Approximation of the Euclidean ball by polytopes

by

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Abstract. There is a constant \( c \) such that for every \( n \in \mathbb{N} \), there is an \( N_n \) so that for every \( N \geq N_n \) there is a polytope \( P \) in \( \mathbb{R}^n \) with \( N \) vertices and

\[
\text{vol}_n(B_n^2 \triangle P) \leq c \text{vol}_n(B_n^2)N^{-\frac{2}{n-1}}
\]

where \( B_n^2 \) denotes the Euclidean unit ball of dimension \( n \).

1. Main results. Let \( C \) and \( K \) be two convex bodies in \( \mathbb{R}^n \). The Euclidean ball with center 0 and radius \( r \) is denoted by \( B_n^2(r) \). The ball \( B_n^2(1) \) is denoted by \( B_n^2 \). Let \( K \) be a convex body in \( \mathbb{R}^n \) with \( C^2 \)-boundary \( \partial K \) and everywhere strictly positive curvature \( \kappa \). Then

\[
\lim_{N \to \infty} \frac{\inf\{\text{vol}_n(K \setminus P) \mid P \subseteq K \text{ and } P \text{ has at most } N \text{ vertices}\}}{N^{-\frac{2}{n-1}}} = \frac{1}{2} \text{del}_n \left( \int_{\partial K} \kappa(x) \frac{1}{n+1} d\mu_{\partial K}(x) \right)^{\frac{n+1}{n-1}}
\]

where \( \mu_{\partial K} \) denotes the surface measure of \( \partial K \). This theorem gives asymptotically the order of best approximation of a convex body \( K \) by polytopes contained in \( K \) with a fixed number of vertices. It was proved by McClure and Vitale [McV] in dimension 2 and by Gruber [Gr2] for general \( n \). The constant \( \text{del}_n \) is positive and depends on the dimension \( n \) only. Its order of magnitude can be computed by considering the case \( K = B_n^2 \). This has been done in [GRS1] and [GRS2] by Gordon, Reisner and Schütt, namely there are numerical constants \( a \) and \( b \) such that

\[
an \leq \text{del}_n \leq bn.
\]

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The constant $\text{del}_{n-1}$ was determined more precisely by Mankiewicz and Schütt [MaS1], [MaS2]. They showed that

$$(2) \quad \frac{n-1}{n+1} (\text{vol}_{n-1}(B_2^{n-1}))^{-\frac{2}{n-1}} \leq \text{del}_{n-1} \leq \left(1 + \frac{c \ln n}{n+1}\right) (\text{vol}_{n-1}(B_2^{n-1}))^{-\frac{2}{n-1}}$$

where $c$ is a numerical constant. In particular,

$$\lim_{n \to \infty} \frac{\text{del}_{n-1}}{n} = \frac{1}{2\pi e}.$$

What happens if we drop the condition that the polytopes have to be contained in the convex body and allow all polytopes have at most $N$ vertices? How much better can we approximate the Euclidean ball?

In [Lud] it was shown that for all convex bodies $K$ whose boundary is twice continuously differentiable and whose curvature is everywhere strictly positive,

$$\lim_{N \to \infty} \inf \{\text{vol}_n(K \triangle P) \mid P \text{ is a polytope with at most } N \text{ vertices}\} = \frac{1}{2} \text{idel}_{n-1} \left( \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x) \right)^{\frac{n+1}{n-1}}.$$

The constant $\text{idel}_{n-1}$ is positive and depends only on $n$. Clearly, from the above mentioned results it follows that $\text{idel}_{n-1} \leq cn$. On the other hand, it has been shown in [Bö] that for a polytope $P$ with at most $N$ vertices,

$$\text{vol}_n(B_2^n \triangle P) \geq \frac{1}{67e^2 \pi} \frac{1}{n} \text{vol}_n(B_2^n) N^{-\frac{2}{n-1}}.$$ 

Thus between the upper and lower estimate for $\text{idel}_{n-1}$ there is a gap of order $n^2$. In this paper we narrow this gap by showing that $\text{idel}_{n-1} \leq c$ where $c$ is a numerical constant.

**Theorem 1.** There is a constant $c$ such that for every $n \in \mathbb{N}$ there is an $N_n$ so that for every $N \geq N_n$ there is a polytope $P$ in $\mathbb{R}^n$ with $N$ vertices such that

$$(3) \quad \text{vol}_n(B_2^n \triangle P) \leq c \text{vol}_n(B_2^n) N^{-\frac{2}{n-1}}.$$

Gruber [Gr2] also showed

$$\lim_{N \to \infty} \inf \{\text{vol}_n(K \triangle P) \mid K \subseteq P \text{ and } P \text{ is a polytope with at most } N \text{ facets}\} = \frac{1}{2} \text{div}_{n-1} \left( \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x) \right)^{\frac{n+1}{n-1}}$$
where \( \text{div}_{n-1} \) is a positive constant that depends on \( n \) only. It is easy to show \([\text{Lud, MaS}1]\) that there are numerical constants \( a \) and \( b \) such that \( an \leq \text{div}_{n-1} \leq bn \).

Ludwig \([\text{Lud}]\) showed that for general polytopes

\[
\lim_{N \to \infty} \inf \{\text{vol}_n(K \triangle P) \mid P \text{ is a polytope with at most } N \text{ facets}\} = \frac{1}{2} \text{ldiv}_{n-1} \left( \int_{\partial K} \kappa(x) \frac{1}{n+1} d\mu_{\partial K}(x) \right)^{\frac{n+1}{n-1}}
\]

where \( \text{ldiv}_{n-1} \) is a positive constant that depends on \( n \) only. Clearly, \( \text{ldiv}_{n-1} \leq cn \) and Böröczky \([\text{Bö}]\) showed that for polytopes \( P \) with \( N \) facets,

\[
\text{vol}_n(B_2^n \triangle P) \geq \frac{1}{67e^2\pi} \frac{1}{n} \text{vol}_n(B_2^n) N^{-\frac{2}{n-1}}.
\]

Thus again, there is a gap between the upper and lower estimates for \( \text{ldiv}_{n-1} \) of order \( n^2 \). We narrow this gap by a factor of \( n \).

