

Poisson kernel and Green function of balls for complex hyperbolic Brownian motion

by

TOMASZ ŻAK (Wrocław)

Abstract. The aim of this paper is to give a description of the Poisson kernel and the Green function of balls in the complex hyperbolic space. The description is in terms of the hypergeometric function and unitary spherical harmonics in \mathbb{C}^n .

1. Introduction. The Poisson kernel and Green function are the main objects in potential theory. Even in the classical context, i.e. for the Laplace operator Δ in \mathbb{R}^n , explicit formulas for the Poisson kernel $P_D(x, y)$ and the Green function $G_D(x, y)$ are known only for a few classes of sets D , including balls and half-spaces. Our main goal is to describe the Poisson kernel and the Green function of balls B_r , $0 < r < 1$, in the complex hyperbolic space, that is, in the unit ball in \mathbb{C}^n equipped with the Bergman metric and the Laplace–Beltrami operator Δ_{LB} . In our computations we use stochastic analysis, hence we first recall the probabilistic interpretation of the Poisson kernel and the Green function.

By the general theory, the Laplace–Beltrami operator Δ_{LB} in the unit ball of \mathbb{C}^n is the generator of a diffusion $(X_t)_{t \geq 0}$ with trajectories in this ball. We call this process the *complex hyperbolic Brownian motion*. For $r \in (0, 1)$ let $B_r = \{z \in \mathbb{C}^n : |z| < r\}$ and let $\tau_r = \inf\{t > 0 : |X_t| = r\}$ be the first time the process $(X_t)_{t \geq 0}$, starting from $x \in B_r$, exits from B_r . Then the distribution of X_{τ_r} is the Poisson kernel for the ball B_r . We denote it by $P_r(x, y)$. Next consider $(X_t^{B_r})_{t \geq 0}$, the process starting from $x \in B_r$ and killed on exiting B_r . Let $p_{B_r}(t, x, y)$ be the density of its transition probability. Then $G_r(x, y) = \int_0^\infty p_{B_r}(t, x, y) dt$ is the Green function of the ball B_r . Observe that Δ_{LB} commutes with the action of the unitary group $U(n)$ ([R, Thm. 4.1.2]). This implies that if $(X_t)_{t \geq 0}$ starts from $x \in B_r$ then both

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$P_r(x, y)$ and $G_r(x, y)$, as functions of y , are invariant with respect to those $g \in U(n)$ that leave x fixed. This property allows us to use unitary spherical harmonics to examine $P_r(x, y)$ and $G_r(x, y)$. We will provide formulas for $P_r(x, y)$ and $G_r(x, y)$ in terms of the hypergeometric function ${}_2F_1$ and complex unitary spherical harmonics.

In the case of real hyperbolic Brownian motion such formulas have recently been found by Byczkowski and Małecko [BM]. For the complex hyperbolic space the Poisson kernel of the whole space (known as the Poisson–Szegő kernel for the unit ball) is well-known ([Kr], [R], [S]). In 1975 Folland [F] gave a description of this kernel in terms of spherical harmonics. The problem of finding the Green function is much harder—knowing the Green function we can easily compute the Poisson kernel by differentiation in the normal direction (cf. Theorem 8, p. 174 in [Ch] and Lemma 11 in the Appendix). The potential of the process $(X_t)_{t \geq 0}$ was computed in 2001 by Matsumoto [M]. Matsumoto computed the λ -potential, that is, $G^\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_t(x, y) dt$ for all $\lambda \geq 0$, but, as he wrote (p. 554 of [M]), “in order to use stochastic analysis, we adopt the upper half-space realizations of the hyperbolic spaces”. In this paper we prefer the ball realization of the complex hyperbolic space because in this case the hyperbolic Brownian motion is invariant with respect to the group $U(n)$ of unitary transformations and we can use spherical harmonics. As we will show, in this model it is also possible to use stochastic analysis.

2. Preliminaries. Consider $\mathbb{C}^n = \{z = (z_1, \dots, z_n) : z_i \in \mathbb{C}, i = 1, \dots, n\}$ with the Hermitian scalar product: for $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$,

$$\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j.$$

Then $|z|^2 = \langle z, z \rangle = \sum_{j=1}^n |z_j|^2$.

A. Complex hyperbolic space and its isometries. Let $B_1 = \{z \in \mathbb{C}^n : |z| < 1\}$ denote the unit ball of \mathbb{C}^n . This set, equipped with the Bergman metric, is a model of a complex hyperbolic space. The Bergman metric is induced by the form $h = -4\partial\bar{\partial} \log K(z)$ with $K(z) = 1 - |z|^2$ (cf. [Kr] or [S]), which means that the metric is given by the matrix $[h_{ij}]$, where for $i, j = 1, \dots, n$,

$$h_{ij} = \frac{(1 - |z|^2)\delta_{ij} + \bar{z}_i z_j}{(1 - |z|^2)^2}.$$

Consider the Laplace–Beltrami operator associated to the Bergman metric on B_1 (we choose a normalization as in [R])

$$\Delta_{\text{LB}} = 4(1 - |z|^2) \sum_{j,k=1}^n (\delta_{jk} - z_j \bar{z}_k) \frac{\partial^2}{\partial \bar{z}_k \partial z_j}.$$

If $n = 1$ then $\Delta_{LB} = (1 - |z|^2)^2 \Delta$, where Δ is the ordinary Laplace operator, and the Bergman metric is the Poincaré metric. In this case the Δ_{LB} -harmonic functions coincide with the functions harmonic in the ordinary sense (see [R]). Henceforth we assume $n \geq 2$.

All transformations of B_1 that are isometries of the Bergman metric are described in [R]. Fix $a \in \mathbb{C}^n$, $a \neq 0$, and denote by P_a the orthogonal projection of \mathbb{C}^n on the complex line $\text{lin}[a]$ and by $Q_a = I - P_a$ the orthogonal projection on the orthogonal complement to $\text{lin}[a]$. More precisely, for $z \in \mathbb{C}^n$ put

$$P_a z = \frac{\langle z, a \rangle}{\langle a, a \rangle} a \quad \text{and} \quad \psi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle},$$

where $s_a = (1 - |a|^2)^{1/2}$. If ψ is any isometry of B_1 and $a = \psi^{-1}(0)$, then there exists $g \in U(n)$ such that $\psi = g\psi_a$ (Theorem 2.2.5 in [R]).

B. Hypergeometric function. For $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$, and $|z| < 1$ the *hypergeometric function* ${}_2F_1(a, b; c; z)$ is defined by the following power series ([E]):

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k,$$

where $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1)$ for $k = 1, 2, \dots$ is the Pochhammer symbol. Observe that the roles of a and b are symmetric: ${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z)$. The function is also defined for $c = 0, -1, -2, \dots$ provided a (or b) is also a negative integer and $c \leq a < 0$. Namely, if a is a negative integer, $a = -m$, then for $c = -m - l$ and $l = 0, 1, 2, \dots$,

$${}_2F_1(-m, b; -m - l; z) = \sum_{k=0}^m \frac{(-m)_k (b)_k}{(-m - l)_k k!} z^k.$$

Let $n \geq 2$ be an integer and let p, q be nonnegative integers. We will use the hypergeometric function for two sets of parameters: $a = p, b = q, c = p + q + n$, and $a = 1 - n - q, b = 1 - n - p, c = 2 - n - p - q$. In the first case the hypergeometric function is well defined for all $z \in \mathbb{C}$ with $|z| \leq 1$. In the second case it is a polynomial, provided $q > 0$ (or $p > 0$), because then $2 - n - p - q < 1 - n - q$. Thus we have to define ${}_2F_1(1 - n - q, 1 - n - p; 2 - n - p - q; x)$ for $p = q = 0$. In that case we additionally put ([E])

$${}_2F_1(1 - n, 1 - n; 2 - n; z) = \sum_{k=0}^{n-2} \frac{(-1)^{n-k}}{n - 1 - k} \binom{n - 1}{k} z^k + z^{n-1} \ln z.$$

If $\Re c > \Re b > 0$ and $\Re(c - a - b) > 0$, then the Gauss ${}_2F_1$ identity states that

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$

Moreover, if $c < b < a < 0$ are integers, the series defining ${}_2F_1(a, b; c; z)$ terminates and an analogue of the Gauss identity holds true (cf. [V]).

Differentiation of the hypergeometric function gives again the hypergeometric function

$$\frac{d^k}{dx^k} {}_2F_1(a, b; c; z) = \frac{(a)_k(b)_k}{(c)_k} {}_2F_1(a + k, b + k; c + k; z)$$

and one of Kummer's relations (see [E, 2.9]) gives the identity

$${}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} F(c - a, c - b; c; z).$$

The function $y(z) = {}_2F_1(a, b; c; z)$ satisfies the hypergeometric equation

$$(1) \quad z(1 - z)y''(z) + (c - (a + b + 1)z)y'(z) - aby(z) = 0.$$

For $a = p, b = q, c = n + p + q$, the general solution of (1) is given by the formula ([E])

$$y(z) = c_1 \cdot {}_2F_1(a, b; c; z) + \frac{c_2}{z^{1-c}} \cdot {}_2F_1(1 + a - c, 1 + b - c; 2 - c; z).$$

C. Jacobi polynomials. For $\alpha, \beta > -1$ and $k = 0, 1, 2, \dots$ the Jacobi polynomial of degree k associated to (α, β) is given by

$$P_k^{\alpha, \beta}(x) = \frac{(-1)^k (1 - x)^{-\alpha} (1 + x)^{-\beta}}{k! 2^k} \frac{d^k}{dx^k} [(1 - x)^{\alpha+k} (1 + x)^{\beta+k}]$$

with

$$P_k^{\alpha, \beta}(1) = \frac{\Gamma(\alpha + k + 1)}{k! \Gamma(\alpha + 1)} = \binom{k + \alpha}{k}.$$

The Jacobi polynomials satisfy the Jacobi equation: for $-1 \leq x \leq 1$,

$$(2) \quad (1 - x)^2 y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x] y'(x) + k(k + \alpha + \beta + 1) y(x) = 0.$$

For fixed α, β they form an orthogonal basis in $L^2([-1, 1], (1 - x)^\alpha (1 + x)^\beta dx)$:

$$\begin{aligned} & \int_{-1}^1 P_k^{\alpha, \beta}(x) P_m^{\alpha, \beta}(x) (1 - x)^\alpha (1 + x)^\beta dx \\ &= \delta_{km} \frac{2^{\alpha+\beta+1} \Gamma(\alpha + k + 1) \Gamma(\beta + k + 1)}{k! (k + \alpha + \beta + 2k + 1) \Gamma(\alpha + \beta + k + 1)}. \end{aligned}$$

D. Spherical harmonics in \mathbb{C}^n . We can identify the set \mathbb{C}^n with \mathbb{R}^{2n} . Then $S_1 = \partial B_1$, the unit sphere in \mathbb{C}^n , coincides with the unit sphere in \mathbb{R}^{2n} . Fix n and let σ_1 denote the surface measure on the unit sphere in \mathbb{R}^{2n} . We denote by $\omega_{2n-1} = 2\pi^n / \Gamma(n)$ the area of the unit sphere in \mathbb{R}^{2n} .

