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L^p spectral multipliers on the free group $N_{3,2}$

by

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Abstract. Let L be a homogeneous sublaplacian on the 6-dimensional free 2-step nilpotent Lie group $N_{3,2}$ on three generators. We prove a theorem of Mikhlin–Hörmander type for the functional calculus of L, where the order of differentiability $s > 6/2$ is required on the multiplier.

1. Introduction. The free 2-step nilpotent Lie group $N_{3,2}$ on three generators is the simply connected, connected nilpotent Lie group defined by the relations

$$[X_1, X_2] = Y_3, \quad [X_2, X_3] = Y_1, \quad [X_3, X_1] = Y_2,$$

where $X_1, X_2, X_3, Y_1, Y_2, Y_3$ is a basis of its Lie algebra (that is, the Lie algebra of left-invariant vector fields on $N_{3,2}$). In exponential coordinates, $N_{3,2}$ can be identified with $\mathbb{R}^3_x \times \mathbb{R}^3_y$, where the group law is given by

$$(x, y) \cdot (x', y') = (x + x', y + y' + x \wedge x'/2)$$

and $x \wedge x'$ denotes the usual vector product of $x, x' \in \mathbb{R}^3$. The family $(\delta_t)_{t>0}$ of automorphic dilations of $N_{3,2}$, defined by

$$\delta_t(x, y) = (tx, t^2 y),$$

turns $N_{3,2}$ into a stratified group of homogeneous dimension $Q = 9$.

Let L be a homogeneous sublaplacian on $N_{3,2}$; without loss of generality, we may assume that $L = -(X_1^2 + X_2^2 + X_3^2)$. Since L is a self-adjoint operator on $L^2(N_{3,2})$, a functional calculus for L is defined via spectral integration and, for all Borel functions $F : \mathbb{R} \to \mathbb{C}$, the operator $F(L)$ is bounded on $L^2(N_{3,2})$ whenever the "spectral multiplier" F is a bounded function. Here we are interested in giving a sufficient condition for the L^p-boundedness (for $p \neq 2$) of the operator $F(L)$, in terms of smoothness properties of the multiplier F.

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Let $W_2^s(\mathbb{R})$ denote the L^2 Sobolev space of (fractional) order s. Then our main result reads as follows.

THEOREM 1.1. *Suppose that a function* $F : \mathbb{R} \to \mathbb{C}$ *satisfies*

$$\sup_{t>0} \|\eta F(t\,\cdot)\|_{W_2^s} < \infty$$

for some $s > 6/2$ *and some nonzero* $\eta \in C_c^\infty(]0,\infty[)$. *Then the operator* $F(L)$ *is of weak type* $(1,1)$ *and bounded on* $L^p(N_{3,2})$ *for all* $p \in]1,\infty[$.

Observe that the general multiplier theorem for homogeneous sublaplacians on stratified Lie groups by Christ [3] and Mauceri and Meda [16] requires the stronger regularity condition $s > Q/2 = 9/2$. To the best of our knowledge, in the case of $N_{3,2}$ none of the results and techniques known so far allowed one to go below the condition $s > Q/2$. Our result pushes the regularity assumption down to $s > d/2 = 6/2$, where $d = 6$ is the topological dimension of $N_{3,2}$. We conjecture that this condition is sharp.

The problem of L^p-boundedness for spectral multipliers on nilpotent Lie groups has a long history, and the theorem by Christ and Mauceri and Meda is itself an improvement of a series of previous results (see, e.g., [4, 9, 5]). Nevertheless it is still an open question whether the homogeneous dimension in the smoothness condition may always be replaced by the topological dimension.

It has been known for a long time [10, 19] that such an improvement of the multiplier theorem holds true in the case of the Heisenberg and related groups (more precisely, for direct products of Métivier and abelian groups; see also [11, 14]). This class of groups, however, does not include $N_{3,2}$, or any free 2-step nilpotent group $N_{n,2}$ on n generators (see [20, §3] for a definition), except for the smallest one, $N_{2,2}$, which is the 3-dimensional Heisenberg group. The free groups $N_{n,2}$ have in a sense the maximal structural complexity among 2-step groups, since every 2-step nilpotent Lie group is a quotient of a free one. Our result should then hopefully shed some new light and contribute to the understanding of the problem for general 2-step nilpotent Lie groups.

2. Strategy of the proof. The sublaplacian L is a left-invariant operator on $N_{3,2}$, hence any operator of the form $F(L)$ is left-invariant too. Let $\mathcal{K}_{F(L)}$ then denote the convolution kernel of $F(L)$. As shown, e.g., in [14, Theorem 4.6], the previous Theorem 1.1 is a consequence of the following L^1-estimate.

PROPOSITION 2.1. *For all* $s > 6/2$, *for all compact sets* $K \subseteq]0,\infty[$, *and for all functions* $F : \mathbb{R} \to \mathbb{C}$ *such that* $\operatorname{supp} F \subseteq K$,

$$\|\mathcal{K}_{F(L)}\|_1 \le C_{K,s} \|F\|_{W_2^s}. \tag{2.1}$$

Let $|\cdot|_\delta$ be any δ_t-homogeneous norm on $N_{3,2}$; take, e.g., $|(x,y)|_\delta = |x| + |y|^{1/2}$. The crucial estimate in the proof in [16] of the general theorem for stratified groups, that is,

$$\|(1+|\cdot|_\delta)^\alpha \mathcal{K}_{F(L)}\|_2 \le C_{K,\alpha,\beta}\|F\|_{W_2^\beta} \tag{2.2}$$

for all $\alpha \ge 0$ and $\beta > \alpha$, implies (2.1) when $s > 9/2$, by Hölder's inequality. In order to push the condition down to $s > 6/2$, here we prove an enhanced version of (2.2), namely

$$\|(1+|\cdot|_\delta)^\alpha w^r \mathcal{K}_{F(L)}\|_2 \le C_{K,\alpha,\beta,r}\|F\|_{W_2^\beta}, \tag{2.3}$$

for some "extra weight" function w on $N_{3,2}$, and suitable constraints on the exponents α, β, r.