**Theorem 2.** There is a constant \( c \) such that for every \( n \in \mathbb{N} \) and for every \( M \geq 10^{(n-1)/2} \) and all polytopes \( P \) in \( \mathbb{R}^n \) with \( M \) facets we have

\[
\text{vol}_n(B_2^n \triangle P) \geq c \text{vol}_n(B_2^n) M^{-\frac{2}{n-1}}.
\]

For additional information on asymptotic approximation see \([\text{Gr}1, \text{Gr}3, \text{Sch}]\).

**2. Proof of Theorem 1.** We need the following lemmas.

**Lemma 3 (Stirling’s formula).** For all \( x > 0 \),

\[
\sqrt{2\pi x} x^x e^{-x} < \Gamma(x+1) < \sqrt{2\pi x} x^x e^{-x} e^{1/12x}.
\]

The following lemma is due to J. Müller \([\text{Mü}]\).

**Lemma 4 ([Mü]).** Let \( \mathbb{E}(\partial B_2^n, N) \) be the expected volume of a random polytope of \( N \) points that are independently chosen on the boundary of the Euclidean ball \( B_2^n \) with respect to the normalized surface measure. Then

\[
\lim_{N \to \infty} \frac{\text{vol}_n(B_2^n) - \mathbb{E}(\partial B_2^n, N)}{N^{-\frac{2}{n-1}}} = \frac{(n-1)^{\frac{n+1}{n-1}}(\text{vol}_{n-1}(\partial B_2^n))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}} \frac{\Gamma(n+1+\frac{2}{n-1})}{2(n+1)!}.
\]

The following lemma can be found in \([\text{Mil}, \text{SW, p. 317}]\), and \([\text{Zä}]\). Let \([x_n, \ldots, x_n] \) be the convex hull of \( x_1, \ldots, x_n \).
Lemma 5 ([Mil]).
\[
(n - 1)! \frac{{\operatorname{vol}_{n-1}([x_1, \ldots, x_n])}}{{(1 - p^2)^{n/2}}} d\mu_{\partial B_2^n \cap H}(x_1) \cdots d\mu_{\partial B_2^n \cap H}(x_n) \, dp \, d\mu_{\partial B_2^n}(\xi)
\]
where \( \xi \) is the normal to the plane \( H \) through \( x_1, \ldots, x_n \) and \( p \) is the distance of the plane \( H \) to the origin.

Lemma 6 ([Mil]).
\[
\int_{\partial B_2^n(r)} \cdots \int_{\partial B_2^n(r)} (\operatorname{vol}_n([x_1, \ldots, x_{n+1}]))^2 d\mu_{\partial B_2^n(r)}(x_1) \cdots d\mu_{\partial B_2^n(r)}(x_{n+1}) = \frac{(n+1) r^{n^2+2n-1}}{n!n^n} (\operatorname{vol}_{n-1}(\partial B_2^n))^{n+1}.
\]

For a given hyperplane \( H \) that does not contain the origin we denote by \( H^+ \) the halfspace containing the origin and by \( H^- \) the halfspace not containing the origin. A cap \( C \) of the Euclidean ball \( B_2^n \) is the intersection of a halfspace \( H^- \) with \( B_2^n \). The radius of such a cap is the radius of the \((n-1)\)-dimensional ball \( B_2^{n-1} \cap H \).

Lemma 7 ([SW]). Let \( H \) be a hyperplane, \( p \) its distance from the origin and \( s \) the normalized surface area of \( \partial B_2^n \cap H^- \), i.e.
\[
s = \frac{\operatorname{vol}_{n-1}(\partial B_2^n \cap H^-)}{\operatorname{vol}_{n-1}(\partial B_2^n)}.
\]
Then
\[
\frac{dp}{ds} = -\frac{1}{(1 - p^2)^{n-3}} \frac{\operatorname{vol}_{n-1}(\partial B_2^n)}{\operatorname{vol}_{n-2}(\partial B_2^{n-1})}.
\]

Lemma 8 ([SW, Lemma 3.13]). Let \( C \) be a cap of a Euclidean ball with radius 1. Let \( u \) be the surface area of this cap and \( r \) its radius. Then
\[
\left(\frac{u}{\operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} - \frac{1}{2(n+1)} \left(\frac{u}{\operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{3}{n-1}} - c \left(\frac{u}{\operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{5}{n-1}} \leq r(u) \leq \left(\frac{u}{\operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}
\]
where \( c \) is a numerical constant.

The right hand inequality is immediate, since \( u \geq r^{n-1} \operatorname{vol}_{n-1}(B_2^{n-1}) \).

Proof of Theorem 1. The approximating polytope is obtained in a probabilistic way. We consider a Euclidean ball that is slightly bigger than the ball with radius 1, by a carefully chosen factor. We choose \( N \) points randomly in the bigger ball and we take their convex hull. With large probability there is a random polytope that fits our requirements.
For technical reasons we choose random points in a Euclidean ball of radius 1 and we approximate a slightly smaller Euclidean ball, say with radius $1 - c$ where $c = c_{n,N}$ depends on $n$ and $N$ only.

