Local completeness of locally pseudoconvex spaces and Borwein–Preiss variational principle

by

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Abstract. The notion of local completeness is extended to locally pseudoconvex spaces. Then a general version of the Borwein–Preiss variational principle in locally complete locally pseudoconvex spaces is given, where the perturbation is an infinite sum involving differentiable real-valued functions and subadditive functionals. From this, some particular versions of the Borwein–Preiss variational principle are derived. In particular, a version with respect to the Minkowski gauge of a bounded closed convex set in a locally convex space is presented. In locally convex spaces it can be shown that the relevant perturbation only consists of a single summand if and only if the bounded closed convex set has the quasi-weak drop property if and only if it is weakly compact. From this, a new description of reflexive locally convex spaces is obtained.

1. Introduction. Borwein and Preiss gave a smooth variational principle in Banach spaces, which has many applications in nonsmooth analysis and optimization (see [1] and [9]). In terms of Φ-subgradient, Pallaschke and Rolewicz presented a general version of the Borwein–Preiss variational principle in complete metric spaces and gave its interesting applications (see [7, Chapter 2]). In this paper, we first extend the notion of local completeness to locally pseudoconvex spaces. Then we give an extension of the Borwein–Preiss variational principle to locally complete locally pseudoconvex spaces, where the perturbation is an infinite sum involving differentiable real-valued functions and subadditive functionals. From the extension, we deduce several versions of the Borwein–Preiss variational principle in various kinds of locally complete locally pseudoconvex spaces. Any closed convex set in a locally convex space yields a subadditive functional, the Minkowski gauge of the set (or of some translation of it). In particular, we give a version of the variational principle with respect to the Minkowski gauge of a bounded

closed convex set in a locally convex space. We consider the problem: under which conditions can the perturbation appearing in the principle consist of a single summand only? In the framework of quasi-complete locally convex spaces, we show that this happens exactly when the bounded closed convex set is weakly compact, or equivalently, has the quasi-weak drop property (on quasi-weak drop property, see [12, 13]). From this we obtain a characterization of reflexive locally convex spaces by using the Borwein–Preiss variational principle.

2. Locally complete locally pseudoconvex spaces. In this paper, all the spaces X are assumed to be real Hausdorff topological linear spaces (briefly called topological linear spaces). Let A be a nonempty subset of the space X. We say that A is *starlike* if for all $x \in A$ and $0 < t \leq 1$, $tx \in A$. The *modulus of concavity* (see [15]) of a starlike set A is defined by $c(A) = \inf\{s > 0 : A + A \subset sA\}$, with the convention that the infimum of an empty set is equal to $+\infty$. Obviously $c(A) \geq 2$ and for a convex set A, $c(A) = 2$. A starlike set A with a finite modulus of concavity, $c(A) < \infty$, is called *pseudoconvex*.

A topological linear space X is called *locally pseudoconvex* if there is a basis $\{U_\alpha\}$ of neighborhoods of zero consisting of pseudoconvex sets. Let $0 < p \leq 1$. We say that a set $A \subset X$ is *absolutely p-convex* if for all $x, y \in A$ and $0 \leq |s|^p + |t|^p \leq 1$, $tx + sy \in A$ (see [6]). If $p = 1$, then A is called *absolutely convex*. Obviously, any intersection of absolutely p-convex sets is still absolutely p-convex. The *absolutely p-convex hull* $\Gamma_p(A)$ of a set A is the intersection of all the absolutely p-convex sets containing A. We observe that $\Gamma_p(A)$ consists of all the elements of the form $\sum_{i=1}^n t_i x_i$, where $n \in \mathbb{N}$, $x_i \in A$ for $i = 1, \ldots, n$, and $\sum_{i=1}^n |t_i|^p \leq 1$ (see [16, p. 298]). It is easy to see that if A is absolutely p-convex and $0 < p' < p \leq 1$, then A is automatically absolutely p'-convex; and hence for any set A, $\Gamma_{p'}(A) \subset \Gamma_p(A)$. For a closed (or open) balanced absorbing set U in X, U is absolutely p-convex if and only if $\| \ \|_U$ is a p-homogeneous F-pseudonorm, where $\|x\|_U := \inf\{\lambda > 0 : x \in \lambda^{1/p} U\}$ for all $x \in X$. It can be shown (see for example [16, pp. 90–93]) that for a locally pseudoconvex space X, there is a family $\{\| \ \|_\alpha\}$ of p_α-homogeneous F-pseudonorms, which determines a topology equivalent to the original one on X. Thus the sets of the form $U = \bigcap_{i=1}^n \{x \in X : \|x\|_{\alpha_i} \leq r_i\}$ constitute a basis of neighborhoods of zero in X, where $n \in \mathbb{N}$, $\| \ \|_{\alpha_i}$ is a p_{α_i}-homogeneous F-pseudonorm, $0 < p_{\alpha_i} \leq 1$ and $r_i > 0$ for $i = 1, \ldots, n$. Since the set $\{x \in X : \|x\|_{\alpha_i} \leq r_i\}$ is absolutely p_{α_i}-convex, it is also absolutely $\min\{p_{\alpha_1}, \ldots, p_{\alpha_n}\}$-convex and hence $U = \bigcap_{i=1}^n \{x \in X : \|x\|_{\alpha_i} \leq r_i\}$ is absolutely $\min\{p_{\alpha_1}, \ldots, p_{\alpha_n}\}$-convex. Thus for a topological linear space X, the following statements are equivalent:

(i) X is locally pseudoconvex.
(ii) There is a basis $\{U_\alpha\}$ of neighborhoods of zero consisting of absolutely p_α-convex sets, where $0 < p_\alpha \leq 1$.
(iii) There is a family $\{\|\ \|_\alpha\}$ of p_α-homogeneous F-pseudonorms which determines a topology equivalent to the original one on X.

Let the space X be locally pseudoconvex and let $\{U_\alpha\}$ be a basis of neighborhoods of zero which consists of absolutely p_α-convex sets. If $p := \inf\{p_\alpha\} > 0$, then every U_α is absolutely p-convex. We say that the space X is *locally p-convex* if there is a basis of neighborhoods of zero consisting of absolutely p-convex sets for some fixed p, $0 < p \leq 1$. In particular, if $p = 1$, then X is called *locally convex*. For a locally convex space X, we always denote by X^* the topological dual of X. As is well known, for locally convex spaces there are various kinds of completeness, for example, completeness, quasi-completeness, sequential completeness, $\sum$-completeness, l^∞-completeness, local completeness and so on; for details, we refer to [8, pp. 164–165] and [10]. Up to now, local completeness is the weakest known kind of completeness.

