STUDIA MATHEMATICA 217 (1) (2013)

Rademacher functions in weighted Cesàro spaces

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Abstract. We study the behaviour of the Rademacher functions in the weighted Cesàro spaces $\mathrm{Ces}(\omega, p)$, for $\omega(x)$ a weight and $1 \le p \le \infty$. In particular, the case when the Rademacher functions generate in $\mathrm{Ces}(\omega, p)$ a closed linear subspace isomorphic to ℓ^2 is considered.

1. Introduction. The Cesàro function spaces $\mathrm{Ces}(p)$ are defined by

$$\|f\|_{\mathrm{Ces}(p)} = \bigg(\int_0^1 \bigg(\frac{1}{x}\int_0^x |f(t)|\,dt\bigg)^p dx\bigg)^{1/p} < \infty \quad \text{for} \quad 1 \le p < \infty,$$

$$\|f\|_{\mathrm{Ces}(\infty)} = \sup_{0<x\le 1} \frac{1}{x}\int_0^x |f(t)|\,dt < \infty \quad \text{for} \quad p = \infty.$$

These spaces are the continuous counterpart to the classical Cesàro sequence spaces, which have been thoroughly studied; see, for example, [6], [8], and the references therein. Functional and geometrical properties of $\mathrm{Ces}(p)$ have been studied in detail, including: duality and reflexivity; isomorphic copies of classical sequence and function spaces; type and cotype; fixed point, Dunford–Pettis, Banach–Saks, and Radon–Nikodym properties; see [2], [3], [5], [8].

More recently, weighted Cesàro function spaces have been considered; in [8] their dual space has been identified. For $\omega(x)$ a weight, i.e., a measurable function with $0 < \omega(x) < \infty$ a.e., and $1 \le p \le \infty$, the weighted Cesàro spaces $\mathrm{Ces}(\omega, p)$ are defined by

$$\|f\|_{\mathrm{Ces}(\omega,p)} := \bigg(\int_0^1 \bigg(\frac{1}{\omega(x)}\int_0^x |f(t)|\,dt\bigg)^p dx\bigg)^{1/p} < \infty \qquad \text{for } 1 \le p < \infty,$$

2010 *Mathematics Subject Classification*: Primary 46E30; Secondary 46B20, 46B42.
Key words and phrases: Cesàro function spaces, Rademacher functions, Banach function spaces, rearrangement invariant spaces.

DOI: 10.4064/sm217-1-2

$$\|f\|_{\mathrm{Ces}(\omega,\infty)} := \sup_{0\le x\le 1} \frac{1}{\omega(x)} \int_0^x |f(t)|\,dt < \infty \quad \text{for} \quad p=\infty.$$

The Rademacher functions are defined by

$$r_k(t) := \mathrm{sign}(\sin(2^k\pi t)), \quad t\in[0,1],\ k\ge 1.$$

Recall that a Rademacher series $\sum_{k=1}^{\infty} a_k r_k$ converges a.e. if and only if $(a_k)_{k=1}^{\infty}\in\ell^2$. For the set of Rademacher series we write

$$\mathcal{R} = \Big\{ \sum_{k=1}^{\infty} a_k r_k : (a_k)_{k=1}^{\infty} \in \ell^2 \Big\}.$$

The study of the Rademacher series in function spaces is classical. The Khintchine inequalities state, for $0<p<\infty$, that $\{r_k\}$ generates in $L^p([0,1])$ a closed linear subspace $\mathcal{R}\cap L^p([0,1])$ isomorphic to ℓ^2. The behaviour of $\{r_k\}$ in rearrangement invariant spaces was studied in a celebrated result of V. A. Rodin and E. M. Semenov: for X an r.i. space on $[0,1]$, we have $\mathcal{R}\cap X$ isomorphic to ℓ^2 if and only if the closure of $L^\infty([0,1])$ in L^M is continuously embedded into X, where L^M is the Orlicz space generated by the Young function $M(t)=\exp(t^2)-1$ (see [11, Theorem 6]).

For the Cesàro spaces it was proved in [4], for the unweighted case $\omega(x)=x$ and for $1\le p<\infty$, that $\{r_k\}$ generates in $\mathrm{Ces}(p)$ a non-complemented closed linear subspace isomorphic to ℓ^2. For $p=\infty$ and $\omega(x)$ a quasiconcave weight, it was also shown that

$$\Big\|\sum_{k=1}^{m} a_k r_k\Big\|_{\mathrm{Ces}(\omega,\infty)} \asymp \|(a_k)_{k=1}^{m}\|_2 + \max_{1\le n\le m} \frac{2^{-n}}{\omega(2^{-n})}\Big|\sum_{k=1}^{n} a_k\Big|,$$

where $A\asymp B$ stands for $c_1A\le B\le c_2A$ for some constants $c_1,c_2>0$. The case when $\mathcal{R}\cap\mathrm{Ces}(\omega,\infty)$ is isomorphic to ℓ^2 was characterized by means of a condition on $\omega(x)$; namely, $\omega(x)\ge cx\log_2^{1/2}(2/x)$ for $0<x\le 1$ and some constant $c>0$. We will consider this condition, which we denote (P3) for $p=\infty$, in more generality.

In this paper we study, by means of conditions on $\omega(x)$ and p, the behaviour of the Rademacher functions $\{r_k\}$ in the spaces $\mathrm{Ces}(\omega,p)$.

After the preliminaries in Section 2, we start in Section 3 discussing several conditions, (P1) to (P5), on the weight $\omega(x)$ and the index $1\le p\le\infty$, which are naturally related to the behaviour of the Rademacher series in the spaces $\mathrm{Ces}(\omega,p)$.

In Section 4 we compute, under a certain condition on the weight $\omega(x)$, the norm in $\mathrm{Ces}(\omega,p)$ of a Rademacher series, showing, for $1\le p<\infty$, that

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{\mathrm{Ces}(\omega,p)} \asymp \Big(\sum_{n=0}^{\infty} \omega_{p,n}\Big(\Big|\sum_{k=1}^{n+1} a_k\Big| + \|(a_k)_{k=n+2}^{\infty}\|_2\Big)^p\Big)^{1/p},$$

and, for $p = \infty$, that

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,\infty)} \asymp \sup_{n\ge 0} \omega_{\infty,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{k=n+2}^{\infty}\|_2 \Big),$$

where, for $J_n = (1/2^{n+1}, 1/2^n)$, $n \ge 0$, we have

$$\omega_{p,n} = \int_{J_n} \left(\frac{x}{\omega(x)} \right)^p dx, \qquad \omega_{\infty,n} = \sup_{x\in J_n} \frac{x}{\omega(x)}.$$

These inequalities allow describing $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$, studying when $\{r_k\}$ is a basic sequence, studying the complementability of $\mathcal{R}\cap\mathrm{Ces}(\omega,p)$ in $\mathrm{Ces}(\omega,p)$, and studying the extremal cases when the individual Rademacher functions do not belong to $\mathrm{Ces}(\omega,p)$ and $\mathcal{R}\cap\mathrm{Ces}(\omega,p)$ consists only of certain (finite) Rademacher polynomials.

In Section 5 we consider the case when $\mathcal{R}\cap\mathrm{Ces}(\omega,p)$ is isomorphic to ℓ^2. By means of determining the norm in $\mathrm{Ces}(\omega,p)$ of the decreasing rearrangement of a Rademacher series, we prove that $(x/\omega(x))\log_2^{1/2}(2/x) \in L^p([0,1])$ is a sufficient condition for $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ to be isomorphic to ℓ^2, for all $1 \le p \le \infty$, which is necessary in the case $p = \infty$, and 'almost' necessary for $1 \le p < \infty$.

Particular attention is given to the power weights $\omega(x) = x^{\lambda}$, for $\lambda \in \mathbb{R}$, which illustrate many of the features appearing throughout the paper.

2. Preliminaries. Following Luxemburg and Zaanen (see [13]), a *Banach function space* on $[0,1]$ is a linear space X of (classes of) measurable functions on $[0,1]$, endowed with a complete norm $\|\cdot\|_X$, such that $g \in X$ and $|f| \le |g|$ a.e. implies $f \in X$ and $\|f\|_X \le \|g\|_X$. The associated space X' consists of all measurable functions g on $[0,1]$ such that the associated functional

$$\|g\|_{X'} := \sup \Big\{ \Big| \int_0^1 f(t)g(t)\,dt \Big| : f \in X,\ \|f\|_X \le 1 \Big\} < \infty.$$

A Banach function space X is *saturated* if for every set E with $m(E) > 0$ there exists $F \subset E$ such that $m(F) > 0$ and $\chi_F \in X$. This property is equivalent to the associated functional $\|\cdot\|_{X'}$ being a norm in X' (see [13, Ch. 15, §68, Theorem 4]). A Banach function space is *order continuous* when order bounded increasing sequences are norm convergent. In this case, the associated space X' coincides with the topological dual space X^*. Note that other authors use more restrictive definitions of Banach function space [7], [10].

We denote by m the Lebesgue measure on $[0,1]$. The *distribution function* of a measurable function f is $m_f(\lambda) := m(\{x \in [0,1] : |f(x)| > \lambda\})$, $\lambda \ge 0$.