Consider the unitary group $U(n)$ acting on S_1 . The subgroup of unitary transformations which leave one point of S_1 fixed is $U(n - 1)$, and the sphere

can be identified with the homogeneous space $U(n)/U(n-1)$. Functions on S_1 invariant under $U(n-1)$ are called *zonal functions*. It turns out ([K], see also Proposition 1 in [F]) that there exists an orthogonal decomposition of the function space $L^2(S_1, \sigma_1)$ into subspaces $\mathcal{H}^{p,q}$, $p, q = 0, 1, 2, \dots$, which are invariant and irreducible under $U(n)$. The functions belonging to $\mathcal{H}^{p,q}$ are called *spherical harmonics*. The *zonal harmonics* $H_n^{p,q} \in \mathcal{H}^{p,q}$ are the following: for $z = Re^{i\theta} \in \mathbb{C}$,

$$(3) \quad H_n^{p,q}(Re^{i\theta}) = \frac{D(p, q, n)}{\omega_{2n-1} P_{p \wedge q}^{n-2, |p-q|}(1)} e^{i(p-q)\theta} R^{|p-q|} P_{p \wedge q}^{n-2, |p-q|}(2R^2 - 1),$$

where $P_{p \wedge q}^{n-2, |p-q|}(2R^2 - 1)$ is the appropriate Jacobi polynomial, $p \wedge q = \min(p, q)$ and

$$D(p, q, n) = \frac{(p + q + n - 1)(p + n - 2)!(q + n - 2)!}{p!q!(n - 1)!(n - 2)!}$$

is the dimension of the space $\mathcal{H}^{p,q}$. Observe also that $\overline{H_n^{p,q}(Re^{i\theta})} = H_n^{q,p}(Re^{i\theta})$ and $H_n^{p,q}(1) = H_n^{q,p}(1) = D(p, q, n)\omega_{2n-1}^{-1}$.

It is also well-known (cf. e.g. [K]) that $(H_n^{p,q})_{p,q=0}^\infty$ is an orthogonal basis in $L^2(\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}, (1 - x^2 - y^2)^{n-2} dx dy)$.

The orthogonal projections from $L^2(S_1, \sigma_1)$ onto $\mathcal{H}^{p,q}$ were described by Koornwinder [K]; the formulation below is taken from [F]:

PROPOSITION 1. *If $f_1, \dots, f_{D(p,q,n)}$ is any orthonormal basis for $\mathcal{H}^{p,q} \subset L^2(S_1, \sigma_1)$, then for $x, y \in S_1$,*

$$\sum_{j=1}^{D(p,q,n)} f_j(x) \overline{f_j(y)} = H_n^{p,q}(\langle x, y \rangle).$$

Consequently, the orthogonal projection $\pi_{p,q} : L^2(S_1, \sigma_1) \rightarrow \mathcal{H}^{p,q}$ is given by

$$(4) \quad \pi_{p,q} f(x) = \int_{S_1} H_n^{p,q}(\langle x, y \rangle) f(y) d\sigma_1(y).$$

Fix $x \in B_1$, $x \neq 0$, and let $a = Re^{i\theta}$ with $R \leq 1$. Put $S_{x,a} = \{z \in S_1 : \langle z, x \rangle / (|z||x|) = a\}$. Observe that $S_{x,a}$ is a sphere in \mathbb{C}^{n-1} . In this way we decompose the unit sphere as $S_1 = D \times S_{x,a}$, where D is the unit disc in the complex plane.

If $g_x \in U(n)$ is such that $g_x x = x$, then of course $g_x(S_{x,a}) = S_{x,a}$. Let $\sigma_{x,a}$ denote the spherical measure on $S_{x,a} \subset \mathbb{R}^{2n-2}$. The above decomposition of S_1 induces a decomposition of σ_1 : if $z = (a, z_a)$ with $a = Re^{i\theta}$ and $z_a \in S_{x,a}$, then

$$d\sigma_1(z) = d\sigma_{x,a}(z_a) R(1 - R^2)^{n-2} dR d\theta.$$

Let f_x be a function defined on S_1 , continuous and invariant under those $g_x \in U(n)$ which leave x fixed. Such functions f_x are uniquely determined

by their expansions with respect to $(H_n^{p,q})_{p,q=0}^\infty$. Indeed, if in view of (4) we define the (p, q) -coefficient of f_x as

$$(\widehat{f_x})_{p,q} = \int_{S_1} H_n^{p,q}(\langle x, z \rangle) f_x(z) d\sigma_1(z) = \int_{S_1} H_n^{q,p}(\langle z, x \rangle) f_x(z) d\sigma_1(z),$$

then, by the above decomposition of the unit sphere, we have

$$\begin{aligned} (\widehat{f_x})_{p,q} &= \int_D \left(\int_{S_{x,a}} H_n^{p,q}(\langle x, z_a \rangle) f_x((a, z_a)) d\sigma_{x,a}(z_a) \right) R(1 - R^2)^{n-2} dR d\theta \\ &= \int_D H_n^{p,q}(Re^{i\theta}) \widetilde{f_x}(Re^{i\theta}) R(1 - R^2)^{n-2} dR d\theta, \end{aligned}$$

where $\widetilde{f_x}(Re^{i\theta}) = \widetilde{f_x}(a) = f_x(a, z_a)$ for some (and hence all) $z_a \in S_{x,a}$. The family of harmonics $(H_n^{p,q}(Re^{i\theta}))_{p,q=0,1,2,\dots}$ is an orthogonal basis in $L^2(D, R(1 - R^2)^{n-2} dR d\theta)$, hence the coefficients $(\widehat{f_x})_{p,q}$, $p, q = 0, 1, 2, \dots$, uniquely determine the values of f_x on every $S_{x,a}$, whence on the whole sphere S_1 .

If f_x is defined on the sphere $S_r = \{z \in \mathbb{C}^n : |z| = r\}$ and σ_r denotes the spherical measure on S_r , then we define

$$\begin{aligned} (5) \quad (\widehat{f_x})_{p,q} &= \int_{S_r} H_n^{p,q} \left(\frac{\langle x, z \rangle}{|x||z|} \right) f_x(z) d\sigma_r(z) \\ &= \int_{S_r} H_n^{q,p} \left(\frac{\langle z, x \rangle}{|z||x|} \right) f_x(z) d\sigma_r(z). \end{aligned}$$

E. Diffusion: relation between its generator and a stochastic differential equation. The following fact is well-known. The proof can be found in many textbooks, e.g. [RW]. Let $(X_t)_{t \geq 0}$ be a real-valued diffusion with the following generator: for f twice differentiable,

$$Af(x) = \sigma(x)f''(x) + b(x)f'(x),$$

where σ, b are continuous functions and $\sigma > 0$ is locally Lipschitz. Then (X_t) satisfies the stochastic differential equation

$$dX_t = \sqrt{\sigma(X_t)} dW_t + b(X_t) dt,$$

where $(W_t)_{t \geq 0}$ is a real-valued Brownian motion with $\mathbb{E}(W_t) = 2t$.

3. The process $Y_t = \langle X_t, a \rangle / (|X_t||a|)$ and its polar decomposition.

Fix a nonzero point $a \in B_1$ and consider the isometry ψ_a of B_1 equipped with the Bergman metric. From the description of such isometries in item A of Section 2 we know that, modulo unitary transformations, ψ_a is a combination of two orthogonal projections. If we consider $\psi_a(X_t)$, it turns out that the

projection

$$P_a(X_t) = \frac{\langle X_t, a \rangle}{\langle a, a \rangle} a = \left\langle X_t, \frac{a}{|a|} \right\rangle \cdot \frac{a}{|a|}$$

contains essential information about the behaviour of the process (X_t) .

More precisely: consider the complex hyperbolic Brownian motion $(X_t)_{t \geq 0}$ starting from the point $a \in B_1, a \neq 0$. With probability one, this process does not visit zero. Indeed, consider the projection of X_t onto $\{z \in \mathbb{C}^n : z = (z_1, 0, \dots, 0)\}$. Computing the action of Δ_{LB} on functions $f(z_1, 0, \dots, 0) = g(z_1)$, depending on z_1 only, we easily see that the generator of this two-dimensional process is $(1 - |z|^2)(1 - |z_1|^2)\Delta_{\mathbb{R}^2}$, hence the process is a time-changed planar Brownian motion. With probability one, the planar Brownian motion does not visit zero, hence neither does $(X_t)_{t \geq 0}$. This implies that the process

$$Y_t = \frac{\langle X_t, a \rangle}{|X_t| |a|}$$

is well-defined. Observe that Y_t is a complex-valued process with values in the unit disc of the complex plane, hence we can decompose it using polar coordinates:

$$Y_t = \frac{\langle X_t, a \rangle}{|X_t| |a|} = R_t e^{i\theta_t},$$

where

$$(6) \quad R_t = \frac{|\langle X_t, a \rangle|}{|X_t| |a|}$$

is the length of the projection of the vector $X_t/|X_t|$ onto $\text{lin}[a]$, and

$$(7) \quad e^{i\theta_t} = \frac{\langle X_t, a \rangle}{|\langle X_t, a \rangle|}$$

measures the angle (on the plane $\text{lin}[a]$) between this projection and a .

4. The radial process $(r_t)_{t \geq 0}$ and its generator. We will need one more process, the length of X_t . As the norm is not differentiable at zero, instead of $|X_t|$ we will consider its square:

$$r_t = |X_t|^2.$$

For $z \in \mathbb{C}^n$ put $r = |z|^2 = \sum_{j=1}^n z_j \bar{z}_j$. To compute the generator of the process $(r_t)_{t \geq 0}$ it is enough to compute $\Delta_{LB} f(r)$ for twice differentiable real functions f . Because

$$\frac{\partial r}{\partial \bar{z}_k} = z_k, \quad \frac{\partial r}{\partial z_j} = \bar{z}_j, \quad \frac{\partial^2 r}{\partial \bar{z}_k \partial z_j} = \delta_{jk},$$

for any function $f : (0, \infty) \rightarrow \mathbb{R}$ with two continuous derivatives we have

$$\begin{aligned}
 (8) \quad \Delta_{\text{LB}}f(r) &= 4(1 - |z|^2) \cdot \sum_{j,k=1}^n (\delta_{jk} - z_j\bar{z}_k) \frac{\partial^2 f(r)}{\partial \bar{z}_k \partial z_j} \\
 &= 4(1 - |z|^2) \cdot \sum_{j,k=1}^n (\delta_{jk} - z_j\bar{z}_k) \left(f''(r) \frac{\partial r}{\partial \bar{z}_k} \cdot \frac{\partial r}{\partial z_j} + f'(r) \frac{\partial^2 r}{\partial \bar{z}_k \partial z_j} \right) \\
 &= 4r(1 - r)^2 f''(r) + 4(1 - r)(n - r)f'(r).
 \end{aligned}$$

This implies that the process $(r_t)_{t \geq 0}$ is a solution of the stochastic differential equation

$$(9) \quad dr_t = 2\sqrt{r_t}(1 - r_t) dW_t^{(1)} + 4(1 - r_t)(n - r_t) dt,$$

where $(W_t^{(1)})_{t \geq 0}$ is a real-valued Brownian motion normalized in such a way that $\mathbb{E}(W_t^{(1)}) = 2t$.

