

Gagliardo–Nirenberg inequalities in weighted Orlicz spaces

by

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Abstract. We derive inequalities of Gagliardo–Nirenberg type in weighted Orlicz spaces on \mathbb{R}^n , for maximal functions of derivatives and for the derivatives themselves. This is done by an application of pointwise interpolation inequalities obtained previously by the first author and of Muckenhoupt–Bloom–Kerman-type theorems for maximal functions.

1. Introduction. Interpolation inequalities for derivatives have been studied for a long time. Beginning with the pioneering works by Hadamard [19–21], Kneser [32], Landau [38–40], Hardy, Littlewood and Landau [22–24], Kolmogorov’s celebrated article [34], and the famous results of Gagliardo [16] and Nirenberg [45], now there is a large body of literature on this subject (see e.g. [3, 7, 11, 12, 18, 25, 27, 36, 41–43, 46, 49] and their references).

Despite intensive investigation carried out in this area, there are only a few articles about interpolation inequalities in Orlicz spaces. The only ones we know of are contributions by Bang and coauthors [3–6] and our recent papers [26–28]. Here we continue the research in this direction.

Our main goal is to obtain inequalities of the form

$$(1.1) \quad \int \Psi(|\nabla^{(k)}u|) d\mu \leq \left(\int \Psi_1(|u|) d\mu + \int \Psi_2(|\nabla^{(m)}u|) d\mu \right)$$

and also

$$(1.2) \quad \|\nabla^{(k)}u\|_{L^\Psi(\mu)} \leq c \|u\|_{L^{\Psi_1}(\mu)}^{1-k/m} \|\nabla^{(m)}u\|_{L^{\Psi_2}(\mu)}^{k/m},$$

where Ψ, Ψ_1 and Ψ_2 are N-functions satisfying certain consistency conditions, $0 < k < m$ are positive integers, and μ belongs to some class of weighted Radon measures.

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Our results extend those of Bang [3] who dealt with inequalities (1.2) for one-variable function and the Lebesgue measure. In our previous papers [26–28], we have discussed inequalities (1.1), (1.2), together with their generalizations, for the Lebesgue measure. More precisely, in [26] and [28] we have obtained variants of (1.1) and (1.2) in various logarithmic Zygmund spaces (for $\mu = dx$) and our paper [27] was devoted to similar inequalities in general Orlicz spaces, but for the Lebesgue measure and $k = 1$, $m = 2$ only. They have arisen there as special cases of more general inequalities

$$(1.3) \quad \int \Psi(|\nabla u|) dx \leq \left(\int \Psi_1(R_1(|u|, |\nabla^{(2)}u|)) dx + \int \Psi_2(R_2(|u|, |\nabla^{(2)}u|)) dx \right)$$

and also

$$(1.4) \quad \|\nabla u\|_{L^\Psi(\mu)} \leq C \|R_1(|u|, |\nabla^{(2)}u|)\|_{L^{\Psi_1}(\mu)}^{1/2} \|R_2(|u|, |\nabla^{(2)}u|)\|_{L^{\Psi_2}(\mu)}^{1/2},$$

where $R_1, R_2 : [0, \infty)^2 \rightarrow \mathbb{R}_+$ were continuous functions such that

$$R_1(\lambda_1, \lambda_2)R_2(\lambda_1, \lambda_2) = \lambda_1\lambda_2.$$

Note that inequalities (1.1) and (1.2) for $k = 1, m = 2$ and the Lebesgue measure correspond to $R_1(\lambda_1, \lambda_2) = \lambda_1$ and $R_2(\lambda_1, \lambda_2) = \lambda_2$. The results of [26–28] were based on the crucial observation that for a smooth function u with bounded support the integral $\int \Psi(|\nabla u|) dx$ can be estimated by the quantity $\int N(|\nabla u|)|u| |\nabla^{(2)}u| dx$ (with a certain function N), which was then the object of further analysis.

Here we also deal with inequalities in the form (1.1) and (1.2), but our present approach is essentially different and brings new results. Namely, our starting point is the following pointwise multiplicative inequality expressed in terms of the Hardy–Littlewood maximal functions, obtained by the first author in [25]:

$$\mathcal{M}(\nabla^{(k)}f)(x) \leq C_0(\mathcal{M}f(x))^{1-k/m}(\mathcal{M}(\nabla^{(m)}f)(x))^{k/m},$$

holding true for every $f \in W_{\text{loc}}^{m,1}(\mathbb{R}^n)$ and almost all $x \in \mathbb{R}^n$. Then, using Young inequalities in Orlicz spaces, we obtain counterparts of (1.1) and (1.2) for maximal functions (see Theorem 3.1):

$$(1.5) \quad \int_{\mathbb{R}^n} \Phi(C_s \mathcal{M}(\nabla^{(k)}f)) d\mu \leq \int_{\mathbb{R}^n} \Phi_1(s_1 \mathcal{M}f) d\mu + \int_{\mathbb{R}^n} \Phi_2(s_2 \mathcal{M}(\nabla^{(m)}f)) d\mu$$

and

$$(1.6) \quad \|\mathcal{M}(\nabla^{(k)}f)\|_{L^\Phi(\mu)} \leq C_1 \|\mathcal{M}f\|_{L^{\Phi_1}(\mu)}^{1-k/m} \|\mathcal{M}(\nabla^{(m)}f)\|_{L^{\Phi_2}(\mu)}^{k/m},$$

where Φ, Φ_1, Φ_2 are certain functions, μ is an arbitrary nonnegative Radon measure, and s, s_1, s_2 are certain positive numbers. Inequality (1.6) requires Φ, Φ_1, Φ_2 to be N-functions, but (1.5) is valid for some nonconvex functions as well. In Section 4 inequalities (1.5) and (1.6) are then transformed into

(1.1) and (1.2). This is done by an application of various Muckenhoupt–Bloom–Kerman-type direct and reverse inequalities for maximal functions from [8, 30, 31], within appropriate classes of measures.

Although inequalities (1.3) and (1.4) do not follow from our new methods, those derived now extend some of the previously obtained ones (for $\mu = dx$, $k = 1$, $m = 2$) to wider classes of admissible functions and measures. More precisely, when we restrict ourselves to $k = 1$, $m = 2$, $\mu = dx$, then the Gagliardo–Nirenberg inequalities (1.1) and (1.2) are a special case of (1.3) and (1.4). These are the inequalities that are generalized presently. See Section 5 for a detailed discussion.

We hope that the results presented here will contribute to the development of regularity theory for PDE’s in Orlicz spaces, similarly to the classical case. For various regularity results in Orlicz–Sobolev spaces and motivations we refer to the papers [10, 14, 47, 51] and their references.

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2. Preliminaries and notation. We start by recalling preliminary facts about Orlicz spaces, referring e.g. to [35] for details.

Suppose that μ is a positive Radon measure on \mathbb{R}^n and let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be an *N-function*, i.e. a strictly convex function satisfying $\lim_{\lambda \rightarrow 0} \Phi(\lambda)/\lambda = 0$ and $\lim_{\lambda \rightarrow \infty} \Phi(\lambda)/\lambda = \infty$.

The *weighted Orlicz space* $L^\Phi(\mu)$ is the space

$$L^\Phi(\mu) := \left\{ f \text{ measurable} : \int_{\mathbb{R}^n} \Phi(|f(x)|/K) d\mu(x) \leq 1 \text{ for some } K > 0 \right\},$$

equipped with the Luxemburg norm

$$\|f\|_{L^\Phi(\mu)} = \inf \left\{ K > 0 : \int_{\mathbb{R}^n} \Phi(|f(x)|/K) d\mu(x) \leq 1 \right\}.$$

This norm is complete and turns $L^\Phi(\mu)$ into a Banach space. When μ is the Lebesgue measure, it is dropped from the notation. For $\Phi(\lambda) = \lambda^p$ with $p > 1$, the space $L^\Phi(\mu)$ coincides with the usual $L^p(\mu)$ space.

The symbol Φ^* denotes the *Legendre transform* of an N-function Φ , i.e. $\Phi^*(y) = \sup_{x>0} [xy - \Phi(x)]$, defined for $y \geq 0$. It is again an N-function and from its definition we get the *Young inequality*:

$$(2.1) \quad xy \leq \Phi(x) + \Phi^*(y) \quad \text{for } x, y \geq 0.$$

Φ is said to satisfy the Δ_2 -condition if, for some constant $c > 0$ and every $\lambda > 0$, we have

$$(2.2) \quad \Phi(2\lambda) \leq c\Phi(\lambda).$$

In the class of differentiable convex functions the Δ_2 -condition is equivalent to

$$(2.3) \quad \lambda\Phi'(\lambda) \leq c\Phi(\lambda),$$

satisfied for every $\lambda > 0$, with a constant c independent of λ (see e.g. [35, Theorem 4.1]).

