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Examples of k-iterated spreading models

by

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Abstract. It is shown that for every $k \in \mathbb{N}$ and every spreading sequence $\{e_n\}_n$ that generates a uniformly convex Banach space E, there exists a uniformly convex Banach space X_{k+1} admitting $\{e_n\}_n$ as a k+1-iterated spreading model, but not as a k-iterated one.

Introduction. The aim of the present note is to continue some research initialized by B. Beauzamy and B. Maurey in [8]. Before we state our result, we need to recall the definition of k-iterated spreading models. As is well known, spreading models are a central concept in Banach space theory, invented by A. Brunel and L. Sucheston in [9]. For $k \geq 2$, the k-iterated spreading models of a Banach space X are inductively defined as the spreading models of the spaces generated by the $k-1$-iterated spreading models of X, where by 1-iterated models, we understand the usual spreading models. For detailed definitions see Section 1.

H. P. Rosenthal asked whether the k-iterated, $k \geq 2$, spreading models of any Banach spaces coincide with the 1-iterated ones. Beauzamy and Maurey answered that question by showing that the 2-iterated spreading models are, in general, different from the 1-iterated ones. More precisely they showed that there exists a Banach space X, generating a spreading model, isomorphically containing ℓ_1 and such that ℓ_1 is not a spreading model of X. A related question is whether every Banach space admits c_0 or some ℓ_p as a spreading model. This was answered in the negative by E. Odell and Th. Schlumprecht [18], who constructed a Banach space failing this property. A result in the same direction is given in [2], where it is shown that there exists a Banach space X such that every non-trivial spreading model of X isomorphically contains ℓ_1 and ℓ_1 is not a spreading model of X. A naturally arising problem, which appeared in [18], is whether there exists a Banach space that does not admit c_0 or ℓ_p as a k-iterated spreading model, for any $k \in \mathbb{N}$. A space with this property is exhibited in [4].

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In the present paper we separate the k-iterated and the $k+1$-iterated spreading models for every $k \in \mathbb{N}$. More precisely, the following is proved.

THEOREM 1. *Let $\{e_n\}_n$ be a spreading sequence* ([1]) *generating a uniformly convex Banach space E. Then there exists a sequence $\{X_k\}_k$ of uniformly convex Banach spaces, each one with a symmetric basis, such that for every $k \in \mathbb{N}$, the space X_k admits a k-iterated spreading model $\{\tilde{e}_n\}_n$ equivalent to $\{e_n\}_n$ and, for every $i < k$, E is not isomorphic to a subspace of the space generated by any i-iterated spreading model of X_k.*

Denoting by $\mathcal{SM}_k^{\mathrm{it}}(X)$ the class of all k-iterated spreading models of a Banach space X, it is an easy observation that these classes form an increasing family with respect to k. The above-mentioned family $\{X_k\}_k$ has the additional property that for every $k \in \mathbb{N}$ the family $\{\mathcal{SM}_i^{\mathrm{it}}(X_k)\}_{i=1}^k$ is strictly increasing.

It is worth pointing out that the k-iterated spreading models of a Banach space X, for $k \geq 2$, are not easily visualized from the structure of the space X, and this is an obstacle for studying the structure of the space generated by them. The key property of the aforementioned sequence $\{X_k\}_k$ is that the space generated by a spreading model of any X_k, $k \geq 2$, is isomorphic either to a subspace of X_k, or to a subspace of X_{k-1} (see Lemma 5.4).

The definition of the sequence $\{X_k\}_k$ relies on well known methods and results, which we combine in order to obtain the desired properties for these spaces. Some features of B. Beauzamy and B. Maurey's construction [8], and also the classical result that every space with an unconditional basis embeds into a space with a symmetric basis [10], [16], [22], are used. In particular, among those three papers, W. J. Davis' approach [10], based on the W. J. Davis, T. Figiel, W. B. Johnson, and A. Pełczyński interpolation method [11], is the one which is the most convenient for our needs. We also make heavy use of results of T. Figiel and W. B. Johnson from [14], in particular those concerning renormings of superreflexive spaces with an unconditional basis. Of independent interest is also Proposition 4.1, characterizing the structure of the spreading models of spaces with a 1-symmetric basis.

1. Preliminaries. Our notation concerning Banach space theory will follow the standard one from [17].

DEFINITION 1.1. Let $(X, \|\cdot\|)$ be a Banach space and $(E, \|\cdot\|_*)$ a seminormed space. Let $\{x_n\}_n$ be a bounded sequence in X and $\{e_n\}_n$ a sequence in E. We say that $\{x_n\}_n$ *generates* $\{e_n\}_n$ *as a spreading model* if there exists

([1]) A sequence $\{e_n\}_n$ in a seminormed space $(E, \|\cdot\|_*)$ is called *spreading* if for every $n \in \mathbb{N}$, $k_1 < \cdots < k_n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$, we have $\|\sum_{j=1}^n a_j e_j\|_* = \|\sum_{j=1}^n a_j e_{k_j}\|_*$.

a sequence $\{\delta_n\}_n$ of positive reals with $\delta_n \searrow 0$ such that for every $n \in \mathbb{N}$, $n \le k_1 < \cdots < k_n$ and every choice $\{a_i\}_{i=1}^n \subset [-1,1]$ the following holds:

$$\left| \left\| \sum_{i=1}^n a_i x_{k_i} \right\| - \left\| \sum_{i=1}^n a_i e_i \right\|_* \right| < \delta_n.$$

We also say that the Banach space X admits $\{e_n\}_n$ as a spreading model, or $\{e_n\}_n$ is a spreading model of X, if there exists a sequence in X which generates $\{e_n\}_n$ as a spreading model.

REMARK. In the literature, the notion of a spreading model is sometimes understood differently, i.e. if $\{x_n\}_n$ and $\{e_n\}_n$ are as in the definition above, the spreading model of $\{x_n\}_n$ is said to be the space $\bar{E}$, where $\bar{E}$ denotes the completion of the linear span of $\{e_n\}_n$ (see [7]).

It is more convenient, in general, to understand the sequence $\{e_n\}_n$ itself as the spreading model of $\{x_k\}_k$ and to refer to $\bar{E}$ as the space generated by the spreading model (see [3], [4] and [5]).

Brunel and Sucheston proved that every bounded sequence in a Banach space has a subsequence which generates a spreading model. The main property of spreading models is that they are spreading sequences, i.e. for every $n \in \mathbb{N}$, $k_1 < \cdots < k_n$ and every choice $\{a_i\}_{i=1}^n \subset \mathbb{R}$ we have $\|\sum_{i=1}^n a_i e_i\|_* = \|\sum_{i=1}^n a_i e_{k_i}\|_*$.

Spreading sequences are classified into four categories with respect to their norm properties. These are the trivial, the unconditional, the singular and the non-unconditional Schauder basic spreading sequences (see [4]).

A spreading sequence $\{e_n\}_n$ is called *trivial* if the seminorm on the space generated by the sequence is not actually a norm. In this case, Proposition 13 from [4] yields the following. If E is the vector space generated by $\{e_n\}_n$ and $\mathcal{N} = \{x \in E : \|x\|_* = 0\}$, then $E/\mathcal{N}$ has dimension at most 1. It is also worth mentioning that a sequence in a Banach space X generates a trivial spreading model if and only if it has a norm convergent subsequence. For more details see [4], [7]. From now on, we will only refer to non-trivial spreading models.

A spreading sequence is called *singular* if it is not trivial and not Schauder basic. A simple example of a singular spreading sequence is the following. Let X be c_0 or ℓ_p, $1 < p < \infty$ and let $\{e_i\}_i$ denote the unit vector basis of X. Then the sequence $\{x_i\}_i$ with $x_i = e_{i+1} - e_1$ is spreading and not Schauder basic.

The definition of the other two cases is the obvious one.

The following notation is from [18].

NOTATION. (1) Let E_0, E be Banach spaces. We write $E_0 \to E$, if E is generated by a spreading sequence, which is a spreading model of some seminormalized sequence in E_0. Also, for $k \in \mathbb{N}$, the notation $E_0 \xrightarrow{k} E$

means that $E_0 \to E_1 \to \cdots \to E_{k-1} \to E$ for some sequence of Banach spaces $E_1, \dots, E_{k-1}$.

(2) Let E_0, E be a Banach spaces such that E_0 has a Schauder basis. We write $E_0 \underset{\mathrm{bl}}{\to} E$ if E is generated by a spreading sequence which is a spreading model of some seminormalized block sequence of the basis of E_0. Also, for $k \in \mathbb{N}$, the notation $E_0 \overset{k}{\underset{\mathrm{bl}}{\to}} E$ means that $E \underset{\mathrm{bl}}{\to} E_1 \underset{\mathrm{bl}}{\to} \dots \underset{\mathrm{bl}}{\to} E_{k-1} \underset{\mathrm{bl}}{\to} E$ for some sequence of Banach spaces $E_1, \dots, E_{k-1}$ with Schauder bases.