Now we extend the notion of local completeness to locally pseudoconvex spaces (concerning local completeness of locally convex spaces, see, for example, [8, Chapter 5]) .

DEFINITION 2.1. A subset B of the space X is called a *p-disc* $(0 < p \leq 1)$ if it is bounded and absolutely p-convex. We denote by E_B the linear span of B, span$[B]$, endowed with the topology induced by the gauge q_B of B, where $q_B(x) = \inf\{\lambda > 0 : x \in \lambda^{1/p}B\}$ for all $x \in$ span$[B]$. If E_B is complete, we call the p-disc B a *self-complete p-disc*. A sequence (x_n) in X is called a *locally Cauchy sequence* if there is a $p \in (0,1]$ and a p-disc B in X such that (x_n) is a Cauchy sequence in E_B. A sequence (x_n) in X is said to be *locally convergent* to a point x_0 if there is a $p \in (0,1]$ and a p-disc B such that $x_n \to x_0$ in E_B. It is easy to see that (x_n) is locally convergent to x_0 if and only if $(x_n - x_0)$ is locally convergent to zero. A point x_0 is called a *local limit point* of a set A in X if there is a sequence (x_n) in A such that (x_n) is locally convergent to x_0. The set A is called *locally closed* if every local limit point of A belongs to A. Obviously every sequentially closed set is locally closed, but the converse is not true (for an example, see [11, Example 3.1]).

PROPOSITION 2.1. *Let X be a locally pseudoconvex space. Then the following statements are equivalent*:

(i) *For each $p \in (0,1]$ and any p-disc A, there is a $p' \in (0,1]$ and a self-complete p'-disc B such that $A \subset B$.*
(ii) *Every locally Cauchy sequence is locally convergent.*
(iii) *For each $p \in (0,1]$, every closed p-disc is a self-complete p-disc.*

Proof. (i)⇒(ii). Let (x_n) be a locally Cauchy sequence. Then there is a $p \in (0,1]$ and a p-disc A such that $(x_n) \subset \mathrm{span}[A]$ and (x_n) is a Cauchy sequence in E_A. By (i), there is a $p' \in (0,1]$ and a self-complete p'-disc B such that $A \subset B$. For any $\varepsilon > 0$, put $\varepsilon_1 = \varepsilon^{p/p'}$. Then there is $n_0 \in \mathbb{N}$ such that $x_n - x_m \in \varepsilon_1^{1/p} A$ for all $m, n \geq n_0$. Since $A \subset B$, we have $x_n - x_m \in \varepsilon_1^{1/p} B = \varepsilon^{1/p'} B$ for all $m, n \geq n_0$. That is, (x_n) is a Cauchy sequence in E_B. Since E_B is complete, there exists $x_0 \in E_B$ such that $x_n \to x_0$ in E_B, so (x_n) is locally convergent.

(ii)⇒(iii). Let $p \in (0,1]$ and A be a closed p-disc and let (x_n) be a Cauchy sequence in E_A. By (ii), (x_n) is locally convergent to some x_0. Clearly $x_n \to x_0$ in X. For any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $x_n - x_m \in \varepsilon^{1/p} A$ for all $m, n \geq n_0$. If $m \to \infty$, then $x_m \to x_0$ in X. And since A is closed, we have $x_n - x_0 \in \varepsilon^{1/p} A$ for all $n \geq n_0$. This means that $x_n \to x_0$ in E_A and hence A is a self-complete p-disc.

(iii)⇒(i). Let A be a p-disc. Then $\overline{A}$ is also a p-disc. By (iii), $\overline{A}$ is a self-complete p-disc. By taking $B = \overline{A}$, we complete the proof. ∎

DEFINITION 2.2. A locally pseudoconvex space X is called *locally complete* if one of the three equivalent statements in Proposition 2.1 is satisfied.

Let X be a locally p-convex space ($0 < p \leq 1$). Then a sequence (x_n) in X locally converges to zero if and only if there is an increasing unbounded sequence (α_n) of positive real numbers such that $(\alpha_n x_n)$ converges to zero in X (the proof is similar to that of [8, Proposition 5.1.3]). Moreover, X is locally complete if and only if for every bounded set A in X there is a self-complete p-disc B such that $A \subset B$. We also observe that most statements (except for those involving topological duals) concerning locally complete locally convex spaces can be extended to locally complete locally p-convex spaces. Here we do not discuss this in detail.

Next we consider local completeness of locally pseudoconvex spaces (not only locally p-convex spaces).

We start with the following obvious

LEMMA 2.1. *Let $(X, \|\ \|)$ be an F^*-space, i.e., a linear space with an F-norm $\|\ \|$ (see [16, p. 5]). Let (x_n) be a sequence in X convergent to zero. Then there is a sequence (b_n) of positive reals tending to infinity such that $(b_n x_n)$ is convergent to zero.*

Proof. Without loss of generality we may assume that $x_n \neq 0$. We put $b_n = E(1/\sqrt{\|x_n\|})$, where $E(t)$ denotes the greatest integer not greater than t. Then by the triangle inequality, (b_n) has the required property. ∎

COROLLARY 2.1. *Let X be a topological linear space. Let (x_n) be a sequence in X locally convergent to zero. Then there is a sequence (b_n) of*

positive reals tending to infinity such that $(b_n x_n)$ is locally convergent to zero.

Proof. Since (x_n) is locally convergent to 0, there is a $p \in (0,1]$ and a p-disc B such that (x_n) is convergent to 0 with respect to the p-homogeneous norm $\| \,\|_B$ induced by B. Thus by Lemma 2.1 there is a sequence (b_n), $b_n > 0$, tending to infinity such that $(b_n x_n)$ is convergent to 0 with respect to the p-homogeneous norm $\| \,\|_B$. Thus, by definition, $(b_n x_n)$ is locally convergent to 0. ∎

PROPOSITION 2.2. *A locally pseudoconvex F-space (i.e. complete metrizable space) X is locally complete.*

Proof. Let (x_n) be a locally Cauchy sequence in X. Then there is a $p \in (0,1]$ and a closed p-disc B such that (x_n) is a Cauchy sequence in $E_B = (\mathrm{span}[B], q_B)$. That is, for any $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that $x_n - x_m \in \varepsilon B$ for all $n, m > n_\varepsilon$. Since B is bounded in X, (x_n) is also a Cauchy sequence in X. And since X is complete, (x_n) is convergent to some x_0 in X. Letting $m \to \infty$ shows that $x_n - x_0 \in \varepsilon B$ for all $n > n_\varepsilon$. This means that (x_n) is convergent to x_0 in E_B, hence (x_n) is locally convergent to x_0 and X is locally complete. ∎

It is of interest to know whether Proposition 2.2 can be reversed. We are able to do it only in locally p-convex spaces.