We will need the following two properties of modular functionals (see [35, formulas (9.20) and (9.4)]): for every $f \in L^\Phi(\mu)$ we have

$$(2.4) \quad \|f\|_{L^\Phi(\mu)} \leq \int_{\mathbb{R}^n} \Phi(|f(x)|) d\mu(x) + 1,$$

and also ([35, formula (9.21)])

$$(2.5) \quad \int_{\mathbb{R}^n} \Phi\left(\frac{f(x)}{\|f\|_{L^\Phi(\mu)}}\right) d\mu(x) \leq 1.$$

If Φ satisfies the Δ_2 -condition, then (2.5) becomes an equality.

The function Φ_1 is said to *dominate* Φ_2 (symbolically: $\Phi_2 \prec \Phi_1$) if there exist two positive constants K_1, K_2 such that $\Phi_2(\lambda) \leq K_1\Phi_1(K_2\lambda)$ for every $\lambda > 0$. In that case we have

$$(2.6) \quad \|\cdot\|_{L_2^\Phi(\mu)} \leq K\|\cdot\|_{L_1^\Phi(\mu)} \quad \text{with } K = K_2(K_1 + 1).$$

Functions Φ_1 and Φ_2 are called *equivalent* (symbolically $\Phi_1 \asymp \Phi_2$) when $\Phi_2 \prec \Phi_1$ and $\Phi_1 \prec \Phi_2$. In particular equivalent N-functions give rise to equivalent Luxemburg norms.

We use the standard notation: $C_0^k(\mathbb{R}^n)$ for compactly supported functions of class C^k defined on \mathbb{R}^n , and $L^p(\mathbb{R}^n)$, $L_{\text{loc}}^p(\mathbb{R}^n)$, $W^{k,p}(\mathbb{R}^n)$, $W_{\text{loc}}^{k,p}(\mathbb{R}^n)$ for the L^p and Sobolev spaces respectively. By $\tilde{C}_0(\mathbb{R}^n)$ we denote the continuous functions on \mathbb{R}^n vanishing at infinity, while $L^{0,\Phi}(\mu)$ stands for the completion of $\tilde{C}_0(\mathbb{R}^n)$ in the space $L^\Phi(\mu)$. By $\nabla^{(k)}f$ we denote the vector $(D^\alpha f)_{|\alpha|=k}$, understood in the classical sense (we assume that f is of class C^k). If w is a vector in Euclidean space then $|w|$ stands for its standard norm.

The letter c is reserved to denote a generic constant, whose value can change from line to line. The relevant constants are denoted by upper-case letters.

3. Inequalities involving maximal functions. Let $f \in L_{\text{loc}}^1(\mathbb{R}^n)$. The *Hardy–Littlewood maximal function* of f is defined as

$$\mathcal{M}f(x) = \sup_{S: x \in S} \frac{1}{|S|} \int_S |f(y)| dy,$$

where the supremum is taken over all cubes S in \mathbb{R}^n containing x with edges parallel to the axes, and $|S|$ denotes their Lebesgue measure (see e.g. [33, 50]).

For a vector-valued function f , the symbol $\mathcal{M}f$ stands for the maximal function of $|f|$.

The main result of this section reads as follows.

THEOREM 3.1. *Let $k, m \in \mathbb{Z}_+$ with $0 < k < m$. Suppose that μ is an arbitrary positive Radon measure on \mathbb{R}^n , $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function, and $F : [0, \infty) \rightarrow [0, \infty)$ an N -function. Set*

$$(3.1) \quad \Phi_1(\lambda) = \Phi(F(\lambda^{1-k/m})), \quad \Phi_2(\lambda) = \Phi(F^*(\lambda^{k/m})).$$

Then there exists a constant $C = C(n) > 0$ such that for every $f \in C_0^m(\mathbb{R}^n)$ and any numbers $s, s_1, s_2 > 0$ which satisfy $s = s_1^{1-k/m} s_2^{k/m}$ one has

$$(3.2) \quad \int_{\mathbb{R}^n} \Phi(C_s \mathcal{M}(\nabla^{(k)} f)) d\mu \leq \int_{\mathbb{R}^n} \Phi_1(s_1 \mathcal{M}f) d\mu + \int_{\mathbb{R}^n} \Phi_2(s_2 \mathcal{M}(\nabla^{(m)} f)) d\mu.$$

Moreover, if Φ , Φ_1 and Φ_2 are N -functions, then also

$$(3.3) \quad \|\mathcal{M}(\nabla^{(k)} f)\|_{L^\Phi(\mu)} \leq C_1 \|\mathcal{M}f\|_{L^{\Phi_1}(\mu)}^{1-k/m} \|\mathcal{M}(\nabla^{(m)} f)\|_{L^{\Phi_2}(\mu)}^{k/m},$$

with a constant C_1 independent of $f \in C_0^m(\mathbb{R}^n)$.

Proof. Our result is based on the following version of the Gagliardo–Nirenberg inequality obtained by the first author in [25] (see also [41, 42] for other related inequalities):

$$(3.4) \quad \mathcal{M}(\nabla^{(k)} f)(x) \leq C_0 (\mathcal{M}f(x))^{1-k/m} (\mathcal{M}(\nabla^{(m)} f)(x))^{k/m},$$

holding for every $f \in W_{\text{loc}}^{m,1}(\mathbb{R}^n)$ and almost every x (with respect to the Lebesgue measure), with a constant $C_0 > 0$ depending on dimension only.

It is not hard to show that for $f \in C_0(\mathbb{R}^n)$ one has $\mathcal{M}f \in \tilde{C}_0(\mathbb{R}^n)$. In particular, for $f \in C_0^m(\mathbb{R}^n)$ all the functions

$$g(x) = \frac{1}{2C_0} \mathcal{M}(\nabla^{(k)} f)(x), \quad g_1(x) = \mathcal{M}f(x), \quad g_2(x) = \mathcal{M}(\nabla^{(m)} f)(x)$$

belong to $\tilde{C}_0(\mathbb{R}^n)$, so that the inequality (3.4) holds true for every $x \in \mathbb{R}^n$.

Let s_1, s_2 be arbitrary positive numbers, $\alpha_1 = 1 - k/m$, $\alpha_2 = k/m$, $s = s_1^{\alpha_1} s_2^{\alpha_2}$, $\tilde{g} = sg$, $\tilde{g}_1 = s_1 g_1$, $\tilde{g}_2 = s_2 g_2$. Then (3.4) can be rewritten as

$$(3.5) \quad \tilde{g}(x) \leq \frac{1}{2} \tilde{g}_1(x)^{\alpha_1} \tilde{g}_2(x)^{\alpha_2}.$$

Since Φ is monotone, (3.4) combined with the Young inequality (2.1) implies that

$$(3.6) \quad \begin{aligned} \Phi(\tilde{g}(x)) &\leq \Phi\left(\frac{1}{2} (F(\tilde{g}_1(x)^{\alpha_1}) + F^*(\tilde{g}_2(x)^{\alpha_2}))\right) \\ &\leq \Phi(\max\{F(\tilde{g}_1(x)^{\alpha_1}), F^*(\tilde{g}_2(x)^{\alpha_2})\}) \\ &\leq \Phi_1(\tilde{g}_1(x)) + \Phi_2(\tilde{g}_2(x)). \end{aligned}$$

Integrating this over \mathbb{R}^n with respect to μ gives

$$(3.7) \quad \int_{\mathbb{R}^n} \Phi \left(\frac{s}{2C_0} \mathcal{M}(\nabla^{(k)} f) \right) d\mu \\ \leq \int_{\mathbb{R}^n} \Phi_1(s_1 \mathcal{M}f) d\mu + \int_{\mathbb{R}^n} \Phi_2(s_2 \mathcal{M}(\nabla^{(m)} f)) d\mu.$$

For $C = (2C_0)^{-1}$, this is (3.2).

To prove (3.3), assume that both $\|\mathcal{M}f\|_{L^{\Phi_1}(\mu)}$ and $\|\mathcal{M}(\nabla^{(m)} f)\|_{L^{\Phi_2}(\mu)}$ are nonzero. This can be done without loss of generality, as otherwise $f \equiv 0$. Then take $s_1 = (\|\mathcal{M}f\|_{L^{\Phi_1}(\mu)})^{-1}$ and $s_2 = (\|\mathcal{M}(\nabla^{(m)} f)\|_{L^{\Phi_2}(\mu)})^{-1}$. Using (2.5), from (3.7) we obtain $\int \Phi(\frac{s}{2C_0} \mathcal{M}(\nabla^{(k)} f)) d\mu \leq 2$, which further gives that $\int \Phi_0(\frac{s}{2C_0} \mathcal{M}(\nabla^{(k)} f)) d\mu \leq 1$, where we put $\Phi_0(\lambda) = \frac{1}{2}\Phi(\lambda)$. Consequently, from the very definition of the Luxemburg norm,

$$(3.8) \quad \|\mathcal{M}(\nabla^{(k)} f)\|_{L^{\Phi_0}(\mu)} \leq 2C_0 s^{-1} = 2C_0 s_1^{-\alpha_1} s_2^{-\alpha_2}.$$

As equivalent N-functions give rise to equivalent norms, this gives (3.3). More precisely, one applies (2.6) to get

$$\|\mathcal{M}(\nabla^{(k)} f)\|_{L^{\Phi}(\mu)} \leq 3\|\mathcal{M}(\nabla^{(k)} f)\|_{L^{\Phi_0}(\mu)},$$

which together with (3.8) implies (3.3) with $C_1 = 6C_0$. ■

The Δ_2 -condition permits to move the constant C in (3.2) outside the integral. In this case we have the following result.