A similar approach is adopted in the above-mentioned works on the Heisenberg and related groups. However, in [19] the extra weight w is the full weight $1 + |\cdot|_\delta$, while [10] employs the weight $w(x,y) = 1 + |x|$. Here instead the weight $w(x,y) = 1 + |y|$ is used, and (2.3) is proved under the conditions $\alpha \ge 0$, $0 \le r < 3/2$, $\beta > \alpha + r$ (see Proposition 4.6 below).

The proof of (2.3) when $\alpha = 0$ is based on a careful analysis exploiting identities for Laguerre polynomials, somewhat in the spirit of [4, 19, 18], but with additional complexity due, inter alia, to the simultaneous use of generalized Laguerre polynomials of different types. The estimate for arbitrary α is then recovered by interpolation with (2.2). An analogous strategy is followed in [15], where identities for Hermite polynomials are used in order to prove a sharp spectral multiplier theorem for Grushin operators.

3. A joint functional calculus. It is convenient for us to embed the functional calculus for the sublaplacian L in a larger functional calculus for a system of commuting left-invariant differential operators on $N_{3,2}$. Specifically, the operators

$$L, -iY_1, -iY_2, -iY_3 \tag{3.1}$$

are essentially self-adjoint and commute strongly, hence they admit a joint functional calculus (see, e.g., [13]).

If $\mathbf{Y}$ denotes the "vector of operators" $(-iY_1, -iY_2, -iY_3)$, then we can express the convolution kernel $\mathcal{K}_{G(L,\mathbf{Y})}$ of the operator $G(L,\mathbf{Y})$ in terms of Laguerre functions (cf. [7]). Namely, for all $n, k \in \mathbb{N}$, let

$$L_n^{(k)}(u) = \frac{u^{-k}e^u}{n!}\left(\frac{d}{du}\right)^n (u^{k+n}e^{-u})$$

be the nth Laguerre polynomial of type k, and define

$$\mathcal{L}_n^{(k)}(t) = (-1)^n e^{-t} L_n^{(k)}(2t).$$

Further, for all $\eta \in \mathbb{R}^3 \setminus \{0\}$ and $\xi \in \mathbb{R}^3$, define $\xi_\parallel^\eta$ and $\xi_\perp^\eta$ by

$$\xi_\parallel^\eta = \langle \xi, \eta/|\eta|\rangle, \qquad \xi_\perp^\eta = \xi - \xi_\parallel^\eta \eta/|\eta|.$$

PROPOSITION 3.1. *Let $G : \mathbb{R}^4 \to \mathbb{C}$ be in the Schwartz class, and set*

$$m(n,\mu,\eta) = G((2n+1)|\eta| + \mu^2, \eta) \tag{3.2}$$

for all $n \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\eta \in \mathbb{R}^3$ with $\eta \neq 0$. Then

$$\mathcal{K}_{G(L,\mathbf{Y})}(x,y) = \frac{2}{(2\pi)^6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{n\in\mathbb{N}} m(n, \xi_\parallel^\eta, \eta) \mathcal{L}_n^{(0)}(|\xi_\perp^\eta|^2/|\eta|) e^{i\langle \xi, x\rangle} e^{i\langle \eta, y\rangle} \, d\xi \, d\eta.$$

Proof. For all $\eta \in \mathbb{R}^3 \setminus \{0\}$, choose a unit vector $E_\eta \in \eta^\perp$, and set $\bar{E}_\eta = (\eta/|\eta|) \wedge E_\eta$; moreover, for all $x \in \mathbb{R}^3$, denote by x_1^η, x_2^η, $x_\parallel^\eta$ the components of x with respect to the positive orthonormal basis E^η, $\bar{E}^\eta$, $\eta/|\eta|$ of $\mathbb{R}^3$.

For all $\eta \in \mathbb{R}^3 \setminus \{0\}$ and all $\mu \in \mathbb{R}$, an irreducible unitary representation $\pi_{\eta,\mu}$ of $N_{3,2}$ on $L^2(\mathbb{R})$ is defined by

$$\pi_{\eta,\mu}(x,y)\phi(u) = e^{i\langle \eta, y\rangle} e^{i|\eta|(u + x_1^\eta/2)x_2^\eta} e^{i\mu x_\parallel^\eta} \phi(x_1^\eta + u)$$

for all $(x,y) \in N_{3,2}$, $u \in \mathbb{R}$, $\phi \in L^2(\mathbb{R})$. Following e.g. [1, §2], one can see that these representations are sufficient to write the Plancherel formula for the group Fourier transform of $N_{3,2}$, and the corresponding Fourier inversion formula:

$$f(x,y) = (2\pi)^{-5} \int_{\mathbb{R}^3\setminus\{0\}} \int_{\mathbb{R}} \operatorname{tr}(\pi_{\eta,\mu}(x,y)\pi_{\eta,\mu}(f)) |\eta| \, d\mu \, d\eta \tag{3.3}$$

for all $f : N_{3,2} \to \mathbb{C}$ in the Schwartz class and all $(x,y) \in N_{3,2}$, where $\pi_{\eta,\mu}(f) = \int_{N_{3,2}} f(z)\pi_{\eta,\mu}(z^{-1})\, dz$.