Definition 1.2. (1) Let E_0 be a Banach space, $\{e_n\}_n$ be a spreading sequence in a seminormed space, $k \in \mathbb{N}$. Then $\{e_n\}_n$ is said to be a *k-iterated spreading model of* E_0 if there exists a Banach space E such that $E_0 \xrightarrow{k-1} E$ and $\{e_n\}_n$ is the spreading model of some seminormalized sequence in E.

(2) Let E_0 be a Banach space with a Schauder basis, $\{e_n\}_n$ be a spreading sequence in a seminormed space, and $k \in \mathbb{N}$. Then $\{e_n\}_n$ is said to be a *block k-iterated spreading model of E* if there exists a Banach space E with a Schauder basis such that $E_0 \overset{k-1}{\underset{\mathrm{bl}}{\longrightarrow}} E$ and $\{e_n\}_n$ is the spreading model of some seminormalized block sequence of the basis of E.

Remark. If $\{e_n\}_n, \{\tilde{e}_n\}_n$ are non-trivial spreading sequences which generate the Banach spaces E and $\tilde{E}$ respectively, we shall say that $\{e_n\}_n$ and $\{\tilde{e}_n\}_n$ are *equivalent* if the linear map $e_n \to \tilde{e}_n$ extends to an isomorphism between E and $\tilde{E}$.

Clearly, if X and Y are isomorphic Banach spaces, then any non-trivial spreading model admitted by X is equivalent to one admitted by Y and vice versa.

In accordance with the above, we shall say that a sequence $\{x_n\}_n$ *isomorphically generates* $\{e_n\}_n$ *as a spreading model* if $\{x_n\}_n$ generates $\{\tilde{e}_n\}_n$ as a spreading model and $\{\tilde{e}_n\}_n$ is equivalent to $\{e_n\}_n$.

2. Interpolating spaces with a symmetric basis. We begin by presenting some estimations concerning sequences of $\|\cdot\|_{q,p}^{m_k}$ norms, next defined.

Definition 2.1. Let $1 \le q < p$. For a real number $m \ge 1$, define $\|\cdot\|_{q,p}^m$ on ℓ_p as follows:

$$\|x\|_{q,p}^m = \inf\Big\{\lambda > 0 : \frac{x}{\lambda} \in mB_{\ell_q} + \frac{1}{m}B_{\ell_p}\Big\}.$$

Remarks. The following statements are true for all real numbers $m \ge 1$:

(i) $\frac{1}{m+1/m}\|\cdot\|_p \le \|\cdot\|_{q,p}^m \le m\|\cdot\|_p$, thus $\|\cdot\|_{q,p}^m \sim \|\cdot\|_p$.

(ii) If $x \in \ell_q$, then $\|x\|_{q,p}^m \le \frac{\|x\|_q}{m+1/m}$.

(iii) $\|\cdot\|_{q,p}^m$ is a *symmetric* norm, i.e. if $\{a_i\}_i \in \ell_p$, then

$$\Big\|\sum_{i=1}^{\infty} a_i e_i\Big\|_{q,p}^m = \Big\|\sum_{i=1}^{\infty} \varepsilon_i a_i e_{\pi(i)}\Big\|_{q,p}^m$$

for any choice $\{\varepsilon_i\}_i$ of signs and any permutation π of the naturals.

LEMMA 2.2. *Let $1 \le q < p$, $\{x_n\}_n$ be a sequence in ℓ_p, $\varepsilon > 0$, and $\{m_k\}_k$ be an unbounded sequence of real numbers, greater than or equal to one, such that $\|x_n\|_p > \varepsilon$ for all $n \in \mathbb{N}$ and $\lim_n \|x_n\|_\infty = 0$. Then $\sup\{\|x_n\|_{q,p}^{m_k} : n, k \in \mathbb{N}\} = \infty$.*

Proof. Towards a contradiction, suppose that $\sup\{\|x_n\|_{q,p}^{m_k} : n, k \in \mathbb{N}\} < C$. Then for all $n, k \in \mathbb{N}$ there exist $0 < \lambda_n^k < C$, $y_n^k \in B_{\ell_q}$, $z_n^k \in B_{\ell_p}$ such that

$$x_n = \lambda_n^k\Big(m_k y_n^k + \frac{1}{m_k} z_n^k\Big).$$

Choose $k_0 \in \mathbb{N}$ such that $(\lambda_n^{n_0}/m_{k_0})\|z_n^{k_0}\|_p < \varepsilon/2$ for all $n \in \mathbb{N}$. Then

$$\lambda_n^{k_0} m_{k_0} \|y_n^{k_0}\|_p > \varepsilon/2 \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Since the norms are symmetric, we may assume that if $x_n = \sum_{i=1}^{\infty} a_i e_i$, then $a_i \ge 0$ for all $i \in \mathbb{N}$. Moreover, if $y_n^k = \sum_{i=1}^{\infty} b_i e_i$, $z_n^k = \sum_{i=1}^{\infty} c_i e_i$, we may assume that $0 \le \lambda_n^k m_k b_i$, $(\lambda_n^k m_k) c_i \le a_i$ for all $i \in \mathbb{N}$.

Otherwise, with simple calculations one may find $y_n^{k\prime} = \sum_{i=1}^{\infty} b_i' e_i$, $z_n^{k\prime} = \sum_{i=1}^{\infty} c_i' e_i$, satisfying this condition, such that $x_n = \lambda_n^k(m_k y_n^{k\prime} + (1/m_k) z_n^{m_k\prime})$ and $y_n^{k\prime} \in B_{\ell_q}, z_n^{k\prime} \in B_{\ell_p}$. This means that $\lambda_n^{k_0} m_{k_0} \|y_n^{k_0}\|_\infty \le \|x_n\|_\infty \to 0$ as $n \to \infty$. Since $\lambda_n^{k_0} m_{k_0} \|y_n^{m_{k_0}}\|_q < C m_{k_0}$ for all $n \in \mathbb{N}$, by using the Hölder inequality, it is easy to see that $\lambda_n^{k_0} m_{k_0} \|y_n^{m_{k_0}}\|_p \to 0$ as $n \to \infty$. This contradicts (1), which completes the proof. ∎

LEMMA 2.3. *Let $1 \le q < p$, $\{x_n\}_n$ be a sequence in ℓ_p, and $\{m_k\}_k$ be an unbounded sequence of real numbers, greater than or equal to one, such that $\lim_n \|x_n\|_\infty = 0$ and $\sup\{\|x_n\|_{q,p}^{m_k} : n, k \in \mathbb{N}\} < \infty$. Then for every $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $\max\{\|x_n\|_{q,p}^{m_k} : k \in \mathbb{N}, k \le k_0\} < \varepsilon$.*

Proof. Towards a contradiction, suppose that there exist $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there exists $j_n \ge n$ with $\max\{\|x_{j_n}\|_{q,p}^{m_k} : k \in \mathbb{N}, k \le k_0\} \ge \varepsilon$. By passing to a subsequence of $\{x_n\}_n$, we can find $k \le k_0$ such that $\|x_n\|_{q,p}^{m_k} \ge \varepsilon$ for all $n \in \mathbb{N}$. But $\|\cdot\|_{q,p}^{m_k} \sim \|\cdot\|_p$, hence $\|x_n\|_p \ge \varepsilon'$ for all $n \in \mathbb{N}$. By Lemma 2.2, this means that $\sup\{\|x_n\|_{q,p}^{m_k} : n, k \in \mathbb{N}\} = \infty$. Since this cannot be the case, the proof is complete. ∎

The following theorem is due to W. J. Davis [10]. See also [17, Theorem 3.b.2, p. 124].

THEOREM 2.4. *Let X be a* (*reflexive, uniformly convex*) *Banach space with a* 1*-unconditional basis. Then there exists a* (*reflexive, uniformly convex*) *Banach space D with a* 1*-symmetric basis such that X is isomorphic to a complemented subspace of D.*

Moreover D is saturated with subspaces of X, i.e. if Z is a subspace of D, then there exists a further subspace of Z which is isomorphic to a subspace of X.

Given a Banach space X with a 1-unconditional basis, D is defined to be the diagonal subspace of $\mathfrak{X} = \big(\sum_{k=1}^{\infty} \bigoplus(\ell_p, \|\!|\cdot\|\!|_{q,p}^{m_k})\big)_X$, where the norms $\|\!|\cdot\|\!|_{q,p}^{m}$ are defined on ℓ_p and are of the form

$$\|\!|x\|\!|_{q,p}^{m} = \inf\bigg\{(\|y\|_{\ell_q}^2 + \|z\|_{\ell_p}^2)^{1/2} : x = my + \frac{1}{m}z,\ y\in \ell_q,\ z\in\ell_p\bigg\}$$

for $1 < q < p$, and the sequence $\{m_k\}_k$ is chosen to satisfy a condition found in [17, p. 126], namely the following. Choose a sequence $\{n_k\}_k$ of natural numbers such that if $m_k = n_k^{(p-q)/(2pq)}$, the inequality

$$\frac{1}{m_k}\sum_{i=1}^{k-1} m_i + m_k \sum_{i=k+1}^{\infty}\frac{1}{m_i} < \frac{1}{2^{k+1}} \quad \text{for all } k\in\mathbb{N} \tag{2}$$

is satisfied. Then $\{m_k\}_k$ is the desired sequence.