An *r.i. space* on $[0,1]$ is a Banach function space X such that $\|f\|_X = \|g\|_X$ whenever $m_f = m_g$. If X is an r.i. space, then so is X'. The *decreasing rearrangement* of a measurable function f is $f^*(t) := \inf\{\lambda \geq 0 : m_f(\lambda) < t\}$, $0 \leq t \leq 1$. Since f and f^* have the same distribution function, we see that $f \in X$ if and only if $f^* \in X$, and in that case $\|f\|_X = \|f^*\|_X$ for X an r.i. space. The *fundamental function* of an r.i. space X is $\varphi_X(t) := \|\chi_E\|_X$, $0 \leq t \leq 1$, where E is any set with $m(E) = t$.

A function $\omega(x)$ is *quasiconcave* if $\omega(0) = 0$, $\omega(x)$ is non-decreasing, and $\omega(x)/x$ is non-increasing.

For further details on function spaces and r.i. spaces, see [7], [9], and [10].

3. Conditions on the weight $\omega(x)$. Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0,1]$. In order to study the Rademacher functions in $\mathrm{Ces}(\omega,p)$ it is convenient to write the norm in $\mathrm{Ces}(\omega,p)$ in the following way:

$$\|f\|_{\mathrm{Ces}(\omega,p)} = \left(\int_0^1 \left(\frac{x}{\omega(x)}\right)^p \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx\right)^{1/p} \qquad \text{for } 1 \leq p < \infty,$$

$$\|f\|_{\mathrm{Ces}(\omega,\infty)} = \sup_{0<x\leq 1} \frac{x}{\omega(x)}\,\frac{1}{x}\int_0^x |f(t)|\,dt \qquad \text{for } p = \infty.$$

Let $J_n := (1/2^{n+1}, 1/2^n)$ for $n \geq 0$. We say that $\omega(x)$ satisfies *condition* (P1) if, for $n \geq 0$,

$$\text{(P1)} \qquad \begin{aligned} \omega_{p,n} &:= \int_{J_n} \left(\frac{x}{\omega(x)}\right)^p dx < \infty && \text{for } 1 \leq p < \infty, \\ \omega_{\infty,n} &:= \sup_{x\in J_n} \frac{x}{\omega(x)} < \infty && \text{for } p = \infty. \end{aligned}$$

Since $\omega(x)$ is finite a.e. we find that $\omega_{p,n} > 0$ for $n \geq 0$.

Since a Banach function space, as defined in this paper, need not contain all characteristic functions, the following result is meaningful.

PROPOSITION 3.1. *Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0,1]$. If condition* (P1) *is satisfied, then the space* $\mathrm{Ces}(\omega,p)$ *has a saturated norm. In particular, the associated functional $\|\cdot\|_{\mathrm{Ces}(\omega,p)'}$ is a norm in* $\mathrm{Ces}(\omega,p)'$.

Proof. Since the average of χ_{J_n} on $[0,x]$ vanishes for $0 < x < 1/2^{n+1}$ and it is at most 1 for $1/2^{n+1} \leq x \leq 1$, for $1 \leq p < \infty$ we have

$$\|\chi_{J_n}\|^p_{\mathrm{Ces}(\omega,p)} \leq \int_{1/2^{n+1}}^1 \left(\frac{x}{\omega(x)}\right)^p dx = \sum_{k=0}^n \omega_{p,k}.$$

Analogously, $\|\chi_{J_n}\|_{\mathrm{Ces}(\omega,\infty)} \le \sup_{0\le k\le n} \omega_{\infty,k}$ for $p = \infty$. It follows that $\chi_{J_n} \in \mathrm{Ces}(\omega, p)$ for $n \ge 0$ and $1 \le p \le \infty$.

For $E \subset [0,1]$ a set with $m(E) > 0$, there exists J_n such that $m(E \cap J_n) > 0$. Noting that $\|\chi_{E\cap J_n}\|_{\mathrm{Ces}(\omega,p)} \le \|\chi_{J_n}\|_{\mathrm{Ces}(\omega,p)}$, we deduce that $\mathrm{Ces}(\omega, p)$ is saturated. ∎

We say that $\omega(x)$ satisfies *condition* (P2) if $x/\omega(x) \in L^p([0,1])$, i.e.,

$$\text{(P2)} \qquad \begin{aligned} &\int_0^1 \left(\frac{x}{\omega(x)}\right)^p dx < \infty \qquad &&\text{for } 1 \le p < \infty,\\ &\sup_{0\le x\le 1} \frac{x}{\omega(x)} < \infty &&\text{for } p = \infty. \end{aligned}$$

Note that (P2) is equivalent to $r_k \in \mathrm{Ces}(\omega, p)$ for all $k \in \mathbb{N}$. Moreover, (P2) can be written via the coefficients $\omega_{p,n}$, namely it is equivalent to

$$\begin{aligned} &\sum_{n=0}^{\infty} \omega_{p,n} < \infty \qquad &&\text{for } 1 \le p < \infty,\\ &\sup_{n\ge 0} \omega_{\infty,n} < \infty &&\text{for } p = \infty. \end{aligned}$$

We say that $\omega(x)$ satisfies *condition* (P3) if

$$\text{(P3)} \qquad \begin{aligned} &\int_0^1 \left(\frac{x}{\omega(x)}\right)^p \log_2^{p/2}(2/x)\, dx < \infty \qquad &&\text{for } 1 \le p < \infty,\\ &\sup_{0< x\le 1} \frac{x}{\omega(x)} \log_2^{1/2}(2/x) < \infty &&\text{for } p = \infty. \end{aligned}$$

Condition (P3) is equivalent to

$$\begin{aligned} &\sum_{n=0}^{\infty} \omega_{p,n}(n+1)^{p/2} < \infty \qquad &&\text{for } 1 \le p < \infty,\\ &\sup_{n\ge 0} \omega_{\infty,n}(n+1)^{1/2} < \infty &&\text{for } p = \infty. \end{aligned}$$

For $\omega(x)$ a non-decreasing function, condition (P3) can be stated in terms of the Lorentz–Zygmund spaces $L^{p,q}(\log L)^{\alpha}$ (see [7, §4.6]). Namely, it is equivalent to $1/\omega(x) \in L^{p/(p+1),p}(\log L)^{1/2}$. In particular, condition (P3) holds for $1/\omega(x) \in L^{r,s}([0,1])$ with $p/(p+1) < r$ and $1 < s \le \infty$, or $r = p/(p+1)$ and $1 \le s < p$.

We say that $\omega(x)$ satisfies *condition* (P4*) if there exists $C > 0$ such that

$$\text{(P4*)} \qquad \sup_{n\ge 0} \frac{\omega_{p,n+1}}{\omega_{p,n}} \le C.$$

Condition (P4*) is a particular case of a more general condition. We say that $\omega(x)$ satisfies *condition* (P4) if there exist $C > 0$ and $M \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists $n' \in \mathbb{N}$ with

$$0 < n - n' \le M \quad \text{and} \quad \sup_{n\ge 1} \frac{\omega_{p,n}}{\omega_{p,n'}} \le C. \tag{P4}$$

Remark 3.2. Condition (P4) holds in the following situations.

(a) If $\omega(x)$ is a quasiconcave function, then it satisfies (P4*). Since $1/\omega(x)$ is non-increasing it follows that $\omega_{p,0}$ is finite; since $x/\omega(x)$ is non-decreasing, we have $\omega_{p,n+1}/\omega_{p,n} \le 1$ for $n \ge 0$.

(b) If $\omega(x)$ is non-increasing, we have $\omega_{p,n+1} \le 2^{-p}\omega_{p,n}$. Hence, (P4*) holds provided that $\omega_{p,0}$ is finite.

(c) If $x/\omega(x)$ is non-increasing, then condition (P4*) depends on the slope of the function $x/\omega(x)$. In particular, it holds for $1 \le p \le \infty$ when $\omega(x)$ satisfies, for some $C > 0$,

$$\sup_{n\ge 0} \frac{\omega(1/2^n)}{\omega(1/2^{n+1})} \le C.$$

(d) A weight $\omega(x)$ has the *doubling property* if there exists a positive constant C such that $\omega(I) \le C\omega(2I)$ for every interval I, where $2I$ denotes the interval with the same center as I and twice its radius, and $\omega(I) = \int_I \omega(x)\,dx$. If $(x/\omega(x))^p$ has the doubling property, then condition (P4*) is satisfied. Namely, since $J_{n+1} \subset 2J_n$, we have

$$\int\limits_{J_{n+1}} \left(\frac{x}{\omega(x)}\right)^p dx \le \int\limits_{2J_n} \left(\frac{x}{\omega(x)}\right)^p dx \le C \int\limits_{J_n} \left(\frac{x}{\omega(x)}\right)^p dx.$$

Hence, $\omega_{p,n+1} \le C\omega_{p,n}$. In particular, $(x/\omega(x))^p$ has the doubling property if it belongs to the Muckenhoupt weight class A_r for some $1 < r < \infty$.