COROLLARY 3.1. *Let $k, m \in \mathbb{Z}$ with $0 < k < m$. Suppose that μ is an arbitrary nonnegative Radon measure on \mathbb{R}^n , $F : [0, \infty) \rightarrow [0, \infty)$ is an N-function, $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function, Φ_1 and Φ_2 are given by (3.1), and additionally at least one of the functions Φ, Φ_1, Φ_2 satisfies the Δ_2 -condition. Then there exists a constant $\tilde{C} > 0$ such that for every $f \in C_0^m(\mathbb{R}^n)$ one has*

$$(3.9) \quad \int_{\mathbb{R}^n} \Phi(\mathcal{M}(\nabla^{(k)} f)) d\mu \leq \tilde{C} \left(\int_{\mathbb{R}^n} \Phi_1(\mathcal{M}f) d\mu + \int_{\mathbb{R}^n} \Phi_2(\mathcal{M}(\nabla^{(m)} f)) d\mu \right).$$

Proof. If Φ satisfies the Δ_2 -condition, then we apply (3.7) with $s = s_1 = s_2 = 1$ and then the Δ_2 -condition. On the other hand, when one of the functions Φ_1, Φ_2 , say Φ_1 , satisfies the Δ_2 -condition, one applies (3.7) for $s_1 = (2C_0)^{m/(m-k)}$, $s_2 = 1$, $s = 2C_0$. To conclude, one uses the Δ_2 -condition (2.2) for the function Φ_1 . ■

REMARK 3.1. Note that the Radon measure admissible in inequalities (3.2), (3.3), and (3.9) can be taken arbitrary. In particular it need not be absolutely continuous with respect to the Lebesgue measure. Also, we do not

require the function Φ to be convex in order for (3.2) and (3.9) to hold. Its monotonicity is sufficient.

REMARK 3.2. We already know from the proof of Theorem 3.1 that (3.4) implies (3.2). On the other hand, if (3.2) is satisfied for an arbitrary Radon measure μ and arbitrary Φ, F, s, s_1, s_2 as in the statement of Theorem 3.1, then (3.2) implies (3.4). Indeed, once we take $\mu = \delta_{\{x\}}$ to be the Dirac measure, $\Phi = \text{id}$, $s = s_1 = s_2 = 1$ and $F(\lambda) = \lambda^{m/(m-k)}$, we arrive at

$$\mathcal{M}(\nabla^{(k)}f)(x) \leq C(\mathcal{M}f(x) + \mathcal{M}(\nabla^{(m)}f)(x)).$$

Then (3.4) is obtained by the classical rescaling argument: first substitute $f_t(x) = f(tx)$ in the inequality above and then optimize with respect to $t > 0$.

REMARK 3.3. An obvious corollary of (3.4) is the following inequality, holding for an arbitrary nondecreasing function Φ and $f \in C_0^m(\mathbb{R}^n)$:

$$(3.10) \quad \int_{\mathbb{R}^n} \Phi(\mathcal{M}(\nabla^{(k)}f)) d\mu \leq \int_{\mathbb{R}^n} \Phi(C_0(\mathcal{M}f)^{1-k/m}(\mathcal{M}(\nabla^{(m)}f))^{k/m}) d\mu,$$

where μ is an arbitrary Radon measure, $k, m \in \mathbb{N}$, $0 < k < m$, and the constant C_0 is independent of f . In [26] we have shown that if Φ satisfies the “differential” Δ_2 -condition (2.3) (but is not necessarily convex) and moreover $\Phi(\lambda)/\lambda^2$ is nondecreasing, then for every $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$(3.11) \quad \int_{\mathbb{R}^n} \Phi(|\nabla f|) dx \leq \int_{\mathbb{R}^n} \Phi(C\sqrt{|f| |\nabla^{(2)}f|}) dx.$$

Obviously, as $|h| \leq \mathcal{M}h$ almost everywhere, we see that both inequalities (3.10) (considered for $\mu = dx$, $k = 1$, $m = 2$) and (3.11) imply

$$\int_{\mathbb{R}^n} \Phi(|\nabla f|) dx \leq \int_{\mathbb{R}^n} \Phi(C\sqrt{\mathcal{M}f \cdot \mathcal{M}(\nabla^{(2)}f)}) dx,$$

but (3.10) (for $\mu = dx$, $k = 1$, $m = 2$) and (3.11) do not seem to be equivalent.

4. Inequalities for derivatives. We now aim at transforming inequalities (3.2) and (3.3) into corresponding inequalities of Gagliardo–Nirenberg type involving derivatives rather than maximal functions.

As a tool we present below some strong type inequalities for the Hardy–Littlewood maximal operator between Orlicz spaces. The results summarized in Section 4.1 are known.

4.1. Strong-type inequalities for maximal functions. Summary of known results. In this section we deal with positive absolutely continuous Radon measures on \mathbb{R}^n , $\mu(dx) = w(x)dx$, where $w(x)$ is a *weight function* (a non-negative, locally integrable function $w : \mathbb{R}^n \rightarrow [0, \infty)$) satisfying certain conditions described below.

The following definition and the subsequent theorem are due to Bloom and Kerman (see [8]).

DEFINITION 4.1 (the class W_Φ). Suppose that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is an N -function. We say that a weight $w : \mathbb{R}^n \rightarrow [0, \infty)$ of a measure $\mu(dx) = w(x)dx$ belongs to the class W_Φ if for all cubes $S \subset \mathbb{R}^n$ and all $\lambda > 0$,

$$\int_S \Phi^* \left(\frac{\Phi(\lambda)\mu(S)}{c\lambda|S|w(x)} \right) w(x) dx \leq \Phi(\lambda)\mu(S) < \infty,$$

with a constant $c > 0$ independent of S .

If $\Phi(\lambda) = \lambda^p$, $p > 1$, then the class W_Φ coincides with the class of A_p -weights (see e.g. [50] for the definition).

THEOREM 4.1. *Suppose that Φ is an N -function and let w be a nonnegative weight on \mathbb{R}^n . Then the following two statements are equivalent:*

(a) *For all measurable u ,*

$$(4.1) \quad \int_{\mathbb{R}^n} \Phi(\mathcal{M}u(x))w(x) dx \leq \int_{\mathbb{R}^n} \Phi(B|u(x)|)w(x) dx,$$

with some constant $B = B_\Phi$ not depending on u .

(b) *Φ^* satisfies the Δ_2 -condition and $w \in W_\Phi$.*

Following [8], if Φ^* satisfies the Δ_2 -condition, then we write $\Phi \in \Delta_2^c$.

This theorem completely describes the class of weights for which the inequality (4.1) holds. In particular, if Φ^* does not satisfy the Δ_2 -condition, then there is no weight w for which (4.1) could possibly hold for all measurable functions u . Theorem 4.1 generalizes the Muckenhoupt theorem (see e.g. [44]), which asserts that the maximal operator is of strong type (p, p) if and only if the weight considered is an A_p -weight. After Muckenhoupt, variants of Theorem 4.1 were addressed also by Kerman and Torchinsky under the assumption that both Φ and Φ^* satisfy the Δ_2 -condition (see [29]), Kokilashvili and Krbeč (Theorem 8.3.1 on page 339 in [33]) and Lai [37]. See also the references in those papers.