Fix $\eta \in \mathbb{R}^3 \setminus \{0\}$ and $\mu \in \mathbb{R}$. The operators (3.1) are represented in $\pi_{\eta,\mu}$ as

$$d\pi_{\eta,\mu}(L) = -\partial_u^2 + |\eta|^2 u^2 + \mu^2, \qquad d\pi_{\eta,\mu}(-iY_j) = \eta_j. \tag{3.4}$$

If h_n is the nth Hermite function, that is,

$$h_n(t) = (-1)^n (n! 2^n \sqrt{\pi})^{-1/2} e^{t^2/2} \left(\frac{d}{dt}\right)^n e^{-t^2},$$

and $\tilde{h}_{\eta,n}$ is defined by

$$\tilde{h}_{\eta,n}(u) = |\eta|^{1/4} h_n(|\eta|^{1/2} u),$$

then $\{\tilde{h}_{\eta,n}\}_{n\in\mathbb{N}}$ is a complete orthonormal system for $L^2(\mathbb{R})$, made of joint eigenfunctions of the operators (3.4); in fact,

$$d\pi_{\eta,\mu}(L)\tilde{h}_{\eta,n} = (|\eta|(2n+1) + \mu^2)\tilde{h}_{\eta,n}, \quad d\pi_{\eta,\mu}(-iY_j)\tilde{h}_{\eta,n} = \eta_j \tilde{h}_{\eta,n}. \tag{3.5}$$

Moreover the corresponding diagonal matrix coefficients $\varphi_{\eta,\mu,n}$ of $\pi_{\eta,\mu}$ are given by

$$\begin{aligned}\varphi_{\eta,\mu,n}(x,y) &= \langle \pi_{\eta,\mu}(x,y)\tilde{h}_{\eta,n}, \tilde{h}_{\eta,n}\rangle \\ &= e^{i\langle\eta,y\rangle} e^{i\mu x_\parallel^\eta} |\eta|^{1/2} \int_{\mathbb{R}} e^{i|\eta| u x_2^\eta} h_n(|\eta|^{1/2}(u + x_1^\eta/2)) h_n(|\eta|^{1/2}(u - x_1^\eta/2))\, du.\end{aligned}$$

The last integral is essentially the Fourier–Wigner transform of the pair (h_n, h_n), whose Fourier transform has a particularly simple expression (cf. [8, formula (1.90)]); the parity of the Hermite functions then yields

$$\begin{aligned}\varphi_{\eta,\mu,n}(x,y) = e^{i\langle\eta,y\rangle} e^{i\mu x_\parallel^\eta} \frac{(-1)^n}{\pi|\eta|} \int_{\mathbb{R}^2} e^{iv_2 x_2^\eta} e^{iv_1 x_1^\eta} \\ \times \int_{\mathbb{R}} e^{-it(2v_1/|\eta|^{1/2})} h_n(t + v_2/|\eta|^{1/2}) h_n(t - v_2/|\eta|^{1/2})\, dt\, dv,\end{aligned}$$

that is,

$$\varphi_{\eta,\mu,n}(x,y) = \frac{1}{\pi|\eta|} e^{i\langle\eta,y\rangle} e^{i\mu x_\parallel^\eta} \int_{\mathbb{R}^2} e^{iv_1 x_1^\eta} e^{iv_2 x_2^\eta} \mathcal{L}_n^{(0)}(|v|^2/|\eta|)\, dv \tag{3.6}$$

(see [21, Theorem 1.3.4] or [8, Theorem 1.104]).

Note that $\mathcal{K}_{G(L,\mathbf{Y})} \in \mathcal{S}(N_{3,2})$ since $G \in \mathcal{S}(\mathbb{R}^4)$ (see [2, Theorem 5.2] or [12, §4.2]). Moreover

$$\pi_{\eta,\mu}(\mathcal{K}_{G(L,\mathbf{Y})})\tilde{h}_{\eta,n} = G(|\eta|(2n+1) + \mu^2, \eta)\tilde{h}_{\eta,n}$$

by (3.5) and [17, Proposition 1.1], hence

$$\langle \pi_{\eta,\mu}(x,y)\pi_{\eta,\mu}(\mathcal{K}_{G(L,\mathbf{Y})})\tilde{h}_{\eta,n}, \tilde{h}_{\eta,n}\rangle = m(n,\mu,\eta)\varphi_{\eta,\mu,n}(x,y).$$

Therefore, by (3.3) and (3.6),

$$\begin{aligned}&\mathcal{K}_{G(L,\mathbf{Y})}(x,y) \\ &\quad = (2\pi)^{-5} \int_{\mathbb{R}^3\setminus\{0\}} \int_{\mathbb{R}} \sum_{n\in\mathbb{N}} m(n,\mu,\eta)\varphi_{\eta,\mu,n}(x,y)|\eta|\, d\mu\, d\eta \\ &\quad = \frac{2}{(2\pi)^6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{n\in\mathbb{N}} m(n,\xi_3,\eta) e^{i\langle\eta,y\rangle} e^{i\langle\xi,(x_1^\eta,x_2^\eta,x_\parallel^\eta)\rangle} \mathcal{L}_n^{(0)}((\xi_1^2+\xi_2^2)/|\eta|)\, d\xi\, d\eta.\end{aligned}$$

The conclusion follows by a change of variable in the inner integral. ∎

4. Weighted estimates. For convenience, set $\mathcal{L}_n^{(k)} = 0$ for all $n < 0$. The following identities are easily obtained from the properties of Laguerre polynomials (see, e.g., [6, §10.12]).