We say that $\omega(x)$ satisfies *condition* (P5) if there exists a constant $C > 0$ such that for every $m \ge 0$,

$$\text{(P5)} \qquad \begin{aligned} &\sum_{n=m}^{\infty} \omega_{p,n} \le C\omega_{p,m} && \text{for } 1 \le p < \infty, \\ &\sup_{n\ge m} \omega_{\infty,n} \le C\omega_{\infty,m} && \text{for } p = \infty. \end{aligned}$$

Condition (P5) is satisfied whenever $\omega(x)$ is quasiconcave, since

$$\sum_{n=m}^{\infty} \omega_{p,n} = \int\limits_0^{1/2^m} \left(\frac{x}{\omega(x)}\right)^p dx \le 2 \int\limits_{1/2^{m+1}}^{1/2^m} \left(\frac{x}{\omega(x)}\right)^p dx = 2\omega_{m,p}.$$

4. Rademacher functions in $\mathrm{Ces}(\omega, p)$**.** In this section we study the space $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$. The following sequence space is useful to describe the norm of a Rademacher series in $\mathrm{Ces}(\omega, p)$.

Definition 4.1. Let $1 \le p \le \infty$ and $\omega(x)$ be a weight on $[0,1]$. Assume that condition (P1) holds. Let $\mathscr{R}(\omega,p)$ be the space of all sequences $(a_k)_{k=1}^\infty \in \ell^2$ such that, for $1 \le p < \infty$,

$$\|(a_k)_{k=1}^\infty\|_{\mathscr{R}(\omega,p)} := \Big(\sum_{n=0}^\infty \omega_{p,n}\Big(\Big|\sum_{k=1}^{n+1} a_k\Big| + \|(a_k)_{k=n+2}^\infty\|_2\Big)^p\Big)^{1/p} < \infty,$$

and, for $p=\infty$,

$$\|(a_k)_{k=1}^\infty\|_{\mathscr{R}(\omega,\infty)} := \sup_{n\ge 0} \omega_{\infty,n}\Big(\Big|\sum_{k=1}^{n+1} a_k\Big| + \|(a_k)_{k=n+2}^\infty\|_2\Big) < \infty.$$

The space $\mathscr{R}(\omega,p)$ with the norm $\|\cdot\|_{\mathscr{R}(\omega,p)}$ is a Banach space.

The following result gives an equivalent expression for the average of the absolute value of a Rademacher series on a dyadic interval. We denote the dyadic intervals of order n by $I_j^n := ((j-1)/2^n, j/2^n)$ for $1 \le j \le 2^n$ and $n \ge 0$.

Proposition 4.2. *For $(a_k)_{k=1}^\infty \in \ell^2$, $1 \le j \le 2^n$, and $n \ge 0$, we have*

$$\frac{1}{3\sqrt{2}}\Big(\Big|\sum_{k=1}^n \varepsilon_{k,j}a_k\Big| + \|(a_k)_{k=n+1}^\infty\|_2\Big) \le \frac{1}{m(I_j^n)}\int_{I_j^n}\Big|\sum_{k=1}^\infty a_k r_k(t)\Big|\,dt$$
$$\le \Big|\sum_{k=1}^n \varepsilon_{k,j}a_k\Big| + \|(a_k)_{k=n+1}^\infty\|_2,$$

where $\varepsilon_{k,j} := \operatorname{sign} r_{k|I_j^n}$.

Proof. We can suppose that $j=1$, which gives $\varepsilon_{k,j}=1$; the proof in the general case is similar.

Note that, due to the dilation properties of the Rademacher functions,

$$\frac{1}{m(I_j^n)}\int_{I_j^n}\Big|\sum_{k=n+1}^\infty a_k r_k(t)\Big|\,dt = \int_0^1\Big|\sum_{k=1}^\infty a_{n+k} r_k(t)\Big|\,dt.$$

Consequently,

$$\frac{1}{2^{-n}}\int_0^{2^{-n}}\Big|\sum_{k=1}^\infty a_k r_k(t)\Big|\,dt \le \frac{1}{2^{-n}}\int_0^{2^{-n}}\Big(\Big|\sum_{k=1}^n a_k r_k(t)\Big| + \Big|\sum_{k=n+1}^\infty a_k r_k(t)\Big|\Big)\,dt$$
$$\le \Big|\sum_{k=1}^n a_k\Big| + \|(a_k)_{k=n+1}^\infty\|_2.$$

Concerning the lower bound, we obtain it by combining two inequalities. The first one relies on the fact that, for $k \ge n+1$, the integral of r_k on

$[0, 2^{-n}]$ vanishes. Thus,

$$\frac{1}{2^{-n}} \int_0^{2^{-n}} \Big| \sum_{k=1}^{\infty} a_k r_k(t) \Big| \, dt \geq \Big| \frac{1}{2^{-n}} \int_0^{2^{-n}} \sum_{k=1}^{\infty} a_k r_k(t) \, dt \Big| = \Big| \sum_{k=1}^{n} a_k \Big|.$$

On the other hand, from the inverse triangle inequality and the Khintchine inequality for $L^1([0,1])$ it follows that

$$\begin{aligned} \frac{1}{2^{-n}} \int_0^{2^{-n}} \Big| \sum_{k=1}^{\infty} a_k r_k(t) \Big| \, dt &\geq \frac{1}{2^{-n}} \int_0^{2^{-n}} \Big(\Big| \sum_{k=n+1}^{\infty} a_k r_k(t) \Big| - \Big| \sum_{k=1}^{n} a_k r_k(t) \Big| \Big) \, dt \\ &\geq C \|(a_k)_{k=n+1}^{\infty}\|_2 - \Big| \sum_{k=1}^{n} a_k \Big|. \end{aligned}$$

Hence,

$$3 \frac{1}{2^{-n}} \int_0^{2^{-n}} \Big| \sum_{k=1}^{\infty} a_k r_k(t) \Big| \, dt \geq \Big| \sum_{k=1}^{n} a_k \Big| + C \|(a_k)_{k=n+1}^{\infty}\|_2,$$

and the proof is complete. The optimal constant in the previous inequality is $C = 1/\sqrt{2}$ (see [12]). ∎

For $(a_k)_{k=1}^{\infty} \in \ell^2$, we denote $A_0 := \|(a_k)_{k=1}^{\infty}\|_2$, and

$$A_n := \Big| \sum_{k=1}^{n} a_k \Big| + \|(a_k)_{k=n+1}^{\infty}\|_2, \quad n \in \mathbb{N}.$$

Theorem 4.3. *Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0,1]$. Assume that condition (P4) holds. Then the space $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is isomorphic to $\mathscr{R}(\omega, p)$ with equivalent norms. Consequently, $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is a Banach space.*

In particular, for $(a_k)_{k=1}^{\infty} \in \mathscr{R}(\omega, p)$ and $1 \leq p < \infty$, we have

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \asymp \Big(\sum_{n=0}^{\infty} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{k=n+2}^{\infty}\|_2 \Big)^p \Big)^{1/p},$$

and for $p = \infty$,

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,\infty)} \asymp \sup_{n \geq 0} \omega_{\infty,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{k=n+2}^{\infty}\|_2 \Big),$$

with constants depending on p and $\omega(x)$.

Proof. We prove the result for $1 \leq p < \infty$; the case $p = \infty$ is analogous. For $x \in J_n = (1/2^{n+1}, 1/2^n)$ and $n \geq 0$, from Proposition 4.2 we have

$$\frac{1}{6\sqrt{2}} A_{n+1} \leq \frac{1}{1/2^n} \int_0^{1/2^{n+1}} \Big| \sum_{k \geq 1} a_k r_k(t) \Big| \, dt \leq \frac{1}{x} \int_0^{x} \Big| \sum_{k \geq 1} a_k r_k(t) \Big| \, dt.$$

In an analogous way we obtain an upper bound:

$$\frac{1}{x}\int_0^x \Big|\sum_{k\geq 1} a_k r_k(t)\Big|\,dt \leq \frac{1}{1/2^{n+1}}\int_0^{1/2^n} \Big|\sum_{k\geq 1} a_k r_k(t)\Big|\,dt \leq 2A_n.$$

Thus, for $n \geq 0$,

$$\frac{1}{6\sqrt{2}}A_{n+1} \leq \frac{1}{x}\int_0^x \Big|\sum_{k\geq 1} a_k r_k(t)\Big|\,dt \leq 2A_n, \quad x \in J_n. \tag{1}$$

By splitting the interval $[0,1]$ into the intervals J_n, from (1) we have

$$\begin{aligned} \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|^p_{\mathrm{Ces}(\omega,p)} &= \sum_{n=0}^{\infty}\int_{J_n}\Big(\frac{x}{\omega(x)}\Big)^p\Big(\frac{1}{x}\int_0^x\Big|\sum_{k=1}^{\infty} a_k r_k(t)\Big|\,dt\Big)^p dx \\ &\geq \frac{1}{(6\sqrt{2})^p}\sum_{n=0}^{\infty}\omega_{p,n}A^p_{n+1}, \end{aligned} \tag{2}$$

and

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|^p_{\mathrm{Ces}(\omega,p)} \leq 2^p\sum_{n=0}^{\infty}\omega_{p,n}A^p_n. \tag{3}$$