REMARK 4.1. If $\Phi_1 \asymp \Phi_2$, then also $\Phi_1^* \asymp \Phi_2^*$, and so the Δ_2^c -property is shared by all equivalent functions. It is also easy to see that if $\Phi_1 \asymp \Phi_2$ and $\Phi_1 \in \Delta_2^c$, then $W_{\Phi_1} = W_{\Phi_2}$. Indeed, suppose that $\Phi_1 \asymp \Phi_2$ and $w \in W_{\Phi_1}$. Then according to Theorem 4.1 we have

$$\int_{\mathbb{R}^n} \Phi_1(\mathcal{M}u(x))w(x) dx \leq \int_{\mathbb{R}^n} \Phi_1(Bu(x))w(x) dx,$$

with a constant B independent of u . This implies

$$\int_{\mathbb{R}^n} \Phi_2(C_1\mathcal{M}u(x))w(x) dx \leq C_2 \int_{\mathbb{R}^n} \Phi_2(C_3u(x))w(x) dx,$$

with constants C_1, C_2, C_3 independent of u . Substitution $\tilde{u} = C_1 u$ leads to a similar inequality with $C_1 = 1$ (and possibly different C_3). We may also assume that $C_2 > 1$. Next, the convexity of Φ_2 together with the property $\Phi_2(0) = 0$ give $C_2 \Phi_2(\lambda) \leq \Phi_2(C_2 \lambda)$, which implies

$$\int_{\mathbb{R}^n} \Phi_2(\mathcal{M}u(x))w(x) dx \leq \int_{\mathbb{R}^n} \Phi_2(Cu(x))w(x) dx,$$

with a constant C independent of u . Finally, again from Theorem 4.1, we get $w \in W_{\Phi_2}$ (and $\Phi_2 \in \Delta_2^c$), which is exactly what we claimed.

As it is usually not possible to express Φ^* in a closed form, below we present a quantitative tool which permits one to decide whether a given function belongs to the class Δ_2^c . It is basically taken from [35, Theorem 4.3]. Since slight modifications are needed, the proof is given in the Appendix.

PROPOSITION 4.1. *Suppose that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a continuously differentiable N -function. Then the following three statements are equivalent:*

- (D1) $\Phi \in \Delta_2^c$,
- (D2) *there exists a constant $\alpha > 1$ such that for all $t > 0$ we have $t\Phi'(t)/\Phi(t) \geq \alpha$,*
- (D3) *there exists a constant $\alpha > 1$ such that $\Phi(t)/t^\alpha$ is nondecreasing.*

Results similar to Theorem 4.1 are also available when Φ^* does not satisfy the Δ_2 -condition. In this case one has to deal with two distinct Orlicz spaces L_Φ and L_Ψ . For related results we refer to [2, 8, 13, 17, 29–31, 33, 37, 48].

Below (Theorem 4.2) we present two results which were included in Kita's papers [30, 31]. For completeness, also the proof of Theorem 4.2 is given in the Appendix.

To continue, we recall two further classes of weights.

DEFINITION 4.2. We say that a weight function $w : \mathbb{R}^n \rightarrow [0, \infty)$ belongs to the A_1 -class ($w \in A_1$) if there exists a constant $C > 0$ such that for every open cube $S \subseteq \mathbb{R}^n$ we have

$$\frac{1}{|S|} \int_S w(y) dy \leq C \operatorname{ess\,inf}_{x \in S} w(x).$$

DEFINITION 4.3. We say that a weight function $w : \mathbb{R}^n \rightarrow [0, \infty)$ belongs to the A'_∞ -class ($w \in A'_\infty$) if there exists a constant $C > 0$ such that for every open cube $S \subseteq \mathbb{R}^n$ we have

$$\frac{1}{|S|} \int_{2S} w(y) dy \geq C \operatorname{ess\,sup}_{x \in S} w(x).$$

Following Kita, for a weight function $w : \mathbb{R}^n \rightarrow [0, \infty)$, we write

$$(4.2) \quad \mathcal{R}_0(w) = \left\{ f \text{ measurable: } \int_{\{|f|>t\}} |f(x)|w(x) dx < \infty \text{ for every } t > 0 \right\}.$$

THEOREM 4.2. *Suppose that $a, b : [0, \infty) \rightarrow [0, \infty)$ are functions such that $a(s)/s$ is locally integrable on $(0, \infty)$,*

$$\int_0^1 \frac{a(t)}{t} dt < \infty \quad \text{and} \quad b(t) = \int_0^t \frac{a(s)}{s} ds.$$

Set

$$(4.3) \quad \Phi(t) = \int_0^t a(s) ds \quad \text{and} \quad \Psi(t) = \int_0^t b(s) ds.$$

Then the following two statements hold true:

1. If $w \in A_1$, then for every $f \in L^1_{\text{loc}} \cap \mathcal{R}_0(w)$,

$$(4.4) \quad \int_{\mathbb{R}^n} \Phi(\mathcal{M}f(x))w(x) dx \leq K_1 \int_{\mathbb{R}^n} \Psi(|f(x)|)w(x) dx.$$

2. If $w \in A'_\infty$, then for every $f \in L^1_{\text{loc}} \cap \mathcal{R}_0(w)$,

$$(4.5) \quad \int_{\mathbb{R}^n} \Psi(|f(x)|)w(x) dx \leq K_2 \int_{\mathbb{R}^n} \Phi(\mathcal{M}f(x))w(x) dx.$$

The constants $K_1, K_2 > 0$ do not depend on f nor on Φ and Ψ , but only on the weight function w .

Inequality (4.5) is an example of a reverse strong-type inequality for the Hardy–Littlewood maximal function. Observe that since $|u| \leq |\mathcal{M}u|$ a.e., it is always true with $\Phi = \Psi$ and an arbitrary weight function w . Other results in this direction, for logarithmic spaces $L(\log L)^k$, were previously obtained by Favo, Gatto and Gutiérrez [15] and by Anderson and Young [1].

It remains an open problem to describe the class of weights that appear in (4.4) with Φ and Ψ as in (4.3).

4.2. Interpolation inequalities for derivatives. Using Theorems 4.1 and 4.2 we can now obtain the Gagliardo–Nirenberg inequalities for derivatives in Orlicz spaces.

Our first theorem fits the Bloom–Kerman setting.

THEOREM 4.3. *Suppose that $\Phi : [0, \infty) \rightarrow [0, \infty)$ and $F : [0, \infty) \rightarrow [0, \infty)$ are two N -functions. Let $\mu(dx) = w(x)dx$, where w is a nonnegative weight on \mathbb{R}^n . Take $k, m \in \mathbb{Z}_+$, $0 < k < m$, and define*

$$(4.6) \quad \Phi_1(\lambda) = \Phi(F(\lambda^{1-k/m})), \quad \Phi_2(\lambda) = \Phi(F^*(\lambda^{k/m})).$$

If Φ_1, Φ_2 are N -functions of class Δ_2^c and $w \in W_{\Phi_1} \cap W_{\Phi_2}$ (see Definition 4.1), then for every $f \in C_0^m(\mathbb{R}^n)$, and arbitrary positive numbers s, s_1, s_2 such that

$s = s_1^{1-k/m} s_2^{k/m}$, one has

$$(4.7) \quad \int_{\mathbb{R}^n} \Phi(s|\nabla^{(k)} f|) d\mu \leq \int_{\mathbb{R}^n} \Phi_1(s_1 B_1 |f|) d\mu + \int_{\mathbb{R}^n} \Phi_2(s_2 B_2 |\nabla^{(m)} f|) d\mu,$$

$$(4.8) \quad \|\nabla^{(k)} f\|_{L^\Phi(\mu)} \leq B_3 \|f\|_{L^{\Phi_1}(\mu)}^{1-k/m} \|\nabla^{(m)} f\|_{L^{\Phi_2}(\mu)}^{k/m},$$

where the constants B_i are independent of f and s_i .

Proof. As Φ is increasing, and for every measurable function u one has $|u| < \mathcal{M}u$ a.e., using (3.7) we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi\left(\frac{s}{2C_0} |\nabla^{(k)} f|\right) d\mu &\leq \int_{\mathbb{R}^n} \Phi\left(\frac{s}{2C_0} \mathcal{M}(\nabla^{(k)} f)\right) d\mu \\ &\leq \int_{\mathbb{R}^n} \Phi_1(s_1 \mathcal{M}f) d\mu + \int_{\mathbb{R}^n} \Phi_2(s_2 \mathcal{M}(\nabla^{(m)} f)) d\mu, \end{aligned}$$

where $s_1, s_2 > 0$ are arbitrary and $s = s_1^{1-k/m} s_2^{k/m}$. According to Theorem 4.1, for all measurable u we have $\int_{\mathbb{R}^n} \Phi_i(\mathcal{M}u) d\mu \leq \int_{\mathbb{R}^n} \Phi_i(B_{\Phi_i}|u|) d\mu$ for $i = 1, 2$, which gives

$$\int_{\mathbb{R}^n} \Phi\left(\frac{s}{2C_0} |\nabla^{(k)} f|\right) d\mu \leq \int_{\mathbb{R}^n} \Phi_1(s_1 B_{\Phi_1} |f|) d\mu + \int_{\mathbb{R}^n} \Phi_2(s_2 B_{\Phi_2} |\nabla^{(m)} f|) d\mu.$$

Now it suffices to substitute $\tilde{f} = 2C_0 f$ in the equation above, which results in (4.7) with $B_i = 2C_0 B_{\Phi_i}$ where B_{Φ_i} are the constants from Theorem 4.1 and C_0 is taken from (3.4).