LEMMA 4.1. *For all $k, n, n' \in \mathbb{N}$ and $t \in \mathbb{R}$,*

$$\mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) + \mathcal{L}_n^{(k+1)}(t), \tag{4.1}$$

$$\frac{d}{dt}\mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) - \mathcal{L}_n^{(k+1)}(t), \tag{4.2}$$

$$\int_0^\infty \mathcal{L}_n^{(k)}(t)\mathcal{L}_{n'}^{(k)}(t) t^k\,dt = \begin{cases} \dfrac{(n+k)!}{2^{k+1}n!} & \text{if } n = n', \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

We introduce some operators on functions $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$:

$$\begin{aligned} \tau f(n, \mu, \eta) &= f(n+1, \mu, \eta), \\ \delta f(n, \mu, \eta) &= f(n+1, \mu, \eta) - f(n, \mu, \eta), \\ \partial_\mu f(n, \mu, \eta) &= \frac{\partial}{\partial \mu} f(n, \mu, \eta), \\ \partial_\eta^\alpha f(n, \mu, \eta) &= \left(\frac{\partial}{\partial \eta}\right)^\alpha f(n, \mu, \eta), \end{aligned}$$

for all $\alpha \in \mathbb{N}^3$. For each multiindex $\alpha \in \mathbb{N}^3$, we denote by $|\alpha|$ its length $\alpha_1 + \alpha_2 + \alpha_3$. We set moreover $\langle t \rangle = 2|t| + 1$ for all $t \in \mathbb{R}$.

Note that, for all compactly supported $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$, $\tau^l f$ is null for all sufficiently large $l \in \mathbb{N}$; hence the operator $1 + \tau$, when restricted to the set of compactly supported functions, is invertible, with inverse given by

$$(1+\tau)^{-1} f = \sum_{l \in \mathbb{N}} (-1)^l \tau^l f,$$

and therefore the operator $(1+\tau)^q$ is well-defined for all $q \in \mathbb{Z}$.

PROPOSITION 4.2. *Let $G : \mathbb{R}^4 \to \mathbb{C}$ be smooth and compactly supported in $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, and let $m(n, \mu, \eta)$ be defined by (3.2). Then for all $\alpha \in \mathbb{N}^3$,*

$$\begin{aligned} &\int_{N_{3,2}} |y^\alpha \mathcal{K}_{G(L,\mathbf{Y})}(x,y)|^2\,dx\,dy \\ &\quad \le C_\alpha \sum_{\iota \in I_\alpha} \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1+\tau)^{|\beta^\iota| - k_\iota} m(n, \mu, \eta)|^2 \\ &\qquad\qquad \times \mu^{2b_\iota} |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2k_\iota + |\beta^\iota| + 1} \langle n \rangle^{|\beta^\iota|}\,d\mu\,d\eta, \end{aligned} \tag{4.4}$$

where I_α is a finite set and, for all $\iota \in I_\alpha$,

- $\gamma^\iota \in \mathbb{N}^3$, $l_\iota, k_\iota \in \mathbb{N}$, $\gamma^\iota \le \alpha$, $\min\{1, |\alpha|\} \le |\gamma^\iota| + l_\iota + k_\iota \le |\alpha|$,
- $b_\iota \in \mathbb{N}$, $\beta^\iota \in \mathbb{N}^3$, $b_\iota + |\beta^\iota| = l_\iota + 2k_\iota$, $|\gamma^\iota| + l_\iota + b_\iota \le |\alpha|$.

Proof. Proposition 3.1 and integration by parts allow us to write

$$(4.5)\qquad y^\alpha \mathcal{K}_{G(L,\mathbf{Y})}(x,y) = \frac{2i^{|\alpha|}}{(2\pi)^6} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[\left(\frac{\partial}{\partial\eta}\right)^\alpha \sum_{n\in\mathbb{N}} m(n,\xi_\parallel^\eta,\eta)\mathcal{L}_n^{(0)}(|\xi_\perp^\eta|^2/|\eta|)\right] e^{i\langle\xi,x\rangle} e^{i\langle\eta,y\rangle}\,d\xi\,d\eta.$$

From the definition of $\xi_\parallel^\eta$ and $\xi_\perp^\eta$, the following identities are not difficult to obtain:

$$(4.6)\qquad \begin{aligned} &\frac{\partial}{\partial\eta_j}\xi_\parallel^\eta = (\xi_\perp^\eta)_j \frac{1}{|\eta|}, \qquad \frac{\partial}{\partial\eta_j}(\xi_\perp^\eta)_k = -\xi_\parallel^\eta \frac{\partial}{\partial\eta_j}\frac{\eta_k}{|\eta|} - (\xi_\perp^\eta)_j \frac{\eta_k}{|\eta|^2},\\ &\frac{\partial}{\partial\eta_j}\frac{|\xi_\perp^\eta|^2}{|\eta|} = -\xi_\parallel^\eta(\xi_\perp^\eta)_j \frac{2}{|\eta|^2} - |\xi_\perp^\eta|^2 \frac{\eta_j}{|\eta|^3}.\end{aligned}$$

The multiindex notation will also be used as follows:

$$(\xi_\perp^\eta)^\beta = (\xi_\perp^\eta)_1^{\beta_1}(\xi_\perp^\eta)_2^{\beta_2}(\xi_\perp^\eta)_3^{\beta_3}$$

for all $\xi,\eta\in\mathbb{R}$, with $\eta\neq 0$, and all $\beta\in\mathbb{N}^3$; consequently,

$$|\xi_\perp^\eta|^2 = (\xi_\perp^\eta)^{(2,0,0)} + (\xi_\perp^\eta)^{(0,2,0)} + (\xi_\perp^\eta)^{(0,0,2)}.$$

Via these identities, one can prove inductively that, for all $\alpha\in\mathbb{N}^3$,

$$(4.7)\qquad \left(\frac{\partial}{\partial\eta}\right)^\alpha \sum_{n\in\mathbb{N}} m(n,\xi_\parallel^\eta,\eta)\mathcal{L}_n^{(0)}(|\xi_\perp^\eta|^2/|\eta|) = \sum_{\iota\in I_\alpha}\sum_{n\in\mathbb{N}} \partial_\eta^{\gamma^\iota}\partial_\mu^{l_\iota}\delta^{k_\iota} m(n,\xi_\parallel^\eta,\eta)(\xi_\parallel^\eta)^{b_\iota}(\xi_\perp^\eta)^{\beta^\iota}\Theta_\iota(\eta)\mathcal{L}_n^{(k_\iota)}(|\xi_\perp^\eta|^2/|\eta|),$$

where I_α, γ^ι, l_ι, k_ι, b_ι, β^ι are as in the statement above, while $\Theta_\iota : \mathbb{R}^3\setminus\{0\} \to \mathbb{R}$ is smooth and homogeneous of degree $|\gamma^\iota| - |\alpha| - k_\iota$. For the inductive step, one employs Leibniz' rule, and when a derivative hits a Laguerre function, the identity (4.2) together with summation by parts is used.