We now compute the expected volume difference between $B^n_2(1 - c)$ and a random polytope $[x_1, \ldots, x_N]$ whose vertices are chosen randomly from the boundary of $B^n_2$. Note that random polytopes are simplicial with probability 1. We want to estimate the expected volume difference

$$E \text{vol}_n(B^n_2(1 - c) \triangle P_N) = \int_{\partial B^n_2} \cdots \int_{\partial B^n_2} \text{vol}_n(B^n_2(1 - c) \triangle [x_1, \ldots, x_N]) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N)$$

where $\mathbb{P}$ denotes the uniform probability measure on $\partial B^n_2$. Since the volume difference between $B^n_2(1 - c)$ and a polytope $P_N = [x_1, \ldots, x_N]$ is

$$\text{vol}_n(B^n_2(1 - c) \triangle P_N) = \text{vol}_n(B^n_2 \setminus B^n_2(1 - c)) - \text{vol}_n(B^n_2 \setminus P_N) + 2 \text{vol}_n(B^n_2(1 - c) \cap P_N^n),$$

the above expression equals

$$\text{vol}_n(B^n_2 \setminus B^n_2(1 - c)) - \int_{\partial B^n_2} \cdots \int_{\partial B^n_2} \text{vol}_n(B^n_2 \setminus [x_1, \ldots, x_N]) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N) + 2 \int_{\partial B^n_2} \cdots \int_{\partial B^n_2} \text{vol}_n(B^n_2(1 - c) \cap [x_1, \ldots, x_N]^c) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N).$$

For given $N$ we choose $c$ such that

$$\text{vol}_n(B^n_2 \setminus B^n_2(1 - c)) = \int_{\partial B^n_2} \cdots \int_{\partial B^n_2} \text{vol}_n(B^n_2 \setminus [x_1, \ldots, x_N]) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N).$$

For this particular $c$ we have

$$\int_{\partial B^n_2} \cdots \int_{\partial B^n_2} \text{vol}_n(B^n_2(1 - c) \triangle [x_1, \ldots, x_N]) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N) = 2 \int_{\partial B^n_2} \cdots \int_{\partial B^n_2} \text{vol}_n(B^n_2(1 - c) \cap [x_1, \ldots, x_N]^c) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N).$$

By Lemma 4 the quantity $c$ is for large $N$ asymptotically equal to

$$N^{-2\frac{n}{n-1}}(n - 1)^{\frac{n+1}{n-1}}\left(\frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-2}(\partial B^{n-1}_2)}\right)^{\frac{2}{n-1}} \Gamma\left(n + 1 + \frac{2}{n-1}\right) \frac{2(n + 1)!}{2(n + 1)!}.$$
In particular, for large enough $N$,

$$c \leq \left(1 + \frac{1}{n^2}\right) N^{-\frac{2}{n-1}} (n-1)^{\frac{n+1}{n-1}}$$

(12)

$$\times \left(\frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma(n + 1 + \frac{2}{n-1})}{2(n+1)!}$$

and

$$\left(1 - \frac{1}{n^2}\right) N^{-\frac{2}{n-1}} (n-1)^{\frac{n+1}{n-1}}$$

(13)

$$\times \left(\frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma(n + 1 + \frac{2}{n-1})}{2(n+1)!} \leq c.$$ 

Thus there are constants $a$ and $b$ such that

$$aN^{-\frac{2}{n-1}} \leq c \leq bN^{-\frac{2}{n-1}}.$$ 

(14)

We continue the computation of the expected volume difference:

$$\int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_n(B_2^n(1-c) \triangle [x_1, \ldots, x_N]) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N)$$

$$= 2 \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_n(B_2^n(1-c) \cap [x_1, \ldots, x_N]^c)$$

$$\times \chi_{\{0 \in [x_1, \ldots, x_N]^c\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N)$$

$$+ 2 \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_n(B_2^n(1-c) \cap [x_1, \ldots, x_N]^c)$$

$$\times \chi_{\{0 \notin [x_1, \ldots, x_N]^c\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N)$$

$$\leq 2 \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_n(B_2^n(1-c) \cap [x_1, \ldots, x_N]^c)$$

$$\times \chi_{\{0 \notin [x_1, \ldots, x_N]^c\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N)$$

$$+ \text{vol}_n(B_2^n) \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \chi_{\{0 \notin [x_1, \ldots, x_N]^c\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N).$$

By a result of [Wen] the second summand equals

$$\text{vol}_n(B_2^n) 2^{-N+1} \sum_{k=0}^{n-1} \binom{N-1}{k} \leq \text{vol}_n(B_2^n) 2^{-N+1} n N^n,$$

so it is of much smaller order (essentially $2^{-N}$) than the first summand, which, as we shall see, is of the order of $N^{-2/(n-1)}$. Therefore, in what follows we consider the first summand.

We introduce $\Phi_{j_1, \ldots, j_n} : \partial B_2^n \times \cdots \times \partial B_2^n \to \mathbb{R}$ where

$$\Phi_{j_1, \ldots, j_n}(x_1, \ldots, x_N) = 0$$
if \([x_j, \ldots, x_{jn}]\) is not an \((n - 1)\)-dimensional face of \([x_1, \ldots, x_N]\) or if 0 is not in \([x_1, \ldots, x_N]\), and

\[
\Phi_{j_1,\ldots,j_n}(x_1, \ldots, x_N) = \text{vol}_n(B_2^n(1 - c) \cap [x_1, \ldots, x_N]^c \cap \text{cone}(x_{j_1}, \ldots, x_{jn})) \chi_{\{0 \in [x_1, \ldots, x_N]^c\}}
\]

if \([x_j, \ldots, x_{jn}]\) is a facet of \([x_1, \ldots, x_N]\) and if 0 \(\not\in [x_1, \ldots, x_N]\). Here

\[
\text{cone}(x_1, \ldots, x_n) = \left\{ \sum_{i=1}^{n} a_i x_i \mid \forall i : a_i \geq 0 \right\}.
\]

For all random polytopes \([x_1, \ldots, x_N]\) that contain 0 as an interior point,

\[
\mathbb{R}^n = \bigcup_{[x_{j_1}, \ldots, x_{jn}] \text{ is a facet of } [x_1, \ldots, x_n]} \text{cone}(x_{j_1}, \ldots, x_{jn}).
\]