For the proof of (4.8) first note that the inequality $|u| \leq \mathcal{M}u$ implies $\|u\|_{L^\Phi(\mu)} \leq \|\mathcal{M}u\|_{L^\Phi(\mu)}$. Next, it can be derived from Theorem 4.1 that for an arbitrary measurable function u one has $\|\mathcal{M}u\|_{L^{\Phi_i}(\mu)} \leq \tilde{B}_i \|u\|_{L^{\Phi_i}(\mu)}$, with \tilde{B}_i independent of u , provided $\Phi_i \in \Delta_2^c$ and $\mu \in W_\Phi$. Now (4.8) follows directly from (3.3). ■

REMARK 4.2.

- (i) Choosing $s = s_1 = s_2 = 1$ (resp. $s_1 = B_1^{-1}, s_2 = B_2^{-1}$) in (4.7) under the assumptions of Theorem 4.3 we get, for all $f \in C_0^m(\mathbb{R}^n)$,

$$(4.9) \quad \int_{\mathbb{R}^n} \Phi(|\nabla^{(k)} f|) d\mu \leq \int_{\mathbb{R}^n} \Phi_1(B_1 |f|) d\mu + \int_{\mathbb{R}^n} \Phi_2(B_2 |\nabla^{(m)} f|) d\mu$$

and

$$(4.10) \quad \int_{\mathbb{R}^n} \Phi(\tilde{B} |\nabla^{(k)} f|) d\mu \leq \int_{\mathbb{R}^n} \Phi_1(|f|) d\mu + \int_{\mathbb{R}^n} \Phi_2(|\nabla^{(m)} f|) d\mu.$$

The constants \tilde{B}, B_1, B_2 do not depend on f .

(ii) If additionally at least one of the functions Φ , Φ_1 or Φ_2 satisfies the Δ_2 -condition, then

$$(4.11) \quad \int_{\mathbb{R}^n} \Phi(|\nabla^{(k)} f|) d\mu \leq C \left(\int_{\mathbb{R}^n} \Phi_1(|f|) d\mu + \int_{\mathbb{R}^n} \Phi_2(|\nabla^{(m)} f|) d\mu \right),$$

with a constant C independent of f . See the proof of Corollary 3.1.

If either Φ_1^* or Φ_2^* does not satisfy the Δ_2 -condition, then the Bloom–Kerman theorem is not applicable. Instead, we use Kita’s results summarized in Theorem 4.2. This leads to the following theorem.

THEOREM 4.4. *Let $k, m \in \mathbb{Z}_+$ with $0 < k < m$ and $\mu(dx) = w(x)dx$, where $w \in A_1 \cap A'_\infty$. Suppose that $\Phi_0 : [0, \infty) \rightarrow [0, \infty)$ is an increasing function of class $C^1((0, \infty))$ such that $\Phi_0(0) = (\Phi_0)'_+(0) = 0$, and that $F : [0, \infty) \rightarrow [0, \infty)$ is an N -function of class C^1 . Set*

$$\Phi_1(\lambda) = \Phi_0(F(\lambda^{1-k/m})), \quad \Phi_2(\lambda) = \Phi_0(F^*(\lambda^{k/m})).$$

Assume further that $\int_0^1 \frac{\Phi_i(v)}{v^2} dv < \infty$ for $i = 0, 1, 2$, and define

$$(4.12) \quad \Psi_i(\lambda) = \int_0^1 \frac{\Phi_i(\lambda v)}{v^2} dv, \quad i = 0, 1, 2.$$

Then there exist constants $C, K > 0$ such that for every $f \in C_0^m(\mathbb{R}^n)$ and for any positive numbers s, s_1, s_2 such that $s = s_1^{1-k/m} s_2^{k/m}$ one has

$$(4.13) \quad \int_{\mathbb{R}^n} \Psi_0(Cs|\nabla^{(k)} f|) d\mu \leq K \left(\int_{\mathbb{R}^n} \Psi_1(s_1|f|) d\mu + \int_{\mathbb{R}^n} \Psi_2(s_2|\nabla^{(m)} f|) d\mu \right).$$

Moreover, if Ψ_1 and Ψ_2 are N -functions, then also

$$(4.14) \quad \|\nabla^{(k)} f\|_{L^{\Psi_0}(\mu)} \leq \tilde{K} \|f\|_{L^{\Psi_1}(\mu)}^{1-k/m} \|\nabla^{(m)} f\|_{L^{\Psi_2}(\mu)}^{k/m},$$

with the constant \tilde{K} independent of f .

REMARK 4.3. The constant C in (4.13) is taken from (3.2) and depends on n only.

Proof. By (3.7), we have

$$\begin{aligned} I &:= \int_{\mathbb{R}^n} \Phi_0 \left(\frac{s}{2C_0} \mathcal{M}(\nabla^{(k)} f) \right) d\mu \\ &\leq \int_{\mathbb{R}^n} \Phi_1(s_1 \mathcal{M}f) d\mu + \int_{\mathbb{R}^n} \Phi_2(s_2 \mathcal{M}(\nabla^{(m)} f)) d\mu =: I_1 + I_2, \end{aligned}$$

where $s_1, s_2 > 0$ are arbitrary and $s = s_1^{1-k/m} s_2^{k/m}$. An easy computation shows that $\Psi'_i(\lambda) = \int_0^\lambda (\Phi'_i(\tau)/\tau) d\tau$, and $\Psi''_i(\lambda) = \Phi'_i(\lambda)/\lambda$ for $i = 0, 1, 2$.

Therefore the assumptions of Theorem 4.2 are satisfied for pairs (Φ_i, Ψ_i) with $i = 0, 1, 2$. This implies

$$\int_{\mathbb{R}^n} \Psi_0 \left(\frac{s}{2C_0} |\nabla^{(k)} f| \right) d\mu \leq K_2 I,$$

$$I_1 \leq K_1 \int_{\mathbb{R}^n} \Psi_1(s_1 |f|) d\mu, \quad I_2 \leq K_1 \int_{\mathbb{R}^n} \Psi_2(s_2 |\nabla^{(m)} f|) d\mu,$$

and (4.13) follows with $K = K_1 K_2$, where K_1 and K_2 are the constants from (4.4) and (4.5). To get (4.14), we assume that both summands on the right-hand side of (4.13) are nonzero; then we use (4.13), choosing $s_1 = (\|f\|_{L^{\Psi_1}(\mu)})^{-1}$ and $s_2 = (\|\nabla^{(m)} f\|_{L^{\Psi_2}(\mu)})^{-1}$. This results in

$$\frac{1}{2K} \int_{\mathbb{R}^n} \Phi_0 \left(\frac{s |\nabla^{(k)} f|}{2C_0} \right) d\mu \leq 1.$$

Clearly $\frac{1}{2K} \Psi_0(\frac{\lambda}{2C_0}) \asymp \Psi_0(\lambda)$, and so (4.14) follows directly from (2.6), with $\tilde{K} = 2C_0(2K + 1)$. ■

REMARK 4.4.

(i) Taking $s_1 = s_2 = 1$ in (4.13) we get

$$(4.15) \quad \int_{\mathbb{R}^n} \Psi_0(C |\nabla^{(k)} f|) d\mu \leq K \left(\int_{\mathbb{R}^n} \Psi_1(|f|) d\mu + \int_{\mathbb{R}^n} \Psi_2(|\nabla^{(m)} f|) d\mu \right),$$

where C and K are independent of f .

(ii) If the assumptions of Theorem 4.4 are satisfied, and additionally at least one of the functions Ψ_0, Ψ_1 or Ψ_2 satisfies the Δ_2 -condition, then

$$(4.16) \quad \int_{\mathbb{R}^n} \Psi_0(|\nabla^{(k)} f|) d\mu \leq C \left(\int_{\mathbb{R}^n} \Psi_1(|f|) d\mu + \int_{\mathbb{R}^n} \Psi_2(|\nabla^{(m)} f|) d\mu \right),$$

with a constant C independent of f . Arguments are similar to those in Corollary 3.1 and Remark 4.2.

Examples of triples (Ψ_0, Ψ_1, Ψ_2) satisfying (4.13) and (4.14) are given in Section 6.

5. Further discussion. Comments and remarks

5.1. Additional comments concerning Theorems 4.3 and 4.4. We start with the following auxiliary observations concerning functions that appear in Kita’s theorem (Theorem 4.2).

PROPOSITION 5.1. *Suppose that Φ and Ψ are defined by (4.3), with a and b satisfying the assumptions of Theorem 4.2. Then we have:*

- (1) Φ is nondecreasing, Ψ is convex, and $\Phi(0) = \Psi(0) = 0$. For every $t > 0$ we have $\Phi \in W^{1,1}(0, t)$ and $\Psi \in W^{2,1}(0, t)$.