Note that, for all compactly supported $f : \mathbb{N}\times\mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$,

$$\sum_{n\in\mathbb{N}} f(n,\mu,\eta)\mathcal{L}_n^{(k)}(t) = \sum_{n\in\mathbb{N}} (1+\tau)f(n,\mu,\eta)\mathcal{L}_n^{(k+1)}(t),$$

by (4.1). Since $1+\tau$ is invertible, simple manipulations and iteration yield the more general identity

$$\sum_{n\in\mathbb{N}} f(n,\mu,\eta)\mathcal{L}_n^{(k)}(t) = \sum_{n\in\mathbb{N}} (1+\tau)^{k'-k} f(n,\mu,\eta)\mathcal{L}_n^{(k')}(t)$$

for all $k,k'\in\mathbb{N}$. This formula allows us to adjust in (4.7) the type of the

Laguerre functions to the exponent of $\xi_\perp$, and to deduce that

$$\left(\frac{\partial}{\partial\eta}\right)^\alpha \sum_{n\in\mathbb{N}} m(n,\xi_\parallel^\eta,\eta)\mathcal{L}_n^{(0)}(|\xi_\perp^\eta|^2/|\eta|) = \sum_{\iota\in I_\alpha}\sum_{n\in\mathbb{N}} \partial_\eta^{\gamma^\iota}\partial_\mu^{l_\iota}\delta^{k_\iota}(1+\tau)^{|\beta^\iota|-k_\iota} m(n,\xi_\parallel^\eta,\eta) \times (\xi_\parallel^\eta)^{b_\iota}(\xi_\perp^\eta)^{\beta^\iota}\Theta_\iota(\eta)\mathcal{L}_n^{(|\beta^\iota|)}(|\xi_\perp^\eta|^2/|\eta|).$$

By plugging this identity into (4.5) and exploiting Plancherel's formula for the Fourier transform, the finiteness of I_α and the triangular inequality, we get

$$\int_{N_{3,2}} |y^\alpha \mathcal{K}_{G(L,\mathbf{Y})}(x,y)|^2\,dx\,dy \le C_\alpha \sum_{\iota\in I_\alpha}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\Big|\sum_{n\in\mathbb{N}} \partial_\eta^{\gamma^\iota}\partial_\mu^{l_\iota}\delta^{k_\iota}(1+\tau)^{|\beta^\iota|-k_\iota} m(n,\mu,\eta)\mathcal{L}_n^{(|\beta^\iota|)}(|\zeta|^2/|\eta|)\Big|^2 \times \mu^{2b_\iota}|\zeta|^{2|\beta^\iota|}|\eta|^{2|\gamma^\iota|-2|\alpha|-2k_\iota}\,d\zeta\,d\mu\,d\eta.$$

The passage to polar coordinates in the ζ-integral and rescaling then give

$$\int_{N_{3,2}} |y^\alpha \mathcal{K}_{G(L,\mathbf{Y})}(x,y)|^2\,dx\,dy \le C_\alpha \sum_{\iota\in I_\alpha}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_0^\infty\Big|\sum_{n\in\mathbb{N}} \partial_\eta^{\gamma^\iota}\partial_\mu^{l_\iota}\delta^{k_\iota}(1+\tau)^{|\beta^\iota|-k_\iota} m(n,\mu,\eta)\mathcal{L}_n^{(|\beta^\iota|)}(s)\Big|^2 s^{|\beta^\iota|}\,ds \times \mu^{2b_\iota}|\eta|^{2|\gamma^\iota|-2|\alpha|-2k_\iota+|\beta^\iota|+1}\,d\mu\,d\eta,$$

and the conclusion follows by applying the orthogonality relations (4.3) for the Laguerre functions to the inner integral. ■

Note that $\tau f(\cdot,\mu,\eta)$, $\delta f(\cdot,\mu,\eta)$ depend only on $f(\cdot,\mu,\eta)$; in other words, τ and δ can be considered as operators on functions $\mathbb{N}\to\mathbb{C}$. The next lemma will be useful in converting finite differences into continuous derivatives.

LEMMA 4.3. *Let $f:\mathbb{N}\to\mathbb{C}$ have a smooth extension $\tilde f:[0,\infty[\,\to\mathbb{C}$, and let $k\in\mathbb{N}$. Then*

$$\delta^k f(n) = \int_{J_k} \tilde f^{(k)}(n+s)\,d\nu_k(s)$$

for all $n\in\mathbb{N}$, where $J_k=[0,k]$ and ν_k is a Borel probability measure on J_k. In particular

$$|\delta^k f(n)|^2 \le \int_{J_k} |\tilde f^{(k)}(n+s)|^2\,d\nu_k(s)$$

for all $n\in\mathbb{N}$.