5. Generators of the processes $(R_t)_{t \geq 0}$ and $(\theta_t)_{t \geq 0}$. Fix $a \neq 0$ and put

$$A = \langle a, z \rangle = \sum_{j=1}^n a_j \bar{z}_j, \quad \bar{A} = \langle z, a \rangle = \sum_{j=1}^n z_j \bar{a}_j.$$

By the formulas (6) and (7) we have

$$R = \frac{|\langle z, a \rangle|}{|z||a|} = \frac{|A|}{|a||z|} \quad \text{and} \quad e^{i\theta} = \frac{\langle z, a \rangle}{|\langle z, a \rangle|} = \frac{\langle z, a \rangle}{\sqrt{\langle a, z \rangle \langle z, a \rangle}} = \frac{\bar{A}}{\sqrt{A\bar{A}}},$$

hence $\theta = -i \ln(\bar{A}/\sqrt{A\bar{A}})$, where $\ln z$ denotes the principal branch of the logarithm, i.e. the continuous branch with $\ln 1 = 0$.

We will need the partial derivatives of these functions:

$$(10) \quad \frac{\partial \theta}{\partial \bar{z}_k} = \frac{ia_k}{2A}, \quad \frac{\partial \theta}{\partial z_j} = \frac{-i\bar{a}_j}{2\bar{A}}, \quad \frac{\partial^2 \theta}{\partial \bar{z}_k \partial z_j} = 0,$$

$$(11) \quad \frac{\partial R}{\partial \bar{z}_k} = \frac{a_k \bar{A}}{2|a||z||A|} - \frac{z_k |A|}{2|a||z|^3}, \quad \frac{\partial R}{\partial z_j} = \frac{\bar{a}_j A}{2|a||z||A|} - \frac{\bar{z}_j |A|}{2|a||z|^3}$$

and

$$\begin{aligned}
 (12) \quad \frac{\partial^2 R}{\partial \bar{z}_k \partial z_j} &= \frac{a_k \bar{a}_j}{4|a||z||A|} - \frac{a_k \bar{z}_j \bar{A}}{4|a||z|^3|A|} \\
 &\quad - \frac{\delta_{jk}|A|}{2|a||z|^3} - \frac{\bar{a}_j z_k A}{4|a||z|^3|A|} + \frac{3z_k \bar{z}_j |A|}{4|a||z|^5}.
 \end{aligned}$$

Let f be a twice differentiable real function. We compute the action of Δ_{LB} on $f(R)$:

$$\begin{aligned}
 (13) \quad \Delta_{\text{LB}}f(R) &= 4(1 - |z|^2) \sum_{j,k=1}^n (\delta_{jk} - z_j \bar{z}_k) \frac{\partial^2 f(R)}{\partial \bar{z}_k \partial z_j} \\
 &= 4(1 - |z|^2) \sum_{j,k=1}^n (\delta_{jk} - z_j \bar{z}_k) \left(f''(R) \frac{\partial R}{\partial \bar{z}_k} \frac{\partial R}{\partial z_j} + f'(R) \frac{\partial^2 R}{\partial \bar{z}_k \partial z_j} \right).
 \end{aligned}$$

Using (11), we get

$$\begin{aligned}
 (14) \quad 4(1 - |z|^2) \sum_{j,k=1}^n (\delta_{jk} - z_j \bar{z}_k) \frac{\partial R}{\partial \bar{z}_k} \frac{\partial R}{\partial z_j} &= 4(1 - |z|^2) \sum_{j,k=1}^n (\delta_{jk} - z_j \bar{z}_k) \left(\frac{a_k \bar{A}}{2|a||z||A|} - \frac{z_k |A|}{2|a||z|^3} \right) \\
 &\quad \times \left(\frac{\bar{a}_j A}{2|a||z||A|} - \frac{\bar{z}_j |A|}{2|a||z|^3} \right) \\
 &= \frac{1 - |z|^2}{|z|^2} \left(1 - \frac{|A|^2}{|a|^2 |z|^2} \right) = \frac{1 - |z|^2}{|z|^2} (1 - R^2)
 \end{aligned}$$

and using (12) we obtain

$$\begin{aligned}
 (15) \quad 4(1 - |z|^2) \sum_{j,k=1}^n (\delta_{jk} - z_j \bar{z}_k) \frac{\partial^2 R}{\partial \bar{z}_k \partial z_j} &= 4(1 - |z|^2) \sum_{j,k=1}^n (\delta_{jk} - z_j \bar{z}_k) \left(\frac{a_k \bar{a}_j}{4|a||z||A|} - \frac{a_k \bar{z}_j \bar{A}}{4|a||z|^3|A|} \right. \\
 &\quad \left. - \frac{\delta_{jk} |A|}{2|a||z|^3} - \frac{\bar{a}_j z_k A}{4|a||z|^3|A|} + \frac{3z_k \bar{z}_j |A|}{4|a||z|^5} \right) \\
 &= \frac{1 - |z|^2}{|z|^2} \left(\frac{1}{R} - (2n - 1)R \right),
 \end{aligned}$$

hence

$$(16) \quad \Delta_{\text{LB}}f(R) = \frac{1 - |z|^2}{|z|^2} \left((1 - R^2) f''(R) + \left(\frac{1}{R} - (2n - 1)R \right) f'(R) \right).$$

This means that the real-valued process R_t is a diffusion, described by the stochastic equation

$$(17) \quad dR_t = \sqrt{\frac{1 - r_t}{r_t}} \sqrt{1 - R_t^2} dW_t^{(2)} + \frac{1 - r_t}{r_t} \left(\frac{1}{R_t} - (2n - 1)R_t \right) dt.$$

In a similar way, computing the action of Δ_{LB} on $g(\theta)$ and using (10), we find that

$$(18) \quad \Delta_{\text{LB}}g(\theta) = \frac{1 - |z|^2}{|z|^2} \left(\frac{1}{R^2} - |z|^2 \right) g''(\theta),$$

hence

$$(19) \quad d\theta_t = \sqrt{\frac{1 - r_t}{r_t}} \sqrt{\frac{1}{R_t^2} - r_t} dW_t^{(3)}.$$

6. Independence of $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$. Now we will show that the Brownian motions $W_t^{(1)}$, $W_t^{(2)}$ and $W_t^{(3)}$ can be chosen to be independent. We are interested in the martingale parts of the processes r_t , R_t and θ_t ; for convenience denote them by superscript m. For instance, $(r_t)^m$ will denote the martingale part of r_t , i.e. the stochastic integral $\int_0^t 2\sqrt{r_s}(1 - r_s) dW_s^{(1)}$.

The group $U(n)$ acts transitively on the unit sphere, so we may assume that $a = (a_1, 0, \dots, 0)$. Then (6), (7) and the definition of r_t imply that $X_1(t)$, the first coordinate of the hyperbolic Brownian motion $X(t)$, is $\sqrt{r_t}R_t e^{i\theta_t}$. First we will show the independence of $W_t^{(1)}$ and $W_t^{(2)}$, the Brownian motions from formulas (9) and (17).

On the one hand, as shown in Section 3, the process $X_1(t)$ has the generator

$$(20) \quad (1 - r)(1 - R^2 r) \Delta_{\mathbb{R}^2}.$$

This implies that the process $|X_1(t)|$, being a time-changed Bessel process of dimension two, has the following bracket:

$$(21) \quad \langle |X_1|, |X_1| \rangle_t = 2(1 - r_t)(1 - R_t^2 r_t).$$

On the other hand, $|X_1(t)| = \sqrt{r_t}R_t$, thus (9) and (17) give another way to compute the bracket of $|X_1(t)|$. By the Itô formula,

$$d(|X_1(t)|)^m = R_t d(\sqrt{r_t})^m + \sqrt{r_t} d(R_t)^m,$$

hence first we have to find the stochastic equation describing the martingale part of the process $\sqrt{r_t}$. We make use of the Itô formula once again to get $d(\sqrt{r_t})^m = (1 - r_t) dW_t^{(1)}$. From this and (17), we get

$$\begin{aligned} d(|X_1(t)|) &= R_t(1 - r_t) dW_t^{(1)} + \sqrt{r_t} d(R_t)^m \\ &= R_t(1 - r_t) dW_t^{(1)} + \sqrt{1 - r_t} \sqrt{1 - R_t^2} dW_t^{(2)}. \end{aligned}$$

We can now compute the bracket of $|X_1(t)|$:

$$\begin{aligned} \langle |X_1|, |X_1| \rangle_t &= 2(R_t^2(1 - r_t)^2 + (\sqrt{1 - r_t})^2(\sqrt{1 - R_t^2})^2) \\ &\quad + R_t \sqrt{1 - R_t^2} (\sqrt{1 - r_t})^3 \langle W^{(1)}, W^{(2)} \rangle_t \\ &= 2(1 - r_t)(1 - r_t R_t^2) + R_t \sqrt{1 - R_t^2} (\sqrt{1 - r_t})^3 \langle W^{(1)}, W^{(2)} \rangle_t. \end{aligned}$$

Comparing this with (21) we see that $\langle W^{(1)}, W^{(2)} \rangle_t = 0$, hence the Brownian motions $W^{(1)}$ and $W^{(2)}$ are independent.

We know that $X_1(t) = \sqrt{r_t} R_t e^{i\theta_t}$ and X_1 is a two-dimensional Brownian motion with random clock, because its generator, given in (20), is the Laplace operator in \mathbb{R}^2 , multiplied by a function.

Let (u, θ) denote the polar coordinates on the plane, $0 \leq u < \infty$, $0 \leq \theta < 2\pi$. The polar decomposition of the Laplace operator on the plane,

$$\Delta_{\mathbb{R}^2} = \frac{\partial^2}{\partial u^2} + \frac{1}{u} \frac{\partial}{\partial u} + \frac{1}{u^2} \frac{\partial^2}{\partial \theta^2},$$

implies a representation of $X_1(t)$ as a skew-product of a radial part (which is a Bessel process) and an angular (spherical) part (cf. [PR] or [IMcK, Chapter 7.15]). More precisely, if $(B_t^{(1)}, B_t^{(2)})$ is the standard two-dimensional Brownian motion then there exist a Bessel process $Bes(t)$ and a real Brownian motion $W(t)$, independent of $Bes(t)$, such that

$$(B_t^{(1)}, B_t^{(2)}) \stackrel{d}{=} [Bes(t), e^{iW(A(t))}],$$

where $A(t) = \int_0^t Bes^{-2}(s) ds$. The same applies in our situation, because our process $X_1(t)$ is a time-changed two-dimensional Brownian motion.

As we have already described the process $\sqrt{r_t} R_t$, which is the radial part of $X_1(t)$, we now focus on the spherical part. In our case $u = R\sqrt{r}$, hence the spherical part of (20) is

$$(1 - r)(1 - R^2 r) \frac{1}{R^2 r} \frac{\partial^2}{\partial \theta^2} = \frac{1 - r}{r} \left(\frac{1}{R^2} - r \right) \frac{\partial^2}{\partial \theta^2},$$

which gives the following representation for the process $e^{i\theta_t}$: there exists a real Brownian motion $W^{(3)}$, independent of $W^{(1)}$ and $W^{(2)}$, such that

$$d\theta_t = \sqrt{\frac{1 - r_t}{r_t}} \sqrt{\frac{1}{R_t^2} - r_t} dW_t^{(3)}.$$

This is precisely equation (19), computed there by use of the generator.