If we denote $\tilde{e}_n = \{e_n, e_n, \ldots\}$, where $\{e_n\}_n$ is the natural basis of ℓ_p, then $\tilde{e}_n \in \mathfrak{X}$ and $\{\tilde{e}_n\}_n$ is the 1-symmetric basis of D. Observe that for every real number $m \ge 1$, we have $\|\cdot\|_{q,p}^m \le \|\!|\cdot\|\!|_{q,p}^m \le \sqrt{2}\,\|\cdot\|_{q,p}^m$. It easily follows that the spaces $\mathfrak{X}$ and $\mathfrak{X}'$ are isomorphic, where $\mathfrak{X}' = \big(\sum_{k=1}^{\infty} \bigoplus(\ell_p, \|\cdot\|_{q,p}^{m_k})\big)_X$.

As shown in [17, Proposition 3.b.4], X embeds into D as a complemented subspace and every subspace of D contains a further subspace isomorphic to a subspace of X. The latter is shown in [14, Lemma 2.2], but a proof also follows from the above and the following.

LEMMA 2.5. *Let Y be a block subspace of D. Then there exists a further block subspace Z of Y such that $\lim_n \|z_n\|_\infty = 0$, where $\{z_n\}_n$ denotes the normalized block basis of Z.*

Proof. Let $\{y_n\}_n$ be a normalized block basis of Y. If, after passing to a subsequence, $\|y_n\|_\infty \to 0$, then there is nothing more to prove. Otherwise, again after passing to a subsequence, there is $\varepsilon > 0$ such that $\|y_n\|_\infty > \varepsilon$ for all $n\in\mathbb{N}$.

Denote by j the map $j : D \to \ell_p$ with $j\big(\sum_{i=1}^{\infty} a_i\tilde{e}_i\big) = \sum_{i=1}^{\infty} a_i e_i$. Notice that the natural projection $P_1 : \mathfrak{X}' \to (\ell_p, \|\cdot\|_{q,p}^{m_1})$ is of course bounded. Then j is the restriction of P_1 to D and hence it is also bounded.

Choose finite subsets $I_1 < \cdots < I_k < \cdots$ of the natural numbers with $|I_k| \geq (\|j\| k/\varepsilon)^p$ for all $k \in \mathbb{N}$. Then

$$\|j\| \Big\| \frac{1}{k} \sum_{i \in I_k} y_i \Big\|_D \geq \Big\| \frac{1}{k} \sum_{i \in I_k} j(y_i) \Big\|_p = \frac{1}{k} \Big(\sum_{i \in I_k} \|j(y_i)\|_p^p \Big)^{1/p}$$

$$\geq \frac{1}{k} \Big(\sum_{i \in I_k} \|y_i\|_\infty^p \Big)^{1/p} > \frac{\varepsilon}{k} |I_k|^{1/p}.$$

Thus $\|(1/k)\sum_{i\in I_k} y_i\|_D \geq 1$ and of course $\|(1/k)\sum_{i\in I} y_i\|_\infty \leq 1/k$.

Set $z_k = \|(1/k)\sum_{i\in I_k} y_i\|_D^{-1}((1/k)\sum_{i\in I_k} y_i)$; then it easily follows that $\{z_k\}_k$ is normalized and $\|z_k\|_\infty \to 0$. ∎

PROPOSITION 2.6. *Let $\{y_n\}_n$ be a normalized bounded sequence in D such that* $\lim_n \|y_n\|_\infty = 0$. *Then $\{y_n\}_n$ has a subsequence which is equivalent to a block sequence in X.*

Proof. Use Lemma 2.3 and a sliding hump argument with respect to the decomposition $\{(\ell_p, \|\cdot\|_{q,p}^{m_k})\}_k$ of $\mathfrak{X}'$. ∎

Since every subspace of D contains a further subspace which is isomorphic to a block subspace of D, it follows that D is saturated with subspaces of X.

3. Uniformly convex Schreier–Baernstein spaces. We begin by presenting some key definitions and results from [14].

DEFINITION 3.1. Let X be a Banach space with a 1-unconditional basis $\{e_n\}_n$. The norm on X is said to be *p-convex* if for every $n \in \mathbb{N}$ and real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$,

$$\Big\| \sum_{i=1}^n (|a_i|^p + |b_i|^p)^{1/p} e_i \Big\| \leq \Big(\Big\| \sum_{i=1}^n a_i e_i \Big\|^p + \Big\| \sum_{i=1}^n b_i e_i \Big\|^p \Big)^{1/p}.$$

Analogously, it is called *q-concave* if for every $n \in \mathbb{N}$ and real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$,

$$\Big\| \sum_{i=1}^n (|a_i|^q + |b_i|^q)^{1/q} e_i \Big\| \geq \Big(\Big\| \sum_{i=1}^n a_i e_i \Big\|^q + \Big\| \sum_{i=1}^n b_i e_i \Big\|^q \Big)^{1/q}.$$

The norm on X is said to satisfy an *upper ℓ_p estimate* if

$$\|x + y\| \leq (\|x\|^p + \|y\|^p)^{1/p}$$

whenever x and y are disjointly supported with respect to the basis $\{e_n\}_n$. Analogously, it is said to satisfy a *lower ℓ_q estimate* if

$$\|x + y\| \geq (\|x\|^q + \|y\|^q)^{1/q}$$

whenever x and y are disjointly supported with respect to the basis $\{e_n\}_n$.

It is immediate that if the norm on X is p-convex (resp. q-concave), then it satisfies an upper ℓ_p estimate (resp. a lower ℓ_q estimate).

The following two results are restatements of Remark 3.2 and Theorem 3.1, respectively, of [14].

THEOREM 3.2. *Let X be a uniformly convex Banach space with an unconditional basis. Then there exists an equivalent 1-unconditional norm on X which is p-convex and q-concave for some $1 < p \le q < \infty$. Moreover, if the initial norm on X is spreading or 1-symmetric, the same is true for the equivalent norm.*

LEMMA 3.3. *Let X be a Banach space with a 1-unconditional basis. If for some $1 < p \le q < \infty$ the norm on X is p-convex and satisfies a lower ℓ_q estimate, then X is uniformly convex.*

Let X be a Banach space with a 1-unconditional basis $\{e_n\}_n$. We denote by $\mathbb{S}$ the Schreier family $\mathbb{S} = \{F \subset \mathbb{N} : \min F \ge |F|\}$. Let $1 \le r < \infty$. Define the following norm on $c_{00}(\mathbb{N})$:

$$\|x\|_{X,r} = \sup\Big\{ \Big(\sum_{j=1}^{d} \|F_j x\|_X^r \Big)^{1/r} \Big\}$$

where the supremum is taken over all finite sequences $\{F_j\}_{j=1}^d \subset \mathbb{S}$ which are pairwise disjoint. Define the *Schreier–Baernstein space* $\mathrm{SB}_{X,r}$ to be the completion of $c_{00}(\mathbb{N})$ with the aforementioned norm.

It can be easily seen that the usual basis of $c_{00}(\mathbb{N})$ forms a 1-unconditional basis of $\mathrm{SB}_{X,r}$.