Then

\[
\int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_n(B_2^n(1 - c) \cap [x_1, \ldots, x_N]^c) \chi_{\{0 \in [x_1, \ldots, x_N]^c\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N)
\]

\[
= \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \sum_{\{j_1, \ldots, j_n\} \subseteq \{1, \ldots, N\}} \Phi_{j_1,\ldots,j_n}(x_1, \ldots, x_N) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N)
\]

where we sum over all different subsets \(\{j_1, \ldots, j_n\}\). The latter expression equals

\[
\binom{N}{n} \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \Phi_{1,\ldots,n}(x_1, \ldots, x_N) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N).
\]

Let \(H\) be the hyperplane containing the points \(x_1, \ldots, x_n\). The set of points where \(H\) is not well defined has measure 0 and

\[
\mathbb{P}^{N-n}(\{(x_{n+1}, \ldots, x_N) \mid [x_1, \ldots, x_n] \text{ is a facet of } [x_1, \ldots, x_N] \text{ and } 0 \in [x_1, \ldots, x_N]\}) = \left(\frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}.
\]

Therefore the above expression equals

\[
\binom{N}{n} \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \left(\frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)}\right)^{N-n} \times \text{vol}_n(B_2^n(1 - c) \cap H^- \cap \text{cone}(x_1, \ldots, x_n)) \chi_{\{0 \in [x_1, \ldots, x_N]^c\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_n).
\]

Since \(H^-\) does not contain 0 this can be estimated by

\[
\binom{N}{n} \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \left(\frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)}\right)^{N-n} \times \text{vol}_n(B_2^n(1 - c) \cap H^- \cap \text{cone}(x_1, \ldots, x_n)) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_n).
\]
By Lemma 5 the latter expression equals

$$\left( \frac{N}{n} \right) \frac{(n-1)!}{(\text{vol}_{n-1}(\partial B^n_2))^n} \int_{\partial B^n_2 \cap H} \cdots \int_{\partial B^n_2 \cap H} \left( \frac{\text{vol}_{n-1}(\partial B^n_2 \cap H^+)}{\text{vol}_{n-1}(\partial B^n_2)} \right)^{N-n} \times \text{vol}_n(B^n_2(1-c) \cap H^- \cap \text{cone}(x_1, \ldots, x_n))$$

$$\times \frac{\text{vol}_{n-1}([x_1, \ldots, x_n])}{(1-p^2)^{n/2}} d\mu_{\partial B^n_2 \cap H}(x_1) \cdots d\mu_{\partial B^n_2 \cap H}(x_n) dp d\mu_{\partial B^n_2}(\xi).$$

This in turn can be estimated by

(15) $$\left( \frac{N}{n} \right) \frac{(n-1)!}{(\text{vol}_{n-1}(\partial B^n_2))^n} \times \int_{\partial B^n_2 1-1/n \cap \partial B^n_2 \cap H} \cdots \int_{\partial B^n_2 \cap H} \left( \frac{\text{vol}_{n-1}(\partial B^n_2 \cap H^+)}{\text{vol}_{n-1}(\partial B^n_2)} \right)^{N-n} \times \text{vol}_n(B^n_2(1-c) \cap H^- \cap \text{cone}(x_1, \ldots, x_n))$$

$$\times \frac{\text{vol}_{n-1}([x_1, \ldots, x_n])}{(1-p^2)^{n/2}} d\mu_{\partial B^n_2 \cap H}(x_1) \cdots d\mu_{\partial B^n_2 \cap H}(x_n) dp d\mu_{\partial B^n_2}(\xi)$$

times a factor that is less than 2 provided that $N$ is sufficiently large. Indeed, for $p \leq 1 - 1/n$, 

$$\left( \frac{\text{vol}_{n-1}(\partial B^n_2 \cap H^+)}{\text{vol}_{n-1}(\partial B^n_2)} \right)^{N-n} \leq \exp \left( -(N-n) \frac{\text{vol}_{n-1}(\partial B^n_2 \cap H^-)}{\text{vol}_{n-1}(\partial B^n_2)} \right)$$

$$\leq \exp \left( -(N-n) \left( \frac{2}{n} - \frac{1}{n^2} \right)^{n-1} \frac{\text{vol}_{n-1}(B^{n-1}_2)}{n \text{vol}_n(B^n_2)} \right)$$

$$\leq \exp \left( -\frac{N-n}{n(n+1)/2} \right)$$

and the rest of the expression is bounded. We have

$$\text{vol}_n(B^n_2(1-c) \cap H^- \cap \text{cone}(x_1, \ldots, x_n))$$

$$\leq \frac{p}{n} \max \left\{ 0, \left( \frac{1-c}{p} \right)^n - 1 \right\} \text{vol}_{n-1}([x_1, \ldots, x_n]).$$

This holds since $B^n_2(1-c) \cap H^- \cap \text{cone}(x_1, \ldots, x_n)$ is contained in the cone $\text{cone}(x_1, \ldots, x_n)$, truncated between $H$ and the hyperplane parallel to $H$ at distance $1-c$ from 0. Therefore, as $p \leq 1$ the above is at most
\[
\binom{N}{n} \frac{(n-1)!}{\left(\text{vol}_{n-1}(\partial B^n_2)\right)^n} \int_{\partial B^n_2 \cap \partial B^n_2 \cap H} \left( \frac{\text{vol}_{n-1}(\partial B^n_2 \cap H^+)}{\text{vol}_{n-1}(\partial B^n_2)} \right)^{N-n} \times \frac{1}{n} \max \left\{ 0, \left( \frac{1-c}{p} \right)^n - 1 \right\} \frac{\text{vol}_{n-1}([x_1, \ldots, x_n])^2}{(1-p^2)^{n/2}} \times d\mu_{\partial B^n_2 \cap H}(x_1) \cdots d\mu_{\partial B^n_2 \cap H}(x_n) \, dp \, d\mu_{\partial B^n_2}(\xi).
\]