Proof. Iterated application of the fundamental theorem of integral calculus gives

$$\delta^k f(n) = \int_{[0,1]^k} \tilde{f}^{(k)}(n + s_1 + \cdots + s_k)\, ds.$$

The conclusion follows by taking as ν_k the push-forward of the uniform distribution on $[0,1]^k$ via the map $(s_1, \ldots, s_k) \mapsto s_1 + \cdots + s_k$, and by Hölder's inequality. ∎

We now give a simplified version of the right-hand side of (4.4), in the case where we restrict to the functional calculus for the sublaplacian L alone. In order to avoid divergent series, however, it is convenient to first truncate the multiplier along the spectrum of $\mathbf{Y}$.

Lemma 4.4. *Let $\chi \in C_c^\infty(\mathbb{R})$ be supported in $[1/2, 2]$, $K \subseteq]0,\infty[$ be compact and $M \in]0,\infty[$. If $F : \mathbb{R} \to \mathbb{C}$ is smooth and supported in K, and $F_M : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is given by*

$$F_M(\lambda, \eta) = F(\lambda)\chi(|\eta|/M),$$

then, for all $r \in [0,\infty[$,

$$\int_{N_{3,2}} \big||y|^r \mathcal{K}_{F_M(L,\mathbf{Y})}(x,y)\big|^2\, dx\, dy \le C_{K,\chi,r} M^{3-2r} \|F\|^2_{W_2^r}.$$

Proof. We may restrict to the case $r \in \mathbb{N}$, the other cases being recovered a posteriori by interpolation. Hence we need to prove that

$$\int_{N_{3,2}} |y^\alpha \mathcal{K}_{F_M(L,\mathbf{Y})}(x,y)|^2\, dx\, dy \le C_{K,\chi,\alpha} M^{3-2|\alpha|} \|F\|^2_{W_2^{|\alpha|}} \tag{4.8}$$

for all $\alpha \in \mathbb{N}^3$. On the other hand, if

$$m(n, \mu, \eta) = F(|\eta|\langle n\rangle + \mu^2)\chi(|\eta|/M),$$

then the left-hand side of (4.8) can be majorized by (4.4), and we are reduced to proving that

$$\sum_{n\in\mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1+\tau)^{|\beta^\iota| - k_\iota} m(n,\mu,\eta)|^2 \mu^{2b_\iota} |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2k_\iota + |\beta^\iota| + 1} \times \langle n\rangle^{|\beta^\iota|}\, d\mu\, d\eta \le C_{K,\chi,\alpha} M^{3-2|\alpha|} \|F\|^2_{W_2^{|\alpha|}} \tag{4.9}$$

for all $\iota \in I_\alpha$.

Consider first the case $|\beta^\iota| \ge k_\iota$. A smooth extension $\tilde{m} : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ of m is defined by

$$\tilde{m}(t,\mu,\eta) = F(|\eta|(2t+1) + \mu^2)\chi(|\eta|/M).$$

Then, by Lemma 4.3,

$$\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1+\tau)^{|\beta^\iota|-k_\iota} m(n,\mu,\eta) = \sum_{j=0}^{|\beta^\iota|-k_\iota} \binom{|\beta^\iota|-k_\iota}{j} \int_{J_\iota} \partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(n+j+s,\mu,\eta)\, d\nu_\iota(s),$$

where $J_\iota = [0,k_\iota]$ and ν_ι is a suitable probability measure on J_ι; consequently, inequality (4.9) will be proved if we show that

$$\sum_{n\in\mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(n+s,\mu,\eta)|^2 \mu^{2b_\iota} |\eta|^{2|\gamma^\iota|-2|\alpha|-2k_\iota+|\beta^\iota|+1} \times \langle n\rangle^{|\beta^\iota|}\, d\mu\, d\eta \le C_{K,\chi,\alpha} M^{3-2|\alpha|} \|F\|^2_{W_2^{|\alpha|}} \tag{4.10}$$

for all $s\in[0,|\beta^\iota|]$. On the other hand, it is easily proved inductively that

$$\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(t,\mu,\eta) = \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \sum_{q=0}^{|\gamma^\iota|-v} \Psi_{\iota,v,q}(\eta) \langle t\rangle^v \mu^{2r-l_\iota} M^{-q} F^{(k_\iota+v+r)}(|\eta|\langle t\rangle + \mu^2) \chi^{(q)}(|\eta|/M)$$

for all $t\ge 0$, where $\Psi_{\iota,v,q} : \mathbb{R}^3\setminus\{0\} \to \mathbb{R}$ is smooth and homogeneous of degree $k_\iota+v+q-|\gamma^\iota|$; hence

$$|\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(t,\mu,\eta)|^2 \le C_{\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{2k_\iota+2v-2|\gamma^\iota|} \langle t\rangle^{2v} \mu^{4r-2l_\iota} \times |F^{(k_\iota+v+r)}(|\eta|\langle t\rangle+\mu^2)|^2 \tilde{\chi}(|\eta|/M), \tag{4.11}$$

where $\tilde{\chi}$ is the characteristic function of $[1/2,2]$, and we are using the fact that $|\eta|\sim M$ in the region where $\tilde{\chi}(|\eta|/M)\neq 0$. Consequently, the left-hand side of (4.10) is majorized by

$$\begin{aligned}
&C_{\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{2v-2|\alpha|+|\beta^\iota|+1} \sum_{n\in\mathbb{N}} \langle n\rangle^{|\beta^\iota|} \langle n+s\rangle^{2v} \\
&\qquad \times \int_{\mathbb{R}^3} \int_{\mathbb{R}} |F^{(k_\iota+v+r)}(|\eta|\langle n+s\rangle+\mu^2)|^2 \mu^{2b_\iota+4r-2l_\iota} \tilde{\chi}(|\eta|/M)\, d\mu\, d\eta \\
&\quad \le C_{\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{2v-2|\alpha|+|\beta^\iota|+3} \sum_{n\in\mathbb{N}} \langle n+s\rangle^{|\beta^\iota|+2v} \\
&\qquad \times \int_0^\infty \int_0^\infty |F^{(k_\iota+v+r)}(\rho\langle n+s\rangle+\mu^2)|^2 \mu^{2b_\iota+4r-2l_\iota} \tilde{\chi}(\rho/M)\, d\mu\, d\rho
\end{aligned}$$

