (M).*

Proof. Let $P \in B_{\mathcal{P}(^nE)}$, $\lambda \in \mathbb{K}$, $v \in E$ with $|\lambda| \le \|v\|^n$, and let $\{x_\alpha\}_\alpha \subset E$ be a bounded weakly null net. We want to prove that

$$\limsup_\alpha |\lambda + P(x_\alpha)| \le \limsup_\alpha \|v + x_\alpha\|^n.$$

Suppose first that $\|P\| = 1$ and that E is a complex Banach space. Given $\varepsilon > 0$, there exists $u_\varepsilon \in E$ such that $P(u_\varepsilon) = \lambda(1-\varepsilon)$ and $\|u_\varepsilon\| \le |\lambda|^{1/n}$ (and so $\|u_\varepsilon\| < \|v\|$). Thus,

$$\begin{aligned}\limsup_\alpha |\lambda + P(x_\alpha)| &\le \limsup_\alpha |P(u_\varepsilon) + P(x_\alpha)| + |\lambda|\varepsilon\\ &= \limsup_\alpha |P(u_\varepsilon + x_\alpha)| + |\lambda|\varepsilon\\ &\qquad\qquad (\text{since } \mathcal{P}(^kE) = \mathcal{P}_w(^kE) \text{ for } k < n)\\ &\le \|P\| \limsup_\alpha \|u_\varepsilon + x_\alpha\|^n + |\lambda|\varepsilon\\ &\le \limsup_\alpha \|v + x_\alpha\|^n + |\lambda|\varepsilon.\end{aligned}$$

Since ε is arbitrary, the desired inequality is proved.

If E is real and $\|P\| = 1$, the same argument works, except if n even and we are in one of the following two cases:

- $P(x) \geq 0$ for all $x \in E$, and $\lambda < 0$ (or, analogously, $P(x) \leq 0$ for all $x \in E$, and $\lambda > 0$).
- $\inf\{P(x) : \|x\| \leq 1\} = -1$, $\sup\{P(x) : \|x\| \leq 1\} = a < 1$ and $\lambda > 0$ (or, analogously, $\inf\{P(x) : \|x\| \leq 1\} = -a > -1$, $\sup\{P(x) : \|x\| \leq 1\} = 1$ and $\lambda < 0$).

In the first case,

$$|\lambda + P(x_\alpha)| \leq |-\lambda + P(x_\alpha)|,$$

and the above steps prove that $\limsup_\alpha |-\lambda + P(x_\alpha)| \leq \limsup_\alpha \|v + x_\alpha\|^n$.

In the second case, we can suppose without loss of generality that $P(x_\alpha) \geq 0$ for all α, or $P(x_\alpha) < 0$ for all α.

If $P(x_\alpha) < 0$ for all α, it is clear that

$$\limsup_\alpha |\lambda + P(x_\alpha)| \leq \limsup_\alpha |\lambda| \leq \limsup_\alpha \|v + x_\alpha\|^n.$$

Hence, the case we have to consider is the following:

$$\lambda > 0 \quad \text{and} \quad P(x_\alpha) \geq 0 \quad \text{for all } \alpha.$$

Given $0 < \varepsilon < 1$, there exists $u_\varepsilon \in E$ such that $P(u_\varepsilon) = \lambda(1-\varepsilon)$ and $\|u_\varepsilon\| \leq |\lambda/a|^{1/n} \leq \|v\|/a^{1/n}$. Then

$$\begin{aligned}
\limsup_\alpha |\lambda + P(x_\alpha)| &= \limsup_\alpha (\lambda + P(x_\alpha)) \\
&= \limsup_\alpha (P(u_\varepsilon) + P(x_\alpha) + \lambda\varepsilon) \\
&\leq \limsup_\alpha \left(P(u_\varepsilon) + P\left(\frac{x_\alpha}{a^{1/n}}\right) + \lambda\varepsilon \right) \\
&= \limsup_\alpha P\left(u_\varepsilon + \frac{x_\alpha}{a^{1/n}} \right) + \lambda\varepsilon \\
&\leq a \limsup_\alpha \left\| u_\varepsilon + \frac{x_\alpha}{a^{1/n}} \right\|^n + \lambda\varepsilon \\
&\qquad\qquad (\text{since } \limsup_\alpha P(u_\varepsilon + x_\alpha/a^{1/n}) > 0) \\
&\leq a \limsup_\alpha \left\| \frac{v}{a^{1/n}} + \frac{x_\alpha}{a^{1/n}} \right\|^n + \lambda\varepsilon \\
&= \limsup_\alpha \|v + x_\alpha\|^n + \lambda\varepsilon.
\end{aligned}$$