REMARK. Independence of $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$ can also be obtained by computing the action of Δ_{LB} on the product $f(r)g(R)h(\theta)$, but this requires extensive computations.

7. A martingale. Our main tools in computing the Poisson kernel and Green function of the balls B_r , $0 < r < 1$, are unitary spherical harmonics and killing X_t when it reaches the boundary of B_r . In order to use unitary spherical harmonics we must have a complex-valued process; by analogy to the real case we consider $H_n^{q,p}(\langle X_t, a \rangle / (|X_t| |a|))$. For this reason in this

section we examine the properties of the process

$$Z_t = H_n^{q,p} \left(\frac{\langle X_t, a \rangle}{|X_t||a|} \right) \cdot \exp \left(\int_0^t Q(r_s) ds \right),$$

where $(X_t)_{t \geq 0}$ is the complex hyperbolic Brownian motion, $a = X_0 \neq 0$ is its starting point and $Q(r)$ is a function to be defined later.

We will find a positive, continuous function $Q(|z|^2)$ (a potential in the sense of [ChZ]) such that

$$(22) \quad (\Delta_{LB} - Q(|z|^2)) \left(H_n^{q,p} \left(\frac{\langle z, a \rangle}{|z||a|} \right) \right) = 0.$$

It turns out (see e.g. [W, III.39, p. 159]) that for such a Q the process Z_t is a martingale.

Finding Q by analytical methods requires tedious computations so we prefer using stochastic calculus and the Itô formula, which makes the calculations a little shorter.

Let $z = Re^{i\theta}$ be a complex number with modulus $R \leq 1$. Recall that by (3),

$$H_n^{p,q}(Re^{i\theta}) = Ce^{i(p-q)\theta} R^{|p-q|} P(2R^2 - 1),$$

where C is some constant and $P = P_{p \wedge q}^{n-2, |p-q|}$ denotes the appropriate Jacobi polynomial. This implies that $H_n^{q,p}(Re^{i\theta}) = Ce^{i(q-p)\theta} R^{|p-q|} P(2R^2 - 1) = \overline{H_n^{p,q}(Re^{i\theta})}$.

For $n = 2, 3, \dots$ and $p, q = 0, 1, 2, \dots$ consider the process

$$Z_t = Ce^{i(q-p)\theta t} \cdot R_t^{|p-q|} P(2R_t^2 - 1) \cdot \exp \left(\int_0^t Q(r_s) ds \right).$$

To simplify notation, write

$$g(\theta_t) = Ce^{i(q-p)\theta t}, \quad f(R_t) = R_t^{|p-q|} P(2R_t^2 - 1), \quad V_t = \exp \left(\int_0^t Q(r_s) ds \right).$$

Then

$$(23) \quad Z_t = g(\theta_t) f(R_t) V_t, \quad Z_0 = g(0) f(1) V_0 = H_n^{p,q}(1) = D(p, q, n) \omega_{2n-1}^{-1}$$

and we have the following.

THEOREM 2. *Set*

$$Q(r) = 4(1-r) \left[\frac{\frac{p+q}{2} \left(\frac{p+q}{2} + n - 1 \right)}{r} - \left(\frac{p-q}{2} \right)^2 \right].$$

Then

$$dZ_t = V_t \left\{ g(\theta_t) f'(R_t) \sqrt{\frac{1-r_t}{r_t}} \sqrt{1-R_t^2} dW_t^{(2)} + f(R_t) g'(\theta_t) \sqrt{\frac{1-r_t}{r_t}} \sqrt{\frac{1}{R_t^2} - r_t} dW_t^{(3)} \right\},$$

hence $(Z_t)_{t \geq 0}$ is a martingale and

$$(24) \quad Z_t = H_n^{p,q}(1) + \int_0^t V_s g(\theta_s) f'(R_s) \sqrt{\frac{1-r_s}{r_s}} \sqrt{1-R_s^2} dW_s^{(2)} + \int_0^t V_s f(R_s) g'(\theta_s) \sqrt{\frac{1-r_s}{r_s}} \sqrt{\frac{1}{R_s^2} - r_s} dW_s^{(3)}.$$

Proof. Observe that Q is always positive, because $r \in (0, 1)$ and $n \geq 2$. Below we use the Itô formula several times. Because V_t is a process of bounded variation, we have $dV_t = Q(r_t)V_t dt$ and $\langle g(\theta_t)f(R_t), V_t \rangle = 0$. Thus, differentiating Z_t given by (23), we obtain

$$dZ_t = d(g(\theta_t)f(R_t)V_t) = d(g(\theta_t)f(R_t)) \cdot V_t + g(\theta_t)f(R_t) \cdot dV_t.$$

Now

$$(26) \quad d(g(\theta_t)f(R_t)) = d(g(\theta_t))f(R_t) + d(f(R_t))g(\theta_t) + d\langle g(\theta_t), f(R_t) \rangle.$$

By the Itô formula and (17),

$$(27) \quad d(f(R_t)) = f'(R_t) dR_t + \frac{1}{2} f''(R_t) d\langle R_t, R_t \rangle = f'(R_t) \left[\sqrt{\frac{1-r_t}{r_t}} \sqrt{1-R_t^2} dW_t^{(2)} + \frac{1-r_t}{r_t} \left(\frac{1}{R_t} - (2n-1)R_t \right) dt \right] + f''(R_t) \left[\frac{1-r_t}{r_t} (1-R_t^2) \right] dt,$$

because $d\langle W_t^{(2)}, W_t^{(2)} \rangle = 2t dt$.

In a similar way, using (19), we get

$$(28) \quad d(g(\theta_t)) = g'(\theta_t) d\theta_t + \frac{1}{2} g''(\theta_t) d\langle \theta_t, \theta_t \rangle = g'(\theta_t) \sqrt{\frac{1-r_t}{r_t}} \sqrt{\frac{1}{R_t^2} - r_t} dW_t^{(3)} + g''(\theta_t) \cdot \frac{1-r_t}{r_t} \cdot \left(\frac{1}{R_t^2} - r_t \right) dt.$$

Both $g(\theta_t)$ and $f(R_t)$ are continuous semimartingales, so that

$$\langle g(\theta_t), f(R_t) \rangle = \langle g^m(\theta_t), f^m(R_t) \rangle,$$

where the superscript “m” denotes the martingale part of the process ([RW, p. 58]).

Moreover, because $W^{(2)}$ and $W^{(3)}$ in (17) and (19) are independent, we have $\langle g(\theta_t), f(R_t) \rangle = 0$.

Substituting (27) and (28) into (26) and next (26) to (25), we get

$$\begin{aligned}
 (29) \quad dZ_t &= V_t \{ d(g(\theta_t))f(R_t) + g(\theta_t)d(f(R_t)) + g(\theta_t)f(R_t)Q(r_t) \} \\
 &= V_t \left\{ f(R_t)g'(\theta_t) \sqrt{\frac{1-r_t}{r_t}} \sqrt{\frac{1}{R_t^2} - r_t} dW_t^{(3)} \right. \\
 &\quad + f(R_t)g''(\theta_t) \cdot \frac{1-r_t}{r_t} \cdot \left(\frac{1}{R_t^2} - r_t \right) dt \\
 &\quad + g(\theta_t)f'(R_t) \left[\sqrt{\frac{1-r_t}{r_t}} \sqrt{1-R_t^2} dW_t^{(2)} + \frac{1-r_t}{r_t} \left(\frac{1}{R_t} - (2n-1)R_t \right) dt \right] \\
 &\quad \left. + g(\theta_t)f''(R_t) \left[\frac{1-r_t}{r_t} (1-R_t^2) \right] dt + g(\theta_t)f(R_t)Q(r_t) dt \right\}.
 \end{aligned}$$

The right-hand side of (29) is a sum of two martingales,

$$\begin{aligned}
 (30) \quad dM_t^{(1)} &= V_t f(R_t)g'(\theta_t) \sqrt{\frac{1-r_t}{r_t}} \sqrt{\frac{1}{R_t^2} - r_t} dW_t^{(3)}, \\
 dM_t^{(2)} &= V_t g(\theta_t)f'(R_t) \sqrt{\frac{1-r_t}{r_t}} \sqrt{1-R_t^2} dW_t^{(2)},
 \end{aligned}$$

and the following quantity:

$$\begin{aligned}
 (31) \quad V_t \left\{ f(R_t)g''(\theta_t) \frac{1-r_t}{r_t} \left(\frac{1}{R_t^2} - r_t \right) \right. \\
 + g(\theta_t)f'(R_t) \frac{1-r_t}{r_t} \left(\frac{1}{R_t} - (2n-1)R_t \right) \\
 \left. + g(\theta_t)f''(R_t) \frac{1-r_t}{r_t} (1-R_t^2) + g(\theta_t)f(R_t)Q(r_t) \right\} dt = V_t \cdot J_t dt.
 \end{aligned}$$

Observe now that

$$g''(\theta) = (C e^{i(q-p)\theta})'' = -C(q-p)^2 e^{i(q-p)\theta} = -(p-q)^2 g(\theta),$$

hence J_t , the quantity in braces in (31), is equal to

$$\begin{aligned}
 (32) \quad J_t &= \frac{1-r_t}{r_t} g(\theta) \left[(1-R_t^2)f''(R_t) + \left(\frac{1}{R_t} - (2n-1)R_t \right) f'(R_t) \right. \\
 &\quad \left. - (p-q)^2 \left(\frac{1}{R_t^2} - r_t \right) f(R_t) + \frac{r_t}{1-r_t} Q(r_t) f(R_t) \right].
 \end{aligned}$$

But $f(R) = R^{|p-q|} P(2R^2 - 1)$, hence

$$(33) \quad f'(R) = |p-q|R^{|p-q|-1} P(2R^2 - 1) + 4R^{|p-q|+1} P'(2R^2 - 1),$$

$$(34) \quad f''(R) = |p - q|(|p - q| - 1)R^{|p-q|-2}P(2R^2 - 1) + (8|p - q| + 4)R^{|p-q|}P'(2R^2 - 1) + 16R^{|p-q|+2}P''(2R^2 - 1).$$

We want to find Q that makes $J_t = 0$. Substituting (33) and (34) into (32) we get a quantity which turns into the left-hand side of (2) if we put

$$Q(r_t) = 4(1 - r_t) \left[\frac{\frac{p+q}{2} \left(\frac{p+q}{2} + n - 1 \right)}{r_t} - \left(\frac{p - q}{2} \right)^2 \right].$$