In general, for an arbitrary weight $\omega(x)$, the lower and upper bounds in (2) and (3) are not equivalent. Condition (P4) provides the equivalence. For $n \in \mathbb{N}$, let n' be given by (P4). From the triangle inequality and Cauchy–Schwarz inequality we have

$$\begin{aligned} A_n &\leq \Big|\sum_{k=1}^{n'+1} a_k\Big| + (n-n'-1)^{1/2}\|(a_k)^n_{k=n'+2}\|_2 + \|(a_k)^{\infty}_{k=n+1}\|_2 \\ &\leq 2(M-1)^{1/2}A_{n'+1}. \end{aligned} \tag{4}$$

From the fact that $0 < n - n' \leq M$, it follows that, for each $m \geq 0$, there are at most M indices $n \in \mathbb{N}$ such that $n' = m$. Hence,

$$\sum_{n=1}^{\infty}\omega_{p,n'}A^p_{n'+1} \leq M\sum_{n=1}^{\infty}\omega_{p,n}A^p_{n+1}.$$

This, together with $A_0 \leq A_1$, inequality (4), and condition (P4) gives

$$\begin{aligned} \sum_{n=0}^{\infty}\omega_{p,n}\,A^p_n &\leq \omega_{p,0}\,A^p_0 + C\sum_{n=1}^{\infty}\omega_{p,n'}\,A^p_n \\ &\leq \omega_{p,0}\,A^p_1 + 2^pC(M-1)^{p/2}\sum_{n=1}^{\infty}\omega_{p,n'}\,A^p_{n'+1} \end{aligned}$$

$$\leq \omega_{p,0}\, A_1^p + 2^p CM(M-1)^{p/2} \sum_{n=0}^{\infty} \omega_{p,n}\, A_{n+1}^p$$
$$\leq B_{\omega,p} \sum_{n=0}^{\infty} \omega_{p,n}\, A_{n+1}^p,$$

with $B_{\omega,p} = \max\{\omega_{p,0}, 2^p CM(M-1)^{p/2}\}$. This establishes the equivalence between the upper and lower bounds.

Completeness of $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ follows since it is isomorphic to $\mathscr{R}(\omega,p)$. ∎

Next we consider when $\{r_k\}$ is a basic sequence in $\mathrm{Ces}(\omega,p)$. In particular, this will be the case when $\omega(x)$ is quasiconcave.

Corollary 4.4. *Let* $1 \leq p \leq \infty$ *and* $\omega(x)$ *be a weight on* $[0,1]$*. If conditions* (P4) *and* (P5) *are satisfied, then* $\{r_k\}$ *is a basic sequence in* $\mathrm{Ces}(\omega,p)$*.*

Proof. Suppose that $1 \leq p < \infty$; the case $p = \infty$ is analogous.

Let $m_1 < m_2$. From Theorem 4.3 we have

$$\Big\| \sum_{k=1}^{m_1} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \leq A_{\omega,p} \Big(\sum_{n=0}^{m_1-2} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{n+2}^{m_1}\|_2 \Big)^p + \Big| \sum_{k=1}^{m_1} a_k \Big|^p \sum_{n=m_1-1}^{\infty} \omega_{p,n} \Big)^{1/p},$$

and

$$\Big\| \sum_{k=1}^{m_2} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \geq B_{\omega,p} \Big(\sum_{n=0}^{m_1-2} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{k=n+2}^{m_2}\|_2 \Big)^p + \sum_{n=m_1-1}^{\infty} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{k=n+2}^{m_2}\|_2 \Big)^p \Big)^{1/p}$$
$$\geq B_{\omega,p} \Big(\sum_{n=0}^{m_1-2} \omega_{p,n} \Big(\Big| \sum_{k=1}^{n+1} a_k \Big| + \|(a_k)_{k=n+2}^{m_1}\|_2 \Big)^p + \omega_{p,m_1-1} \Big| \sum_{k=1}^{m_1} a_k \Big|^p \Big)^{1/p},$$

where $A_{\omega,p}$ and $B_{\omega,p}$ are the equivalence constants appearing in Theorem 4.3.

Condition (P5), together with the previous inequalities, yields

$$\Big\| \sum_{k=1}^{m_1} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \leq C^{1/p} \frac{A_{\omega,p}}{B_{\omega,p}} \Big\| \sum_{k=1}^{m_2} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)},$$

which proves that $\{r_k\}$ is a basic sequence. ∎

Theorem 4.3 allows studying the behaviour of the Rademacher polynomials in $\mathrm{Ces}(\omega,p)$ even in the case when $r_k \notin \mathrm{Ces}(\omega,p)$ for (all) $k \in \mathbb{N}$. In particular, we will see that if some of the coefficients $\omega_{p,n}$ fail to be finite, then $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ is a finite-dimensional vector space consisting of Rademacher polynomials. Let $\mathcal{P}$ be the space of all Rademacher polynomials, and set $\mathcal{P}^0 := \bigcup_{m\geq 1} \mathcal{P}^0_m$, where, for $m \in \mathbb{N}$,

$$\mathcal{P}^0_m := \Big\{ \sum_{k=1}^{m} a_k r_k : a_k \in \mathbb{R} \text{ with } \sum_{k=1}^{m} a_k = 0 \Big\}.$$

PROPOSITION 4.5. *Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0,1]$.*

(i) *Condition* (P2) *holds if and only if $\mathcal{P} \subset \mathcal{R} \cap \mathrm{Ces}(\omega,p)$.*

(ii) *Assume that condition* (P1) *holds but condition* (P2) *is not satisfied. Then*

$$\mathcal{P} \cap \mathrm{Ces}(\omega,p) = \mathcal{P}^0.$$

In this case, $r_k \notin \mathrm{Ces}(\omega,p)$ for all $k \in \mathbb{N}$.

(iii) *Assume that condition* (P1) *fails. If $\omega_{p,m} = \infty$ and $\omega_{p,n}$ is finite for $0 \leq n \leq m-1$, then*

$$\mathcal{P}^0_m \subset \mathcal{R} \cap \mathrm{Ces}(\omega,p) \subset \mathcal{P}^0_{m+1}.$$

Moreover, $\mathcal{R} \cap \mathrm{Ces}(\omega,p) = \mathcal{P}^0_{m+1}$ if and only if

$$\int_{J_m} \Big(\frac{x - 1/2^{m+1}}{\omega(x)}\Big)^p dx < \infty \qquad \text{for } 1 \leq p < \infty,$$

$$\sup_{x\in J_m} \frac{x - 1/2^{m+1}}{\omega(x)} < \infty \qquad \text{for } p = \infty.$$

Otherwise, $\mathcal{R} \cap \mathrm{Ces}(\omega,p) = \mathcal{P}^0_m$.

(iv) *If $\omega_{p,0} = \infty$, then $\mathcal{R} \cap \mathrm{Ces}(\omega,p) = \{0\}$.*

Proof. We suppose that $1 \leq p < \infty$; the proof in the case $p = \infty$ is analogous.

For a Rademacher polynomial $\sum_{k=1}^m a_k r_k$ we have $A_n = |\sum_{k=1}^m a_k|$ for $n \geq m$. It follows from (3) that

$$\Big\| \sum_{k=1}^{m} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \leq 2\Big(\sum_{n=1}^{m-1} \omega_{p,n} A_n^p + \Big| \sum_{k=1}^{m} a_k \Big|^p \sum_{n=m}^{\infty} \omega_{p,n} \Big)^{1/p}. \tag{5}$$

(i) Condition (P2) holds if $r_k \in \mathcal{R} \cap \mathrm{Ces}(\omega,p)$ for all $k \in \mathbb{N}$. Conversely, since (P2) is equivalent to $\sum_{n=0}^{\infty} \omega_{p,n} < \infty$, from (5) it follows that $\mathcal{P} \subset \mathcal{R} \cap \mathrm{Ces}(\omega,p)$.

(ii) Since $\sum_{k=1}^m a_k = 0$ for $\sum_{k=1}^m a_k r_k \in \mathcal{P}^0$, from (5) we deduce that $\mathcal{P}^0 \subset \mathcal{P} \cap \mathrm{Ces}(\omega,p)$. On the other hand, if (P2) fails, then $\sum_{n=0}^{\infty} \omega_{p,n} = \infty$.

From the corresponding version of (2) for polynomials, we have

$$\Big\|\sum_{k=1}^{m}a_kr_k\Big\|_{\mathrm{Ces}(\omega,p)}\ge\frac{1}{6\sqrt{2}}\Big(\sum_{n=1}^{m-1}\omega_{p,n}A_{n+1}^p+\Big|\sum_{k=1}^{m}a_k\Big|^p\sum_{n=m}^{\infty}\omega_{p,n}\Big)^{1/p},$$

which shows that the space $\mathrm{Ces}(\omega,p)$ only contains those Rademacher polynomials $\sum_{k=1}^m a_kr_k$ such that $\sum_{k=1}^m a_k=0$.