Let us now assume that $\|P\| < 1$. Since $\lambda + P(x_\alpha)$ is a convex combination of $\lambda + \frac{P}{\|P\|}(x_\alpha)$ and $\lambda - \frac{P}{\|P\|}(x_\alpha)$, we get

$$|\lambda + P(x_\alpha)| \leq \max\left\{ \left|\lambda + \frac{P}{\|P\|}(x_\alpha)\right|, \left|\lambda - \frac{P}{\|P\|}(x_\alpha)\right| \right\}.$$

Therefore,

$$\limsup_\alpha |\lambda + P(x_\alpha)| \le \max\Big\{\limsup_\alpha \Big|\lambda + \frac{P}{\|P\|}(x_\alpha)\Big|, \limsup_\alpha \Big|\lambda - \frac{P}{\|P\|}(x_\alpha)\Big|\Big\}$$
$$\le \limsup_\alpha \|v + x_\alpha\|^n. \ \blacksquare$$

Now we derive a relationship with the linear theory that enables us to produce more examples of polynomial M-structures.

COROLLARY 3.11. *Let E be a Banach space and let $n = \mathrm{cd}(E)$. If $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$, then $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$.*

Proof. By [23, Theorem VI.4.17], if $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$, then E has property (M) and there exists a net $\{K_\alpha\}_\alpha \subset \mathcal{K}(E)$ satisfying $K_\alpha x \to x$ for all $x \in E$, $K_\alpha^*\gamma \to \gamma$ for all $\gamma \in E^*$ and $\|\mathrm{Id}-2K_\alpha\| \underset{\alpha}{\to} 1$. From Theorem 3.9 and Proposition 3.10, it follows that $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$. $\blacksquare$

REMARK 3.12. The converse of the previous corollary does not hold. Indeed, in Example 2.8, we see that for $1 < p < 2$, there is a renorming E of $L_p[0,1]$ such that $\mathcal{P}_w(^2E)$ is an M-ideal in $\mathcal{P}(^2E)$. But, from [23, Corollary VI.6.10], we know that $L_p[0,1]$ cannot be renormed to a Banach space E which makes $\mathcal{K}(E)$ an M-ideal in $\mathcal{L}(E)$.

Note that, by the previous corollary, all the known examples of spaces E such that $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$ would provide polynomial examples, once we find the critical degree. Recall also that the critical degree is preserved by isomorphism.

EXAMPLE 3.13. If $\mathbb{D}$ is the complex disc, the *Bergman space* B_p is the space of all holomorphic functions in $L_p(\mathbb{D}, dxdy)$. If $1 < p < \infty$, B_p is isomorphic to ℓ_p [33, Theorem III.A.11] and so $\mathrm{cd}(B_p) = \mathrm{cd}(\ell_p)$. By [27, Corollary 4.8], $\mathcal{K}(B_p)$ is an M-ideal in $\mathcal{L}(B_p)$. Thus, $\mathcal{P}_w(^nB_p)$ is an M-ideal in $\mathcal{P}(^nB_p)$ for $p \le n < p+1$.

EXAMPLE 3.14. By [23, Corollary VI.6.12], an Orlicz sequence space h_M can be renormed to a space E for which $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$ if and only if $(h_M)^*$ is separable. Also, $\mathcal{P}_w(^kh_M) = \mathcal{P}(^kh_M)$ if $k < \alpha_M$ and $\mathcal{P}_w(^kh_M) \ne \mathcal{P}(^kh_M)$ if $k > \beta_M$ (see [21]), where α_M and β_M are the lower and upper Boyd indices associated to M. So, for certain values of α_M and β_M the critical degree can be established. Thus, if $(h_M)^*$ is separable and n is the critical degree, $\mathcal{P}_w(^nE)$ is an M-ideal in $\mathcal{P}(^nE)$.