Indeed,

$$\begin{aligned} J_t &= \frac{1 - r_t}{r_t} g(\theta_t) R_t^{|p-q|} \left\{ 16(R_t^2 - R_t^4)P''(2R_t^2 - 1) \right. \\ &\quad + 8[1 + |p - q| - (|p - q| + n)R_t^2]P'(2R_t^2 - 1) \\ &\quad + \left[-|p - q|^2 - (2n - 2)|p - q| + (p - q)^2 r_t \right. \\ &\quad \left. + 4r_t \left(\frac{\frac{p+q}{2} \left(\frac{p+q}{2} + n - 1 \right)}{r_t} - \left(\frac{p - q}{2} \right)^2 \right) \right] P(2R_t^2 - 1) \left. \right\} \\ &= 4 \cdot \frac{1 - r_t}{r_t} g(\theta_t) R_t^{|p-q|} \{ 4(R_t^2 - R_t^4)P''(2R_t^2 - 1) \\ &\quad + 2[|p - q| + 1 - (|p - q| + n)R_t^2]P'(2R_t^2 - 1) \\ &\quad + [pq + (n - 1) \min(p, q)]P(2R_t^2 - 1) \}. \end{aligned}$$

Substituting $x = 2R^2 - 1$ in the last braces above, we get

$$(1 - x^2)P''(x) + [|p - q| - n + 2 - (|p - q| + n)x]P'(x) + [\min(p, q)(\min(p, q) + n - 1 + |p - q|)]P(x).$$

But this is precisely the left-hand side of (2), with $y(x)$ being the Jacobi polynomial $P(x) = P_{p \wedge q}^{n-2, |p-q|}(x)$. This implies that $J_t = 0$, hence the quantity (31) is zero,

$$\begin{aligned} dZ_t &= V_t f(R_t) g'(\theta_t) \sqrt{\frac{1 - r_t}{r_t}} \sqrt{\frac{1}{R_t^2} - r_t} dW_t^{(3)} \\ &\quad + V_t g(\theta_t) f'(R_t) \sqrt{\frac{1 - r_t}{r_t}} \sqrt{1 - R_t^2} dW_t^{(2)} \end{aligned}$$

is a martingale and $Z_0 = H_n^{q,p}(\langle X_0, a \rangle / (|X_0| |a|)) = H_n^{q,p}(1)$, because $X_0 = a$. ■

8. Expectation of $H_n^{q,p}(\langle X_t, x \rangle / (|X_t| |x|))$. Fix $0 < r < 1$ and let $B_r = \{z \in \mathbb{C}^n : |z| < r\}$. Let $(X_t)_{t \geq 0}$ be the complex hyperbolic Brownian motion with $X_0 = x \neq 0$. Recall that $r_t = |X_t|^2$ and $Z_t = H_n^{q,p}(\langle X_t, x \rangle / (|X_t| |x|)) V_t$. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be Borel measurable and bounded.

THEOREM 3. *Under the above assumptions,*

$$\mathbb{E}^x \left[h(|X_t|^2) H_n^{q,p} \left(\frac{\langle X_t, x \rangle}{|X_t| |x|} \right) \right] = H_n^{q,p}(1) \cdot \mathbb{E}^{|x|^2} \left[h(r_t) \exp \left(- \int_0^t Q(r_s) ds \right) \right].$$

Proof. The process Z_t is a martingale, hence its expectation is constant. In order to get Z_t we multiply $H_n^{q,p}(\langle X_t, x \rangle / (|X_t| |x|))$ by $1 = V_t V_t^{-1}$:

$$\begin{aligned} \mathbb{E}^x \left[h(|X_t|^2) H_n^{q,p} \left(\frac{\langle X_t, x \rangle}{|X_t| |x|} \right) \right] &= \mathbb{E}^x \left[h(r_t) H_n^{q,p} \left(\frac{\langle X_t, x \rangle}{|X_t| |x|} \right) V_t V_t^{-1} \right] \\ &= \mathbb{E}^x [h(r_t) Z_t V_t^{-1}] = \mathbb{E}^x \left[h(r_t) Z_t \exp \left(- \int_0^t Q(r_s) ds \right) \right]. \end{aligned}$$

But we know the structure of Z_t : by (24) we have

$$\begin{aligned} \mathbb{E}^x \left[h(r_t) Z_t \exp \left(- \int_0^t Q(r_s) ds \right) \right] &= \mathbb{E}^x [h(r_t) V_t^{-1} H_n^{q,p}(1)] \\ &+ \mathbb{E}^x \left[h(r_t) V_t^{-1} \left(\int_0^t V_s g(\theta_s) f'(R_s) \sqrt{\frac{1-r_s}{r_s}} \sqrt{1-R_s^2} dW_s^{(2)} \right) \right] \\ &+ \mathbb{E}^x \left[h(r_t) V_t^{-1} \left(\int_0^t V_s f(R_s) g'(\theta_s) \sqrt{\frac{1-r_s}{r_s}} \sqrt{\frac{1}{R_s^2} - r_s} dW_s^{(3)} \right) \right], \end{aligned}$$

where $V_t^{-1} = \exp(-\int_0^t Q(r_s) ds)$. The process $h(r_t) \exp(-\int_0^t Q(r_s) ds)$ depends only on $(r_s)_{0 \leq s \leq t}$, hence it is measurable with respect to the filtration of the Brownian motion $W^{(1)}$. The stochastic integrals in the last two terms of the right-hand side are integrals with respect to the Brownian motions $W^{(2)}$ and $W^{(3)}$, respectively, and these Brownian motions are independent of $W^{(1)}$. Hence the last two terms in the above sum are zero (cf. also Lemma 2.4 in [BM]) and we finally have

$$\mathbb{E}^x \left[h(r_t) H_n^{q,p} \left(\frac{\langle X_t, x \rangle}{|X_t| |x|} \right) \right] = H_n^{q,p}(1) \mathbb{E}^{|x|^2} \left[h(r_t) \exp \left(- \int_0^t Q(r_s) ds \right) \right]. \blacksquare$$

An analogue of Theorem 3 holds true if, instead of t , we take τ_r , the first time the process $(X_t)_{t \geq 0}$ exits from B_r .

COROLLARY 4. *Let $r \in (0, 1)$ and $\tau_r = \inf\{t > 0 : |X_t| = r\}$. If $X_0 = x$ with $|x| < r$, then*

$$\mathbb{E}^x \left[H_n^{q,p} \left(\frac{\langle X_{\tau_r}, x \rangle}{|X_{\tau_r}| |x|} \right) \right] = H_n^{q,p}(1) \cdot \mathbb{E}^{|x|^2} \left[\exp \left(- \int_0^{\tau_r} Q(r_s) ds \right) \right].$$

Proof. Let $T_k = \inf\{t > 0 : r_t \leq 1/k\}$. Because $r_t > 0$ for all $t > 0$ a.s. and this process has continuous trajectories, $T_k \rightarrow \infty$ as $k \rightarrow \infty$. To

simplify notation write $\tau(t, k, r) = t \wedge T_k \wedge \tau_r$. Because Z_t is a martingale, arguing as in the proof of Theorem 3, for $h \equiv 1$ and the bounded stopping time $\tau(t, k, r)$ we get

$$\mathbb{E}^x \left[H_n^{q,p} \left(\frac{\langle X_{\tau(t,k,r)}, x \rangle}{|X_{\tau(t,k,r)}||x|} \right) \right] = H_n^{q,p}(1) \mathbb{E}^{|x|^2} \left[\exp \left(- \int_0^{\tau(t,k,r)} Q(r_s) ds \right) \right].$$

The function $\exp(-\int_0^t Q(r_s) ds)$ is bounded, because Q is positive. If $k \rightarrow \infty$, then $T_k \rightarrow \infty$. If we also let $t \rightarrow \infty$, using the dominated convergence theorem, we get the assertion. ■

9. Poisson kernel of a ball. Now we can compute the coefficients of the expansion of $P_r(x, y)$, the Poisson kernel of a ball B_r . If $X_0 = x$, then $P_r(x, \cdot)$ is the density (with respect to σ_r) of the distribution of X_{τ_r} . This fact and formula (5) give

$$(\widehat{P}_r)_{p,q} = \int_{S_r} H_n^{q,p} \left(\frac{\langle y, x \rangle}{|y||x|} \right) P_r(x, y) d\sigma_r(y) = \mathbb{E}^x H_n^{q,p} \left(\frac{\langle X_{\tau_r}, x \rangle}{|X_{\tau_r}||x|} \right).$$

The theorem below determines the coefficients $(\widehat{P}_r)_{p,q}$.

THEOREM 5. *Let $x \neq 0$ be the starting point of the complex hyperbolic Brownian motion $(X_t)_{t \geq 0}$. For $|x| < r$,*

$$\mathbb{E}^x \left[H_n^{q,p} \left(\frac{\langle X_{\tau_r}, x \rangle}{|X_{\tau_r}||x|} \right) \right] = H_n^{q,p}(1) \cdot \left(\frac{|x|}{r} \right)^{p+q} \cdot \frac{{}_2F_1(p, q; p+q+n; |x|^2)}{{}_2F_1(p, q; p+q+n; r^2)}.$$

Proof. By Corollary 4 we have

$$\mathbb{E}^x \left[H_n^{q,p} \left(\frac{\langle X_{\tau_r}, x \rangle}{|X_{\tau_r}||x|} \right) \right] = H_n^{q,p}(1) \mathbb{E}^{|x|^2} \left[\exp \left(- \int_0^{\tau_r} Q(r_s) ds \right) \right].$$

Observe that the process under the last expectation is real-valued. Let x be a real number, $0 < x < 1$. The function $\phi(x) = \mathbb{E}^x[\exp(-\int_0^{\tau_r} Q(r_s) ds)]$ is the so-called gauge function (see [ChZ, 4.3]) for the Schrödinger operator based on the generator of r_t and potential $-Q$, hence, by the general Feynman–Kac theory for the Schrödinger equation [ChZ], ϕ is a solution of the appropriate Schrödinger equation (cf. (8) and definition of Q):

$$(35) \quad 4x(1-x)^2 \phi''(x) + 4(1-x)(n-x) \phi'(x) - 4(1-x) \left[\frac{\frac{p+q}{2} \left(\frac{p+q}{2} + n - 1 \right)}{x} - \left(\frac{p-q}{2} \right)^2 \right] \phi(x) = 0.$$

If we divide both sides of (35) by $4(1-x)$ and next substitute $\phi(x) = x^{(p+q)/2} y(x)$, we get

$$x(1-x)y''(x) + [p+q+n - (p+q+1)x]y'(x) - pqy(x) = 0.$$

This is the hypergeometric equation (1), hence the general solution of (35) is given in terms of the hypergeometric function ${}_2F_1$ and is the following:

$$(36) \quad y(x) = x^{(p+q)/2} \left[c_1 \cdot {}_2F_1(p, q; p + q + n; x) + c_2 \cdot \frac{{}_2F_1(1 - n - q, 1 - n - p; 2 - n - p - q; x)}{x^{p+q+n-1}} \right].$$

When $x \rightarrow 0$, the expectation $\mathbb{E}^{|x|^2}[\exp(-\int_0^{\tau_r} Q(r_s) ds)]$ remains bounded (because Q is positive), so that we are interested in a solution of (36) which is bounded in the neighbourhood of zero. This implies $c_2 = 0$ in the formula (36) and

$$\phi(x) = c_1 x^{(p+q)/2} {}_2F_1(p, q; p + q + n; x).$$

If X_t starts from x with $|x| = r$ then $\tau_r = 0$, hence $\phi(r^2) = 1$ and this implies

$$c_1 = \frac{1}{r^{p+q} \cdot {}_2F_1(p, q; p + q + n; r^2)}.$$

Finally, for $x \in B_r$, we get

$$\phi(|x|^2) = \frac{|x|^{p+q}}{r^{p+q}} \cdot \frac{{}_2F_1(p, q; p + q + n; |x|^2)}{{}_2F_1(p, q; p + q + n; r^2)}$$

and the proof is finished. ■

REMARK. If $X_0 = 0$ then, by the unitary invariance of the distribution of the complex hyperbolic Brownian motion, the distribution of X_{τ_r} is a unitary invariant probability measure on the sphere S_r . By [R, 1.4.7], σ_r/ω_{2n-1} is the only measure with this property.