PROPOSITION 3.4. *Let X be a Banach space with a 1-unconditional basis $\{e_n\}_n$ and with a p-convex norm $\|\cdot\|_X$, for some $1 < p \le \infty$. Let $r \ge p$. Then the space $\mathrm{SB}_{X,r}$ is uniformly convex.*

Proof. We will show that the demands of Lemma 3.3 are satisfied. First we show that $\|\cdot\|_{X,r}$ is p-convex. Let $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{R}$. Then for some $\{F_j\}_{j=1}^d \subset \mathbb{S}$ and by the p-convexity of the norm on X,

$$\begin{aligned}
\Big\|\sum_{i=1}^n (|a_i|^p + |b_i|^p)^{1/p} e_i\Big\|_{X,r} &= \Big(\sum_{j=1}^d \Big\|\sum_{i\in F_j} (|a_i|^p + |b_i|^p)^{1/p} e_i\Big\|_X^r\Big)^{1/r} \\
&\le \Big(\sum_{j=1}^d \Big(\Big\|\sum_{i\in F_j} a_i e_i\Big\|_X^p + \Big\|\sum_{i\in F_j} b_i e_i\Big\|_X^p\Big)^{r/p}\Big)^{(p/r)(1/p)} \\
&\le \Big(\Big(\sum_{j=1}^d\Big\|\sum_{i\in F_j} a_i e_i\Big\|_X^r\Big)^{p/r} + \Big(\sum_{j=1}^d\Big\|\sum_{i\in F_j} b_i e_i\Big\|_X^r\Big)^{p/r}\Big)^{1/p} \\
&\le \Big(\Big\|\sum_{i=1}^n a_i e_i\Big\|_{X,r}^p + \Big\|\sum_{i=1}^n b_i e_i\Big\|_{X,r}^p\Big)^{1/p}.
\end{aligned}$$

Thus $\|\cdot\|_{X,r}$ is p-convex. Moreover, if x,y are finitely disjointly supported, then there exist finite sets $\{F_j\}_{j=1}^{d_1}, \{E_j\}_{j=1}^{d_2}$ of natural numbers such that

$$\|x\|_{X,r} = \Big(\sum_{j=1}^{d_1} \|F_j x\|_X^r\Big)^{1/r} \quad \text{and} \quad \|y\|_{X,r} = \Big(\sum_{j=1}^{d_2} \|E_j y\|_X^r\Big)^{1/r}.$$

We may clearly assume that $F_i \cap E_j = \emptyset$ for all i,j. Then

$$\|x+y\|_{X,r}^r \geq \sum_{j=1}^{d_1} \|F_j x\|_X^r + \sum_{j=1}^{d_2} \|E_j y\|_X^r = \|x\|_{X,r}^r + \|y\|_{X,r}^r.$$

Thus $\|\cdot\|_{X,r}$ satisfies a lower ℓ_r estimate and the space $\mathrm{SB}_{X,r}$ is uniformly convex. ∎

PROPOSITION 3.5. *Let X be a Banach space with a 1-unconditional basis $\{e_n\}_n$ and with a norm $\|\cdot\|_X$ which satisfies a lower ℓ_q estimate for some $1 \leq q < \infty$. Let $r \geq q$. Then for every $E \in \mathbb{S}$ and real numbers $\{a_i\}_{i\in E}$,*

$$\Big\|\sum_{i\in E} a_i e_i\Big\|_{X,r} = \Big\|\sum_{i\in E} a_i e_i\Big\|_X.$$

Proof. By the definition of the norm on the space $\mathrm{SB}_{X,r}$ it clearly follows that $\|\sum_{i\in E} a_i e_i\|_{X,r} \geq \|\sum_{i\in E} a_i e_i\|_X$. Therefore it is sufficient to show the inverse inequality. For some $\{F_j\}_{j=1}^d \subset \mathbb{S}$ and by the lower ℓ_q estimate of the norm on X,

$$\begin{aligned}\Big\|\sum_{i\in E} a_i e_i\Big\|_{X,r} &= \Big(\sum_{j=1}^{d} \Big\|\sum_{i\in F_j} a_i e_i\Big\|_X^r\Big)^{1/r} \\ &\leq \Big(\sum_{j=1}^{d} \Big\|\sum_{i\in F_j} a_i e_i\Big\|_X^q\Big)^{1/q} \leq \Big\|\sum_{i\in E} a_i e_i\Big\|_X. \ \blacksquare\end{aligned}$$

COROLLARY 3.6. *Let X be a Banach space with a 1-unconditional and spreading basis $\{e_n\}_n$ and with a norm $\|\cdot\|_X$ which satisfies a lower ℓ_q estimate for some $1 \leq q < \infty$. Let $r \geq q$. Then the basis of $\mathrm{SB}_{X,r}$ generates the basis of X as a spreading model.*

This is an immediate consequence of Proposition 3.5 and the spreading property of the basis of X.

PROPOSITION 3.7. *Let X be a Banach space with a 1-unconditional basis $\{e_n\}_n$. Let $1 \leq r < \infty$. Let $\{x_n\}_n$ be a normalized block sequence in $\mathrm{SB}_{X,r}$ such that $\lim_n \|x_n\|_\infty = 0$. Then $\{x_n\}_n$ has a subsequence equivalent to the usual basis of ℓ_r.*

The proof is the same as for the classical Schreier–Baernstein space $\mathrm{SB}_{\ell_1,2}$, where such a sequence has a further subsequence which is equivalent to the basis of ℓ_2. It also follows that the space $\mathrm{SB}_{X,r}$ is ℓ_r-saturated. If the norm on X satisfies a lower ℓ_q estimate and $r > q$, then the space X cannot contain ℓ_r. Thus in this case the spaces X and $\mathrm{SB}_{X,r}$ are totally incomparable.

4. Spreading models of Banach spaces with a symmetric basis. In this section we study the structure of spreading models in Banach spaces with a 1-symmetric basis. We start with the following, which is critical for our proofs. This result and the next proposition can be traced back to [8] and are closely related to [7, Lemma IV.2.A.3].

PROPOSITION 4.1. *Let X be a Banach space with a 1-symmetric and boundedly complete basis. Let $\{x_n\}_n$ be a normalized block sequence in X and assume that there is some $\varepsilon > 0$ such that $\|x_n\|_\infty > \varepsilon$ for all $n \in \mathbb{N}$. Then, after passing to an appropriate subsequence, there exist block sequences $\{y_n\}_n$ and $\{z_n\}_n$ in X and a disjointly supported 1-symmetric sequence $\{u_n\}_n$ in X with the following properties:*

(i) $x_n = y_n + z_n$ *for all* $n \in \mathbb{N}$ *and* $\operatorname{supp} y_n \cap \operatorname{supp} z_m = \emptyset$ *for all* $n, m \in \mathbb{N}$.
(ii) $\lim_n \|z_n\|_\infty = 0$.
(iii) $\{y_n\}_n$ *isometrically generates* $\{u_n\}_n$ *as a spreading model.*
(iv) $\|u_n\|_\infty = \|u_1\|_\infty > 0$ *for all* $n \in \mathbb{N}$.

Proof. Since the basis of X is boundedly complete and symmetric, for every $\delta > 0$, there exists $m(\delta) \in \mathbb{N}$ such that, for every $x \in X$ with $\|x\| = 1$, $\#\{i : |x(i)| \geq \delta\} \leq m(\delta)$. Otherwise the basis of X would be equivalent to the basis of c_0.

Since X has a 1-symmetric basis, we may assume that $x_n(i) \geq 0$ for all $n, i \in \mathbb{N}$ and the non-zero entries of each x_n are in decreasing order. For each x_n, we set $G_n = \operatorname{supp} x_n$. Let $G_n = \{i_1^n, \ldots, i_{d_n}^n\}$; then we put

$$\tilde{x}_n(\ell) = \begin{cases} x_n(i_\ell^n) & \text{if } \ell \leq d_n, \\ 0 & \text{otherwise.} \end{cases}$$

We notice that $\tilde{x}_n$ is the backward shift of all non-zero entries of x_n to an initial interval of the natural numbers.

By passing to a subsequence if necessary, we may assume that for every $i \in \mathbb{N}$ the sequence $\{\tilde{x}_n(i)\}_n$ is convergent to some real number λ_i. Then $\|\sum_{i=1}^m \lambda_i e_i\| = \|\lim_n \sum_{i=1}^m \tilde{x}_n(i) e_i\| = \|\lim_n P_{[1,m]} \tilde{x}_n\| \leq \|\tilde{x}_n\| = \|x_n\| = 1$ for $m \in \mathbb{N}$.

Since the basis of X is assumed to be boundedly complete, we conclude that the series $\sum \lambda_i e_i$ converges in norm to some $x \in X$. Observe that for every $m \in \mathbb{N}$,

$$\lim_n \|P_{[1,m]}(\tilde{x}_n - x)\| = 0. \tag{3}$$

Choose sequences $\{\delta_k\}_k$, $\{\varepsilon_k\}_k$ of positive reals, both strictly decreasing to zero.

Inductively choose a decreasing sequence $\{L_k\}_k$ of infinite subsets of the natural numbers and a (not necessarily strictly) increasing sequence $\{m_k\}_k$ of natural numbers such that, for every $k_0 \in \mathbb{N}$ and $k \in L_{k_0}$, we have:

(α) $\#\{i : \tilde{x}_k(i) \geq \delta_{k_0}\} = m_{k_0}$.
(β) $\|P_{[1,m_{k_0}]}(\tilde{x}_k - x)\| < \varepsilon_{k_0}$.

We only present here the first step of the induction, as the general step is identical to the first one.

For every $k \in \mathbb{N}$, we have $\#\{i : \tilde{x}_{n_k}(i) \geq \delta_1\} \leq m(\delta_1)$. Using the pigeonhole principle, there exists an infinite set M_1 of natural numbers and $m_1 \in \mathbb{N}$ with $\#\{i : \tilde{x}_{n_k}(i) \geq \delta_1\} = m_1$ for all $k \in M_1$. Using (3), we may choose an infinite subset L_1 of M_1 such that (β) is also satisfied.