PROPOSITION 2.3. *A locally complete locally p-convex F^*-space $(X, \| \,\|)$ is complete, i.e. it is an F-space.*

Proof. Let (x_n) be a Cauchy sequence in X and let $U_1 \supset U_2 \supset \cdots$ be a 0-neighborhood basis consisting of absolutely p-convex sets. There is a sequence $n_1 < n_2 < \cdots$ such that

$$x_{n_{i+1}} - x_{n_i} \in \frac{1}{2^i} U_i \quad \text{for all } i \in \mathbb{N}.$$

Obviously the sequence $(2^i(x_{n_{i+1}} - x_{n_i}))_{i \in \mathbb{N}}$ is convergent to 0 in X. Let B be the closed absolutely p-convex hull of the bounded set

$$\{x_{n_1}, 2(x_{n_2} - x_{n_1}), \ldots, 2^i(x_{n_{i+1}} - x_{n_i}), \ldots\}.$$

Then B is a closed p-disc, $\{x_{n_1}, x_{n_2}, \ldots\} \subset \mathrm{span}[B]$ and $x_{n_{i+1}} - x_{n_i} \in 2^{-i}B$. Observe that

$$x_{n_{i+k}} - x_{n_i} = (x_{n_{i+k}} - x_{n_{i+k-1}}) + \cdots + (x_{n_{i+1}} - x_{n_i})$$

$$\in \frac{1}{2^{i+k-1}} B + \cdots + \frac{1}{2^i} B \subset \left[\left(\frac{1}{2^{i+k-1}} \right)^p + \cdots + \left(\frac{1}{2^i} \right)^p \right]^{1/p} B$$

$$= \frac{1}{2^i} \cdot \left(\frac{1 - (1/2^p)^k}{1 - 1/2^p} \right)^{1/p} B \subset \frac{1}{2^i} \cdot \frac{2}{(2^p - 1)^{1/p}} B.$$

Since $p > 0$, $(x_{n_i})_{i \in \mathbb{N}}$ is a Cauchy sequence in $E_B = (\mathrm{span}[B], q_B)$. By the hypothesis that $(X, \| \, \|)$ is locally complete, E_B is complete. Hence there is a point x_0 in E_B such that (x_{n_i}) is convergent to x_0 in E_B; then clearly (x_{n_i}) is convergent to x_0 in $(X, \| \, \|)$. Combining this with the assumption that (x_n) is a Cauchy sequence in $(X, \| \, \|)$, we conclude that (x_n) is convergent to x_0 in $(X, \| \, \|)$, i.e. $(X, \| \, \|)$ is complete. ∎

Unfortunately this proof does not work in locally pseudoconvex F^*-spaces as the following example shows.

EXAMPLE 2.1. Let
$$X = L^1[0,1] \times L^{1/2}[0,1] \times \cdots \times L^{1/n}[0,1] \times \cdots$$
with topology given by pseudonorms
$$\|x\|_n = \|\{x_1(t), \ldots, x_n(t), \ldots\}\|_n = \sum_{i=1}^{n} \int_0^1 |x_i(t)|^{1/i} \, dt.$$

Take arbitrary p, $0 < p < 1$. In a similar way to the proof that there is no continuous linear functional on $L^p[0,1]$ for $p < 1$ we can construct a sequence $(x^n)_{n \in \mathbb{N}} \subset X$ converging to 0 such that its absolute p-convex hull contains the subspace
$$X_n = \{0\} \times \cdots \times \{0\} \times L^{1/n}[0,1] \times L^{1/(n+1)}[0,1] \times \cdots$$
provided $1/n < p$. Of course X_n is not bounded.

Thus we propose the following open problem to end this section.

PROBLEM 2.1. Is every locally complete locally pseudoconvex F^*-space X complete?

3. A version of the Borwein–Preiss principle in locally pseudoconvex spaces. Let X be a locally pseudoconvex space. For any function $f : X \to (-\infty, \infty]$, we always assume that $\mathrm{dom}\, f = \{x \in X : f(x) \neq \infty\}$ is nonempty. A function $f : X \to (-\infty, \infty]$ is called *lower semicontinuous* (respectively, *sequentially lower semicontinuous*) if for any net (x_n) converging to a (respectively, for any sequence (x_n) converging to a) in X, we have $f(a) \leq \liminf_n f(x_n)$ (see [5, p. 40] and [18, pp. 149–151]).

As in [14], we call a function $f : X \to (-\infty, \infty]$ *locally lower semicontinuous* if for any $r \in \mathbb{R}$, the set $\{x \in X : f(x) \leq r\}$ is locally closed in X. This is equivalent to the condition that for any sequence (x_n) locally converging to a, $f(a) \leq \liminf_{n \to \infty} f(x_n)$. Obviously every (sequentially) lower semicontinuous function is locally lower semicontinuous, and the converse is not true. We denote by Φ_0 the class of all strictly increasing, continuous functions $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$, and by Φ_1 the class of all functions $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = \varphi'(0) = 0$, $\varphi'(t)$ exists

and $\varphi'(t) > 0$ on $(0, \infty)$. Clearly $\Phi_1 \subset \Phi_0$. Moreover $\varphi(\infty)$ always means $\lim_{t\to\infty} \varphi(t)$ (clearly, it may be ∞). By modifying the proof of Borwein and Preiss (see [1] or [9, pp. 66–69]), we first give a general version of the Borwein–Preiss variational principle in locally pseudoconvex spaces.

THEOREM 3.1. *Let X be a locally complete locally pseudoconvex space, and $p_0(\cdot) \leq p_1(\cdot) \leq p_2(\cdot) \leq \cdots$ be a sequence of locally lower semicontinuous, subadditive, nonnegative functions on X with $p_i(0) = 0$ such that whenever a sequence (x_n) in X satisfies $p_i(x_m - x_n) \to 0$ $(m \geq n \to \infty)$ for each i, then (x_n) is a locally Cauchy sequence in X. Let $g : X \to (-\infty, \infty]$ be a locally lower semicontinuous function, bounded below, and let $x_0 \in \operatorname{dom} g$, $\varepsilon > g(x_0) - \inf g(X)$, $\lambda > 0$ and $\{\varphi_i : i = 0, 1, 2, \ldots\} \subset \Phi_0$. Then there exist a sequence (v_n) locally converging to some v in X and $0 < \mu < 1$ such that*