By Lemma 6 this equals
\[
\binom{N}{n} \frac{(n-2)(\partial B^n_2)}{(\text{vol}_{n-1}(\partial B^n_2))^n} \frac{n}{(n-1)^{n-1}} \int_{\partial B^n_2 \cap H^+} \left( \frac{\text{vol}_{n-1}(\partial B^n_2 \cap H^+)}{\text{vol}_{n-1}(\partial B^n_2)} \right)^{N-n} \times \frac{1}{n} \max \left\{ 0, \left( \frac{1-c}{p} \right)^n - 1 \right\} \frac{r^{n^2-2}}{(1-p^2)^{n/2}} \, dp \, d\mu_{\partial B^n_2}(\xi)
\]

where \( r \) denotes the radius of \( B^n_2 \cap H \). Since the integral does not depend on the direction \( \xi \) and \( r^2 + p^2 = 1 \) this is
\[
\binom{N}{n} \frac{(n-2)(\partial B^n_2)}{(\text{vol}_{n-1}(\partial B^n_2))^n} \frac{n}{(n-1)^{n-1}} \int_{\partial B^n_2 \cap H^+} \left( \frac{\text{vol}_{n-1}(\partial B^n_2 \cap H^+)}{\text{vol}_{n-1}(\partial B^n_2)} \right)^{N-n} \frac{1}{n} \max \left\{ 0, \left( \frac{1-c}{p} \right)^n - 1 \right\} r^{n^2-n-2} \, dp.
\]

This equals
\[
\binom{N}{n} \frac{(n-2)(\partial B^n_2)}{(\text{vol}_{n-1}(\partial B^n_2))^n} \frac{n}{(n-1)^{n-1}} \int_{\partial B^n_2 \cap H^+} \left( \frac{\text{vol}_{n-1}(\partial B^n_2 \cap H^+)}{\text{vol}_{n-1}(\partial B^n_2)} \right)^{N-n} \frac{1}{n} \left\{ \left( \frac{1-c}{p} \right)^n - 1 \right\} r^{n^2-n-2} \, dp.
\]

Since \( p \geq 1 - 1/n \) and \( c \) is of the order \( N^{-2/(n-1)} \), we have, for sufficiently large \( N \),
\[
\frac{1}{n} \left\{ \left( \frac{1-c}{p} \right)^n - 1 \right\} \leq 3(1-c-p).
\]

Therefore, the previous expression can be estimated by an absolute constant times
\[
\binom{N}{n} \frac{(n-2)(\partial B^n_2)}{(\text{vol}_{n-1}(\partial B^n_2))^n} \frac{n}{(n-1)^{n-1}} \int_{\partial B^n_2 \cap H^+} \left( \frac{\text{vol}_{n-1}(\partial B^n_2 \cap H^+)}{\text{vol}_{n-1}(\partial B^n_2)} \right)^{N-n} \left(1-c-p\right)r^{n^2-n-2} \, dp.
\]

Approximation of the ball by polytopes
We choose

\[ s = \frac{\text{vol}_{n-1}(\partial B^n_2 \cap H^-)}{\text{vol}_{n-1}(\partial B^n_2)} \]

as our new variable under the integral. We apply Lemma 7 in order to change the variable under the integral

\[
\left( \frac{N}{n} \right) \frac{\left( \text{vol}_{n-2}(\partial B^{n-1}_2) \right)^{n-1}}{\left( \text{vol}_{n-1}(\partial B^n_2) \right)^{n-2}} \frac{n}{(n-1)^{n-1}} \times \frac{1}{\sqrt{1-r^2}} \int_{s(1-c)}^{1} (1-s)^{N-n}(1-c-p)r^{(n-1)^2} ds
\]

where the normalized surface area \( s \) of the cap is a function of the distance \( p \) of the hyperplane to 0. Before we proceed we want to estimate \( s(1-c) \).

The radius \( r \) and the distance \( p \) satisfy \( 1 = p^2 + r^2 \). We have

\[
r^{n-1} \frac{\text{vol}_{n-1}(B^n_2)}{\text{vol}_{n-1}(\partial B^n_2)} \leq s(\sqrt{1-r^2}) \leq \frac{1}{\sqrt{1-r^2}} r^{n-1} \frac{\text{vol}_{n-1}(B^n_2)}{\text{vol}_{n-1}(\partial B^n_2)}.
\]

To show this, we compare \( s \) with the surface area of the intersection \( B^n_2 \cap H \) of the Euclidean ball and the hyperplane \( H \). We have

\[
\frac{\text{vol}_{n-1}(B^n_2 \cap H)}{\text{vol}_{n-1}(\partial B^n_2)} = r^{n-1} \frac{\text{vol}_{n-1}(B^n_2)}{\text{vol}_{n-1}(\partial B^n_2)}.
\]

Since the orthogonal projection onto \( H \) maps \( \partial B^n_2 \cap H^- \) onto \( B^n_2 \cap H \) the left hand inequality follows.

The right hand inequality follows again by considering the orthogonal projection onto \( H \). The surface area of a surface element of \( \partial B^n_2 \cap H^- \) equals the surface area of the one it is mapped to in \( B^n_2 \cap H \) divided by the cosine of the angle between the normal to \( H \) and the normal to \( \partial B^n_2 \) at the given point. The cosine is always greater than \( \sqrt{1-r^2} \).