4. Block diagonal polynomials. In [15] the concept of "block diagonal polynomials" is introduced for spaces with an unconditional finite-dimensional decomposition, and the relationship between the equality $\mathcal{P}(^kE) = \mathcal{P}_w(^kE)$ and its analogue for block diagonal polynomials is studied. Here we discuss the problem of whether the space of block diagonal polynomials that are weakly continuous on bounded sets is an M-ideal in the space of all block diagonal polynomials. First, recall the definition:

Definition 4.1. Let E be a Banach space with an unconditional finite-dimensional decomposition with associated projections $\{\pi_m\}_m$ and let $J = \{m_j\}_j$ be an increasing sequence of positive integers. For each $j \in \mathbb{N}$, let $\sigma_j = \pi_{m_j} - \pi_{m_{j-1}}$. The class of *block diagonal n-homogeneous polynomials with respect to J* is the set

$$\mathcal{D}_J({}^nE) = \Big\{P \in \mathcal{P}({}^nE) : P(x) = \sum_{j=1}^{\infty} P(\sigma_j(x)),\ \forall x \in E\Big\}.$$

Observe that if $J = \mathbb{N}$ and E has an unconditional basis, then $\mathcal{D}_J({}^nE)$ is the space of n-homogeneous diagonal polynomials. Also, if E is a real Banach space with a 1-unconditional basis, diagonal polynomials coincide with orthogonally additive polynomials defined on a Banach lattice.

We want to state conditions that ensure that, for a fixed sequence J, the space $\mathcal{D}_{J,w}({}^nE)$ is an M-ideal in $\mathcal{D}_J({}^nE)$. Note that this problem makes sense, because by [15, Proposition 1.6], for spaces E with a shrinking unconditional finite-dimensional decomposition, the existence of a polynomial which is not weakly continuous on bounded sets implies the existence of a block diagonal polynomial with respect to some J which is not weakly continuous on bounded sets. And we have more values of n where to look for M-structures (not just a critical degree), because, by [7, Proposition 13], $\mathcal{D}_{J,w}({}^kE) = \mathcal{D}_{J,w0}({}^kE)$ for every k.

Observe that, if E has an unconditional finite-dimensional decomposition with associated projections $\{\pi_m\}_m$ and $J = \{m_j\}_j$, then for $Q \in \mathcal{D}_J({}^nE)$, the polynomial

$$P(x) = Q(x) - Q(x - \pi_{m_j}x) = Q(\pi_{m_j}x)$$

is also in $\mathcal{D}_J({}^nE)$. So, the proof of Proposition 2.2 easily implies the following proposition.

Proposition 4.2. *Let E be a Banach space with a shrinking unconditional finite-dimensional decomposition with associated projections $\{\pi_m\}_m$, and let $J = \{m_j\}_j$ be an increasing sequence of positive integers. Suppose that for all $\varepsilon > 0$ and $j_0 \in \mathbb{N}$ there exists $j > j_0$ such that for every $x \in E$,*

$$\|\pi_{m_j}x\|^n + \|x - \pi_{m_j}x\|^n \le (1+\varepsilon)\|x\|^n.$$

Then, for every $k \ge n$, $\mathcal{D}_{J,w}({}^kE)$ is an M-ideal in $\mathcal{D}_J({}^kE)$.

Example 4.3. Let H be a separable Hilbert space. Then, for every $J \subset \mathbb{N}$ and every $n \ge 2$, $\mathcal{D}_{J,w}({}^nH)$ is an M-ideal in $\mathcal{D}_J({}^nH)$.

EXAMPLE 4.4. Let $E = \bigoplus_{\ell_p} X_m$, where each X_m is a finite-dimensional space and $1 < p < \infty$. Then, for every $J \subset \mathbb{N}$ and every $n \geq p$, $\mathcal{D}_{J,w}({}^n E)$ is an M-ideal in $\mathcal{D}_J({}^n E)$. Recall that $\mathcal{D}_{J,w}({}^n E) = \mathcal{D}_J({}^n E)$ for $n < p$. In particular, $\mathcal{D}_{J,w}({}^n \ell_p)$ is an M-ideal in $\mathcal{D}_J({}^n \ell_p)$ for every $n \geq p$.

EXAMPLE 4.5. Let $E = d^*(w,p)$ with $1 < p < \infty$ and $J \subset \mathbb{N}$. Then $\mathcal{D}_{J,w}({}^n d^*(w,p))$ is an M-ideal in $\mathcal{D}_J({}^n d^*(w,p))$ for every $n \geq p^*$. Recall that $\mathcal{D}_{J,w}({}^n d^*(w,p)) = \mathcal{D}_J({}^n d^*(w,p))$ for $n < p^*$ and $w \notin \ell_s$, where $s = ((n-1)^*/p)^*$. For $n < p^*$ and $w \in \ell_s$ we do not know if there is an M-ideal structure in this space.