$$\leq C_{\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{2v-2|\alpha|+|\beta^\iota|+3} \int_0^\infty \int_0^\infty |F^{(k_\iota+v+r)}(\rho+\mu^2)|^2 \times \mu^{2b_\iota+4r-2l_\iota} \sum_{n\in\mathbb{N}} \langle n+s\rangle^{|\beta^\iota|+2v-1} \tilde{\chi}(\rho/(\langle n+s\rangle M))\, d\mu\, d\rho,$$

by passing to polar coordinates and rescaling. The last sum in n is easily controlled by $(\rho/M)^{|\beta^\iota|+2v}$, hence the left-hand side of (4.10) is majorized by

$$C_{\chi,\alpha} M^{3-2|\alpha|} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \int_0^\infty \int_0^\infty |F^{(k_\iota+v+r)}(\rho+\mu^2)|^2 \mu^{2b_\iota+4r-2l_\iota} \rho^{|\beta^\iota|+2v}\, d\mu\, d\rho$$
$$\leq C_{K,\chi,\alpha} M^{3-2|\alpha|} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \sup_{u\in[0,\max K]} \int_0^\infty |F^{(k_\iota+v+r)}(\rho+u)|^2\, d\rho,$$

because $2b_\iota + 4r - 2l_\iota \geq 0$ and $|\beta^\iota| + 2v \geq 0$ if r and v are in the range of summation, and $\operatorname{supp} F \subseteq K$. Since moreover $k_\iota + v + r \leq k_\iota + |\gamma^\iota| + l_\iota \leq |\alpha|$, the last integral is dominated by $\|F\|^2_{W_2^{|\alpha|}}$ uniformly in r, v, u, and (4.10) follows.

Consider now the case $|\beta^\iota| < k_\iota$. Via the identity

$$(1+\tau)^{-1} = (1-\tau)(1-\tau^2)^{-1} = -\delta(1-\tau^2)^{-1} = -\delta \sum_{j=0}^\infty \tau^{2j},$$

together with Lemma 4.3, we obtain

$$\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1+\tau)^{|\beta^\iota|-k_\iota} m(n,\mu,\eta) = (-1)^{k_\iota-|\beta^\iota|} \sum_{j=0}^\infty \binom{j+k_\iota-|\beta^\iota|-1}{k_\iota-|\beta^\iota|-1} \int_{J_\iota} \partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{2k_\iota-|\beta^\iota|} \tilde{m}(n+2j+s,\mu,\eta)\, d\nu_\iota(s), \tag{4.12}$$

where $J_\iota = [0, 2k_\iota - |\beta^\iota|]$ and ν_ι is a suitable probability measure on J_ι. Note that, because of the assumptions on the supports of F and χ, the sum on j on the right-hand side of (4.12) is a finite sum, that is, the jth summand is nonzero only if $\langle n+2j\rangle \leq 2M^{-1} \max K$; consequently, by applying the Cauchy–Schwarz inequality to the sum in j, and by (4.11),

$$|\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1+\tau)^{|\beta^\iota|-k_\iota} m(n,\mu,\eta)|^2 \leq C_{K,\alpha} M^{1+2|\beta^\iota|-2k_\iota} \sum_{j=0}^\infty \int_{J_\iota} |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{2k_\iota-|\beta^\iota|} \tilde{m}(n+2j+s,\mu,\eta)|^2\, d\nu_\iota(s)$$

$$\leq C_{K,\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{1+2k_\iota+2v-2|\gamma^\iota|} \sum_{j=0}^{\infty} \int_{J_\iota} \langle n+2j+s\rangle^{2v} \mu^{4r-2l_\iota}$$
$$\times |F^{(2k_\iota-|\beta^\iota|+v+r)}(|\eta|\langle n+2j+s\rangle+\mu^2)|^2 \tilde\chi(|\eta|/M)\, d\nu_\iota(s).$$

Remember that $|\eta| \sim M$ in the region where $\tilde\chi(|\eta|/M) \neq 0$. Hence the left-hand side of (4.9) is majorized by

$$C_{K,\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \int_{J_\iota} \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{N}} \langle n+2j+s\rangle^{2v} \langle n\rangle^{|\beta^\iota|} \int_{\mathbb{R}^3}\int_{\mathbb{R}} M^{2+2v-2|\alpha|+|\beta^\iota|}$$
$$\times \mu^{2b_\iota+4r-2l_\iota} |F^{(2k_\iota-|\beta^\iota|+v+r)}(|\eta|\langle n+2j+s\rangle+\mu^2)|^2 \tilde\chi(|\eta|/M)\, d\mu\, d\eta\, d\nu_\iota(s)$$
$$\leq C_{K,\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \int_{J_\iota} \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{N}} \langle n+2j+s\rangle^{2v+|\beta^\iota|} \int_0^\infty\int_0^\infty M^{4+2v-2|\alpha|+|\beta^\iota|}$$
$$\times \mu^{2b_\iota+4r-2l_\iota} |F^{(2k_\iota-|\beta^\iota|+v+r)}(\rho\langle n+2j+s\rangle+\mu^2)|^2 \tilde\chi(\rho/M)\, d\mu\, d\rho\, d\nu_\iota(s)$$
$$\leq C_{K,\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{4+2v-2|\alpha|+|\beta^\iota|} \int_0^\infty\int_0^\infty |F^{(2k_\iota-|\beta^\iota|+v+r)}(\rho+\mu^2)|^2$$
$$\times \mu^{2b_\iota+4r-2l_\iota} \int_{J_\iota} \sum_{(n,j)\in\mathbb{N}^2} \frac{\tilde\chi(\rho/(\langle n+2j+s\rangle M))}{\langle n+2j+s\rangle^{1-2v-|\beta^\iota|}}\, d\nu_\iota(s)\, d\mu\, d\rho,$$

by passing to polar coordinates and rescaling. The sum in (n,j) is dominated by $(\rho/M)^{2v+|\beta^\iota|+1}$, uniformly in $s\in J_\iota$, and moreover $\operatorname{supp} F \subseteq K$. Therefore the left-hand side of (4.9) is majorized by

$$C_{K,\chi,\alpha} M^{3-2|\alpha|} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \sup_{u\in[0,\max K]} \int_0^\infty |F^{(2k_\iota-|\beta^\iota|+v+r)}(\rho+u)|^2\, d\rho.$$