(2) Φ and Ψ can be related through

$$(5.1) \quad \Phi(t) = t\Psi'(t) - \Psi(t).$$

If Ψ is strictly convex, then $\Phi(t) = \Psi^*(\Psi'(t))$.

(3) Φ and Ψ can also be related through an integral identity

$$(5.2) \quad \Psi(\lambda) = \lambda \int_0^\lambda \frac{\Phi(v)}{v^2} dv = \int_0^1 \frac{\Phi(\lambda v)}{v^2} dv.$$

(4)

$$(5.3) \quad \Phi(t) \leq \Psi(2t) \text{ for all } t > 0.$$

(5) If $a(t) > 0$ for $t > 0$, then Ψ^* satisfies the Δ_2 -condition if and only if $\Phi(t) \geq \beta\Psi(t)$ for some $\beta > 0$.

(6) If $a(t) > 0$ for $t > 0$, then $\Psi \in \Delta_2^c$ if and only if Φ and Ψ are equivalent N -functions.

(7) If $a(t) > 0$ for $t > 0$, then $\Phi \in \Delta_2^c$ if and only if $\Psi \in \Delta_2^c$.

Proof. (1) follows directly from the definition and (2) is verified by elementary differentiation.

Let us prove (3). Reversing the order of integration in the definition of Ψ , we have

$$(5.4) \quad \Psi(\lambda) = \int_0^\lambda \int_0^t \frac{\Phi'(u)}{u} du dt = \lambda \int_0^\lambda \frac{\Phi'(u)}{u} du - \Phi(\lambda).$$

As

$$\frac{\Phi(\lambda)}{\lambda} = \frac{1}{\lambda} \int_0^\lambda a(\tau) d\tau \leq \int_0^\lambda \frac{a(\tau)}{\tau} d\tau,$$

we see that $\lim_{\lambda \rightarrow 0} \Phi(\lambda)/\lambda = 0$. Integrating (5.4) by parts we obtain $\Psi(\lambda) = \lambda \int_0^\lambda (\Phi(u)/u^2) du$. A simple substitution $u = \lambda v$ in this integral gives (5.2).

To see (4), observe that

$$\Phi(t) = t\Psi'(t) - \Psi(t) \leq \int_t^{2t} \Psi'(u) du - \Psi(t) = \Psi(2t) - 2\Psi(t),$$

i.e. $\Phi(t) + 2\Psi(t) \leq \Psi(2t)$.

(5) follows from the identity

$$\frac{t\Psi'(t)}{\Psi(t)} = \frac{\Phi(t) + \Psi(t)}{\Psi(t)} = 1 + \frac{\Phi(t)}{\Psi(t)}$$

and Proposition 4.1.

The “if” part of (6) is obtained directly from (4) and (5). For the converse, assume that $\Phi \asymp \Psi$. Since the Lebesgue measure belongs to A_1 , from Kita’s

theorem we get

$$\int \Phi(\mathcal{M}f(x)) dx \leq K_1 \int \Psi(|f(x)|) dx,$$

which yields

$$\int \Phi(\mathcal{M}f(x)) dx \leq \tilde{K}_1 \int \Phi(C|f(x)|) dx.$$

Next, as for every convex function such that $\Phi(0) = 0$ and for every $N > 1$ one has $\Phi(N\lambda) \geq N\Phi(\lambda)$, the constant \tilde{K}_1 can be incorporated into the constant C inside the argument of Φ , and we obtain exactly (4.1). Therefore from the Bloom–Kerman theorem (Theorem 4.1) we can infer that $\Phi \in \Delta_2^c$. As $\Phi \asymp \Psi$, we have $\Psi \in \Delta_2^c$, which follows from (D3) of Proposition 4.1. So (6) is proven.

To see (7), we only need to show that $\Phi \in \Delta_2^c$ implies $\Phi \asymp \Psi$. But this is clear: the inequality $\Phi(t) \leq \Psi(2t)$ is always true, and the reverse domination follows from Proposition 4.1. Indeed, $\Phi \in \Delta_2^c$ is equivalent to the fact that $\Phi(\lambda)/\lambda^\alpha$ is nondecreasing for some $\alpha > 1$. Therefore

$$\Psi(\lambda) = \lambda \int_0^\lambda \frac{\Psi(t)}{t^\alpha} \frac{1}{t^{2-\alpha}} dt \leq \lambda^{1-\alpha} \Phi(\lambda) \int_0^\lambda \frac{1}{t^{2-\alpha}} dt = \frac{1}{\alpha-1} \Phi(\lambda). \quad \blacksquare$$

Now we make the following remarks.

REMARK 5.1. The representation (5.2) was used in formula (4.12).

REMARK 5.2. Let $d\mu(x) = w(x)dx$ be a weighted measure and let $\Psi \in \Delta_2^c$ be an increasing function. Then from Proposition 5.1(6), we can infer that Φ and Ψ are equivalent. According to Kita's results (Theorem 4.2) we see that the Hardy–Littlewood maximal operator satisfies an inequality of the form

$$(5.5) \quad \int_{\mathbb{R}^n} \Psi(\mathcal{M}f(x))w(x) dx \leq C_1 \int_{\mathbb{R}^n} \Psi(C_2|f(x)|)w(x) dx \\ \leq \int_{\mathbb{R}^n} \Psi(C_3|f(x)|)w(x) dx,$$

with constants C_1, C_2, C_3 independent of f , provided $w \in A_1 \cap A'_\infty$. Theorem 4.1 gives more, namely that (5.5) is satisfied whenever $w \in W_\Psi$.

Note that, as $A_1 = \bigcap_\Psi W_\Psi$, the class W_Ψ is substantially larger than $A_1 \cap A'_\infty$. Therefore, when $\Psi \in \Delta_2^c$, Theorem 4.1 extends Theorem 4.2 to a wider class of measures.

REMARK 5.3. For measures $\mu \in A_1 \cap A'_\infty$, Theorem 4.4 implies the results of Theorem 4.3.

Indeed, suppose that (Φ_0, Φ_1, Φ_2) and (Ψ_0, Ψ_1, Ψ_2) are two triples of functions that appear in Theorem 4.4. Assume $\mu(dx) = w(x)dx$, $w \in A_1 \cap A'_\infty$,

$1 < k < m$, $s_1, s_1 > 0$, $s = s_1^{1-k/m} s_2^{k/m}$. Then the inequalities

$$\int_{\mathbb{R}^n} \Psi_0(Cs|\nabla^{(k)}f|) d\mu \leq K \left(\int_{\mathbb{R}^n} \Psi_1(s_1|f|) d\mu + \int_{\mathbb{R}^n} \Psi_2(s_2|\nabla^{(m)}f|) d\mu \right)$$

and

$$\|\nabla^{(k)}f\|_{L^{\Psi_0}(\mu)} \leq \tilde{K} \|f\|_{L^{\Psi_1}(\mu)}^{1-k/m} \|\nabla^{(m)}f\|_{L^{\Psi_2}(\mu)}^{k/m},$$

hold with constants C, K, \tilde{K} independent of f . By Proposition 5.1(4), for every $\lambda > 0$ one has $\Phi_0(\lambda) \leq \Psi_0(2\lambda)$, and so

$$\int \Phi_0\left(\frac{Cs}{2}|\nabla^{(k)}f|\right) d\mu \leq \int \Psi_0(Cs|\nabla^{(k)}f|) d\mu.$$

Suppose additionally that $\Phi_1, \Phi_2 \in \Delta_2^c$, so that the assumptions of Theorem 4.4 are satisfied. By Proposition 5.1(6) we have $\Psi_i \asymp \Phi_i$, thus $\Psi_i \in \Delta_2^c$ as well. Therefore, by Proposition 5.1(5), we get $\beta_i \Psi_i(\lambda) \leq \Phi_i(\lambda)$. These observations (used for $(2/C)f$ instead of f) lead to the inequalities

$$\int \Phi_0(s|\nabla^{(k)}f|) d\mu \leq K_1 \left(\int \Phi_1(s_1\tilde{B}_1|f|) d\mu + \int \Phi_2(s_2\tilde{B}_2|\nabla^{(m)}f|) d\mu \right)$$

and

$$\|\nabla^{(k)}f\|_{L^{\Phi_0}(\mu)} \leq \tilde{K}_1 \|f\|_{L^{\Phi_1}(\mu)}^{1-k/m} \|\nabla^{(m)}f\|_{L^{\Phi_2}(\mu)}^{k/m},$$

which are the same as (4.7) and (4.8), up to multiplicative constants.