(iii) Assume that $\omega_{p,m}=\infty$. The inclusion $\mathcal{P}_m^0\subset\mathcal{R}\cap\mathrm{Ces}(\omega,p)$ follows from (5) and the fact that $\sum_{k=1}^m a_k=0$ for $\sum_{k=1}^m a_kr_k\in\mathcal{P}_m^0$. On the other hand, from (2) we have

$$\Big\|\sum_{k=1}^{\infty}a_kr_k\Big\|_{\mathrm{Ces}(\omega,p)}^p\ge\frac{1}{(6\sqrt{2})^p}\omega_{p,m}A_{m+1}^p.$$

Since $\omega_{p,m}=\infty$, if $\sum_{k=1}^\infty a_kr_k\in\mathrm{Ces}(\omega,p)$ then we necessarily have $A_{m+1}=|\sum_{k=1}^{m+1}a_k|+\|(a_k)_{m+2}^\infty\|_2=0$, that is, $\mathcal{R}\cap\mathrm{Ces}(\omega,p)\subset\mathcal{P}_{m+1}^0$.

Set $\sum_{k=1}^{m+1}a_kr_k\in\mathcal{P}_{m+1}^0\setminus\mathcal{P}_m^0$, where $a_k=1$, for $1\le k\le m$, and $a_{m+1}=-m$. Since the inclusions $\mathcal{P}_m^0\subset\mathcal{R}\cap\mathrm{Ces}(\omega,p)\subset\mathcal{P}_{m+1}^0$ involve finite-dimensional vector spaces, we have $\mathcal{R}\cap\mathrm{Ces}(\omega,p)=\mathcal{P}_{m+1}^0$ if and only if $\sum_{k=1}^{m+1}a_kr_k\in\mathrm{Ces}(\omega,p)$; otherwise, $\mathcal{R}\cap\mathrm{Ces}(\omega,p)=\mathcal{P}_m^0$. Note that, for $x\in(0,1/2^{m+1}]$,

$$\frac{1}{x}\int_0^x\Big|\sum_{k=1}^{m+1}a_kr_k(t)\Big|\,dt=\Big|\sum_{k=1}^{m+1}a_k\Big|=0,$$

and for $x\in J_m=(1/2^{m+1},1/2^m)$,

$$\int_0^x\Big|\sum_{k=1}^{m+1}a_kr_k(t)\Big|\,dt=2m(x-1/2^{m+1}).$$

Hence,

$$\begin{aligned}\Big\|\sum_{k=1}^{m+1}a_kr_k\Big\|_{\mathrm{Ces}(\omega,p)}^p&=(2m)^p\int_{J_m}\Big(\frac{x-1/2^{m+1}}{\omega(x)}\Big)^p\,dx\\&\quad+\sum_{n=0}^{m-1}\int_{J_n}\Big(\frac{x}{\omega(x)}\Big)^p\Big(\frac{1}{x}\int_0^x\Big|\sum_{k=1}^{m+1}a_kr_k(t)\Big|\,dt\Big)^p\,dx.\end{aligned}$$

Since, for $0\le n\le m$, we have $A_n=n+(m-n+m^2)^{1/2}$, there exist constants $C_1,C_2>0$, depending only on m, such that for $0\le n\le m-1$,

$$C_1\le A_{n+1}\le 2A_n\le C_2.$$

This, together with (1) and the fact that $\omega_{p,n}$ is finite for $0\le n\le m-1$, implies that $\sum_{k=1}^{m+1} a_k r_k \in \mathrm{Ces}(\omega, p)$ if and only if

$$\int_{J_m} \left(\frac{x - 1/2^{m+1}}{\omega(x)}\right)^p dx < \infty,$$

which proves the equivalence.

(iv) follows from (iii) and from $\mathcal{P}_1^0 = \{0\}$. ∎

Next, we consider the problem of the complementability of $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ in $\mathrm{Ces}(\omega, p)$. In [4] it was proved, for $1 \le p < \infty$ and $\omega(x) = x$, that $\mathcal{R} \cap \mathrm{Ces}(x, p)$ is not complemented in $\mathrm{Ces}(x, p)$, and, for $\omega(x)$ a quasiconcave function, that $\mathcal{R} \cap \mathrm{Ces}(\omega, \infty)$ is not complemented in $\mathrm{Ces}(\omega, \infty)$. We extend this result to spaces $\mathrm{Ces}(\omega, p)$ under the sole assumption that $\{r_k\}$ is a basic sequence in $\mathrm{Ces}(\omega, p)$. In particular, this result applies for $\omega(x)$ a quasiconcave weight, and for the power weights $\omega(x) = x^\lambda$ with $\lambda < 1 + 1/p$ (see Example 4.9 below).

We need the following lemma, which is related to the study of when $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is isomorphic to ℓ^2 (see Section 5). Recall, for $\omega(x)$ a weight such that $\omega_{p,0} = \infty$, that from Proposition 4.5 we have $\mathcal{R} \cap \mathrm{Ces}(\omega, p) = \{0\}$.

Lemma 4.6. *Let $1 \le p \le \infty$ and $\omega(x)$ be a weight on $[0, 1]$. Assume that $\omega_{p,0}$ is finite. There exists a constant $A_{\omega,p} > 0$ such that*

$$A_{\omega,p} \|(a_k)_{k=1}^\infty\|_2 \le \Big\| \sum_{k=1}^\infty a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)}, \quad (a_k)_{k=1}^\infty \in \ell^2.$$

Proof. Let $1 \le p < \infty$. From (2), we have

$$\frac{\omega_{p,0}^{1/p}}{6\sqrt{2}} \|(a_k)_{k=1}^\infty\|_2 \le \frac{\omega_{p,0}^{1/p}}{6\sqrt{2}} (|a_1| + \|(a_k)_{k=2}^\infty\|_2) \le \Big\| \sum_{k=1}^\infty a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)}.$$

The case $p = \infty$ is analogous. ∎

The proof of the next result follows the steps of [4, Theorem 4], where the case when $p = \infty$ and $\omega(x)$ is quasiconcave is treated, with suitable and necessary adaptations. For the sake of completeness, we include a full sketch of the proof.

Theorem 4.7. *Let $1 \le p \le \infty$ and $\omega(x)$ be a weight on $[0, 1]$. Assume that $\{r_k\}$ is a basic sequence in $\mathrm{Ces}(\omega, p)$. Then, the space $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is not complemented in $\mathrm{Ces}(\omega, p)$.*

Proof. Since $\{r_k\}$ is a basic sequence in $\mathrm{Ces}(\omega, p)$, we know that, for all $k \in \mathbb{N}$, $r_k \in \mathrm{Ces}(\omega, p)$. Thus, condition (P2) is satisfied. From Proposition 3.1, $\mathrm{Ces}(\omega, p)$ has a saturated norm, and so $\mathrm{Ces}(\omega, p)'$ is a normed space. It also follows from (P2) that $L^\infty([0, 1]) \subset \mathrm{Ces}(\omega, p)$. Hence, $\mathrm{Ces}(\omega, p)' \subset L^1([0, 1])$.

Let P be a projection from $\mathrm{Ces}(\omega,p)$ onto $\mathcal{R}\cap\mathrm{Ces}(\omega,p)$. Then $Pf=\sum_{n\geq 1}\langle\phi_n,f\rangle r_n$ with $\phi_n\in\mathrm{Ces}(\omega,p)^*$. For $1\leq p<\infty$, since $\mathrm{Ces}(\omega,p)^*=\mathrm{Ces}(\omega,p)'$, we have

$$Pf=\sum_{n=1}^{\infty}\Big(\int_0^1 g_n(t)f(t)\,dt\Big)r_n,\quad f\in\mathrm{Ces}(\omega,p), \tag{6}$$

where $g_n\in\mathrm{Ces}(\omega,p)'\subset L^1([0,1])$. Since P is a projection, $\langle g_i,r_j\rangle=\delta_{ij}$.

For $p=\infty$, since $\mathrm{Ces}(\omega,\infty)$ is not separable, the situation is different. However, we will see that for f in the separable part of $\mathrm{Ces}(\omega,\infty)$, denoted by $\mathrm{Ces}(\omega,\infty)_0$, we still have the projection P represented as in (6). From [13, Ch. 15, §70, Theorem 2] we have the decomposition

$$\mathrm{Ces}(\omega,\infty)^*=\mathrm{Ces}(\omega,\infty)'\oplus(\mathrm{Ces}(\omega,\infty)')^d,$$

where $(\mathrm{Ces}(\omega,\infty)')^d$ is the space of all singular bounded linear functionals on $\mathrm{Ces}(\omega,\infty)$. It follows that $\phi_n=\psi_n+\theta_n$, $n\geq 1$, where $\psi_n\in\mathrm{Ces}(\omega,\infty)'$ and $\theta_n\in(\mathrm{Ces}(\omega,\infty)')^d$. In particular,

$$\theta_n(f)=0,\quad f\in\mathrm{Ces}(\omega,\infty)_0,$$

and, for some $g_n\in\mathrm{Ces}(\omega,\infty)'\subset L^1([0,1])$,

$$\psi_n(f)=\int_0^1 f(t)g_n(t)\,dt,\quad f\in\mathrm{Ces}(\omega,\infty).$$

Note that, since we do not necessarily have $r_k\in\mathrm{Ces}(\omega,\infty)_0$, it does not follow immediately that $\langle g_i,r_j\rangle=\delta_{ij}$. From the fact that $r_k-\chi_{[0,1]}\in\mathrm{Ces}(\omega,\infty)_0$, we have $\theta_n(r_k-\chi_{[0,1]})=0$, that is,

$$\theta_n(r_k)=\theta_n(\chi_{[0,1]}),\quad k\geq 1.$$

Since P is a projection,

$$\begin{aligned}\psi_n(r_n)+\theta_n(r_n)&=1,\\ \psi_n(r_k)+\theta_n(r_k)&=0,\quad k\neq n.\end{aligned} \tag{7}$$

Hence, for $k>n$, we have $\theta_n(\chi_{[0,1]})=-\psi_n(r_k)$. Moreover, since $(g_n)\subset L^1([0,1])$,

$$\lim_{k\to\infty}\psi_n(r_k)=\lim_{k\to\infty}\int_0^1 g_n(t)r_k(t)\,dt=0.$$

Thus, $\theta_n(r_k)=\theta_n(\chi_{[0,1]})=0$ for all $k\geq 1$, which together with (7) implies that $\langle g_i,r_j\rangle=\delta_{ij}$.