On the other hand, $b_\iota + |\beta^\iota| = l_\iota + 2k_\iota$, hence $2k_\iota - |\beta^\iota| + v + r \leq 2k_\iota - |\beta^\iota| + |\gamma^\iota| + l_\iota = b_\iota + |\gamma^\iota| \leq |\alpha|$ if r and v are in the range of summation, therefore the last integral is dominated by $\|F\|^2_{W_2^{|\alpha|}}$ uniformly in r, v, u, and (4.9) follows. ∎

Proposition 4.5. *Let $F:\mathbb{R}\to\mathbb{C}$ be smooth and such that* $\operatorname{supp} F \subseteq K$ *for some compact set $K \subseteq\,]0,\infty[$. Then for all $r \in [0, 3/2[$,*

$$\int_{N_{3,2}} |(1+|y|)^r \mathcal{K}_{F(L)}(x,y)|^2\, dx\, dy \leq C_{K,r} \|F\|^2_{W_2^r}.$$

Proof. Take $\chi \in C_c^\infty(]0,\infty[)$ such that $\operatorname{supp}\chi \subseteq [1/2,2]$ and $\sum_{k\in\mathbb{Z}} \chi(2^{-k}t) = 1$ for all $t \in\,]0,\infty[$. Note that, if (λ,η) belongs to the joint spectrum of $L, \mathbf{Y}$, then $|\eta| \leq \lambda$. Therefore, if $k_K \in \mathbb{Z}$ is sufficiently large so that

$2^{k_K-1} > \max K$, and if F_M is defined for all $M \in]0,\infty[$ as in Lemma 4.4, then

$$F(L) = \sum_{k\in\mathbb{Z},\, k\le k_K} F_{2^k}(L,\mathbf{Y})$$

(with convergence in the strong sense). Hence an estimate for $\mathcal{K}_{F(L)}$ can be obtained, via Minkowski's inequality, by summing the corresponding estimates for $\mathcal{K}_{F_{2^k}(L,\mathbf{Y})}$ given by Lemma 4.4. If $r < 3/2$, then the series $\sum_{k\le k_K}(2^k)^{3/2-r}$ converges, thus

$$\int_{N_{3,2}} \big||y|^r \mathcal{K}_{F(L)}(x,y)\big|^2\,dx\,dy \le C_{K,r}\|F\|_{W_2^r}^2.$$

The conclusion follows by combining the last inequality with the corresponding one for $r = 0$. ∎

Recall that $|\cdot|_\delta$ denotes a δ_t-homogeneous norm on $N_{3,2}$, thus $|(x,y)|_\delta \sim |x| + |y|^{1/2}$. Interpolation then allows us to improve the standard weighted estimate for a homogeneous sublaplacian on a stratified group.

PROPOSITION 4.6. *Let $F : \mathbb{R} \to \mathbb{C}$ be smooth and such that* $\operatorname{supp} F \subseteq K$ *for some compact set $K \subseteq]0,\infty[$. Then for all $r \in [0,3/2[$, $\alpha \ge 0$ and $\beta > \alpha + r$,*

$$\int_{N_{3,2}} |(1+|(x,y)|_\delta)^\alpha (1+|y|)^r \mathcal{K}_{F(L)}(x,y)|^2\,dx\,dy \le C_{K,\alpha,\beta,r}\|F\|_{W_2^\beta}^2. \tag{4.13}$$

Proof. Note that $1+|y| \le C(1+|(x,y)|_\delta)^2$. Hence, in the case $\alpha \ge 0$, $\beta > \alpha + 2r$, the inequality (4.13) follows by the standard estimate of [16, Lemma 1.2]. On the other hand, if $\alpha = 0$ and $\beta \ge r$, then (4.13) is given by Proposition 4.5. The full range of α and β is then obtained by interpolation (cf. the proof of [16, Lemma 1.2]). ∎

We can finally prove the fundamental L^1-estimate, and consequently Theorem 1.1.

Proof of Proposition 2.1. Take $r \in]9/2-s, 3/2[$. Then $s-r > 3/2+3-2r$, hence we can find $\alpha_1 > 3/2$ and $\alpha_2 > 3-2r$ such that $s-r > \alpha_1+\alpha_2$. Therefore, by Proposition 4.6 and Hölder's inequality,

$$\|\mathcal{K}_{F(L)}\|_1^2 \le C_{k,s}\|F\|_{W_2^s}^2 \int_{N_{3,2}} (1+|(x,y)|_\delta)^{-2\alpha_1-2\alpha_2}(1+|y|)^{-2r}\,dx\,dy.$$

The integral on the right-hand side is finite, because $2\alpha_1 > 3$, $\alpha_2 + 2r > 3$, and

$$(1+|(x,y)|_\delta)^{-2\alpha_1-2\alpha_2}(1+|y|)^{-2r} \le C_s(1+|x|)^{-2\alpha_1}(1+|y|)^{-\alpha_2-2r},$$

and we are done. ∎

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