REMARK 5.4. For some functions Φ for which the assumptions of Theorem 4.3 fail, Theorem 4.4 can still be applied. For example, suppose that the function Φ in Theorem 4.3 does not satisfy the Δ_2^c -condition. Choose $F(\lambda) = \lambda^{m/(m-k)}$. Then $\Phi_1(\lambda) = \Phi(\lambda)$ and $\Phi_2(\lambda) = \Phi(c\lambda)$ do not satisfy the Δ_2^c -condition either. Therefore Theorem 4.3 cannot be applied to obtain Gagliardo–Nirenberg inequalities within the single Orlicz space $L^\Phi(\mu)$ (the Landau–Kolmogorov inequalities). But in this case Theorem 4.4 can be used and yields the desired inequality, at least for measures $\mu \in A_1 \cap A'_\infty$.

6. Examples.

We start with a single Orlicz space $L^\Psi(\mu)$.

EXAMPLE 6.1 (Kolmogorov–Stein inequalities). Suppose that $\mu(dx) = w(x)dx$ and either $\{\Psi \in \Delta_2^c$ and $w \in W_\Psi\}$ or $\{\Psi$ is an arbitrary N-function and $w \in A_1 \cap A'_\infty\}$. Take $F(\lambda) = \lambda^{m/(m-k)}$, so that $F^*(\lambda) = c\lambda^{m/k}$ with $c = c_{k,m} < 1$. Then we get $\Psi_1(\lambda) = \Psi(\lambda)$ and $\Psi_2(\lambda) = \Psi(c\lambda)$ with a constant $c < 1$. Therefore it follows from Theorems 4.3 and 4.4 that in either case we have

$$(6.1) \quad \int_{\mathbb{R}^n} \Psi(Cs|\nabla^{(k)}f|) d\mu \leq K \left(\int_{\mathbb{R}^n} \Psi(s_1|f|) d\mu + \int_{\mathbb{R}^n} \Psi(s_2|\nabla^{(m)}f|) d\mu \right)$$

for every $f \in C_0^m(\mathbb{R}^n)$ and arbitrary $s, s_1, s_2 > 0$ such that $s = s_1^{1-k/m} s_2^{k/m}$, with constants C, K independent of f . Furthermore,

$$(6.2) \quad \|\nabla^{(k)} f\|_{L^\psi(\mu)} \leq \tilde{C} \|f\|_{L^\psi(\mu)}^{1-k/m} \|\nabla^{(m)} f\|_{L^\psi(\mu)}^{k/m}$$

with a constant \tilde{C} independent of f . Inequalities (6.2) for a one-variable function f and the Lebesgue measure were recently obtained by Bang and coauthors [3] directly from the classical Kolmogorov inequality, by means of the convolution technique due to Stein.

EXAMPLE 6.2 (Gagliardo–Nirenberg inequalities). Take $\Phi(\lambda) = \lambda^q$, let $p, r > 1$ be such that

$$\frac{1}{q} = \left(1 - \frac{k}{m}\right) \frac{1}{p} + \frac{k}{m} \frac{1}{r},$$

and let

$$F(\lambda) = \frac{1}{s} \lambda^s \quad \text{with} \quad s = \frac{m}{m-k} \frac{p}{q}.$$

Then

$$F^*(\lambda) = \frac{1}{s^*} \lambda^{s^*} \quad \text{and} \quad s^* = \frac{m}{k} \frac{r}{q}.$$

According to Theorem 4.3 we have $\Phi_1(\lambda) \asymp \lambda^p$, $\Phi_2(\lambda) \asymp \lambda^r$ and all these functions, together with their Legendre transforms, satisfy the Δ_2 -condition. Recalling that for $\Phi(\lambda) = |\lambda|^\kappa$ we have $W_\Phi = A_\kappa$ where A_κ is the Muckenhoupt class (see e.g. [44]), by Theorem 4.3 and Remark 4.2 we obtain

$$\int_{\mathbb{R}^n} |\nabla f|^q d\mu \leq C \left(\int_{\mathbb{R}^n} |f|^p d\mu + \int_{\mathbb{R}^n} |\nabla^{(2)} f|^r d\mu \right)$$

and also

$$\|\nabla^{(k)} f\|_{L^q(\mu)} \leq C \|f\|_{L^p(\mu)}^{1-k/m} \|\nabla^{(m)} f\|_{L^r(\mu)}^{k/m},$$

provided that $\mu \in A_p \cap A_r$, with a constant C independent of f . These and other general interpolation inequalities in weighted L^κ spaces equipped with Muckenhoupt weights were previously obtained by the first author in [25].

Our next example deals with the case when $\Phi \in \Delta_2 \cap \Delta_2^S$ and it is not homogeneous.

EXAMPLE 6.3 (logarithmic inequalities). For $s > 1$ and $\kappa \in \mathbb{R}$ let

$$M_{s,\kappa}(\lambda) = \lambda^s \ln(2 + \lambda)^\kappa.$$

Suppose that $\mu(dx) = w(x)dx$ is a nonnegative weighted measure on \mathbb{R}^n . Write $\|\cdot\|_{(s,\kappa;\mu)} = \|\cdot\|_{(M_{s,\kappa};\mu)}$, where μ is dropped from the notation if it is the Lebesgue measure. In this case we obtain the following result.

THEOREM 6.1. *Let $k, m \in \mathbb{Z}_+$ with $0 < k < m$. Suppose that $p, q, r > 1$ and $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy the conditions*

$$(6.3) \quad \frac{1}{q} = \left(1 - \frac{k}{m}\right) \frac{1}{p} + \frac{k}{m} \frac{1}{r}, \quad \frac{\alpha}{q} = \left(1 - \frac{k}{m}\right) \frac{\beta}{p} + \frac{k}{m} \frac{\gamma}{r},$$

and let $\mu(dx) = w(x)dx$ be a weighted measure with $w \in W_{M_{p,\beta}} \cap W_{M_{r,\gamma}}$. Then for every function $f \in C_0^m(\mathbb{R}^n)$ we have

$$(6.4) \quad \int |\nabla^{(k)} f|^q \ln(2 + |\nabla^{(k)} f|)^\alpha d\mu \\ \leq C \left(\int |f|^p \ln(2 + |f|)^\beta d\mu + \int |\nabla^{(m)} f|^r \ln(2 + |\nabla^{(m)} f|)^\gamma d\mu \right),$$

and also

$$\|\nabla^{(k)} f\|_{(q,\alpha,\mu)} \leq C \|f\|_{(p,\beta,\mu)}^{1-k/m} \|\nabla^{(m)} f\|_{(r,\gamma,\mu)}^{k/m},$$

with a constant C independent of f .

Proof. It is clear that

$$M_{s_1, \kappa_1} \circ M_{s_2, \kappa_2}(\lambda) \asymp \lambda^{s_1 s_2} (\ln(2 + \lambda))^{\kappa_1 + s_1 \kappa_2} = M_{s_1 s_2, \kappa_1 + s_2 \kappa_2}(\lambda).$$

Also, it is elementary to check that $M_{s,\kappa}^* \asymp M_{s^*, -\kappa(s^*-2)}$. This is so because $M_{s,\kappa}(\lambda) \asymp \lambda^s$ for λ small, and $M_{s,\kappa}(\lambda) \asymp \lambda^s (\ln \lambda)^\kappa$ for large values of λ . Consequently, $M_{s,\kappa}^*(\lambda) \asymp \lambda^{s^*}$ for λ small and $M_{s,\kappa}^*(\lambda) \asymp \lambda^{s^*} (\ln \lambda)^{-\kappa(s^*-1)}$ for λ large (see Theorem 7.1 of [35]).

Suppose now that $q, p, r, \alpha, \beta, \gamma$ satisfy (6.3). Then one applies Theorem 4.3 for $\Phi = M_{q,\alpha}$ and $F = M_{s,\kappa}$, which is allowed as all the functions considered belong to the class Δ_2^c . From the just proven properties of the functions $M_{s,\kappa}$ one verifies that

$$\Phi_1(\lambda) \asymp \lambda^{sq(1-k/m)} (\ln(2 + \lambda))^{\alpha + \kappa q}, \quad \Phi_2(\lambda) \asymp \lambda^{s^* q k/m} (\ln(2 + \lambda))^{\alpha - q \kappa (s^* - 1)},$$

and choosing $s = \frac{p-m}{q m-k}$, $\kappa = \frac{\beta-\alpha}{q}$ one gets $\Phi_1(\lambda) \asymp M_{p,\beta}(\lambda)$ and $\Phi_2(\lambda) \asymp M_{r,\gamma}(\lambda)$. The theorem is proven. ■

When $w(x) \equiv 1$ this result, but within a narrower class of parameters, was obtained in [28], as a special case of more general inequalities of the form (1.3), (1.4) adapted to logarithmic spaces.

We also refer to that paper for some details about logarithmic-type N-functions that were only sketched above. Finally, note that for $\alpha = \mu = \gamma = 0$ we get the Gagliardo–Nirenberg inequalities discussed in Example 6.2.

In our next example all the functions Φ, Φ_1 and Φ_2 belong to the class Δ_2^c , but none of them satisfies the Δ_2 -condition.