(a) $\qquad g(x) + \dfrac{\varepsilon}{\varphi_0(\lambda)}\,\theta(x) \geq g(v) + \dfrac{\varepsilon}{\varphi_0(\lambda)}\,\theta(v), \quad \forall x \in X,$

   where $\theta(x) = \sum_{n=0}^{\infty} \mu^n (1 - \mu)\varphi_n(p_n(x - v_n))$ and $v_0 = x_0$;

(b) *$p_0(v - v_0) \leq \lambda$, $p_0(v_n - v_0) \leq \lambda$, $p_n(v - v_n) \leq \lambda$ for all $n \in \mathbb{N}$, and $p_n(v - v_n) \to 0$;*

(c) *$g(v) < \varepsilon + \inf g(X)$.*

Proof. Choose $\varepsilon_1 > 0$ such that

(1) $\qquad\qquad\qquad 0 \leq g(x_0) - \inf g(X) < \varepsilon_1 < \varepsilon.$

Choose $\mu > 0$ such that

(2) $\qquad\quad 0 < \mu < 1 - \varepsilon_1/\varepsilon, \quad \text{equivalently,} \quad 0 < \varepsilon_1/\varepsilon < 1 - \mu.$

Put

(3) $\qquad\qquad\qquad\qquad \delta = \dfrac{(1 - \mu)\varepsilon}{\varphi_0(\lambda)}.$

From (2) and (3), we know that

$$\varepsilon_1/\delta < \varphi_0(\lambda) < \varepsilon/\delta.$$

For $k = 1, 2, \ldots$, choose γ_k, $0 < \gamma_k < \mu$, such that

(4) $\qquad\qquad \sum_{k=1}^{\infty} \varphi_k^{-1}\left(\dfrac{\gamma_1 \cdots \gamma_k}{\mu^k} \cdot \dfrac{\varepsilon_1}{\delta}\right) < \lambda - \varphi_0^{-1}\left(\dfrac{\varepsilon_1}{\delta}\right).$

Put $g_0 = g$ and $v_0 = x_0$. Define $g_1(x) = g_0(x) + \delta\varphi_0(p_0(x - v_0))$, $x \in X$. Clearly, $g_1(x_0) = g(x_0) < \varepsilon + \inf g(X) < \infty$, hence $\operatorname{dom} g_1 \neq \emptyset$. If $g(x_0) = g_1(x_0) = \inf g_1(X)$, then we set $v_1 = v_0$. If $g(x_0) = g_1(x_0) > \inf g_1(X)$, then $\gamma_1\, g(x_0) + (1 - \gamma_1) \inf g_1(X) > \inf g_1(X)$. Hence there exists $v_1 \in X$ such that $g_1(v_1) < \gamma_1 g(x_0) + (1 - \gamma_1) \inf g_1(X)$. Thus in any case, there exists $v_1 \in X$ such that

$$g_1(v_1) \leq \gamma_1 g(x_0) + (1 - \gamma_1) \inf g_1(X).$$

In general, define

(5) $$g_{n+1}(x) = g_n(x) + \delta\mu^n\varphi_n(p_n(x - v_n)), \quad x \in X.$$

Clearly $g_{n+1}(v_n) = g_n(v_n) < \infty$, hence $\operatorname{dom} g_{n+1} \neq \emptyset$. Choose $v_{n+1} \in X$ such that

(6) $$g_{n+1}(v_{n+1}) \leq \gamma_{n+1}g_n(v_n) + (1 - \gamma_{n+1})\inf g_{n+1}(X).$$

Put $s_n = \inf g_n(X)$ and $a_n = g_n(v_n)$. Then

(7) $$\begin{aligned}
a_{n+1} = g_{n+1}(v_{n+1}) &\leq \gamma_{n+1}g_n(v_n) + (1 - \gamma_{n+1})\inf g_{n+1}(X) \\
&\leq \gamma_{n+1}g_n(v_n) + (1 - \gamma_{n+1})g_{n+1}(v_n) \\
&= \gamma_{n+1}g_n(v_n) + (1 - \gamma_{n+1})g_n(v_n) \\
&= g_n(v_n) = a_n.
\end{aligned}$$

Obviously $g_n \leq g_{n+1}$, hence

$$\begin{aligned}
s_n \leq s_{n+1} \leq a_{n+1} = g_{n+1}(v_{n+1}) &\leq \gamma_{n+1}g_n(v_n) + (1 - \gamma_{n+1})\inf g_{n+1}(X) \\
&= \gamma_{n+1}a_n + (1 - \gamma_{n+1})s_{n+1}.
\end{aligned}$$

From this, we have

(8) $$\begin{aligned}
0 \leq a_{n+1} - s_{n+1} &\leq \gamma_{n+1}(a_n - s_{n+1}) \leq \gamma_{n+1}(a_n - s_n) \leq \cdots \\
&\leq \gamma_{n+1}\gamma_n \cdots \gamma_1(a_0 - s_0) < \gamma_{n+1}\gamma_n \cdots \gamma_1\varepsilon_1.
\end{aligned}$$

Taking $x = v_{n+1}$ in (5), we have

(9) $$\begin{aligned}
a_{n+1} = g_{n+1}(v_{n+1}) &= g_n(v_{n+1}) + \delta\mu^n\varphi_n(p_n(v_{n+1} - v_n)) \\
&\geq s_n + \delta\mu^n\varphi_n(p_n(v_{n+1} - v_n)).
\end{aligned}$$

From (7)–(9), we obtain

$$\delta\mu^n\varphi_n(p_n(v_{n+1} - v_n)) \leq a_{n+1} - s_n \leq a_n - s_n < \gamma_n\gamma_{n-1} \cdots \gamma_1\varepsilon_1.$$

Therefore,

$$p_n(v_{n+1} - v_n) \leq \varphi_n^{-1}\left(\frac{\gamma_1 \cdots \gamma_n}{\mu^n} \cdot \frac{\varepsilon_1}{\delta}\right).$$

For each $i \in \mathbb{N}$, $n \geq i$ and $j \in \mathbb{N}$, we have

(10) $$\begin{aligned}
p_i(v_{n+j} - v_n) &\leq p_n(v_{n+j} - v_n) \\
&\leq p_n(v_{n+j} - v_{n+j-1}) + \cdots + p_n(v_{n+1} - v_n) \\
&\leq p_{n+j-1}(v_{n+j} - v_{n+j-1}) + \cdots + p_n(v_{n+1} - v_n) \\
&< \sum_{k=n}^{\infty} \varphi_k^{-1}\left(\frac{\gamma_1 \cdots \gamma_k}{\mu^k} \cdot \frac{\varepsilon_1}{\delta}\right).
\end{aligned}$$