From $\langle g_i,r_j\rangle=\delta_{ij}$ with $g_n\in L^1([0,1])$ it follows, as in [4], that there exist $h\in(0,1)$ and n_0 such that, for $n\geq n_0$,

$$\Big|\int_h^1 g_n(t)r_n(t)\,dt\Big|>\frac{1}{2}. \tag{8}$$

Next, there exists a constant $C > 0$, depending on $\omega(x)$ and h, such that

$$\|f\chi_{[h,1]}\|_{\mathrm{Ces}(\omega,p)} \le C\|f\|_{L^1([h,1])} \tag{9}$$

for $f \in L^1([0,1])$. For $1 \le p < \infty$ we have

$$\begin{aligned}\|f\chi_{[h,1]}\|_{\mathrm{Ces}(\omega,p)} &= \Big(\int_0^1 \Big(\frac{1}{\omega(x)}\int_0^x |f(t)|\chi_{[h,1]}(t)\,dt\Big)^p dx\Big)^{1/p} \\ &= \Big(\int_h^1 \Big(\frac{1}{\omega(x)}\int_0^x |f(t)|\chi_{[h,1]}(t)\,dt\Big)^p dx\Big)^{1/p} \\ &\le \Big(\int_h^1 \frac{1}{\omega(x)^p}\,dx\Big)^{1/p} \|f\|_{L^1([h,1])}.\end{aligned}$$

For $p = \infty$, we have the analogous inequality

$$\|f\chi_{[h,1]}\|_{\mathrm{Ces}(\omega,\infty)} \le \sup_{h\le x\le 1} \frac{1}{\omega(x)} \|f\|_{L^1([h,1])}.$$

The finiteness of the integral for $1 \le p < \infty$ or the supremum for $p = \infty$ follows from condition (P2).

Define $P_h(f) := P(f\chi_{[h,1]})$. Then the operator $P_h\colon L^1([h,1]) \to L^1([0,1])$ is bounded. To see this, from the Khintchine inequalities in $L^1([0,1])$, we have

$$\|P_h f\|_{L^1([0,1])} = \|P(f\chi_{[h,1]})\|_{L^1([0,1])} \asymp \|(\langle f\chi_{[h,1]}, g_n\rangle)_{n=1}^\infty\|_{\ell^2}.$$

The previous equivalence, together with Lemma 4.6, yields

$$A_{\omega,p}\|(\langle f\chi_{[h,1]}, g_n\rangle)_{n=1}^\infty\|_{\ell^2} \le \|P(f\chi_{[h,1]})\|_{\mathrm{Ces}(\omega,p)}.$$

From (9) and the fact that P is a bounded operator it follows that

$$\|P(f\chi_{[h,1]})\|_{\mathrm{Ces}(\omega,p)} \le \|P\|\,\|f\chi_{[h,1]}\|_{\mathrm{Ces}(\omega,p)} \le C\|P\|\,\|f\|_{L^1([h,1])}$$

Thus, $P_h\colon L^1([h,1]) \to L^1([0,1])$ is bounded.

Since P_h is weakly compact and $L^1([h,1])$ has the Dunford–Pettis property, it follows that $\|P_h(r_n\chi_{[h,1]})\|_{L^1([0,1])} \to 0$ as $n \to \infty$. On the other hand, from (8), it follows, for $n \ge n_0$, that

$$\|P_h(r_n\chi_{[h,1]})\|_{L^1([0,1])} \asymp \Big(\sum_{k=1}^\infty \Big(\int_h^1 g_k(t)r_n(t)\,dt\Big)^2\Big)^{1/2} \ge \Big|\int_h^1 g_n(t)r_n(t)\,dt\Big| > \frac{1}{2},$$

which gives a contradiction. ∎

From Theorem 4.7 and Corollary 4.4, we have the following.

Corollary 4.8. *Let $1 \le p \le \infty$ and $\omega(x)$ be a weight on $[0,1]$.*

(a) *If conditions* (P4) *and* (P5) *are satisfied, then $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ is not complemented in* $\mathrm{Ces}(\omega,p)$.

(b) *In particular, if* $\omega(x)$ *is quasiconcave, then* $\mathcal{R}\cap\mathrm{Ces}(\omega,p)$ *is not complemented in* $\mathrm{Ces}(\omega,p)$.

We end this section considering the Cesàro spaces $\mathrm{Ces}(x^\lambda,p)$ corresponding to power weights $\omega(x)=x^\lambda$, for $\lambda\in\mathbb{R}$.

EXAMPLE 4.9. Let $1\le p<\infty$ and consider $\mathrm{Ces}(x^\lambda,p)$ for $\lambda\in\mathbb{R}$, that is,

$$\|f\|_{\mathrm{Ces}(x^\lambda,p)}=\left(\int_0^1\left(\frac{1}{x^\lambda}\int_0^x|f(t)|\,dt\right)^p dx\right)^{1/p}.$$

Set $\delta:=p(1-\lambda)+1$. A straightforward computation shows that for $\delta\neq 0$ we have $\omega_{p,n}=1/\delta 2^{n\delta}$, whereas for $\delta=0$ we have $\omega_{p,n}=\ln 2$. Thus, in both cases,

$$\frac{\omega_{p,n+1}}{\omega_{p,n}}=2^{-\delta}.$$

Hence (P4*) holds for arbitrary $\lambda\in\mathbb{R}$ and $1\le p<\infty$. From Theorem 4.3 it follows that

$$\Big\|\sum_{k=1}^\infty a_k r_k\Big\|_{\mathrm{Ces}(x^\lambda,p)}\asymp\left(\sum_{n=0}^\infty\frac{1}{2^{n\delta}}\Big(\Big|\sum_{k=1}^{n+1}a_k\Big|+\|(a_k)_{k=n+2}^\infty\|_2\Big)^p\right)^{1/p}. \tag{10}$$

Suppose $\delta>0$, that is, $\lambda<1+1/p$. Then condition (P5) is satisfied. From Corollary 4.4, we know that $\{r_k\}$ is a basic sequence in $\mathrm{Ces}(\omega,p)$, and so $\mathcal{R}\cap\mathrm{Ces}(x^\lambda,p)$ is not complemented in $\mathrm{Ces}(x^\lambda,p)$. From the Cauchy–Schwarz inequality, we have

$$\Big|\sum_{k=1}^n a_k\Big|+\|(a_k)_{k=n+1}^\infty\|_2\le 2(n+1)^{1/2}\|(a_k)_{k=1}^\infty\|_2.$$

Hence, from (10) and for $M_{\lambda,p}$ a positive constant,

$$\Big\|\sum_{k=1}^\infty a_k r_k\Big\|_{\mathrm{Ces}(x^\lambda,p)}\le M_{\lambda,p}\left(\sum_{n=0}^\infty\frac{1}{2^{n\delta}}(n+1)^{p/2}\right)^{1/p}\|(a_k)_{k=1}^\infty\|_2.$$

The last series converges, as $\delta>0$. This, together with Lemma 4.6, implies that the closed linear span of $\{r_k\}$ in $\mathrm{Ces}(x^\lambda,p)$ is isomorphic to ℓ^2 (note that in [4] this is proved in the case $\lambda=1$ and $1\le p<\infty$).

Suppose now that $\delta\le 0$, that is, $\lambda\ge 1+1/p$. Then condition (P2) fails. Thus, $\mathrm{Ces}(x^\lambda,p)$ contains no single Rademacher functions, and from Proposition 4.5, it only contains among the Rademacher polynomials those of the form $\sum_{k=1}^m a_k r_k$ with $\sum_{k=1}^m a_k=0$. But there are also infinite Rademacher series in $\mathrm{Ces}(x^\lambda,p)$. To see this, let, for example, $\delta=0$, that is, $\lambda=1+1/p$.