If the unconditional constant of the decomposition equals 1, it is easy to see that:

- $\|\pi_n\| = 1$ for every $n \in \mathbb{N}$.
- $\|\mathrm{Id} - 2\pi_n\| = 1$ for every $n \in \mathbb{N}$.

In this situation we have more results about M-structure for block diagonal polynomials. First, a block diagonal version of Proposition 1.1:

PROPOSITION 4.6. *Let E be a Banach space with a* 1*-unconditional finite-dimensional decomposition with associated projections $\{\pi_m\}_m$, and let $J = \{m_j\}_j$ be an increasing sequence of positive integers. If $\mathcal{D}_{J,w}({}^n E)$ is an M-summand in $\mathcal{D}_J({}^n E)$ then $\mathcal{D}_{J,w}({}^n E) = \mathcal{D}_J({}^n E)$.*

Proof. Suppose that $\mathcal{D}_J({}^n E) = \mathcal{D}_{J,w}({}^n E) \oplus_\infty S$, where $S \neq \{0\}$. For a given $\varepsilon > 0$, let $P \in S$ and $x_0 \in B_E$ be such that $\|P\| = 1$ and $P(x_0) > 1 - \varepsilon$.

Since $P(x_0) = \sum_{j=1}^\infty P(\sigma_j(x_0))$ and this sum is absolutely convergent, there exists $N \in \mathbb{N}$ such that

$$\sum_{j=N+1}^\infty |P(\sigma_j(x_0))| < \varepsilon.$$

The polynomial $Q = P \circ \pi_{m_N}$ belongs to $\mathcal{D}_{J,w}({}^n E)$ and $\|Q\| \leq 1$, so $\|Q + P\| = \max\{\|Q\|, \|P\|\} = 1$. But

$$\begin{aligned}(Q+P)(x_0) &= \sum_{j=1}^N P(\sigma_j(x_0)) + \sum_{j=1}^\infty P(\sigma_j(x_0)) = 2P(x_0) - \sum_{j=N+1}^\infty P(\sigma_j(x_0)) \\ &> 2 - 3\varepsilon,\end{aligned}$$

which is a contradiction. ∎

In order to obtain a condition for $\mathcal{D}_{J,w}({}^n E)$ to be an M-ideal in $\mathcal{D}_J({}^n E)$ involving property (M), we present the following variation of Proposition 3.10.

PROPOSITION 4.7. *Let E be a Banach space with a shrinking* 1*-unconditional finite-dimensional decomposition with associated projections $\{\pi_m\}_m$,* *and let $J = \{m_j\}_j$ be an increasing sequence of positive integers. If E has property (M), then for every $P \in \mathcal{D}_J({}^nE)$ with $\|P\| \le 1$, for every sequence $\{x_k\}_k \subset B_E$ and every sequence $\{\lambda_k\}_k$ of scalars such that $|\lambda_k| \le \|\pi_{m_{j_0}} x_k\|^n$, for any fixed index j_0,*

$$\limsup_k |\lambda_k + P(x_k - \pi_k x_k)| \le \limsup_k \|\pi_{m_{j_0}} x_k + x_k - \pi_k x_k\|^n.$$

Proof. Suppose first that $\|P\| = 1$ and that E is a complex Banach space. Given $\varepsilon > 0$, there exists $z_\varepsilon \in B_E$ such that $P(z_\varepsilon) = 1 - \varepsilon$.