Combining this with (4), we conclude that for each $i \in \mathbb{N}$, $p_i(v_m - v_n) \to 0$ when $m \geq n \to \infty$. This implies that (v_n) is a locally Cauchy sequence in X. Since X is locally complete, there exists $v \in X$ such that (v_n) is locally

convergent to v. Letting $j \to \infty$ on the left of (10) shows that $(v_{n+j})_j$ is locally convergent to v; and since every p_i is locally lower semicontinuous, we have

$$(11) \quad p_n(v - v_n) \le \sum_{k=n}^{\infty} \varphi_k^{-1}\left(\frac{\gamma_1 \cdots \gamma_k}{\mu^k} \cdot \frac{\varepsilon_1}{\delta}\right) < \lambda - \varphi_0^{-1}\left(\frac{\varepsilon_1}{\delta}\right), \quad \forall n \in \mathbb{N}.$$

From (11), we obtain

$$(12) \qquad p_n(v - v_n) < \lambda \quad \text{and} \quad p_n(v - v_n) \to 0 \quad (n \to \infty).$$

Since

$$\delta\varphi_0(p_0(v_1 - v_0)) = g_1(v_1) - g_0(v_1)$$
$$\le g_1(v_1) - \inf g(X) = a_1 - s_0 \le a_0 - s_0 < \varepsilon_1,$$

we have

$$(13) \qquad p_0(v_1 - v_0) < \varphi_0^{-1}(\varepsilon_1/\delta).$$

From (11) and (13), we deduce that

$$(14) \quad p_0(v - v_0) \le p_0(v - v_1) + p_0(v_1 - v_0) \le p_1(v - v_1) + p_0(v_1 - v_0)$$
$$< \lambda - \varphi_0^{-1}(\varepsilon_1/\delta) + \varphi_0^{-1}(\varepsilon_1/\delta) = \lambda.$$

Taking $n = i = 1$ in (10), for any $j \in \mathbb{N}$ we get

$$p_1(v_{1+j} - v_1) < \sum_{k=1}^{\infty} \varphi_k^{-1}\left(\frac{\gamma_1 \cdots \gamma_k}{\mu^k} \cdot \frac{\varepsilon_1}{\delta}\right) < \lambda - \varphi_0^{-1}\left(\frac{\varepsilon_1}{\delta}\right).$$

From this, we have

$$(15) \qquad p_0(v_{1+j} - v_0) \le p_0(v_{1+j} - v_1) + p_0(v_1 - v_0)$$
$$\le p_1(v_{1+j} - v_1) + p_0(v_1 - v_0)$$
$$< \lambda - \varphi_0^{-1}(\varepsilon_1/\delta) + \varphi_0^{-1}(\varepsilon_1/\delta) = \lambda.$$

By (13)–(15), we conclude that

$$p_0(v_n - v_0) \le \lambda \quad \text{and} \quad p_0(v - v_0) \le \lambda.$$

Combining this with (12), we see that (b) holds.

From (7), $a_0 \ge a_1 \ge a_2 \ge \cdots$. Since $a_n = g_n(v_n) \ge \inf g_n(X) > -\infty$, we infer that $\lim_{n \to \infty} a_n$ exists and is finite. By (8),

$$0 \le a_n - s_n < \gamma_1 \cdots \gamma_n \varepsilon_1 < \mu^n \varepsilon_1 \to 0 \quad (n \to \infty),$$

hence $\lim_{n \to \infty} s_n = \lim_{n \to \infty} a_n$ exists and is finite. Put

$$\theta(x) = \sum_{n=0}^{\infty} \mu^n(1 - \mu)\varphi_n(p_n(x - v_n)).$$

Then for any $x \in \operatorname{dom} g$, we have

$$(16) \quad g(x) + \frac{\varepsilon}{\varphi_0(\lambda)} \theta(x) = g(x) + \frac{\varepsilon(1-\mu)}{\varphi_0(\lambda)} \sum_{n=0}^{\infty} \mu^n \varphi_n(p_n(x - v_n))$$

$$= g(x) + \delta \sum_{n=0}^{\infty} \mu^n \varphi_n(p_n(x - v_n))$$

$$= \sup_n g_n(x) \geq \lim_{n \to \infty} s_n = \lim_{n \to \infty} a_n = \lim_{n \to \infty} g_n(v_n).$$

Let $\alpha = \sup_m \liminf_n g_m(v_n)$. Then for any $\eta > 0$, there exists $m \in \mathbb{N}$ such that $\liminf_n g_m(v_n) > \alpha - \eta$. Hence there are $n_1 < n_2 < \cdots$ such that $g_m(v_{n_i}) > \alpha - \eta$. Take i large enough so that $n_i \geq m$. Then

$$g_{n_i}(v_{n_i}) \geq g_m(v_{n_i}) > \alpha - \eta.$$

Also, $\lim_{n \to \infty} a_n = \lim_{n \to \infty} g_n(v_n)$ exists, so we have $\lim_{n \to \infty} g_n(v_n) \geq \alpha$. From this and (16), we obtain

$$(17) \qquad g(x) + \frac{\varepsilon}{\varphi_0(\lambda)} \theta(x) \geq \alpha = \sup_m \liminf_n g_m(v_n).$$

Since g_m is locally lower semicontinuous and (v_n) is locally convergent to v, it follows that $\liminf_n g_m(v_n) \geq g_m(v)$. Combining this with (17), we conclude that

$$g(x) + \frac{\varepsilon}{\varphi_0(\lambda)} \theta(x) \geq \sup_m g_m(v) = g(v) + \frac{\varepsilon}{\varphi_0(\lambda)} \theta(v),$$

i.e. (a) holds.

Finally, we show that (c) holds. Since $g_0(v_0) = g(x_0) < \varepsilon_1 + \inf g(X)$,

$$g_1(v_1) \leq \gamma_1 g(x_0) + (1 - \gamma_1) \inf g_1(X)$$
$$\leq \gamma_1 g(x_0) + (1 - \gamma_1) g_1(x_0) = g(x_0) < \varepsilon_1 + \inf g(X).$$

By (7), (a_n) is decreasing, hence for any $n \in \mathbb{N}$,

$$a_n = g_n(v_n) \leq a_1 < \varepsilon_1 + \inf g(X).$$

Therefore

$$g(v) \leq g(v) + \frac{\varepsilon}{\varphi_0(\lambda)} \theta(v) = \sup_m g_m(v)$$
$$\leq \sup_m \liminf_n g_m(v_n) = \alpha \leq \lim_{n \to \infty} g_n(v_n) < \varepsilon_1 + \inf g(X). \quad \blacksquare$$

As we have seen in Section 2, in a metrizable locally p-convex space $(0 < p \leq 1)$, a sequence is convergent if and only if it is locally convergent. Hence a function defined on the space is (sequentially) lower semicontinuous if and only if it is locally lower semicontinuous. Moreover by Proposition 2.2, a complete metrizable locally pseudoconvex space is locally complete. Now let X be a complete metrizable locally pseudoconvex space and $\| \ \|_0 \leq \| \ \|_1 \leq \| \ \|_2 \leq \cdots$ be a sequence of p_n-homogeneous F-pseudonorms determining the

topology of X (see [16, pp. 90–95]). By substituting the sequence $\|\ \|_0 \leq \|\ \|_1 \leq \|\ \|_2 \leq \cdots$ for the sequence $p_0 \leq p_1 \leq p_2 \leq \cdots$ in Theorem 3.1, we immediately obtain the following result, which may be useful in studying the differentiability of perturbations.