Choosing $n_1 < n_2 < \cdots$ with $n_k \in L_k$ for all $k \in \mathbb{N}$ and relabeling, we see that for every $k_0 \in \mathbb{N}$ and $k \geq k_0$:

(a) $\#\{i : \tilde{x}_k(i) \geq \delta_{k_0}\} = m_{k_0}$.
(b) $\|P_{[1,m_{k_0}]}(\tilde{x}_k - x)\| < \varepsilon_k$.

Define $\{y_k\}_k$ as follows:

$$y_k(i) = \begin{cases} x_k(i) & \text{if } x_k(i) \geq \delta_k, \\ 0 & \text{otherwise,} \end{cases}$$

and set $z_k = x_k - y_k$. Conditions (i) and (ii) are obviously satisfied.

Also observe that if $k_0 \in \mathbb{N}$, then for every $k \geq k_0$, $P_{[1,m_{k_0}]}\tilde{x}_k = P_{[1,m_{k_0}]}\tilde{y}_k$ and $P_{[1,m_{k_0}]}\tilde{x}_{k_0} = \tilde{y}_{k_0}$, where the $\tilde{y}_k$ are defined in the same way as the $\tilde{x}_k$. The above is due to the fact that the non-zero entries of each x_k are assumed to be in decreasing order.

For $\delta > 0$, $y \in X$ define $R^\delta y \in X$ by

$$R^\delta y(i) = \begin{cases} y(i) & \text{if } |y(i)| < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Claim.

$$\lim_{\delta \to 0} \sup\{\|R^\delta y_k\| : k \in \mathbb{N}\} = 0.$$

Let $\varepsilon > 0$. We will first show that we can choose $k_0 \in \mathbb{N}$ such that $\|\sum_{i=m_{k_0}+1}^{\infty} \lambda_i e_i\| < \varepsilon/2$. If the sequence $\{m_k\}_k$ is unbounded, then such a k_0 clearly exists. Otherwise, choose $k_0 \in \mathbb{N}$ with $m_{k_0} = \max\{m_k : k \in \mathbb{N}\}$. Then $\tilde{x}_k(i) = 0$ for all $k \in \mathbb{N}$ and $i > m_{k_0}$. We conclude that $\lambda_i = 0$ for all $i > m_{k_0}$ and hence k_0 has the desired property.

By taking a larger k_0, we may also assume that $\varepsilon_k < \varepsilon/2$ for all $k \geq k_0$.

Let $k \in \mathbb{N}$. We will estimate the norm of $R^{\delta_{k_0}}\tilde{y}_k$. If $k<k_0$, then $R^{\delta_{k_0}}\tilde{y}_k=0$. Otherwise, the fact that $P_{[1,m_{k_0}]}\tilde{x}_k = \tilde{y}_k$ for $k \geq k_0$ yields

$$\begin{aligned} R^{\delta_{k_0}}\tilde{y}_k = \tilde{y}_k - P_{[1,m_{k_0}]}\tilde{y}_k &= P_{[1,m_k]}\tilde{x}_k - P_{[1,m_{k_0}]}\tilde{x}_k \\ &= P_{(m_{k_0},m_k]}\tilde{x}_k = P_{(m_{k_0},m_k]}(\tilde{x}_k - x) + \sum_{i=m_{k_0}+1}^{m_k} \lambda_i e_i. \end{aligned}$$

Hence, for $\delta \leq \delta_{k_0}$ and $k \geq k_0$, using property (b) we have

$$\begin{aligned} \|R^\delta y_k\| = \|R^\delta \tilde{y}_k\| \leq \|R^{\delta_{k_0}}\tilde{y}_k\| &\leq \|P_{(m_{k_0},m_k]}(\tilde{x}_k - x)\| + \Big\|\sum_{i=m_{k_0}+1}^{m_k} \lambda_i e_i\Big\| \\ &\leq \|P_{[1,m_k]}(\tilde{x}_k - x)\| + \Big\|\sum_{i=m_{k_0}+1}^{\infty} \lambda_i e_i\Big\| < \varepsilon_k + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Thus, we have proved the Claim.

Choose a partition $\{N_k\}_k \subset [\mathbb{N}]^\infty$ of the naturals into infinite sets and set $u_k = \sum_{i=1}^\infty \lambda_i e_{N_k(i)}$. It is immediate that $\{u_k\}_k$ is 1-symmetric and $\|u_k\|_\infty = \|u_1\|_\infty > 0$ for all $k \in \mathbb{N}$. We will prove that any spreading model generated by $\{y_k\}_k$ is isometric to $\{u_k\}_k$.

Let $\ell \in \mathbb{N}$, $\{a_i\}_{i=1}^\ell \subset [-1,1]$ and $\varepsilon > 0$. We will find $j_0 \in \mathbb{N}$ such that, for every $j_0 \leq j_1 < \cdots < j_\ell$,

$$\Big|\Big\|\sum_{i=1}^\ell a_i y_{j_i}\Big\| - \Big\|\sum_{i=1}^\ell a_i u_i\Big\|\Big| < \varepsilon.$$

The above is evidently sufficient to complete the proof.

For $k, i \in \mathbb{N}$ we set $P_{N_i(m_k)} u_i = \sum_{i=1}^{m_k} \lambda_i e_{N_k(i)}$, and choose $k_0 \in \mathbb{N}$ such that

$$\Big|\Big\|\sum_{i=1}^\ell a_i P_{N_i(m_k)} u_i\Big\| - \Big\|\sum_{i=1}^\ell a_i u_i\Big\|\Big| < \frac{\varepsilon}{3} \quad \text{for all } k \geq k_0. \tag{4}$$

Observe that if $\{m_k\}_k$ is bounded, then $m_{k_0} \leq \max\{m_k : k \in \mathbb{N}\}$; in this case we may therefore assume that $m_{k_0} = \max\{m_k : k \in \mathbb{N}\}$ and $R^{\delta_{k_0}} y_j = 0$ for all $j \in \mathbb{N}$.

In any case, using the claim and choosing, if necessary, an even larger k_0, we achieve that, for any natural numbers $j_1 < \cdots < j_\ell$,

$$\Big\|\sum_{i=1}^\ell a_i R^{\delta_{k_0}} y_{j_i}\Big\| < \frac{\varepsilon}{3}. \tag{5}$$

Choose $j_0 \geq k_0$ such that $\varepsilon_j < \varepsilon/(3\ell)$ for all $j \geq j_0$. Let now $j_0 \leq j_1 < \cdots < j_\ell$ be natural numbers.

For $i = 1, \ldots, \ell$ we set $y'_i = y_{j_i} - R^{\delta_{k_0}} y_{j_i}$. Then y'_i is a spreading of $P_{[1,m_{k_0}]}\tilde{y}_{j_i} = P_{[1,m_{k_0}]}\tilde{x}_{j_i}$.

For $i = 1, \ldots, \ell$ we also set $u_i' = P_{N_i(m_{k_0})} u_i$. Then u_i' is a spreading of $P_{[1,m_{k_0}]} x$. As the basis of X is symmetric, we may assume for a moment that $\operatorname{supp} y_i' = \operatorname{supp} u_i'$ for $i = 1, \ldots, \ell$. Then by property (b) we have

$$\Big\|\sum_{i=1}^{\ell} a_i (y_i' - u_i')\Big\| < \sum_{i=1}^{\ell} \varepsilon_{j_i} < \frac{\varepsilon}{3}.$$

We conclude that

$$\Big|\Big\|\sum_{i=1}^{\ell} a_i (y_{j_i} - R^{\delta_{k_0}} y_{j_i})\Big\| - \Big\|\sum_{i=1}^{\ell} a_i P_{N_i(M_{k_0})} u_i\Big\|\Big| < \frac{\varepsilon}{3}. \tag{6}$$

By combining (4)–(6), it follows that $\big|\|\sum_{i=1}^{\ell} a_i y_{j_i}\| - \|\sum_{i=1}^{\ell} a_i u_i\|\big| < \varepsilon$, proving the proposition. ∎

The following result corresponds to Proposition 4.1 in the setting of Schreier–Baernstein spaces.

PROPOSITION 4.2. *Let X be a Banach space with a 1-symmetric basis $\{e_n\}_n$ with a norm which satisfies a lower ℓ_q estimate for some $1 \le q < \infty$. Let $r \ge q$. Let $\{x_n\}_n$ be a normalized block sequence in $\mathrm{SB}_{X,r}$ and assume that there is some $\varepsilon > 0$ such that $\|x_n\|_\infty > \varepsilon$ for all $n \in \mathbb{N}$. Then after passing to an appropriate subsequence, there exist block sequences $\{y_n\}_n$ and $\{z_n\}_n$ in $\mathrm{SB}_{X,r}$ and a disjointly supported 1-symmetric sequence $\{u_n\}_n$ in X with the following properties:*

- (i) $x_n = y_n + z_n$ *for all* $n \in \mathbb{N}$ *and* $\operatorname{supp} y_n \cap \operatorname{supp} z_m = \emptyset$ *for all* $n, m \in \mathbb{N}$.
- (ii) $\lim_n \|z_n\|_\infty = 0$.
- (iii) $\{y_n\}_n$ *isometrically generates* $\{u_n\}_n$ *as a spreading model.*
- (iv) $\|u_n\|_\infty = \|u_1\|_\infty > 0$ *for all* $n \in \mathbb{N}$.