For every $k \in \mathbb{N}$, let $u_k^\varepsilon = \lambda_k^{1/n} z_\varepsilon$. So, $P(u_k^\varepsilon) = \lambda_k(1-\varepsilon)$ and $\|u_k^\varepsilon\| \le \|\pi_{m_{j_0}} x_k\|$. Thus,

$$\limsup_k |\lambda_k + P(x_k - \pi_k x_k)| \le \limsup_k (|P(u_k^\varepsilon) + P(x_k - \pi_k x_k)| + |\lambda_k|\varepsilon).$$

Since the series $\sum_j P(\sigma_j(z_\varepsilon))$ converges absolutely, there exists $N \in \mathbb{N}$ such that $\sum_{j=N+1}^\infty |P(\sigma_j(z_\varepsilon))| < \varepsilon$. Thus,

$$|P(u_k^\varepsilon) - P(\pi_{m_N} u_k^\varepsilon)| < |\lambda_k|\varepsilon \le \varepsilon,$$

because $|\lambda_k| \le 1$. This implies that

$$\begin{aligned}
\limsup_k |\lambda_k + P(x_k - \pi_k x_k)| &\le \limsup_k |P(\pi_{m_N} u_k^\varepsilon) + P(x_k - \pi_k x_k)| + 2\varepsilon \\
&= \limsup_k |P(\pi_{m_N} u_k^\varepsilon + x_k - \pi_k x_k)| + 2\varepsilon \\
&\le \limsup_k \|\pi_{m_N} u_k^\varepsilon + x_k - \pi_k x_k\|^n + 2\varepsilon \\
&\le \limsup_k \|\pi_{m_{j_0}} x_k + x_k - \pi_k x_k\|^n + 2\varepsilon;
\end{aligned}$$

the last step follows from property (M) since the sequence $\{x_k - \pi_k x_k\}_k$ is weakly null and $\|\pi_{m_N} u_k^\varepsilon\| \le \|u_k^\varepsilon\| \le \|\pi_{m_{j_0}} x_k\|$.

Since this happens for arbitrary ε, the result follows. If E is real or $\|P\| < 1$, we can repeat the argument of the proof of Proposition 3.10. ∎

THEOREM 4.8. *Let E be a Banach space with a shrinking 1-unconditional finite-dimensional decomposition with associated projections $\{\pi_m\}_m$, and let $J = \{m_j\}_j$ be an increasing sequence of positive integers. If E has property (M), then, for every $n \in \mathbb{N}$, $\mathcal{D}_{J,w}({}^nE)$ is an M-ideal in $\mathcal{D}_J({}^nE)$.*

Proof. To check the 3-ball property, let $P_1, P_2, P_3 \in B_{\mathcal{D}_{J,w}({}^nE)}$, $Q \in B_{\mathcal{D}_J({}^nE)}$, $\varepsilon > 0$. We want to show that, for j large enough, the polynomial $P = Q \circ \pi_{m_j} \in \mathcal{D}_{J,w}({}^nE)$ satisfies $\|Q + P_i - P\| \le 1 + \varepsilon$ for $i = 1, 2, 3$.

Since $P_i - P_i \circ \pi_m \to 0$ as $m \to \infty$, there is $j_0 \in \mathbb{N}$ such that

$$\|P_i - P_i \circ \pi_{m_{j_0}}\| \le \varepsilon \quad \text{for } i = 1, 2, 3.$$

Now we have

$$\begin{aligned}\|Q+P_i-P\| &\le \|Q+P_i\circ\pi_{m_{j_0}}-P\|+\|P_i-P_i\circ\pi_{m_{j_0}}\|\\ &\le \|Q+P_i\circ\pi_{m_{j_0}}-Q\circ\pi_{m_j}\|+\varepsilon.\end{aligned}$$

Let $\{x_j\}_j\subset B_E$ be such that

$$\limsup_j \|Q+P_i\circ\pi_{m_{j_0}}-Q\circ\pi_{m_j}\| = \limsup_j |Q(x_j)+(P_i\circ\pi_{m_{j_0}})(x_j)-(Q\circ\pi_{m_j})(x_j)|.$$

Since the polynomial Q is block diagonal with respect to J, it follows that $Q(x_j)-Q(\pi_{m_j}(x_j))=Q(x_j-\pi_{m_j}(x_j))$. From this and the previous proposition we have

$$\begin{aligned}\limsup_j \|Q+P_i\circ\pi_{m_{j_0}}-Q\circ\pi_{m_j}\| &= \limsup_j |P_i(\pi_{m_{j_0}}(x_j))-Q(x_j-\pi_{m_j}(x_j))|\\ &\le \limsup_j \|\pi_{m_{j_0}}(x_j)+x_j-\pi_{m_j}(x_j)\|^n\\ &\le \limsup_j \|\pi_{m_{j_0}}+\mathrm{Id}-\pi_{m_j}\|^n=1.\end{aligned}$$

With final considerations as in the proof of Theorem 3.9, the result is proved. ■

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