COROLLARY 3.1. *Let X be a complete metrizable locally pseudoconvex space with the topology generated by an increasing sequence $\|\ \|_0 \leq \|\ \|_1 \leq \|\ \|_2 \leq \cdots$ of p_n-homogeneous F-pseudonorms. Let $g : X \to (-\infty, \infty]$ be a lower semicontinuous function bounded below, let x_0, $\varepsilon > 0$, and $\lambda > 0$ be as in Theorem 3.1, and let $\{\varphi_n : n = 0, 1, 2, \ldots\} \subset \Phi_0$. Then there exist a sequence (v_n) converging to v in X and $0 < \mu < 1$ such that*

(a) $$g(x) + \frac{\varepsilon}{\varphi_0(\lambda)}\,\theta(x) \geq g(v) + \frac{\varepsilon}{\varphi_0(\lambda)}\,\theta(v), \quad \forall x \in X,$$

where $\theta(x) = \sum_{n=0}^{\infty} \mu^n (1 - \mu)\varphi_n(\|x - v_n\|_n)$ and $v_0 = x_0$;

(b) $\|v - v_0\|_0 \leq \lambda$, $\|v_n - v_0\|_0 \leq \lambda$, $\|v - v_n\|_n \leq \lambda$ *for all $n \in \mathbb{N}$, and* $\|v - v_n\|_n \to 0$;

(c) $g(v) < \varepsilon + \inf g(X)$.

Let $B \subset X$ be a p-convex set (i.e. $sB + tB \subset B$ for all $s, t \geq 0$ such that $s^p + t^p = 1$) with $0 \in B$ ($0 < p \leq 1$). Then B induces in X a Minkowski gauge $q_B : X \to [0, \infty]$ as follows:

$$q_B(x) = \begin{cases} \inf\{\lambda > 0 : x \in \lambda^{1/p}B\} & \text{if there exists } \lambda > 0 \text{ such that } x \in \lambda^{1/p}B; \\ \infty & \text{else.} \end{cases}$$

The Minkowski gauge q_B is positive p-homogeneous and subadditive.

COROLLARY 3.2. *Let X be a locally complete locally p-convex space ($0 < p \leq 1$), and $B \subset X$ be a locally closed bounded p-convex set with $0 \in B$. Let $g : X \to (-\infty, \infty]$ be a locally lower semicontinuous function bounded below and let $x_0 \in \operatorname{dom} g$, $\varepsilon > g(x_0) - \inf g(X)$, $\lambda > 0$ and $\varphi \in \Phi_0$. Then there exist a sequence (v_n) locally converging to v in X and $0 < \mu < 1$ such that*

(a) $$g(x) + \frac{\varepsilon}{\varphi(\lambda)}\,\theta(x) \geq g(v) + \frac{\varepsilon}{\varphi(\lambda)}\,\theta(v), \quad \forall x \in X,$$

where $\theta(x) = \sum_{n=0}^{\infty} \mu^n (1 - \mu)\varphi(q_B(x - v_n))$ and $v_0 = x_0$;

(b) $q_B(v - v_0) \leq \lambda$, $q_B(v_n - v_0) \leq \lambda$, $q_B(v - v_n) \leq \lambda$ *for all $n \in \mathbb{N}$, and* $q_B(v - v_n) \to 0$;

(c) $g(v) < \varepsilon + \inf g(X)$.

Proof. Clearly, the Minkowski gauge q_B of the locally closed bounded p-convex set B is locally lower semicontinuous and subadditive. Denote the absolutely p-convex hull of B by D; it is a p-disc. Thus if a sequence (x_n)

in X satisfies $q_B(x_m - x_n) \to 0$ $(m \geq n \to \infty)$, then $q_D(x_m - x_n) \to 0$ $(m \geq n \to \infty)$. That is to say, (x_n) is a locally Cauchy sequence in X. Now applying Theorem 3.1, we immediately obtain the result. $\blacksquare$

As a special case of Corollary 3.2, we have the following:

COROLLARY 3.3. *Let X be a quasi-complete (or sequentially complete, or locally complete) locally convex space and $B \subset X$ a bounded closed convex set with $0 \in B$. Let $g : X \to (-\infty, \infty]$ be a lower semicontinuous function bounded below, and let $x_0 \in \mathrm{dom}\, g$, $\varepsilon > g(x_0) - \inf g(X)$, $\lambda > 0$ and $\varphi \in \Phi_0$. Then there exist a sequence (v_n) converging to v in X and $0 < \mu < 1$ such that* (a)–(c) *in Corollary* 3.2 *hold.*

In fact, in view of [7] we see that in Theorem 3.1 and Corollaries 3.1–3.3, the function

$$-\frac{\varepsilon}{\varphi_0(\lambda)}\,\theta(x) \quad \text{or} \quad -\frac{\varepsilon}{\varphi(\lambda)}\,\theta(x)$$

is a Φ-subgradient of g at the point v, where Φ denotes the appropriate class of functions according to different cases.

4. Perturbation only consisting of a single summand and reflexivity of locally convex spaces. As mentioned in [9, p. 71], even in the framework of Banach spaces, it is not always possible to assume that the perturbation function θ used in the variational principle consists of a single summand. Borwein and Preiss [1] have shown that this is possible in reflexive Banach spaces and that these are the only spaces with this property. In this section, we consider the following problem: under which condition, can θ appearing in Corollary 3.3 consist of a single summand only? This problem will be solved in the framework of quasi-complete locally convex spaces. We shall see that this happens exactly when B is weakly compact. From this we obtain a criterion for reflexivity of locally convex spaces. In this section we denote by Φ_2 the class of all functions $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = \varphi'(0) = 0$, $\varphi'(t)$ exists and strictly increases on $[0, \infty)$. Clearly $\Phi_2 \subset \Phi_1 \subset \Phi_0$.