For \( p = 1-c \) we have \( r = \sqrt{2c-c^2} \leq \sqrt{2c} \). Therefore by (12) we get

\[
(19) \quad s(1-c) \leq \frac{e^{1/n}}{1-c} \frac{\text{vol}_{n-1}(B^n_2)}{\text{vol}_{n-1}(\partial B^n_2)} \times \left\{ 2N^{\frac{2}{n-1}}(n-1)^\frac{n+1}{n-1} \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-2}(\partial B^{n-1}_2)} \right) \right. \left. \frac{\Gamma(n+1+\frac{2}{n-1})}{2(n+1)!} \right\}^\frac{n-1}{2} = \frac{e^{1/n}}{1-c} \frac{1}{N} \left\{ \frac{\Gamma(n+1+\frac{2}{n-1})(n-1)}{(n+1)!} \right\}^\frac{n-1}{2}.
\]

The quantity \( c \) is of the order \( N^{-2/(n-1)} \), therefore \( 1/(1-c) \) is as close to 1
as we desire for $N$ large enough. Moreover, for large $n$, 

$$\left(\frac{n-1}{n+1}\right)^{n-1}$$

is asymptotically equal to $1/e$. Therefore, for both $n$ and $N$ large enough, 

$$s(1-c) \leq e^{1/12} \frac{1}{eN} \left(\frac{\Gamma(n+1+\frac{2}{n-1})}{n!}\right)^{n-1}.$$ 

For $n$ sufficiently large, 

$$\left\{\frac{\Gamma(n+1+\frac{2}{n-1})}{n!}\right\}^{n-1} \leq e^{1/12}n.$$ 

Indeed, by Lemma 3, 

$$\frac{\Gamma(n+1+\frac{2}{n-1})}{n!} \leq \left(1 + \frac{2}{n(n-1)}\right)^{n+\frac{1}{2}} \left(n + \frac{2}{n-1}\right)^{\frac{2}{n-1}} e^{-\frac{2}{n-1}e^{\frac{1}{24} (n+\frac{2}{n-1})}}.$$ 

and 

$$\left(\frac{\Gamma(n+1+\frac{2}{n-1})}{n!}\right)^{n-1} \leq \frac{1}{e} \left(1 + \frac{2}{n(n-1)}\right)^{\frac{n-1}{2} (n+\frac{1}{2})} \left(n + \frac{2}{n-1}\right)^{\frac{n-1}{2} e^{24(n+\frac{2}{n-1})}}.$$ 

The right hand expression is asymptotically equal to $ne^{1/24}$. Altogether, 

$$s(1-c) \leq e^{1/6} \frac{n}{eN}.$$ 

Since $p = \sqrt{1 - r^2}$ we get, for all $r$ with $0 \leq r \leq 1$, 

$$1 - c - p = 1 - c - \sqrt{1 - r^2} \leq \frac{1}{2} r^2 + r^4 - c.$$ 

(The estimate is equivalent to $1 - \frac{1}{2} r^2 - r^4 \leq \sqrt{1 - r^2}$. The left hand side is negative for $r \geq .9$ and thus the inequality holds for $r$ with $.9 \leq r \leq 1$. For $r$ with $0 \leq r \leq .9$ we square both sides.) Thus (18) is smaller than or equal to 

$$s(1-c) \leq e^{1/6} \frac{n}{eN}.$$
the fact that $\frac{1}{2} r^2 + r^4 - c$ is nonnegative. Hence
\[
\int_{s(1-c)}^{1} (1 - s)^{N-n} \left( \frac{1}{2} r^2 + r^4 - c \right) r^{(n-1)^2} \, ds 
\]
\[
\leq \frac{1}{2} \int_{0}^{1} (1 - s)^{N-n} \left( s \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{2}{n-1}} \, ds 
\]
\[
+ \int_{0}^{1} (1 - s)^{N-n} \left( s \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{4}{n-1}} \, ds 
\]
\[
- \int_{0}^{1} (1 - s)^{N-n} c \left( s \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1} \, ds 
\]
\[
+ \int_{s(1-c)}^{1} (1 - s)^{N-n} c \left( s \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1} \, ds. 
\]
By (13),
\[
\int_{s(1-c)}^{1} (1 - s)^{N-n} \left( \frac{1}{2} r^2 + r^4 - c \right) r^{(n-1)^2} \, ds 
\]
\[
\leq \frac{1}{2} \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N - n + 1) \Gamma(n + \frac{2}{n-1})}{\Gamma(N + 1 + \frac{2}{n-1})} 
\]
\[
+ \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{4}{n-1}} \frac{\Gamma(N - n + 1) \Gamma(n + \frac{4}{n-1})}{\Gamma(N + 1 + \frac{4}{n-1})} 
\]
\[
- \left( 1 - \frac{1}{n^2} \right) \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1} \frac{\Gamma(N - n + 1) \Gamma(n)}{\Gamma(N + 1)} 
\]
\[
\times \frac{(n - 1)^{\frac{n+1}{n-1}} \text{vol}_{n-2}(\partial B^{n-1}_2)}{(\text{vol}_{n-2}(\partial B^{n-1}_2))^{\frac{2}{n-1}}} \frac{2}{2(n+1)!} \frac{\Gamma(n + 1 + \frac{2}{n-1})}{N^{-\frac{2}{n-1}}} 
\]
\[
+ cs(1-c) \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1} \max_{s\in[0,s(1-c)]} (1 - s)^{N-n} s^{n-1}. 
\]
Thus
\[
(22) \int_{s(1-c)}^{1} (1 - s)^{N-n} \left( \frac{1}{2} r^2 + r^4 - c \right) r^{(n-1)^2} \, ds 
\]
\[
\leq \frac{1}{2} \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N - n + 1) \Gamma(n + \frac{2}{n-1})}{\Gamma(N + 1 + \frac{2}{n-1})} 
\]
\[ + \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{4}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{4}{n-1})}{\Gamma(N+1+\frac{4}{n-1})} \]

\[ - \frac{1}{2} \left( 1 - \frac{1}{n^2} \right) \frac{n-1}{(n+1)n} \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+1+\frac{2}{n-1})}{\Gamma(N+1)} N^{-\frac{2}{n-1}} \]

\[ + cs(1-c) \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1} \max_{s \in [0,s(1-c)]} (1-s)^{N-n} s^{n-1}. \]

The second summand is asymptotically equal to

\[ \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{4}{n-1}} (N-n)!(n-1)! n^{-\frac{4}{n-1}} \]

\[ = \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{4}{n-1}} \frac{n^{-1+\frac{4}{n-1}}}{(N_n)(N+1)^{-\frac{4}{n-1}}}. \]

This summand is of the order $N^{-\frac{4}{n-1}}$ while the others are of the order $N^{-\frac{2}{n-1}}$.