In this case, (10) becomes

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{\mathrm{Ces}(x^{1+1/p},p)} \asymp \Big(\sum_{n=0}^{\infty}\Big(\Big|\sum_{k=1}^{n+1} a_k\Big| + \|(a_k)_{k=n+2}^{\infty}\|_2\Big)^p\Big)^{1/p}.$$

Set $a_{3k} = 1/k^2$ and $a_{3k+1} = a_{3k+2} = -1/2k^2$ for $k \in \mathbb{N}$. Then, for $n \in \mathbb{N}$ and some $C > 0$,

$$\Big|\sum_{k=1}^{n} a_k\Big| \le \frac{1}{n^2}, \qquad \|(a_k)_{k=n}^{\infty}\|_2 \le \frac{C}{n^{3/2}}.$$

Thus, $\sum_{k=1}^{\infty} a_k r_k \in \mathrm{Ces}(x^{1+1/p}, p)$.

In the case $p = \infty$, we have $\omega_{\infty,n} \asymp 2^{n(\lambda-1)}$, and so condition (P4*) holds. Thus, we have the equivalence

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{\mathrm{Ces}(x^{\lambda},\infty)} \asymp \sup_{n\ge 0} 2^{n(\lambda-1)}\Big(\Big|\sum_{k=1}^{n+1} a_k\Big| + \|(a_k)_{k=n+2}^{\infty}\|_2\Big).$$

For $\lambda < 1$ it follows, as in the case $1 \le p < \infty$, that $\mathrm{Ces}(x^{\lambda}, \infty)$ is isomorphic to ℓ^2, and $\mathcal{R} \cap \mathrm{Ces}(x^{\lambda}, \infty)$ is not complemented in $\mathrm{Ces}(x^{\lambda}, \infty)$. For $\lambda \ge 1$, condition (P2) is not satisfied, and so $r_k \notin \mathrm{Ces}(x^{\lambda}, \infty)$ for all $k \ge 1$.

REMARK 4.10. The previous example shows, for power weights $\omega(x) = x^{\lambda}$, that condition (P2) is equivalent to $\mathcal{R} \cap \mathrm{Ces}(x^{\lambda}, p)$ being isomorphic to ℓ^2. This equivalence is not true in general, as can be seen by considering $\omega(x) = x \log_2^{3/2}(2/x)$. For $p = 1$ and $n \ge 0$, we have $\omega_{1,n} \asymp 1/(n+1)^{3/2}$, and so condition (P2) is satisfied. Let $a_k = 1/\sqrt{k}$ for $1 \le k \le N$. Then $\|(a_k)_{k=1}^{N}\|_2 \asymp \log_2^{1/2} N$. On the other hand, from Theorem 4.3 it follows that

$$\Big\|\sum_{k=1}^{N} a_k r_k\Big\|_{\mathrm{Ces}(\omega,1)} \ge A \sum_{n=0}^{N-1} \frac{1}{(n+1)^{3/2}}\Big|\sum_{k=1}^{n+1} a_k\Big| \asymp \log_2 N,$$

with $A > 0$ a constant depending on ω. Hence, $\mathcal{R} \cap \mathrm{Ces}(\omega, 1)$ is not isomorphic to ℓ^2.

5. $\mathcal{R}\cap\mathrm{Ces}(\omega, p)$ isomorphic to ℓ^2. In this section we study the situation when $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is isomorphic to ℓ^2. In Example 4.9 it was shown, for power weights $\omega(x) = x^{\lambda}$ and $1 \le p \le \infty$, that $\mathcal{R} \cap \mathrm{Ces}(x^{\lambda}, p)$ is isomorphic to ℓ^2 precisely when $\lambda < 1 + 1/p$. In [4] it was proved that $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ is isomorphic to ℓ^2 when $\omega(x) = x$ and $1 \le p < \infty$, [4, Theorem 1], while for $p = \infty$ it was shown, for $\omega(x)$ a quasiconcave function, that $\mathcal{R} \cap \mathrm{Ces}(\omega, \infty)$ is isomorphic to ℓ^2 if and only if $\omega(x) \ge cx \log_2^{1/2}(2/x)$ [4, Theorem 3]. Note that this last condition is precisely condition (P3) for $p = \infty$. We prove, for every $1 \le p \le \infty$, that condition (P3) suffices for $\mathcal{R} \cap \mathrm{Ces}(\omega, p)$ to be isomorphic to ℓ^2, thus removing the need for quasiconcavity. However, while condition (P3) is necessary when $p = \infty$, it is not necessary when $1 \le p < \infty$, even though it is very close to being so, as will be shown by considering the decreasing rearrangements of Rademacher series.

THEOREM 5.1. *Let* $1 \le p \le \infty$ *and* $\omega(x)$ *be a weight on* $[0,1]$. *Condition* (P3) *holds if and only if*

$$\Big\| \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^* \Big\|_{\mathrm{Ces}(\omega,p)} \asymp \|(a_k)_{k=1}^{\infty}\|_2.$$

Proof. Assume that condition (P3) holds. From Lemma 4.6 we have

$$A_{\omega,p}\|(a_k)_{k=1}^{\infty}\|_2 \le \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\mathrm{Ces}(\omega,p)} \le \Big\| \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^* \Big\|_{\mathrm{Ces}(\omega,p)}.$$

To prove the reverse inequality, let L^M be the Orlicz space generated by $M(t) := \exp(t^2) - 1$. The fundamental function of its associated space $(L^M)'$ is given by $\varphi_{(L^M)'}(x) = x \log_2^{1/2}(2/x)$.

From

$$\frac{1}{x}\int_0^x |f(t)|\,dt \le \frac{1}{x}\varphi_{(L^M)'}(x)\|f\|_{L^M} = \log_2^{1/2}(2/x)\|f\|_{L^M},$$

and the fact that L^M is an r.i. space where $\{r_k\}$ spans a closed linear subspace isomorphic to ℓ^2, we have, for $0 < x \le 1$ and some $K > 0$,

$$\frac{1}{x}\int_0^x \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^*(t)\,dt \le K \log_2^{1/2}(2/x)\|(a_k)_{k=1}^{\infty}\|_2.$$

Hence, for $1 \le p < \infty$,

$$\Big\| \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^* \Big\|_{\mathrm{Ces}(\omega,p)} \le K \Big(\int_0^1 \Big(\frac{x}{\omega(x)} \Big)^p \log_2^{p/2}(2/x)\,dx \Big)^{1/p} \|(a_k)_{k=1}^{\infty}\|_2,$$

whereas for $p = \infty$,

$$\Big\| \Big(\sum_{k=1}^{\infty} a_k r_k \Big)^* \Big\|_{\mathrm{Ces}(\omega,\infty)} \le K \sup_{0<x\le 1} \frac{x}{\omega(x)} \log_2^{1/2}(2/x)\|(a_k)_{k=1}^{\infty}\|_2.$$

Condition (P3) is precisely the finiteness of the integral or the supremum above.

For the converse, the cases $1 \le p < \infty$ and $p = \infty$ are different. Let $1 \le p < \infty$, and assume that $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ is isomorphic to ℓ^2. Let

$$v_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} r_k.$$

By our assumption, $\|v_n^*\|_{\mathrm{Ces}(\omega,p)} \le B_{\omega,p}\|(1/\sqrt{n})_{k=1}^{n}\|_2 = B_{\omega,p}$ for $n \in \mathbb{N}$. Via the Central Limit Theorem (as can be seen in the proof of [11, Theorem 6], see also [10, Theorem 2.b.4]) we have, for $0 < x \le 1$ and some $C > 0$,

$$\log_2^{1/2}(2/x) \le C \lim_{n\to\infty} v_n^*(x).$$

Hence,

$$\begin{aligned}\int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p \log_2^{p/2}(2/x)\,dx &\le C^p \int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p \Big(\lim_{n\to\infty} v_n^*(x)\Big)^p\,dx \\ &= C^p \lim_{n\to\infty} \int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p v_n^*(x)^p\,dx \\ &\le C^p \lim_{n\to\infty} \int_0^1 \Big(\frac{x}{\omega(x)}\Big)^p \Big(\frac{1}{x}\int_0^x v_n^*(s)ds\Big)^p\,dx \\ &= C^p \lim_{n\to\infty} \|v_n^*\|_{\mathrm{Ces}(\omega,p)} \le C^p B_{\omega,p}.\end{aligned}$$

Thus, condition (P3) is satisfied.