THEOREM 4.1. *Let X be a locally convex space and $B \subset X$ be a bounded closed convex set with $0 \in B$. If for any lower semicontinuous convex function $g : X \to (-\infty, \infty]$ bounded below, any $x_0 \in \mathrm{dom}\, g$, any $\varepsilon > g(x_0) - \inf g(X)$ and any $\lambda > 0$, there exist $\varphi \in \Phi_2$ and $v \in X$ such that*

$$g(x) + \frac{\varepsilon}{\varphi(\lambda)}\,\varphi(q_B(x - x_0)) \geq g(v) + \frac{\varepsilon}{\varphi(\lambda)}\,\varphi(q_B(v - x_0)) \quad \text{for all } x \in X,$$

then for any $x^* \in X^*$, there exists $u \in B$ such that $\langle x^*, u \rangle = \sup\langle x^*, B \rangle$.

Proof. For any fixed $x^* \in X^*$, if $\langle x^*, B \rangle \leq 0$, we may take $u = 0$. If not, there exists $b \in B$ such that $\langle x^*, b \rangle > 0$. Since $\langle x^*, B \rangle$ is bounded, there exists

$\alpha > 0$ such that $|\langle x^*, B \rangle| \leq \alpha$. Without loss of generality, we may assume that $\alpha < 1$, or else we may consider kx^*, where $k > 0$ is small enough. Put $U := \{x \in X : |\langle x^*, x \rangle| \leq 1\}$. Then U is a 0-neighborhood in X and $B \subset U^i$, where U^i denotes the interior of U. Define

$$g(x) = \begin{cases} -\langle x^*, x \rangle & \text{if } x \in U, \\ \infty & \text{if } x \notin U. \end{cases}$$

Clearly, $g : X \to (-\infty, \infty]$ is a lower semicontinuous convex function, bounded below. Set $x_0 = 0$, $\varepsilon = 1$. Then

$$\varepsilon = 1 > \alpha \geq g(x_0) - \inf g(X).$$

By hypothesis, for $\lambda = 1$, there exist $\varphi \in \Phi_2$ and $v \in X$ such that $q_B(v - x_0) = q_B(v) \leq 1$ and

$$(18) \qquad g(x) + \frac{1}{\varphi(1)}\,\varphi(q_B(x)) \geq g(v) + \frac{1}{\varphi(1)}\,\varphi(q_B(v)), \quad x \in X.$$

It is also clear that $x \mapsto \frac{1}{\varphi(1)}\varphi(q_B(x))$ is a lower semicontinuous convex function, bounded below, and g is continuous at $x = 0$. By (18) and the Moreau–Rockafeller theorem (for example, see [18, Theorem 47.B, pp. 389–390]),

$$(19) \qquad 0 \in \partial\left(g + \frac{1}{\varphi(1)}\,\varphi(q_B(\cdot))\right)\Big|_v \subset \partial g|_v + \frac{1}{\varphi(1)}\,\partial\varphi(q_B(\cdot))|_v.$$

Let $z^* \in \partial g|_v$. Then for any $x \in X$,

$$\langle z^*, x \rangle - \langle z^*, v \rangle \leq g(x) - g(v).$$

In particular, when $x \in U$, we have

$$\langle z^*, x \rangle - \langle z^*, v \rangle \leq -\langle x^*, x \rangle + \langle x^*, v \rangle.$$

That is,

$$(20) \qquad \langle z^* + x^*, x - v \rangle \leq 0, \quad \forall x \in U.$$

Since $q_B(v) \leq 1$, we have $v \in B$, which implies that $v \in U^i$. Thus $U - v$ is a 0-neighborhood in X. Since the 0-neighborhood $U - v$ contains a balanced 0-neighborhood, combining this and (20), we conclude that $z^* + x^* = 0$, i.e. $z^* = -x^*$. Hence $\partial g|_v = \{-x^*\}$. Thus (19) becomes

$$0 \in \{-x^*\} + \frac{1}{\varphi(1)}\,\partial\varphi(q_B(\cdot))|_v.$$

From this,

$$x^* \in \frac{1}{\varphi(1)}\,\partial\varphi(q_B(\cdot))|_v.$$

Hence for any $t > 0$, we have

$$\langle x^*, tb \rangle = \langle x^*, v + tb \rangle - \langle x^*, v \rangle \leq \frac{1}{\varphi(1)}\left[\varphi(q_B(v + tb)) - \varphi(q_B(v))\right].$$

Thus

(21) $$\langle x^*, b \rangle \leq \frac{1}{\varphi(1)} \cdot \frac{\varphi(q_B(v + tb)) - \varphi(q_B(v))}{t}.$$

If $q_B(v) = 0$, then $v = 0$ since B is bounded. Thus (21) becomes

$$\langle x^*, b \rangle \leq \frac{1}{\varphi(1)} \cdot \frac{\varphi(q_B(tb)) - \varphi(q_B(0))}{t}$$

$$= \frac{1}{\varphi(1)} \cdot \frac{\varphi(q_B(tb))}{t} = \frac{1}{\varphi(1)} \cdot \frac{\varphi(q_B(tb))}{q_B(tb)} \cdot q_B(b).$$

If $t \to 0^+$, then

$$\frac{\varphi(q_B(tb))}{q_B(tb)} \to \varphi'(0) = 0.$$

Thus we have

$$\langle x^*, b \rangle \leq \frac{1}{\varphi(1)} \varphi'(0) q_B(b) = 0,$$

which contradicts the assumption that $\langle x^*, b \rangle > 0$. Hence we conclude that $q_B(v) > 0$. For any $y \in B$, if there exists $t_1 > 0$ such that $q_B(v + t_1 y) = q_B(v)$, then

$$\langle x^*, y \rangle = \frac{\langle x^*, t_1 y + v \rangle - \langle x^*, v \rangle}{t_1} \leq \frac{1}{\varphi(1)} \cdot \frac{\varphi(q_B(v + t_1 y)) - \varphi(q_B(v))}{t_1} = 0.$$

If not, we have

$$\langle x^*, y \rangle \leq \frac{1}{\varphi(1)} \cdot \frac{\varphi(q_B(v + ty)) - \varphi(q_B(v))}{t}$$

$$\leq \frac{1}{\varphi(1)} \cdot \frac{\varphi(q_B(v + ty)) - \varphi(q_B(v))}{q_B(v + ty) - q_B(v)} \cdot q_B(y)$$

$$\leq \frac{1}{\varphi(1)} \cdot \frac{\varphi(q_B(v + ty)) - \varphi(q_B(v))}{q_B(v + ty) - q_B(v)}$$

$$\to \frac{1}{\varphi(1)} \varphi'(q_B(v)) \quad \text{as } t \to 0^+.$$

In any case, we have

(22) $$\langle x^*, y \rangle \leq \frac{1}{\varphi(1)} \varphi'(q_B(v)), \quad \forall y \in B.$$