Proof. Throughout this proof any finitely supported block vector of $\mathrm{SB}_{X,r}$ will sometimes also be considered as a block vector of X in the natural way and vice versa. We start by making the following remarks which will be used in the proof:

- (I) The basis of $\mathrm{SB}_{X,r}$ satisfies a lower ℓ_r estimate.
- (II) For every finitely supported vector x we have $\|x\|_{X,r} \le \|x\|_X$.

The first one follows from the proof of Proposition 4.1 while the second one follows from the proof of Proposition 3.5.

Using (I) and the fact that the basis of X satisfies a lower ℓ_q estimate, we conclude that the bases of both $\mathrm{SB}_{X,r}$ and X are boundedly complete.

Choose a sequence $\{\delta_k\}_k$ of positive reals strictly decreasing to zero, and arguing as in the proof of Proposition 4.1 find an increasing sequence

$\{m_k\}_k$ of natural numbers such that by passing to a subsequence of $\{x_k\}_k$ if necessary, for every $k_0 \in \mathbb{N}$ and $k \geq k_0$ we have:

(a) $\#\{i : |x_k(i)| \geq \delta_{k_0}\} = m_{k_0}$.
(b) $\min \operatorname{supp} x_{k_0} \geq k_0 m_{k_0}$.

Define $\{y'_k\}_k$ as follows:

$$y'_k(i) = \begin{cases} x_k(i) & \text{if } |x_k(i)| \geq \delta_k, \\ 0 & \text{otherwise.} \end{cases}$$

For $\delta > 0$ and a finitely supported vector y define $R^\delta y$ as in the proof of Proposition 4.1. Then the choice of $\{y'_k\}_k$, Proposition 3.5, and (a), (b) yield:

(α) For every $k \in \mathbb{N}$ and $k \leq j_1 < \cdots < j_k$, $\bigcup_{i=1}^k \operatorname{supp}(y'_{j_i} - R^{\delta_k} y'_{j_i}) \in \mathbb{S}$, and hence, if x is a vector with $\operatorname{supp} x \subset \bigcup_{i=1}^k \operatorname{supp}(y'_{j_i} - R^{\delta_k} y'_{j_i})$, then $\|x\|_{X,r} = \|x\|_X$.
(β) In particular, for every $k \in \mathbb{N}$ and vector x with $\operatorname{supp} x \subset \operatorname{supp} y'_k$, we have $\|x\|_{X,r} = \|x\|_X$.

Apply Proposition 4.1 to the sequence $\{y'_k\}_k$ and the space X and, passing if necessary to a further subsequence, find block sequences $\{y_k\}_k$ and $\{z'_k\}_k$ and a disjointly supported 1-symmetric sequence $\{u_k\}_k$ in X satisfying the conclusion of Proposition 4.1, i.e. such that $y'_k = y_k + z'_k$ for all $k \in \mathbb{N}$, $\bigcup_k \operatorname{supp} y_k \cap \bigcup_k \operatorname{supp} z'_k = \emptyset$, $\lim_k \|z'_k\|_\infty = 0$, $\{y_k\}_k$ as a sequence in X isometrically generates $\{u_k\}_k$ as a spreading model, and $\|u_k\|_\infty = \|u_1\|_\infty > 0$ for all $k \in \mathbb{N}$.

Set $z_k = x_k - y_k$ for all $k \in \mathbb{N}$. Then, by the choice of $\{y'_k\}_k$, it is easy to check that $\{y_k\}_k, \{z_k\}_k$ and $\{u_k\}_k$ satisfy (i), (ii) and (iv) of the conclusion.

In order to complete the proof, it remains to show that $\{y_k\}_k$, as a sequence in $\mathrm{SB}_{X,r}$, generates $\{u_k\}_k$ as a spreading model.

Fix $\varepsilon > 0$, $\ell \in \mathbb{N}$ and $a_1, \ldots, a_\ell \in [-1, 1]$. The Claim in Proposition 4.1 implies that there exists δ_{k_0} such that, for any natural numbers $j_1 < \cdots < j_\ell$,

$$\Big| \Big\| \sum_{i=1}^{\ell} a_i (y_{j_i} - R^{\delta_{k_0}} y_{j_i}) \Big\|_X - \Big\| \sum_{i=1}^{\ell} a_i y_{j_i} \Big\|_X \Big| < \varepsilon. \tag{7}$$

Moreover, (α) shows that for any natural numbers $\max\{k_0, \ell\} \leq j_1 < \cdots < j_\ell$ we have

$$\Big\| \sum_{i=1}^{\ell} a_i (y_{j_i} - R^{\delta_{k_0}} y_{j_i}) \Big\|_{X,r} = \Big\| \sum_{i=1}^{\ell} a_i (y_{j_i} - R^{\delta_{k_0}} y_{j_i}) \Big\|_X. \tag{8}$$

Combining (7), (8) and the unconditionality of the basis of $\mathrm{SB}_{X,r}$, we conclude that for any natural numbers $\max\{k_0, \ell\} \le j_1 < \cdots < j_\ell$ we have

$$\Big\|\sum_{i=1}^{\ell} a_i y_{j_i}\Big\|_{X,r} > \Big\|\sum_{i=1}^{\ell} a_i y_{j_i}\Big\|_X - \varepsilon.$$

On the other hand, (II) yields $\|\sum_{i=1}^{\ell} a_i y_{j_i}\|_{X,r} \le \|\sum_{i=1}^{\ell} a_i y_{j_i}\|_X$.

By the above it easily follows that $\{y_k\}_k$, as a sequence in $\mathrm{SB}_{X,r}$, can only generate the same spreading model as it does when seen as a sequence in X. ∎

Corollary 4.3. *Let X be a Banach space with a 1-symmetric basis $\{e_n\}_n$ and with a norm which satisfies a lower ℓ_q estimate for some $1 \le q < \infty$. Let $r \ge q$. Let $\{x_n\}_n$ be a normalized block sequence in $\mathrm{SB}_{X,r}$ and assume that there is some $\varepsilon > 0$ such that $\|x_n\|_\infty > \varepsilon$ for all $n \in \mathbb{N}$. Then after passing to an appropriate subsequence, there exists a disjointly supported 1-symmetric sequence $\{u_n\}_n$ in X such that:*

(i) *$\{x_n\}_n$ isomorphically generates $\{u_n\}_n$ as a spreading model.*
(ii) *$\|u_n\|_\infty = \|u_1\|_\infty > 0$ for all $n \in \mathbb{N}$.*

Proof. Apply Proposition 4.2, take the decomposition $x_n = y_n + z_n$ and the spreading model $\{u_n\}_n$ of $\{y_n\}_n$. After passing to a further subsequence, by virtue of Proposition 3.7, $\{z_n\}_n$ is equivalent to the basis of ℓ_r. Then for $\ell \in \mathbb{N}, \{a_i\}_{i=1}^{\ell} \subset [-1, 1]$, and so by standard arguments, keeping in mind that the norm on X satisfies a lower ℓ_q estimate and the decomposition's properties, one can see that

$$\Big\|\sum_{i=1}^{\ell} a_i u_i\Big\|_X \le \lim_m \Big\|\sum_{i=1}^{\ell} a_i x_{i+m}\Big\|_{X,r} \le C\Big\|\sum_{i=1}^{\ell} a_i u_i\Big\|_X$$

for some positive constant C. ∎

5. The main result. We start by stating some general facts about spreading models admitted by a super-reflexive Banach space X. As is well known, the class of super-reflexive Banach spaces in a sense coincides with the one of uniformly convex Banach spaces, meaning that every super-reflexive Banach space is isomorphic to a uniformly convex one [13].

Suppose that E is a Banach space such that $X \stackrel{k}{\to} E$ for some $k \in \mathbb{N}$. Any space which is finitely representable in E, is also finitely representable in X, therefore E must be super-reflexive.

Thus any non-trivial k-iterated spreading model $\{e_n\}_n$ of X is weakly convergent. It follows that $\{e_n\}_n$ must be either unconditional and weakly null, or singular (see [1] and [4, Propositions 14, 15] or [7, Propositions I.4.2, I.4.4]).