Let $p = \infty$, and assume that the norm of $(\sum_{k=1}^\infty a_k r_k)^*$ in $\mathrm{Ces}(\omega,\infty)$ is equivalent to $\|(a_k)_{k=1}^\infty\|_2$. In particular, this implies that $r_k \in \mathrm{Ces}(\omega,\infty)$, $k \in \mathbb{N}$, and so all the coefficients $\omega_{\infty,n}$ are finite. Thus, if (P3) does not hold, we have

$$\sup_{n\ge 0} \omega_{\infty,n}(n+1)^{1/2} = \infty,$$

and so there exists $(n_j)_{j=1}^\infty$ such that

$$\lim_{j\to\infty} \omega_{\infty,n_j}(n_j+1)^{1/2} = \infty.$$

Let $a_k^j = (n_j+1)^{-1/2}$ for $1 \le k \le n_j+1$ and $a_k^j = 0$ for $k \ge n_j+2$. It is clear that $\|(a_k^j)_{k=1}^\infty\|_2 = 1$ for $j \ge 1$. From Theorem 4.3 we have, for $A > 0$ a constant depending on $\omega(x)$,

$$\begin{aligned}\Big\|\Big(\sum_{k=1}^\infty a_k^j r_k\Big)^*\Big\|_{\mathrm{Ces}(\omega,\infty)} \ge \Big\|\sum_{k=1}^\infty a_k^j r_k\Big\|_{\mathrm{Ces}(\omega,\infty)} &\ge A\omega_{\infty,n_j}\Big|\sum_{k=1}^{n_j+1} a_k^j\Big| \\ &= A\omega_{\infty,n_j}(n_j+1)^{1/2},\end{aligned}$$

which letting $j \to \infty$ yields a contradiction. ■

In general, the norms in $\mathrm{Ces}(\omega,p)$ of a Rademacher series $\sum_{k=1}^\infty a_k r_k$ and its decreasing rearrangement $(\sum_{k=1}^\infty a_k r_k)^*$ are not equivalent. Consider $\omega(x) = x^{1+1/p}$. From Proposition 4.5 we deduce that $r_1 - r_2 \in \mathrm{Ces}(\omega,p)$. On the other hand, $(r_1-r_2)^* \notin \mathrm{Ces}(\omega,p)$, since $(r_1-r_2)^* = 2\chi_{[0,1/2]}$. This example, together with the following theorem, shows that, for $1 \le p < \infty$, condition (P3) is strictly stronger than $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ being isomorphic to ℓ^2.

Theorem 5.2. *Let $\omega(x)$ be a weight on $[0,1]$.*

(i) *Let $1 \le p < \infty$.*
   (a) *If condition* (P3) *holds, then $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ is isomorphic to ℓ^2.*
   (b) *If $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ is isomorphic to ℓ^2, then for every ε with $0 < \varepsilon < p/2$ we have*

$$\int_0^1 \left(\frac{x}{\omega(x)}\right)^p \log_2^{p/2-\varepsilon}(2/x)\,dx < \infty.$$

(ii) *For $p = \infty$, the space $\mathcal{R} \cap \mathrm{Ces}(\omega,\infty)$ is isomorphic to ℓ^2 if and only if condition* (P3) *holds.*

Proof. (i) If condition (P3) holds, from Theorem 5.1 and Lemma 4.6 we have

$$\begin{aligned} A_{\omega,p}\|(a_k)_{k=1}^\infty\|_2 &\le \Big\|\sum_{k=1}^\infty a_k r_k\Big\|_{\mathrm{Ces}(\omega,p)} \\ &\le \Big\|\Big(\sum_{k=1}^\infty a_k r_k\Big)^*\Big\|_{\mathrm{Ces}(\omega,p)} \le B_{\omega,p}\|(a_k)_{k=1}^\infty\|_2, \end{aligned}$$

which proves (a).

To prove (b), let $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ be isomorphic to ℓ^2. In particular, $\omega_{p,n}$ is finite for $n \ge 0$. Suppose, for some $0 < \varepsilon < p/2$, that

$$\int_0^1 \left(\frac{x}{\omega(x)}\right)^p \log_2^{p/2-\varepsilon}(2/x)\,dx = \infty.$$

Hence, the series $\sum_{n=0}^\infty \omega_{p,n}(n+1)^{p/2-\varepsilon}$ diverges. Set $a_k = k^{-1/2-\varepsilon/p}$ for $k \in \mathbb{N}$. We have $(a_k)_{k=1}^\infty \in \ell^2$. On the other hand, from (2) follows the inequality

$$\Big\|\sum_{k=1}^\infty a_k r_k\Big\|_{\mathrm{Ces}(\omega,p)}^p \ge \frac{1}{(6\sqrt{2})^p}\sum_{n=0}^\infty \omega_{p,n}\Big|\sum_{k=1}^{n+1} a_k\Big|^p,$$

which together with the fact that

$$\Big|\sum_{k=1}^{n+1}\frac{1}{k^{1/2+\varepsilon/p}}\Big|^p \asymp (n+1)^{p/2-\varepsilon}$$

implies that $\sum_{k=1}^\infty a_k r_k \notin \mathrm{Ces}(\omega,p)$. This gives a contradiction.

(ii) If (P3) is satisfied, the equivalence between $\|\sum_{k=1}^\infty a_k r_k\|_{\mathrm{Ces}(\omega,\infty)}$ and $\|(a_k)_{k=1}^\infty\|_2$ can be proved as in the case $1 \le p < \infty$.

Conversely, assume that $\|\sum_{k=1}^{\infty} a_k r_k\|_{\mathrm{Ces}(\omega,\infty)}$ is equivalent to $\|(a_k)_{k=1}^{\infty}\|_2$. In particular, this implies that $\omega_{\infty,n}$ is finite for $n \geq 0$. Suppose that

$$\sup_{0<x\leq 1} \frac{x}{\omega(x)} \log_2^{1/2}(2/x) = \infty.$$

Then we have $\sup_{n\geq 0} \omega_{\infty,n}(n+1)^{1/2} = \infty$, and there exist n_j such that $\lim_{j\to\infty} \omega_{\infty,n_j}(n_j+1)^{1/2} = \infty$. Let $a_k^j = (n_j+1)^{-1/2}$ for $1 \leq k \leq n_j+1$ and $a_k^j = 0$ for $k \geq n_j+2$. It is clear that $\|(a_k^j)_{k=1}^{\infty}\|_2 = 1$ for $j \in \mathbb{N}$. From Theorem 4.3, we have, for some $A > 0$,

$$\Big\|\sum_{k=1}^{\infty} a_k^j r_k\Big\|_{\mathrm{Ces}(\omega,\infty)} \geq A\,\omega_{\infty,n_j}\Big|\sum_{k=1}^{n_j+1} a_k^j\Big| = A\,\omega_{\infty,n_j}(n_j+1)^{1/2},$$

which letting $j \to \infty$ yields a contradiction. ∎

Corollary 5.3. *Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0,1]$. Suppose that $\omega(x)$ satisfies condition* (P3). *Then:*

(i) *The sequence $\{r_k\}$ is basic in* $\mathrm{Ces}(\omega,p)$.
(ii) *The space $\mathcal{R} \cap \mathrm{Ces}(\omega,p)$ is not complemented in* $\mathrm{Ces}(\omega,p)$.
(iii) *For $(a_k)_{k=1}^{\infty} \in \ell^2$, the series $\sum_{k=1}^{\infty} a_k r_k$ converges unconditionally.*

We end by giving an equivalent expression for the norm of $(\sum_{k=1}^{\infty} a_k r_k)^*$ in $\mathrm{Ces}(\omega,p)$. For this, we need the following result, which follows from the proof of [1, Corollary 8.1] with suitable modifications. For $(a_k)_{k=1}^{\infty} \in \ell^2$, let $(a_k^*)_{k=1}^{\infty}$ be the decreasing rearrangement of $(|a_k|)_{k=1}^{\infty}$.

Lemma 5.4. *For $(a_k) \in \ell^2$ and $0 < x \leq 1$,*

$$\frac{1}{x}\int_0^x \Big(\sum_{k=1}^{\infty} a_k r_k\Big)^*(t)\,dt \asymp \sum_{k=1}^{[\log_2(2/x)]} a_k^* + \log_2^{1/2}(2/x)\|(a_k^*)_{k=[\log_2(2/x)]+1}^{\infty}\|_2$$

with absolute constants.

Since $[\log_2(2/x)] = n+1$ for $x \in J_n$, it follows from the previous lemma that

$$\frac{1}{x}\int_0^x \Big(\sum_{k=1}^{\infty} a_k r_k\Big)^*(t)\,dt \asymp \sum_{k=1}^{n+1} a_k^* + (n+1)^{1/2}\|(a_k^*)_{k=n+2}^{\infty}\|_2, \quad x \in J_n.$$

This allows us to obtain an analogous result to Theorem 4.3 (with a similar proof) for the decreasing rearrangement of a Rademacher series.

Theorem 5.5. *Let $1 \leq p \leq \infty$ and $\omega(x)$ be a weight on $[0,1]$. For $1 \leq p < \infty$, we have*

$$\Big\|\Big(\sum_{k=1}^{\infty} a_k r_k\Big)^*\Big\|_{\mathrm{Ces}(\omega,p)} \asymp \Big(\sum_{n\geq 0} \omega_{p,n}\Big(\sum_{k=1}^{n+1} a_k^* + (n+1)^{1/2}\|(a_k^*)_{k=n+2}^{\infty}\|_2\Big)^p\Big)^{1/p},$$

and for $p=\infty$,

$$\Big\|\Big(\sum_{k=1}^{\infty} a_k r_k\Big)^*\Big\|_{\mathrm{Ces}(\omega,\infty)} \asymp \sup_{n\geq 0} \omega_{p,n}\Big(\sum_{k=1}^{n+1} a_k^* + (n+1)^{1/2}\|(a_k^*)_{k=n+2}^{\infty}\|_2\Big).$$

Acknowledgements. This work is part of the Ph.D. Thesis of the author which is being prepared at University of Sevilla under the supervision of Prof. G. P. Curbera.

This research was partially supported by MTM 2012-36732-C03-03 (Ministerio de Economía y Competitividad), FQM-262, FQM-4643 (Junta de Andalucía) and Feder Funds (European Union).

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Received November 13, 2012
Revised version May 25, 2013 (7686)