On the other hand, for $0 < t < 1$, we have

$$\langle x^*, -v \rangle \leq \frac{1}{\varphi(1)} \cdot \frac{\varphi(q_B(v - tv)) - \varphi(q_B(v))}{t}$$

$$= \frac{1}{\varphi(1)} \cdot \frac{\varphi(q_B(v - tv)) - \varphi(q_B(v))}{q_B(v - tv) - q_B(v)} \cdot (-q_B(v)).$$

Letting $t \to 0^+$, we obtain

$$\langle x^*, v \rangle \geq \frac{\varphi'(q_B(v))}{\varphi(1)} \cdot q_B(v).$$

Put $u = v/q_B(v)$. Then $u \in B$ and

$$\langle x^*, u \rangle \geq \frac{\varphi'(q_B(v))}{\varphi(1)}.$$

Combining the above with (22), we conclude that $\langle x^*, u \rangle = \sup\langle x^*, B \rangle$. ∎

THEOREM 4.2. *Let X be a locally convex space and $B \subset X$ be a weakly compact convex set with $0 \in B$. Then for any lower semicontinuous convex function $g : X \to (-\infty, \infty]$ bounded below, any $x_0 \in \operatorname{dom} g$, any $\varepsilon > g(x_0) - \inf g(X)$, any $\varphi \in \Phi_2$ and any $\lambda > 0$, there exists $v \in X$ such that $q_B(v - x_0) \leq \lambda$ and*

$$g(x) + \frac{\varepsilon}{\varphi(\lambda)}\, \varphi(q_B(x - x_0)) \geq g(v) + \frac{\varepsilon}{\varphi(\lambda)}\, \varphi(q_B(v - x_0)) \quad \textit{for all } x \in X.$$

Proof. It is easy to verify that $x \mapsto g(x) + \frac{\varepsilon}{\varphi(\lambda)}\varphi(q_B(x - x_0))$ is a lower semicontinuous convex function. Since $x_0 + \lambda B$ is weakly compact, there exists $v \in x_0 + \lambda B$ such that

$$g(x) + \frac{\varepsilon}{\varphi(\lambda)}\, \varphi(q_B(x - x_0)) \geq g(v) + \frac{\varepsilon}{\varphi(\lambda)}\, \varphi(q_B(v - x_0)), \quad \forall x \in x_0 + \lambda B.$$

When $x \notin x_0 + \lambda B$, we have $q_B(x - x_0) > \lambda$. For those x, we have

$$g(x) + \frac{\varepsilon}{\varphi(\lambda)}\, \varphi(q_B(x - x_0)) > \inf g(E) + \varepsilon > g(x_0) \geq g(v) + \frac{\varepsilon}{\varphi(\lambda)}\, \varphi(q_B(v - x_0)).$$

Thus we see that there exists $v \in X$ such that $q_B(v - x_0) \leq \lambda$ and for every $x \in X$,

$$g(x) + \frac{\varepsilon}{\varphi(\lambda)}\, \varphi(q_B(x - x_0)) \geq g(v) + \frac{\varepsilon}{\varphi(\lambda)}\, \varphi(q_B(v - x_0)). \quad ∎$$

For convenience, if a bounded closed convex set B with $0 \in B$ satisfies the hypothesis in Theorem 4.1, i.e. the perturbation $\theta(x)$ in the Borwein–Preiss principle with respect to q_B only consists of a single summand, then B is said to have the *simplified Borwein–Preiss principle property* (briefly, the *s.b.p. property*). The notion can be extended to general bounded closed convex sets. A bounded closed convex set B is said to have the s.b.p. property if for some $b \in B$, $B - b$ has the s.b.p. property.

For a locally convex space X, we denote by $(X, \tau(X, X^*))$ the associated Mackey space ([5, pp. 260–262]). By the James theorem (see [2, p. 77] or [4]), for any $\tau(X, X^*)$-complete, bounded closed convex set B, every $x^* \in X^*$ attains its supremum on B if and only if B is weakly compact. Thus in a quasi-complete locally convex space (i.e., one where each

bounded closed set is complete, see for example [5, p. 210]), a bounded closed convex set has the s.b.p. property if and only if it is weakly compact.

This prompts us to recall another property of bounded closed convex sets, namely the drop property. Since Rolewicz [17] began the study of the drop property for closed unit balls of Banach spaces, various drop properties have been introduced and studied. For example, Giles and Kutzarova [3] defined a bounded closed convex set B to have the *weak drop property* if for every weakly sequentially closed set A disjoint from B there exists a point $x_0 \in A$ such that $D(x_0, B) \cap A = \{x_0\}$, where $D(x_0, B)$ denotes the convex hull of $\{x_0\} \cup B$. In [12], we defined a bounded closed convex set B to have the *quasi-weak drop property* if for any weakly closed set A disjoint from B there exists $x_0 \in A$ such that $D(x_0, B) \cap A = \{x_0\}$. Later we investigated the relationship between the quasi-weak drop property and weak compactness in the framework of locally convex spaces (for details, see [13]). Now, combining Theorem 4.1, Theorem 4.2 and [13, Theorem 3.2] we have the following.

THEOREM 4.3. *Let X be a quasi-complete locally convex space* (*or, let $(X, \tau(X, X^*))$ be quasi-complete*)*, and $B \subset X$ be a bounded closed convex set. Then the following statements concerning B are equivalent*:

   (i) *B is weakly compact.*
  (ii) *For any $x^* \in X^*$, there exists $u \in B$ such that $\langle x^*, u \rangle = \sup\langle x^*, B \rangle$.*
 (iii) *B has the s.b.p. property.*
 (iv) *B has the quasi-weak drop property.*

From Theorem 4.3 we can deduce the following equivalent descriptions of semireflexivity and reflexivity of locally convex spaces (concerning semireflexive, reflexive and quasi-barrelled spaces, we refer to [5, §23]).

THEOREM 4.4. *Let X be a quasi-complete locally convex space. Then the following statements are equivalent*:

   (i) *X is semireflexive.*
  (ii) *Every bounded closed convex set in X has the quasi-weak drop property.*
 (iii) *Every bounded closed convex set in X has the s.b.p. property.*

THEOREM 4.5. *Let X be a quasi-complete quasi-barrelled space. Then the following statements are equivalent*:

   (i) *X is reflexive.*
  (ii) *Every bounded closed convex set in X has the quasi-weak drop property.*
 (iii) *Every bounded closed convex set in X has the s.b.p. property.*

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