Knowing the coefficients, we can give the expansion of the Poisson kernel.

THEOREM 6 (Poisson kernel). *Let $0 \neq x \in B_r$ and $y \in S_r$. Then*

$$P_r(x, y) = \frac{1}{r^{2n-1}} \sum_{p,q=0}^{\infty} \left(\frac{|x|}{r} \right)^{p+q} \frac{{}_2F_1(p, q; p + q + n; |x|^2)}{{}_2F_1(p, q; p + q + n; r^2)} H_n^{p,q} \left(\frac{\langle y, x \rangle}{|y| |x|} \right).$$

For every fixed $x \in B_r$ the series is convergent absolutely and uniformly for all $y \in S_r$.

REMARK. It is possible to get the same expansion by the purely analytical method of Folland ([F]). This method is even shorter, but depends heavily on the fact that $P_r(x, y)$, as a function of x , is Δ_{LB} -harmonic on the whole B_r . Our method, exploiting the Itô calculus, seems to be more general and also gives the description of the Green function $G_r(x, y)$, which is not continuous on B_r (it tends to infinity as $y \rightarrow x$).

Proof of Theorem 6. We will show that the series

$$(37) \quad \sum_{p,q=0}^{\infty} \left(\frac{|x|}{r}\right)^{p+q} \frac{{}_2F_1(p, q; p+q+n; |x|^2)}{{}_2F_1(p, q; p+q+n; r^2)} H_n^{p,q} \left(\frac{\langle y, x \rangle}{|y||x|}\right)$$

is convergent absolutely and uniformly for all $x \in B_r$ and $y \in S_r$. The proof is a slight modification of that given by Folland [F] for the case $r = 1$. We give it here for completeness.

By Proposition 1, for any orthonormal basis $f_1, \dots, f_{D(p,q,n)}$ of $\mathcal{H}^{p,q} \subset L^2(S_1, \sigma_1)$ and any $x, y \in S_1$,

$$\sum_{j=1}^{D(p,q,n)} f_j(x) \overline{f_j(y)} = H_n^{p,q}(\langle x, y \rangle), \text{ so } \sum_{j=1}^{D(p,q,n)} |f_j(x)|^2 = H_n^{p,q}(1) = \frac{D(p, q, n)}{\omega_{2n-1}}.$$

Orthogonality of f_j 's implies

$$\begin{aligned} \int_{S_1} |H_n^{p,q}(\langle x, y \rangle)|^2 d\sigma_1(y) &= \sum_{j=1}^{D(p,q,n)} \int_{S_1} |f_j(x)|^2 |f_j(y)|^2 d\sigma_1(y) \\ &= \sum_{j=1}^{D(p,q,n)} |f_j(x)|^2 = \frac{D(p, q, n)}{\omega_{2n-1}} = H_n^{p,q}(1). \end{aligned}$$

Moreover, changing variables, we get

$$\begin{aligned} \int_{S_r} \left| H_n^{p,q} \left(\frac{\langle y, x \rangle}{|y||x|} \right) \right|^2 \frac{d\sigma_r(y)}{r^{2n-1}} &= \int_{S_1} \left| H_n^{p,q} \left(\frac{\langle y, x \rangle}{|y||x|} \right) \right|^2 d\sigma_1(y) = \frac{D(p, q, n)}{\omega_{2n-1}} \\ &= H_n^{p,q}(1). \end{aligned}$$

Because $H_n^{p,q}(\langle x, y \rangle)$ is Hermitian symmetric in x and y , and for fixed x it is in $\mathcal{H}^{p,q}$ as a function of y , by the Schwarz inequality we obtain

$$|H_n^{p,q}(\langle y, x \rangle)| = \left| \int_{S_1} H_n^{p,q}(\langle y, z \rangle) H_n^{p,q}(\langle z, x \rangle) d\sigma_1(z) \right| \leq D(p, q, n) \omega_{2n-1}^{-1}.$$

Now,

$$\begin{aligned} D(p, q, n) &= \frac{(p+q+n-1)(p+n-2)!(q+n-2)!}{p!q!(n-1)!(n-2)!} \\ &= \frac{(p+q+n-1)p(p+1)\dots(p+n-2)q(q+1)\dots(q+n-2)!}{(n-1)!(n-2)!} \\ &\leq \frac{(p+q+n-1)}{(n-1)!(n-2)!} (p+n-2)^{n-1} (q+n-2)^{n-1} \\ &\leq \frac{1}{(n-1)!(n-2)!} (p+q+n-1)^{2n-1}. \end{aligned}$$

Because n is fixed, we have

$$1 \leq \frac{p+q+n-1}{p+q+1} \leq n-1 \quad \text{for all } p, q = 0, 1, 2, \dots,$$

which implies that

$$D(p, q, n) \leq \frac{(n-1)^{2n}}{(n-1)!(n-2)!} (p+q+1)^{2n} = c_n (p+q+1)^{2n}.$$

The function ${}_2F_1(p, q; p+q+n; |x|^2)$ is positive and increasing for $|x| \in (0, 1)$ so that

$$\left(\frac{|x|}{r}\right)^{p+q} \frac{{}_2F_1(p, q; p+q+n; |x|^2)}{{}_2F_1(p, q; p+q+n; r^2)} \leq \left(\frac{|x|}{r}\right)^{p+q},$$

thus

$$\begin{aligned} & \sum_{p,q=0}^{\infty} \left(\frac{|x|}{r}\right)^{p+q} \frac{{}_2F_1(p, q; p+q+n; |x|^2)}{{}_2F_1(p, q; p+q+n; r^2)} \left| H_n^{p,q} \left(\frac{\langle x, y \rangle}{|y||x|} \right) \right| \\ & \leq c_n \sum_{p,q=0}^{\infty} \left(\frac{|x|}{r}\right)^{p+q} (p+q+1)^{2n} = c_n \sum_{k=0}^{\infty} \left(\frac{|x|}{r}\right)^k (k+1)^{2n+1} < \infty, \end{aligned}$$

because $p+q=k$ for $k+1$ different values of p , and $|x| < r$.

The functions $(H_n^{p,q}(\cdot))_{p,q=0,1,\dots}$ are orthogonal, hence it is easy to compute the (p, q) -coefficient of $P_r(x, y)$. We have (recall that $H_n^{q,p}(\cdot) = \overline{H_n^{p,q}(\cdot)}$)

$$\begin{aligned} (\widehat{P}_r)_{p,q} &= \int_{S_r} H_n^{q,p} \left(\frac{\langle y, x \rangle}{|y||x|} \right) P_r(x, y) d\sigma_r(y) \\ &= \int_{S_r} H_n^{q,p} \left(\frac{\langle y, x \rangle}{|y||x|} \right) \left(\frac{1}{r^{2n-1}} \sum_{p,q=0}^{\infty} \left(\frac{|x|}{r}\right)^{p+q} \right. \\ & \quad \left. \times \frac{{}_2F_1(p, q; p+q+n; |x|^2)}{{}_2F_1(p, q; p+q+n; r^2)} H_n^{p,q} \left(\frac{\langle y, x \rangle}{|y||x|} \right) \right) d\sigma_r(y) \\ &= \left(\frac{|x|}{r}\right)^{p+q} \frac{{}_2F_1(p, q; p+q+n; |x|^2)}{{}_2F_1(p, q; p+q+n; r^2)} \\ & \quad \times \int_{S_r} H_n^{q,p} \left(\frac{\langle y, x \rangle}{|y||x|} \right) H_n^{p,q} \left(\frac{\langle y, x \rangle}{|y||x|} \right) \frac{d\sigma_r(y)}{r^{2n-1}} \\ &= H_n^{p,q}(1) \left(\frac{|x|}{r}\right)^{p+q} \frac{{}_2F_1(p, q; p+q+n; |x|^2)}{{}_2F_1(p, q; p+q+n; r^2)}. \end{aligned}$$

These coefficients are the same as the (p, q) -coefficients of the distribution of X_{τ_r} , computed in Theorem 5. This implies that the series described in Theorem 6 represents the Poisson kernel of B_r . ■

10. Green function of a ball. In order to compute the Green function of a ball we use the method of Byczkowski and Małecki [BM]. We will compute $G_r(x, y)$, the Green function of the ball B_r , on all spheres S_R , that is, for fixed $x \in B_r$ and all $y \in S_R$, $0 < R < r$. Moreover, because the Green function is symmetric: $G_r(x, y) = G_r(y, x)$, it is enough to consider the case $|x| < R$.

Below we will use several semigroups and Green operators. Recall that (X_t) denotes the complex hyperbolic Brownian motion and $r_t = |X_t|^2$. Let $P_t^{(r)}$ denote the semigroup of transition probabilities for the process $X_t^{B_r}$, the hyperbolic Brownian motion killed at the boundary of B_r . Similarly, let $(r_t^{(r)})$ be the process (r_t) killed on exiting the interval $(0, r^2)$. Because X_t does not hit 0, (r_t) exits $(0, r^2)$ through the point r^2 . We denote by $T_t^{(r)}$ the Feynman–Kac semigroup based on the process r_t and potential $-Q$, with its Feynman–Kac functional $\exp(-\int_0^t Q(r_s) ds)$ stopped at the first time the process (r_t) reaches the point r^2 . Let G_r denote the Green operator for the process X_t and the set B_r , and let $G_r(x, y)$ be the Green function, i.e. the kernel of G_r . Finally, $V^{(r)}$ will denote the $(-Q)$ -Green operator for the semigroup $T_t^{(r)}$. For the details of the above definitions the reader may consult [ChZ].