We consider the sum of the first and third summands:

\[ \frac{1}{2} \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})} \]

\[ \times \left( 1 - \left( 1 - \frac{1}{n^2} \right) \frac{(n-1)\Gamma(n+1+\frac{2}{n-1})\Gamma(N+1+\frac{2}{n-1})}{n(n+1)\Gamma(n+1+\frac{2}{n-1})} \right) \]

Since $\Gamma(n+1+\frac{2}{n-1}) = (n+\frac{2}{n-1})\Gamma(n+\frac{2}{n-1})$ the latter expression equals

\[ \frac{1}{2} \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})} \]

\[ \times \left( 1 - \left( 1 - \frac{1}{n^2} \right) \frac{(n-1)(n+\frac{2}{n-1})\Gamma(N+1+\frac{2}{n-1})}{n(n+1)\Gamma(N+1)N^{-\frac{2}{n-1}}} \right). \]

Since $\Gamma(N+1+\frac{2}{n-1})$ is asymptotically equal to $(N+1)^{\frac{2}{n-1}} \Gamma(N+1)$ the sum of the first and third summands is for large $N$ of the order

\[ \frac{1}{n} \left( \frac{\text{vol}_{n-1}(\partial B^n_2)}{\text{vol}_{n-1}(B^{n-1}_2)} \right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})} \]

\[ \times \left( 1 - \left( 1 - \frac{1}{n^2} \right) \frac{(n-1)(n+\frac{2}{n-1})\Gamma(N+1+\frac{2}{n-1})}{n(n+1)\Gamma(N+1)} \right). \]

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which in turn is of the order

\[
(25) \quad \frac{1}{n^2} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1 + \frac{2}{n-1}} \left( \frac{N}{n} \right)^{-1} N^{-\frac{2}{n-1}}.
\]

We now consider the fourth summand. By (14) and (20) it is less than

\[
(26) \quad bN^{-\frac{2}{n-1}} \frac{n}{e^{5/6}N} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1} \max_{s \in [0, s(1-c)]} (1 - s)^{N-n} s^{n-1}.
\]

The maximum of the function \((1 - s)^{N-n} s^{n-1}\) is attained at \((n-1)/(N-1)\) and the function is increasing on the interval \([0, (n-1)/(N-1)]\). Therefore, by (20) we have \(s(1-c) < (n-1)/(N-1)\) and the maximum of this function over the interval \([0, s(1-c)]\) is attained at \(s(1-c)\). By (20) we have \(s(1-c) \leq \frac{1}{6} \frac{n}{eN}\) and thus for \(N\) sufficiently large

\[
\max_{s \in [0, s(1-c)]} (1 - s)^{N-n} s^{n-1} \leq \left( 1 - \frac{n}{e^{5/6}N} \right)^{N-n} \left( \frac{e^{1/6}}{n} \right)^{n-1} \leq \exp \left( \frac{n-1}{6} - \frac{n(N-n)}{e^{5/6}N} \right) \left( \frac{n}{eN} \right)^{n-1} \leq \exp \left( -\frac{n}{6} \right) \left( \frac{n}{eN} \right)^{n-1}.
\]

Thus we get for (26) the bound, with a new constant \(b\),

\[
(27) \quad bN^{-\frac{2}{n-1}} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1} e^{-n/6} \frac{n^n e^{-n}}{N^n}.
\]

This is asymptotically equal to

\[
(27) \quad bN^{-\frac{2}{n-1}} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1} e^{-n/6} \frac{1}{(\frac{N}{n})^{\sqrt{2\pi n}}}.
\]

Altogether, (15) for \(N\) sufficiently large can be estimated by

\[
\left( \frac{N}{n} \right)^{\binom{n-2}{2}(\partial B_2^n)} \frac{n}{(\text{vol}_{n-1}(\partial B_2^n))^{n-2}} \frac{n}{(n-1)^{n-1}} \times \left\{ \frac{1}{n^2} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1 + \frac{2}{n-1}} \left( \frac{N}{n} \right)^{-1} N^{-\frac{2}{n-1}} + bN^{-\frac{2}{n-1}} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1} e^{-n/6} \frac{1}{(\frac{N}{n})^{\sqrt{2\pi n}}} \right\}.
\]

This can be estimated by a constant times

\[
(28) \quad (\text{vol}_{n-1}(\partial B_2^n)) n \left\{ \frac{1}{n^2} N^{-\frac{2}{n-1}} + bN^{-\frac{2}{n-1}} e^{-n/6} \frac{1}{\sqrt{2\pi n}} \right\}.
\]
Finally, it should be noted that we have been estimating the approximation of $B_2^n(1 - c)$ and not that of $B_2^n$. Therefore, we need to multiply (28) by $(1 - c)^{-n}$. By (14),

$$(1 - c)^n \geq 1 - b \frac{n}{N^{\frac{n}{2-n}}}$$

so that for all $N$ with $N \geq (2bn)^{\frac{n-1}{2}}$ we have $(1 - c)^{-n} \leq 2$. ■