EXAMPLE 6.4 (exponential inequalities). Now we present inequalities for exponential N-functions.

THEOREM 6.2. Let $k, m \in \mathbb{Z}_+$ with $0 < k < m$. Suppose that $p, q, r > 1$ and $\alpha, \beta, \gamma > 0$ satisfy the following conditions:

$$(6.5) \quad \frac{1}{q} = \left(1 - \frac{k}{m}\right) \frac{1}{p} + \frac{k}{m} \frac{1}{r}, \quad \frac{1}{\alpha} = \left(1 - \frac{k}{m}\right) \frac{1}{\beta} + \frac{k}{m} \frac{1}{\gamma}.$$

Then for every $f \in C_0^m(\mathbb{R}^n)$ one has

$$(6.6) \quad \int_{\mathbb{R}^n} |\nabla^{(k)} f|^q \exp(|\nabla^{(k)} f|^\alpha) d\mu \leq C \left(\int_{\mathbb{R}^n} |f|^p \exp(C_1 |f|^\beta) d\mu + \int_{\mathbb{R}^n} |\nabla^{(m)} f|^r \exp(C_2 |\nabla^{(m)} f|^\gamma) d\mu \right);$$

and also

$$(6.7) \quad \|\nabla^{(k)} f\|_{L^\Phi(\mu)} \leq \tilde{C}_3 \|f\|_{L^{\Phi_1}(\mu)}^{1-k/m} \|\nabla^{(m)} f\|_{L^{\Phi_2}(\mu)}^{k/m},$$

where the constants C, C_1, C_2, C_3 are independent of f ,

$$\Phi(\lambda) = \lambda^q \exp(\lambda^\alpha), \quad \Phi_1(\lambda) = \lambda^p \exp(\lambda^\beta), \quad \Phi_2(\lambda) = \lambda^r \exp(\lambda^\gamma),$$

and $\mu(x) = w(x)dx$, where $w \in W_{\Phi_1} \cap W_{\Phi_2}$.

Proof. Take $F(\lambda) = \lambda^{s_1} \exp((s_2 - s_1) \ln(2 + \lambda))$, where the parameters $s_1, s_2 > 1$ will be determined later. Then F is an N-function, and moreover, $F(\lambda) \sim \lambda^{s_1}$ for λ close to 0 and $F(\lambda) \sim \lambda^{s_2}$ for λ close to ∞ . Hence $F^*(\lambda) \sim \lambda^{s_1^*}$ for λ close to 0 and $F^*(\lambda) \sim \lambda^{s_2^*}$ for λ close to ∞ . On the other hand, we have $\Phi(\lambda) \sim \lambda^q$ for λ close to 0 and $\Phi(\lambda) \sim \exp(\lambda^\alpha)$ for λ close to ∞ . Therefore if we take $s_1 = p/q(1 - k/m)$, $s_2 = \beta/(1 - k/m)\alpha$ and use condition (6.5) we obtain

$$\Phi_1(\lambda) \sim \bar{\Phi}_1(\lambda) =: \Phi(F(\lambda^{1-k/m})), \quad \Phi_2(\lambda) \sim \bar{\Phi}_2(\lambda) =: \Phi(F^*(\lambda^{k/m})).$$

Next, as $w \in W_{\Phi_1} \cap W_{\Phi_2}$, according to Remark 4.1, we see that $\omega \in W_{\bar{\Phi}_1} \cap W_{\bar{\Phi}_2}$. Moreover, by Proposition 4.1 we have $\Phi_1, \Phi_2 \in \Delta_2^c$ and therefore $\bar{\Phi}_1, \bar{\Phi}_2 \in \Delta_2^c$ as well. Hence, it suffices to apply Theorem 4.3 with the functions $(\Phi, \bar{\Phi}_1, \bar{\Phi}_2)$ and again use the fact that $\Phi_1 \sim \bar{\Phi}_1$ and $\Phi_2 \sim \bar{\Phi}_2$. ■

REMARK 6.1. In our previous paper (see Example 8.7 in [27]), from more general inequalities we derived inequalities similar to (6.6) and (6.7) for $k = 1, m = 2$, but with parameters $p, q, r, \alpha, \beta, \gamma$ satisfying

$$(6.8) \quad q > 2, \alpha \in (0, 2), \frac{1}{2p} + \frac{1}{2r} = \frac{1}{q}, \beta = \frac{s\alpha}{2-\alpha}, \gamma = \frac{s^*\alpha}{2-\alpha} \text{ and } s = \frac{2p}{q}.$$

In particular, for those parameters we have

$$\frac{1}{2\beta} + \frac{1}{2\gamma} = \frac{2-\alpha}{2\alpha} < \frac{1}{\alpha}.$$

This shows that inequalities (6.6) and (6.7) obtained in Theorem 6.2 (reduced to $k = 1, m = 2, \mu \equiv dx, p > 2, \alpha \in (0, 2)$) are valid for a broader range of parameters than the exponential inequalities from [27].

7. Appendix

7.1. Proof of Proposition 4.1. It is a slight modification of the proof of [35, Theorem 4.3], where the Δ_2 -condition was assumed only for large values of t .

As $\left(\frac{\Phi(t)}{t^\alpha}\right)' = \frac{t\Phi'(t) - \alpha\Phi(t)}{t^{\alpha+1}}$, the equivalence (D2) \Leftrightarrow (D3) is obvious.

Set $\Phi'(t) = p(t)$, $(\Phi^*)'(s) = q(s) = p^{-1}(s)$. Then

$$\Phi(t) = tp(t) - \Phi^*(p(t)) \quad \text{for } t > 0,$$

and similarly

$$(7.1) \quad \Phi^*(s) = sq(s) - \Phi(q(s)) \quad \text{for } s > 0.$$

To see that (D1) \Rightarrow (D2), suppose that Φ^* satisfies the Δ_2 -condition, i.e. there exists $\beta > 0$ such that

$$(7.2) \quad \frac{sq(s)}{\Phi^*(s)} \leq \beta \quad \text{for every } s > 0.$$

As Φ^* is strictly convex, we have $sq(s) > \Phi^*(s)$, so that $\beta > 1$. Inserting (7.1) into (7.2) gives

$$\frac{sq(s)}{sq(s) - \Phi(q(s))} \leq \beta.$$

As for $t = q(s)$ one has $sq(s) = tp(t)$, we get

$$\frac{tp(t)}{tp(t) - \Phi(t)} \leq \beta,$$

and further

$$\frac{t\Phi'(t)}{\Phi(t)} \geq \frac{\beta}{\beta - 1} =: \alpha > 1.$$

It is clear that this reasoning can be reversed, proving also the implication (D2) \Rightarrow (D1). ■

7.2. Proof of Theorem 4.2. We start with two lemmas. For the proof of the first one we refer for example to [31, Lemma 3.2].

LEMMA 7.1. *Suppose $w \in A_1$ and $\mu = \omega dx$. Then for each $f \in \mathcal{R}_0(\mu)$ (see (4.2)) one has*

$$\mu\{\mathcal{M}f > t\} \leq \frac{C}{t} \int_{t/2}^{\infty} \mu\{|f| > s\} ds \quad \text{for all } t > 0,$$

where C is a constant independent of f .

The other lemma is due to Muckenhoupt [44].

LEMMA 7.2. *Suppose $w \in A'_{\infty}$. Then for each $f \in \mathcal{R}_0(\mu)$ one has*

$$\mu\{\mathcal{M}f > t\} \geq \frac{C}{t} \int_t^{\infty} \mu\{|f| > s\} ds \quad \text{for all } t > 0,$$

where C is a constant independent of f .

Proof of Theorem 4.2. For a measurable f we can write

$$\int_{\mathbb{R}^n} \Phi(\mathcal{M}f) d\mu = \int_{(0,\infty)} \Phi'(s) \mu\{\mathcal{M}f > s\} ds$$

(these integrals are either simultaneously infinite, or both finite and equal). By Lemma 7.1 this is further equal to

$$\begin{aligned} \int_{(0,\infty)} a(s) \mu\{\mathcal{M}f > s\} ds &\leq C \int_0^\infty \frac{a(s)}{s} \left(\int_s^\infty \mu\{|f| > t\} dt \right) ds \\ &= C \int_0^\infty \mu\{|f| > t\} \left(\int_0^t \frac{a(s)}{s} ds \right) dt \\ &= C \int_0^\infty b(t) \mu\{|f| > t\} dt \\ &= C \int_0^\infty \Psi'(t) \mu\{|f| > t\} dt = C \int_{\mathbb{R}^n} \Psi(|f|) d\mu, \end{aligned}$$

and (4.4) is proven.

It is clear that for $w \in A'_\infty$ this chain of inequalities can be reversed (use Lemma 7.2 instead of Lemma 7.1), giving (4.5). ■

REMARK 7.1. Note that the constants K_1 and K_2 do not depend on Φ and Ψ .

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