Let $(\widehat{G}_r)_{p,q}(x, R)$ denote the (p, q) -coefficient of $G_r(x, y)$, defined as in (5), considered as a function of $y \in S_R$, with fixed $x \in B_r$. Observe that two such coefficients are equal if $|x_1| = |x_2|$. For this reason we will use the notation $(\widehat{G}_r)_{p,q}(|x|, R)$. By formula (5) we have

$$(38) \quad (\widehat{G}_r)_{p,q}(|x|, R) = \int_{S_R} H_n^{q,p} \left(\frac{\langle y, x \rangle}{|y||x|} \right) G_r(x, y) d\sigma_R(y).$$

Now we compute these coefficients explicitly. To simplify the formulation of the next theorem, write

$$F_{p,q}(x) = {}_2F_1(p, q; p + q + n; x),$$

$$G_{p,q}(x) = {}_2F_1(1 - n - q, 1 - n - p; 2 - n - p - q; x).$$

THEOREM 7. *Let $G_r(x, y)$ be the Green function for the process $(X_t)_{t \geq 0}$ and ball B_r . Then*

$$(\widehat{G}_r)_{p,q}(|x|, R) = \frac{H_n^{p,q}(1)|x|^{p+q}F_{p,q}(|x|^2)}{2(p + q + n - 1)} R^{p+q} \left(\frac{G_{p,q}(R^2)}{R^{2(p+q+n-1)}} - \frac{G_{p,q}(r^2)}{r^{2(p+q+n-1)}} \cdot \frac{F_{p,q}(R^2)}{F_{p,q}(r^2)} \right).$$

Proof. In terms of semigroups of operators Theorem 3 reads as follows: for bounded measurable $h : \mathbb{R} \rightarrow \mathbb{R}$,

$$(39) \quad (P_t^{(r)} h H_n^{q,p})(x) = H_n^{q,p}(1)(T_t^{(r)} h)(|x|^2).$$

Integrating (39) with respect to time, we get

$$\int_0^\infty (P_t^{(r)} h H_n^{q,p})(x) dt = H_n^{q,p}(1) \int_0^\infty (T_t^{(r)} h)(|x|^2) dt,$$

which, in the language of potential operators, is

$$(40) \quad (G_r h H_n^{q,p})(x) = H_n^{q,p}(1)(V^{(r)} h)(|x|^2).$$

Using the fact that $G_r(x, y)$ denotes the kernel of the operator G_r and integrating in polar coordinates in \mathbb{C}^n , we can write the left-hand side of (40) as

$$(41) \quad (G_r h H_n^{q,p})(x) = \int_{B_r} h(|y|^2) H_n^{q,p} \left(\frac{\langle y, x \rangle}{|y| |x|} \right) G_r(x, y) dy \\ = \int_0^r h(R^2) \left(\int_{S_R} H_n^{q,p} \left(\frac{\langle y, x \rangle}{|y| |x|} \right) G_r(x, y) d\sigma_R(y) \right) dR.$$

The quantity $(V^{(r)} h)(|x|^2)$ on the right-hand side of (40) equals

$$(42) \quad (V^{(r)} h)(|x|^2) = \int_0^{r^2} h(y) V^{(r)}(|x|^2, y) dy \\ = \int_0^r h(R^2) V^{(r)}(|x|^2, R^2) 2R dR,$$

where $V^{(r)}(|x|^2, y)$ is the kernel of the operator $V^{(r)}$.

Substituting (42) into the right-hand of (40) and then comparing the right-hand sides of (40) and (41), we get, for measurable bounded functions h ,

$$H_n^{q,p}(1) \int_0^r h(R^2) V^{(r)}(|x|^2, R^2) 2R dR \\ = \int_0^r h(R^2) \left(\int_{S_R} H_n^{q,p} \left(\frac{\langle y, x \rangle}{|y| |x|} \right) G_r(x, y) d\sigma_R(y) \right) dR.$$

This implies

$$2H_n^{q,p}(1) R V^{(r)}(|x|^2, R^2) = \int_{S_R} H_n^{q,p} \left(\frac{\langle y, x \rangle}{|y| |x|} \right) G_r(x, y) d\sigma_R(y).$$

This in turn, together with (38), gives

$$(43) \quad (\widehat{G}_r)_{p,q}(|x|, R) = 2H_n^{q,p}(1) R V^{(r)}(|x|^2, R^2).$$

The $(-Q)$ -Green function $V^{(r)}(y, R^2)$ is $(-Q)$ -harmonic in y , for $0 < y < R$ and for $R < y < r$ (cf. [ChZ]), hence it has to be a solution of (35), the

Schrödinger equation based on the generator of the process $(r_t)_{t \geq 0}$ and potential $-Q$. We have already found this solution: it is given by the formula (36), hence

$$\begin{aligned}
 (44) \quad V^{(r)}(|x|^2, R^2) &= |x|^{p+q} \left[c_1(p, q, R) \cdot {}_2F_1(p, q; p+q+n; |x|^2) \right. \\
 &\quad \left. + \frac{c_2(p, q, R)}{|x|^{2(p+q+n-1)}} \cdot {}_2F_1(1-q-n, 1-p-n; 2-p-q-n; |x|^2) \right] \\
 &= |x|^{p+q} \left(c_1(p, q, R) F_{p,q}(|x|^2) + \frac{c_2(p, q, R)}{|x|^{2(p+q+n-1)}} G_{p,q}(|x|^2) \right).
 \end{aligned}$$

Because $\lim_{|x|^2 \rightarrow r^2} V^{(r)}(|x|^2, R^2) = 0$, we have

$$c_1(p, q, R) = -\frac{c_2(p, q, R)}{r^{2(p+q+n-1)}} \cdot \frac{G_{p,q}(r^2)}{F_{p,q}(r^2)}$$

and

$$\begin{aligned}
 (45) \quad V^{(r)}(|x|^2, R^2) &= c_2(p, q, R) |x|^{p+q} \left(\frac{G_{p,q}(|x|^2)}{|x|^{2(p+q+n-1)}} - \frac{G_{p,q}(r^2)}{r^{2(p+q+n-1)}} \cdot \frac{F_{p,q}(|x|^2)}{F_{p,q}(r^2)} \right).
 \end{aligned}$$

By virtue of (43) and (45),

$$\begin{aligned}
 (\widehat{G}_r)_{p,q}(|x|, R) &= 2H_n^{q,p}(1)R \cdot c_2(p, q, R) |x|^{p+q} \left(\frac{G_{p,q}(|x|^2)}{|x|^{2(p+q+n-1)}} - \frac{G_{p,q}(r^2)}{r^{2(p+q+n-1)}} \cdot \frac{F_{p,q}(|x|^2)}{F_{p,q}(r^2)} \right).
 \end{aligned}$$

The Green function is symmetric so that $(\widehat{G}_r)_{p,q}(|x|, R)$ also has this property. Changing the roles of $|x|$ and R , we get

$$\begin{aligned}
 (46) \quad (\widehat{G}_r)_{p,q}(R, |x|) &= 2H_n^{q,p}(1)|x| \cdot c_2(p, q, |x|) R^{p+q} \left(\frac{G_{p,q}(R^2)}{R^{2(p+q+n-1)}} - \frac{G_{p,q}(r^2)}{r^{2(p+q+n-1)}} \cdot \frac{F_{p,q}(R^2)}{F_{p,q}(r^2)} \right).
 \end{aligned}$$

Now we compute the constant $c_2(p, q, |x|)$, showing first the following:

LEMMA 8. *Let*

$$u(x) = (n+p+q-1)F_{p,q}(x)G_{p,q}(x) + x(F'_{p,q}(x)G_{p,q}(x) - F_{p,q}(x)G'_{p,q}(x)).$$

Then $u(x) = (n+p+q-1)(1-x)^{n-1}$.

Proof. Recall that, by (1) and the definition, $F_{p,q}$ satisfies the equation

$$x(1-x)F''(x) + (p+q+n - (p+q+1)x)F'(x) - pqF(x) = 0,$$

while $G_{p,q}$ satisfies the equation

$$x(1-x)G''(x) + (2-n-p-q - (3-p-q-2n)x)G'(x) - (1-p-n)(1-q-n)G(x) = 0.$$

Let us compute the derivative

$$u'(x) = (n+p+q-1)(F'_{p,q}(x)G_{p,q}(x) + F_{p,q}(x)G'_{p,q}(x)) + F'_{p,q}(x)G_{p,q}(x) - F_{p,q}(x)G'_{p,q}(x) + x(F''_{p,q}(x)G_{p,q}(x) - F_{p,q}(x)G''_{p,q}(x)).$$

We can calculate $x(1-x)F''_{p,q}(x)$ and $x(1-x)G''_{p,q}(x)$ from the above-cited equations. Substituting these quantities, we get

$$\begin{aligned} (1-x)u'(x) &= (1-n)[(n+p+q+1)F_{p,q}(x)G_{p,q}(x) \\ &\quad + x(F'_{p,q}(x)G_{p,q}(x) - F_{p,q}(x)G'_{p,q}(x))] \\ &= (1-n)u(x). \end{aligned}$$

Solving the equation $(1-x)u'(x) = (1-n)u(x)$, we get $u(x) = C(1-x)^{n-1}$. For $x \rightarrow 0$ the asymptotics of the hypergeometric function and its derivative is well-known [E]; using this asymptotics we get $\lim_{x \rightarrow 0+} u(x) = n+p+q-1$, hence finally $u(x) = (n+p+q-1)(1-x)^{n-1}$. ■

Using Lemma 8 we can compute the derivative of the function

$$R^{p+q} \left(\frac{G_{p,q}(R^2)}{R^{2(p+q+n-1)}} - \frac{G_{p,q}(r^2)}{r^{2(p+q+n-1)}} \cdot \frac{F_{p,q}(R^2)}{F_{p,q}(r^2)} \right)$$

with respect to R :

$$\begin{aligned} (47) \quad \left(-\frac{d}{dR} \Big|_{R=r} \right) R^{p+q} \left(\frac{G_{p,q}(R^2)}{R^{2(p+q+n-1)}} - \frac{G_{p,q}(r^2)}{r^{2(p+q+n-1)}} \cdot \frac{F_{p,q}(R^2)}{F_{p,q}(r^2)} \right) \\ = \frac{2(p+q+n-1)(1-r^2)^{n-1}}{r^{p+q+2n-1}F_{p,q}(r^2)}. \end{aligned}$$

On the other hand, by the Green formula (see Lemma 11 in the Appendix), for $y \in S_1$ we have

$$\left(-\frac{d}{dR} \Big|_{R=r} \right) G_r(x, Ry) = (1-r^2)^{n-1} P_r(x, ry).$$

From this fact and (38), for $|y| = 1$ and $|x| < r$ we obtain

$$\begin{aligned} (48) \quad \left(-\frac{d}{dR} \Big|_{R=r} \right) (\widehat{G}_r)_{p,q}(|x|, R) \\ = \left(-\frac{d}{dR} \Big|_{R=r} \right) \int_{S_1} H_n^{q,p} \left(\frac{\langle y, x \rangle}{|x||y|} \right) G_r(x, Ry) d\sigma_1(y) \\ = \int_{S_1} \left(-\frac{d}{dR} \Big|_{R=r} \right) H_n^{q,p} \left(\frac{\langle y, x \rangle}{|x||y|} \right) G_r(x, Ry) d\sigma_1(y) \end{aligned}$$

$$\begin{aligned}
 &= (1 - r^2)^{n-1} \int_{S_1} H_n^{q,p} \left(\frac{\langle y, x \rangle}{|x||y|} \right) P_r(x, ry) d\sigma_1(y) \\
 &= \frac{(1 - r^2)^{n-1}}{r^{2n-1}} \int_{S_r} H_n^{q,p} \left(\frac{\langle y, x \rangle}{|x||y|} \right) P_r(x, y) d\sigma_r(y) \\
 &= \frac{(1 - r^2)^{n-1}}{r^{2n-1}} H_n^{q,p}(1) \cdot \frac{|x|^{p+q}}{r^{p+q}} \cdot \frac{F_{p,q}(|x|^2)}{F_{p,q}(r^2)},
 \end{aligned}$$

where in the last equality we have used Theorem 6.