Also if $\{e_n\}_n$ is singular, then it is weakly convergent to some element e in the Banach space E generated by the sequence $\{e_n\}_n$, and if we set $e'_n = e_n - e$, then $\{e'_n\}_n$ is 1-unconditional, spreading and if $E' = [\{e'_n\}_n]$ then $E = E' \oplus \langle e \rangle$ and E' is isomorphic to E (see [4, Remark 5] or [7]). Moreover, if we take a projection $P : E \to E$ with $P[E] = \langle e \rangle$ and $\ker P = E'$, by doing some calculations we deduce that there exist positive constants c, C such that for every $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$,

$$c \max\Big\{\Big|\sum_{i=1}^{n} \lambda_i\Big|, \Big\|\sum_{i=1}^{n} \lambda_i e'_i\Big\|\Big\} \le \Big\|\sum_{i=1}^{n} \lambda_i e_i\Big\| \le C \max\Big\{\Big|\sum_{i=1}^{n} \lambda_i\Big|, \Big\|\sum_{i=1}^{n} \lambda_i e'_i\Big\|\Big\}. \tag{9}$$

LEMMA 5.1. *Let $\{e_n\}_n$ be a singular and spreading sequence. Then there exists a spreading and weakly null sequence $\{d_n\}_n$ with the following properties:*

(i) *For every Banach space X which admits $\{e_n\}_n$ as a spreading model, X admits $\{d_n\}_n$ as a spreading model.*
(ii) *For every Banach space X and every sequence $\{x_n\}_n$ in X such that $\{x_n\}_n$ generates $\{d_n\}_n$ as a spreading model, there exists a sequence $\{y_n\}_n$ in $[\{x_n\}_n]$ such that $\{y_n\}_n$ isomorphically generates $\{e_n\}_n$ as a spreading model.*

Proof. Set $d_n = e'_n$ as previously defined, i.e. $d_n = e_n - e$, where e is the non-zero weak limit of $\{e_n\}_n$.

For (i), let X be a Banach space and $\{x_n\}_n$ a sequence in X which generates $\{e_n\}_n$ as a spreading model. Since $\{e_n\}_n$ is singular, $\{x_n\}_n$ cannot contain a Schauder basic subsequence. Thus it contains no subsequence which is either equivalent to the basis of ℓ_1, or non-trivial weak-Cauchy (see [19, proof of Proposition 2.2]). By Rosenthal's ℓ_1 theorem (see [20]), this means that it is weakly convergent to some element $x \in X$. By [4, Theorem 38], if we set $x'_n = x_n - x$, then $\{x'_n\}_n$ generates $\{d_n\}_n$ as a spreading model. See also [7].

For (ii) suppose that $\{x_n\}_n$ is a sequence in X that generates $\{d_n\}_n$ as a spreading model. By Rosenthal's criterion for spreading sequences ([4, Proposition 14], see also [7]), $\{d_n\}_n$ is Cesàro summable to zero. Observe that this means that for every infinite subset L of $\mathbb{N}$, for any $\varepsilon > 0$, one may find a finite subset F of L and positive reals $\{\lambda_i\}_{i\in F}$ with $\sum_{i\in F} \lambda_i = 1$ such that $\|\sum_{i\in F} \lambda_i x_i\| < \varepsilon$. This means that $\{x_n\}_n$ is weakly null. Otherwise there would exist $\varepsilon > 0$, $x^* \in S_{X^*}$ and an infinite subset L of $\mathbb{N}$ such that $x^*(x_n) > \varepsilon$ for all $n \in L$. This contradicts our previous observation. Take a non-zero element x in $[\{x_n\}_n]$ and set $y_n = x_n + x$. By combining [4, Theorem 38] and (9) (see also [7]), the result follows. ∎

LEMMA 5.2. *Let X be a super-reflexive Banach space with a basis. Then every k-iterated spreading model of X is equivalent to a spreading sequence in the space generated by a block k-iterated spreading model of X.*

Proof. We prove this lemma by induction on k. Let $\{e_n\}_n$ be a spreading model of X. As previously mentioned, it must be either 1-unconditional and weakly null, or singular.

Suppose it is weakly null. If $\{x_n\}_n$ is a sequence in X which generates $\{e_n\}_n$ as a spreading model, arguing as in the proof of Lemma 5.1, $\{x_n\}_n$ is weakly null, thus has a subsequence equivalent to a block sequence. By the way the block sequence is chosen, it is easy to see that the block sequence actually isometrically generates $\{e_n\}_n$ as a spreading model.

If it is singular, then by (i) of Lemma 5.1, X admits $\{d_n\}_n$ as a spreading model which is 1-unconditional and weakly null. Apply the previous case. Then there is a block sequence in X that generates $\{d_n\}_n$ as a spreading model. Define $d'_n = d_1 + d_{n+1}$. Then $\{d'_n\}_n$ is a spreading sequence in $[\{d_n\}_n]$ and, arguing as in part (ii) of the proof of Lemma 5.1, one may prove that it is equivalent to $\{e_n\}_n$.

Observe that in either case, the space generated by the block sequence in X has a basis. This proves the statement for $k = 1$.

Suppose that it is true for $k \in \mathbb{N}$ and let $\{e_n\}_n$ be a $k+1$-iterated spreading model of X. Thus there exists a super-reflexive Banach space E_k such that $X \overset{k}{\to} E_k$ and $\{e_n\}_n$ is a spreading model of E_k. By the inductive assumption there exists a super-reflexive Banach space E'_k with a basis such that $X \underset{\text{bl}}{\overset{k}{\to}} E'_k$ and $E_k \hookrightarrow E'_k$.

This means that $\{e_n\}_n$ is equivalent to a spreading model admitted by E'_k. By applying the case $k = 1$ for E'_k, the result follows. ■

PROPOSITION 5.3. *Let X be a uniformly convex Banach space with a spreading and unconditional basis $\{e_n\}_n$. Then there exists $q > 1$ such that for every $r > q$ the space ℓ_r does not embed into X and there exists a uniformly convex Banach space X^r with a 1-symmetric basis with the following properties:*

- (i) *The space X^r is ℓ_r-saturated, in particular the spaces X and X^r are totally incomparable.*
- (ii) *There exists a sequence $\{x_n\}_n$ in X^r generating a spreading model which is equivalent to the basis $\{e_n\}_n$ of X.*

If moreover $\{e_n\}_n$ is 1-symmetric, then the following also holds:

- (iii) *Every spreading model admitted by X^r is either equivalent to a spreading sequence in X, or equivalent to a spreading sequence in X^r.*

Proof. Using Theorem 3.2 we may find $1 < p \le q < \infty$ and renorm X in such a way that the basis $\{e_n\}_n$ is 1-unconditional, spreading, p-convex and q-concave. If moreover $\{e_n\}_n$ is 1-symmetric with respect to the original norm, then it retains this property with respect to the new one. The q-concavity of the norm easily implies that for $r > q$, ℓ_r cannot be isomorphic to a subspace of X.

For every $r \ge q$, we will construct a space with the desired properties. We start by defining the space $\mathrm{SB}_{X,r}$, which by Proposition 3.4 is uniformly convex and by some remarks made after Proposition 3.7 it is also ℓ_r-saturated.

Choose $1 < s < t \le p$ and, for a sequence $\{m_k\}_k$ satisfying (2), define $\mathfrak{X} = \big(\sum_{k=1}^{\infty} \bigoplus(\ell_t, \|\!|\cdot|\!\|_{s,t}^{m_k})\big)_{\mathrm{SB}_{X,r}}$. As will become clear later, it is crucial that we choose $t \le p$.

Let $X^r = D$ be the diagonal subspace of $\mathfrak{X}$. Then X^r is uniformly convex, it has a 1-symmetric basis and $\mathrm{SB}_{X,r}$ is isomorphic to a complemented subspace of X^r. Property (i) follows from the fact that $\mathrm{SB}_{X,r}$ is ℓ_r-saturated and X^r is saturated with subspaces of $\mathrm{SB}_{X,r}$, moreover since X does not contain a copy of ℓ_r, it is totally incomparable to X^r. Property (ii) follows from the fact that $\mathrm{SB}_{X,r}$ embeds into X^r and from Corollary 3.6.

It remains to prove that, in the case when $\{e_n\}_n$ is 1-symmetric, (iii) is also satisfied. We will show that if $\{v_n\}_n$ is the spreading model of some block sequence $\{x_n\}_n$ in X^r, then $\{v_n\}_n$ is either equivalent to a spreading sequence in X, or equivalent to a spreading sequence in X^r. If the above is true, using Lemma 5.2 we will conclude that the same is true for every spreading model of X^r.

Let now $\{v_n\}_n$ be the spreading model of a block sequence $\{x_n\}_n$ in X^r. After passing to a subsequence if necessary, one of the following holds: either $\lim_n \|x_n\|_\infty = 0$, or there exists $\varepsilon > 0$ such that $\|x_n\|_\infty > \varepsilon$ for all $n \in \mathbb{N}$. We shall treat these cases separately.