3. Proof of Theorem 2. We need another lemma.

**Lemma 9.** Let $P_M$ be a polytope with $M$ facets $F_1, \ldots, F_M$ that is best approximating for a convex body $K$ in $\mathbb{R}^n$ with respect to the symmetric difference metric. For $k = 1, \ldots, M$, let

$$F_k^i = F_k \cap K, \quad F_k^a = F_k \cap K^c.$$ 

Then, for all $j = 1, \ldots, M$,

$$\text{vol}_{n-1}(F_j^i) = \text{vol}_{n-1}(F_j^a).$$

**Proof.** Let $H_j$, $j = 1, \ldots, M$, be the hyperplane containing the face $F_j$. Then

$$P_M = \bigcap_{j=1}^{M} H_j^+.$$ 

Suppose $H_k = H(x_k; \xi_k)$, i.e. $H_k$ is the hyperplane containing $x_k$ and orthogonal to $\xi_k$. We consider

$$P_t = \bigcap_{j \neq k} H_j^+ \cap H^\perp \left(x_k + \frac{t}{\|x_k\|} x_k, \xi_k \right).$$

We have

$$\text{vol}_{n-1}(P_t \triangle K) = \text{vol}_{n-1}(P_M \triangle K) + t(\text{vol}_{n-1}(F_k^a) - \text{vol}_{n-1}(F_k^i)) + \psi(t)$$

where $\psi(t)/t^2$ is a bounded function. ■

**Proof of Theorem 2.** Let $P_M$ be a best approximating polytope with $M$ facets $F_1, \ldots, F_M$ for $B_2^n$ with respect to the symmetric difference metric. For $k = 1, \ldots, M$, let

$$F_k^i = F_k \cap B_2^n, \quad F_k^a = F_k \cap (B_2^n)^c,$$

let $H_k$ be the hyperplane containing the facet $F_k$ and let $C_k$ be the cap of $B_2^n$ with base $H_k \cap B_2^n$. (There are actually two caps, we take the one whose interior has empty intersection with $P_M$.) For $k = 1, \ldots, M$ we put

$$h_k = \begin{cases} 
\text{height of the cap } C_k & \text{if } F_k \cap (B_2^n)^\circ \neq \emptyset, \\
0, & \text{if } F_k \cap (B_2^n)^\circ = \emptyset.
\end{cases}$$
Then

\[ \text{vol}_{n-1}(P_M \triangle B_2^n) \geq \frac{1}{n} \sum_{k=1}^{M} h_k \text{vol}_{n-1}(F_k^i). \]

Let \( r_k \) be such that \( \text{vol}_{n-1}(r_k B_2^{n-1}) = \text{vol}_{n-1}(F_k^i). \) Thus

\[ r_k = \left( \frac{\text{vol}_{n-1}(F_k^i)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{1}{n-1}}. \]

Let \( \tilde{h}_k \) be the height of the cap of \( B_2^n \) with base \( r_k B_2^{n-1} \). Then

\[ \tilde{h}_k \leq h_k \text{ for all } k, \]

and

\[ \tilde{h}_k \geq \frac{1}{2} r_k^2 \geq \frac{1}{2} \left( \frac{\text{vol}_{n-1}(F_k^i)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{2}{n-1}}. \]

Thus from (29) with (30) we get

\[ \text{vol}_{n-1}(P_M \triangle B_2^n) \geq \frac{1}{2n} \sum_{k=1}^{M} \frac{(\text{vol}_{n-1}(F_k^i))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}} \]

\[ \geq \frac{1}{8\pi e} \sum_{k=1}^{M} \left( \frac{\text{vol}_{n-1}(F_k^i)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{n+1}{n-1}}. \]

We consider two cases. The first case is

\[ \sum_{k=1}^{M} \text{vol}_{n-1}(F_k^i) + \sum_{k=1}^{M} \text{vol}_{n-1}(F_k^n) \geq c \text{vol}_{n-1}(\partial B_2^n), \]

where \( M \geq 10^{(n-1)/2} \) and \( c = 9/10. \) It then follows from Lemma 9 that

\[ \sum_{k=1}^{M} \text{vol}_{n-1}(F_k^i) \geq \frac{c}{2} \text{vol}_{n-1}(\partial B_2^n). \]

By H"older’s inequality

\[ \sum_{k=1}^{M} \text{vol}_{n-1}(F_k^i) \leq \left( \sum_{k=1}^{M} (\text{vol}_{n-1}(F_k^i))^p \right)^{1/p} M^{1/p'}. \]

Therefore from (31) and (33) with \( p = \frac{n+1}{n-1} \) we get

\[ \text{vol}_{n-1}(P_M \triangle B_2^n) \geq \frac{c/2}{8\pi e} \frac{n+1}{n-1} \frac{1}{M^{\frac{2}{n-1}}} (n \text{vol}_{n}(B_2^n))^{\frac{n+1}{n-1}} \geq \frac{c^{n+1}}{8M^{\frac{2}{n-1}}} \text{vol}_{n}(B_2^n). \]
The second case is that (32) does not hold. Thus
\[
\sum_{k=1}^{M} \text{vol}_{n-1}(F_k) = \sum_{k=1}^{M} \text{vol}_{n-1}(F_k^n) + \sum_{k=1}^{M} \text{vol}_{n-1}(F_k^a) < c \text{vol}_{n-1}(\partial B_2^n).
\]
Then, by the isoperimetric inequality,
\[
\text{vol}_n(P_M) \leq \left( \frac{\sum_{k=1}^{M} \text{vol}_{n-1}(F_k)}{\text{vol}_{n-1}(\partial B_2^n)} \right)^{\frac{n}{n-1}} \text{vol}_n(B_2^n) < \bar{c}^{\frac{n}{n-1}} \text{vol}_n(B_2^n)
\]
and thus
\[
\text{vol}_n(P_M \triangle B_2^n) \geq (1 - \bar{c}^{\frac{n}{n-1}}) \text{vol}_n(B_2^n).
\]
Since \(c = 9/10\), this last expression is greater than \(M^{-\frac{2}{n-1}} \text{vol}_n(B_2^n)\), provided \(M \geq 10^{-\frac{n-1}{2}}\), which holds by assumption. ■

References


[GRS2] —, —, —, Erratum to [GRS1], ibid. 95 (1998), 331.


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