Case 1: $\lim_n \|x_n\|_\infty = 0$. Using Proposition 2.6, we may assume that $\{x_n\}_n$ is equivalent to a block sequence $\{y_n\}_n$ in $\mathrm{SB}_{X,r}$. We distinguish two further subcases, namely either $\lim_n \|y_n\|_\infty = 0$, or there is $\varepsilon > 0$ such that $\|y_n\|_\infty > \varepsilon$ for all $n \in \mathbb{N}$.

If the first one holds, then by Proposition 3.7, $\{y_n\}_n$ has a subsequence equivalent to the usual basis of ℓ_r, and therefore $\{v_n\}_n$ is also equivalent to the usual basis of ℓ_r, which embeds into X^r.

If the second one holds, by Corollary 4.3, there exists a symmetric sequence $\{u_n\}_n$ in X such that $\{v_n\}_n$ is equivalent to $\{u_n\}_n$.

Case 2: There exists $\varepsilon > 0$ such that $\|x_n\|_\infty > \varepsilon$ for all $n \in \mathbb{N}$. Apply Proposition 4.1 and find block sequences $\{y_n\}_n$ and $\{z_n\}_n$ in X^r and a disjointly supported block sequence $\{u_n\}_n$ in X^r, satisfying the conclusion of Proposition 4.1. We will show that $\{v_n\}_n$ is equivalent to $\{u_n\}_n$.

If, after passing to some subsequence, $\lim_n \|z_n\| = 0$, then of course $\{v_n\}_n$ is isometric to $\{u_n\}_n$. Otherwise, using Proposition 2.6 once more, $\{z_n\}_n$ may be assumed to be equivalent to a block sequence in $\mathrm{SB}_{X,r}$. The proof of Proposition 3.4 implies that the norm of $\mathrm{SB}_{X,r}$ is p-convex, which in turn shows that $\{z_n\}_n$ is dominated by the usual basis of ℓ_p.

Observe that, since $\lim_n \|z_n\|_\infty = 0$, we may assume that $\|y_n\|_\infty > \varepsilon$ for all $n \in \mathbb{N}$. Arguing as in the proof of Lemma 2.5, we conclude that $\{y_n\}_n$ dominates the usual basis of ℓ_t and, since $t \leq p$, $\{y_n\}_n$ dominates $\{z_n\}_n$. Using the unconditionality of the basis of X^r we finally conclude that there exists a constant $C > 0$ such that, for every $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$,

$$\Big\|\sum_{i=1}^{n} a_i y_i\Big\| \leq \Big\|\sum_{i=1}^{n} a_i x_i\Big\| \leq C\Big\|\sum_{i=1}^{n} a_i y_i\Big\|.$$

By the above it easily follows that $\{v_n\}_n$ is equivalent to $\{u_n\}_n$. ■

5.1. A sequence of uniformly convex Banach spaces with a 1-symmetric basis. Given a uniformly convex Banach space E with a 1-unconditional and spreading basis $\{e_n\}_n$, we shall inductively construct a sequence $\{X_k\}_k$ of Banach spaces with the following properties:

(i) X_k is uniformly convex and has a 1-symmetric basis for all $k \in \mathbb{N}$.
(ii) X_k is ℓ_{r_k}-saturated, where $\{r_k\}_k$ is a strictly increasing sequence of positive reals.
(iii) X_k and E are totally incomparable for all $k \in \mathbb{N}$.
(iv) Any spreading model admitted by X_{k+1} is either equivalent to a spreading sequence in X_k, or equivalent to a spreading sequence in X_{k+1}, for all $k \in \mathbb{N}$.
(v) The basis of E is equivalent to a spreading model of X_1 and the basis of X_k is equivalent to a spreading model of X_{k+1} for all $k \in \mathbb{N}$.

By Proposition 5.3, find $q_0 > 1$ such that ℓ_r does not embed into E for any $r > q_0$, choose $r_1 > q_0$ and define $X_1 = E^{r_1}$ to be the space provided by that proposition. Assume that for some $k \in \mathbb{N}$ we have chosen real numbers $q_0 < r_1 < \cdots < r_k$ and spaces $X_1, \ldots, X_k$ satisfying the desired conditions. Apply once more Proposition 5.3 to the space X_k, which has a 1-symmetric basis, find $q_k > 1$ such that ℓ_r does not embed into X_k for any $r > q_k$, choose $r_{k+1} > \max\{q_k, r_k\}$ and define $X_{k+1} = X_k^{r_{k+1}}$ to be the space provided by the same proposition.

The construction is complete and properties (i) to (v) are clearly satisfied.

LEMMA 5.4. *The sequence $\{X_k\}_k$ satisfies the following additional conditions: for every $k \in \mathbb{N}$, $k \geq 2$, and for every $1 \leq i < k$, if $\{\tilde{e}_n\}_n$ is an i-iterated spreading model of X_k, then there exists $k - i \leq m \leq k$ such that $[\{\tilde{e}_n\}_n]$ is isomorphic to a subspace of X_m.*

Proof. If $k = 2$, then $i = 1$ and the desired result follows from property (iv).

Assume now that the statement holds for some $k \geq 2$ and let $\{x_n\}_n$ be an i-iterated spreading model of X_{k+1} for some $1 \leq i < k+1$. If $[\{x_n\}_n]$ is isomorphic to a subspace of X_{k+1}, then the statement is true for $m = k+1$. If it is not, assume $\{\{x_n^j\}_n\}_{j=1}^i$ is the sequence of spreading models leading to $\{x_n\}_n$, i.e. $\{x_n^1\}_n$ is a spreading model of X_{k+1}. If for $1 \leq j < i$, E_j is the space generated by $\{x_n^j\}_n$, then $\{x_n^{j+1}\}_n$ is a spreading model of E_j and $\{x_n^i\}_n = \{x_n\}_n$.

Set $j_0 = \min\{j : E_j$ is not isomorphic to any subspace of $X_{k+1}\}$. Then if $j_0 > 1$, we see that E_{j_0-1} is isomorphic to a subspace of X_{k+1} and, by property (iv), E_{j_0} is isomorphic to a subspace of X_k. If $j_0 = 1$, then again by property (iv), E_{j_0} is isomorphic to a subspace of X_k. In either case, $\{x_k\}_k$ is equivalent to an $i - j_0$-iterated spreading model of X_k. Since $i - j_0 < k$, by the inductive assumption there is $k+1-i \leq k-i+j_0 \leq m \leq k$ such that $[\{x_n\}_n]$ is isomorphic to a subspace of X_m. ∎

COROLLARY 5.5. *The family $\{\mathcal{SM}_i^{\mathrm{it}}(X_k)\}_{i=1}^k$ is strictly increasing for all $k \in \mathbb{N}$.*

Proof. Let $k \in \mathbb{N}$ and $1 \leq i < k$. It is always true that $\mathcal{SM}_i^{\mathrm{it}}(X_k) \subset \mathcal{SM}_{i+1}^{\mathrm{it}}(X_k)$ and towards a contradiction assume that the inclusion is not proper.

Consider first the case $i < k-1$. Then X_k admits an $i+1$-iterated spreading model equivalent to the basis of X_{k-i-1}. Since we assume that $\mathcal{SM}_i^{\mathrm{it}}(X_k) = \mathcal{SM}_{i+1}^{\mathrm{it}}(X_k)$, the basis of X_{k-i-1} is an i-iterated spreading model of X_k and, by Lemma 5.4, there is $k-i \leq m \leq k$ such that X_{k-i-1} is isomorphic to a subspace of X_m. Recall that by property (ii), the space X_{k-i-1} is $\ell_{r_{k-i-1}}$-saturated and the space X_m is ℓ_{r_m}-saturated. Since $r_{k-i-1} < r_m$, this is obviously not possible.

If on the other hand $i = k$, then X_k admits an $i+1$-iterated spreading model equivalent to the basis of E. Arguing as previously, we conclude that the basis of E is not an i-iterated spreading model of X_k. ∎

Proof of Theorem 1. If $\{e_n\}_n$ is 1-unconditional, then the sequence $\{X_k\}_k$ is the desired one. This is an immediate consequence of properties (i) to (v) and Lemma 5.4.

If $\{e_n\}_n$ is not unconditional, it must be singular. Apply Lemma 5.1 and, keeping in mind that by the way the d_n are chosen, $d_n \in E$ for all $n \in \mathbb{N}$, apply the previous case for $\{d_n\}_n$.

By Lemma 5.1, X_1 isomorphically admits $\{e_n\}_n$ as a spreading model. Thus X_k isomorphically admits $\{e_n\}_n$ as a k-iterated spreading model. Also, E is isomorphic to $[\{d_n\}_n]$ and the space generated by an i-iterated spreading model of X_k for $i < k$ is isomorphic to a subspace of X_m for $k - i \leq m \leq k$. Since the spaces X_m, E are totally incomparable, the result follows. ∎

The methods employed here make heavy use of the nice properties of uniformly convex spaces. Therefore, although our result applies to ℓ_p spaces, $1 < p < \infty$, it remains unknown whether a similar result can be stated for c_0 and ℓ_1.

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