Now we can compare the derivatives: formula (48) gives

$$\left(- \frac{d}{dR} \Big|_{R=r} \right) (\widehat{G}_r)_{p,q}(|x|, R) = \frac{(1 - r^2)^{n-1}}{r^{2n-1}} H_n^{q,p}(1) \frac{|x|^{p+q}}{r^{p+q}} \cdot \frac{F_{p,q}(|x|^2)}{F_{p,q}(r^2)},$$

while (46) and (47) imply

$$\begin{aligned}
 &\left(- \frac{d}{dR} \Big|_{R=r} \right) (\widehat{G}_r)_{p,q}(|x|, R) \\
 &= 2H_n^{q,p}(1)|x|c_2(p, q, |x|) \cdot \frac{2(p + q + n - 1)(1 - r^2)^{n-1}}{r^{p+q+2n-1}F_{p,q}(r^2)}.
 \end{aligned}$$

Comparing the right-hand sides of the last two expressions, we get

$$\begin{aligned}
 &\frac{(1 - r^2)^{n-1} H_n^{q,p}(1) |x|^{p+q} F_{p,q}(|x|^2)}{r^{2n-1} r^{p+q} F_{p,q}(r^2)} \\
 &= \frac{4(p + q + n - 1) H_n^{q,p}(1) |x| c_2(p, q, |x|) (1 - r^2)^{n-1}}{r^{p+q+2n-1} F_{p,q}(r^2)},
 \end{aligned}$$

hence

$$|x|c_2(p, q, |x|) = \frac{|x|^{p+q} F_{p,q}(|x|^2)}{4(p + q + n - 1)},$$

which, by (46), finally gives

$$\begin{aligned}
 &(\widehat{G}_r)_{p,q}(R, |x|) \\
 &= \frac{H_n^{q,p}(1) |x|^{p+q} F_{p,q}(|x|^2)}{2(p + q + n - 1)} R^{p+q} \left(\frac{G_{p,q}(R^2)}{R^{2(p+q+n-1)}} - \frac{G_{p,q}(r^2)}{r^{2(p+q+n-1)}} \cdot \frac{F_{p,q}(R^2)}{F_{p,q}(r^2)} \right). \blacksquare
 \end{aligned}$$

Having the coefficients, we can write down the expansion of the Green function.

THEOREM 9 (Green function). *Let $x, y \in B_r$. If $x \neq 0$ and $|x| < |y|$, then*

$$\begin{aligned}
 G_r(x, y) &= \frac{1}{2r^{2n-1}} \sum_{p,q=0}^{\infty} \frac{|x|^{p+q}|y|^{p+q}}{p + q + n - 1} F_{p,q}(|x|^2) \left(\frac{G_{p,q}(|y|^2)}{|y|^{2(p+q+n-1)}} \right. \\
 &\quad \left. - \frac{G_{p,q}(r^2)}{r^{2(p+q+n-1)}} \cdot \frac{F_{p,q}(|y|^2)}{F_{p,q}(r^2)} \right) H_n^{p,q} \left(\frac{\langle x, y \rangle}{|x||y|} \right).
 \end{aligned}$$

For fixed x the series is uniformly convergent on every sphere S_R with $|x| < R < r$.

Proof. To show the uniform convergence, we can estimate the series as follows:

$$\begin{aligned} & \left| \sum_{p,q=0}^{\infty} \frac{|x|^{p+q}|y|^{p+q}}{p+q+n-1} F_{p,q}(|x|^2) \right. \\ & \quad \times \left. \left(\frac{G_{p,q}(|y|^2)}{|y|^{2(p+q+n-1)}} - \frac{G_{p,q}(r^2)}{r^{2(p+q+n-1)}} \cdot \frac{F_{p,q}(|y|^2)}{F_{p,q}(r^2)} \right) H_n^{p,q} \left(\frac{\langle x, y \rangle}{|x| |y|} \right) \right| \\ & \leq \frac{c}{|y|^{2n-2}} \sum_{p,q=0}^{\infty} \frac{|x|^{p+q}}{|y|^{p+q}} H_n^{p,q}(1) |F_{p,q}(|x|^2) G_{p,q}(|y|^2)| \\ & \quad + \frac{c}{r^{2n-2}} \sum_{p,q=0}^{\infty} \frac{|x|^{p+q}|y|^{p+q}}{r^{2(p+q)}} H_n^{p,q}(1) |F_{p,q}(|x|^2) G_{p,q}(r^2)|, \end{aligned}$$

because $F_{p,q}$ is positive and increasing on $(0, 1)$, which implies that $F_{p,q}(|x|^2)/F_{p,q}(r^2) \leq 1$.

Below in Lemma 10 we will show that, except for the case $pq = 0$, $G'_{p,q} < 0$ on $(0, 1)$ and $G_{p,q}(1) = 0$. These facts immediately imply that $G_{p,q} > 0$ on $(0, 1)$ and $x(F'_{p,q}(x)G_{p,q}(x) - F_{p,q}(x)G'_{p,q}(x)) > 0$, hence by Lemma 8 we have

$$0 < F_{p,q}(x)G_{p,q}(x) < (1-x)^{n-1} \leq 1 \quad \text{for } x \in (0, 1).$$

This, together with the fact that $F_{p,q}$ is positive and increasing, yields

$$\begin{aligned} |F_{p,q}(|x|^2)G_{p,q}(|y|^2)| &< F_{p,q}(|y|^2)G_{p,q}(|y|^2) \leq 1, \\ |F_{p,q}(|x|^2)G_{p,q}(r^2)| &< F_{p,q}(r^2)G_{p,q}(r^2) \leq 1. \end{aligned}$$

By the definition, $G_{0,q}$ is bounded on every interval $[\varepsilon, 1]$ and $F_{0,q} \equiv 1$, hence $F_{0,q}G_{0,q}$ is also bounded on $[\varepsilon, 1]$.

Recall that $0 < |x| < |y| < r$ by our assumption and put $\varrho = |x|/|y|$ in the case of the first series and $\varrho = |x| |y|/r^2$ for the second. In the proof of Theorem 5 we showed that for any $0 < \varrho < 1$ we have $\sum \varrho^{p+q} H_n^{p,q}(1) < \infty$, hence both series are absolutely and uniformly convergent.

The family $(H_n^{p,q}(\cdot))_{p,q=0}^{\infty}$ is orthogonal, hence we can compute (just as we did for the Poisson kernel in the proof of Theorem 6) the (p, q) -coefficients of the function defined by the series. Because the coefficients are equal to those computed in Theorem 7 for the Green function, the series given in Theorem 9 represents $G_r(x, y)$, the Green function of B_r . ■

Now we will prove the above-mentioned properties of $G_{p,q}$.

LEMMA 10. For non-negative integers p, q and $n \geq 2$ let

$$G_{p,q}(x) = {}_2F_1(1 - n - q, 1 - n - p; 2 - n - p - q; x).$$

If $pq > 0$ then $h'_{p,q}(x) < 0$ for $x \in (0, 1)$ and $G_{p,q}(1) = 0$.

Proof. We can assume that $q \leq p$. The Kummer relation mentioned in the Preliminaries implies that

$${}_2F_1(1 - n - q, 1 - n - p; 2 - n - p - q; x) = (1 - x)^n \cdot {}_2F_1(1 - p, 1 - q; 2 - n - p - q; x).$$

Since $f(x) = (1 - x)^n$ is decreasing on $(0, 1)$, $f(1) = 0$ and ${}_2F_1(1 - p, 1 - q; 2 - n - p - q; x)$ is a polynomial, it follows that $G_{p,q}(1) = 0$ and it is enough to show that $h_{p,q}(x) = {}_2F_1(1 - p, 1 - q; 2 - n - p - q; x)$ has negative derivative for $0 < x < 1$.

The function $h_{p,q}(x)$ is a polynomial in x of degree $q - 1$ (recall that we assumed $0 < q \leq p$) and for $k = 1, \dots, q - 1$ its k th derivative is

$$\frac{d^k}{dx^k} h_{p,q}(x) = \frac{(1 - p)_k (1 - q)_k}{(2 - n - p - q)_k} \cdot {}_2F_1(1 - p + k, 1 - q + k; 2 - n - p - q + k; x).$$

In [V, computations of C_2 on p. 132], the following formula was proved:

$$(49) \quad {}_2F_1(1 - p + k, 1 - q + k; 2 - n - p - q + k; 1) = \frac{(n + p - 1)!(n + q - 1)!}{(n + k)!(n + p + q - 2 + k)!}.$$

Now observe that for fixed n, q and p the sequence $\left(\frac{(1-p)_k(1-q)_k}{(2-n-p-q)_k}\right)_{k=1}^{q-1}$ has alternating signs. Indeed, let $q > 0$ be even. Then

$$\frac{d^{q-1}}{dx^{q-1}} h_{p,q}(1) = \frac{(1 - p)_{q-1} (1 - q)_{q-1}}{(2 - n - p - q)_{q-1}} \cdot {}_2F_1(q - p, 0; 1 - n - p; 1) < 0,$$

because there are an odd number of terms in the denominator, they are all negative and ${}_2F_1(q - p, 0; 1 - n - p; 1) = 1$. This implies that $d^{q-2}h_{p,q}/dx^{q-2}$ is decreasing and since its value at $x = 1$ is positive by (49), this derivative is positive. This implies that the derivative of order $q - 3$ is decreasing, and as its value at $x = 1$ is positive, $d^{q-3}h_{p,q}/dx^{q-3}$ is positive. By induction we infer that $dh_{p,q}/dx$ is negative, so that $h_{p,q}$ is decreasing, as required.

If q is odd then $\frac{d^{q-1}}{dx^{q-1}}h_{p,q}(1) > 0$ and the analogous reasoning gives the same result. ■

REMARK. The formula in Theorem 9 is very similar to the one obtained by Byczkowski and Małecki [BM] for the case of the real hyperbolic space \mathbb{D}_n . Instead of our $F_{p,q}$ their representation uses the hypergeometric function ${}_2F_1(p, (2 - n)/2; p + n/2; x)$ and for even n this function is a polynomial. Using this fact they managed to find sums of the analogous series describing $P_r(x, y)$ and $G_r(x, y)$ for $n = 4$ and $n = 6$. It seems hard to get such compact

formulas in the complex hyperbolic case, because for all $n \geq 2$ the functions $F_{p,q}$ are given by infinite series.

Appendix: gradient of a function and directional derivative in the complex hyperbolic space. Consider the unit ball $B_1 = \{z \in \mathbb{C}^n : |z| < 1\}$ with the Bergman (Hermitian) metric, given by the matrix with entries

$$h_{ij} = \frac{(1 - |z|^2)\delta_{ij} + \bar{z}_i z_j}{(1 - |z|^2)^2}, \quad i, j = 1, \dots, n$$

(cf. [K] or [S], where the second exponent in $(1 - |z|^2)^2$ is missing).

The inverse matrix $[h^{ij}]$ is given by the formula [S]

$$h^{ij} = (1 - |z|^2)(\delta_{ij} - \bar{z}_i z_j), \quad i, j = 1, \dots, n.$$

Because we want to use the Green formula, involving derivatives in the normal direction, we describe this metric in the language of the Riemannian metric in the unit ball of \mathbb{R}^{2n} . By the well-known identification between the structures of \mathbb{C}^n and \mathbb{R}^{2n} ([KN], [R]) we have the following:

1) If $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $z_k = x_k + iy_k$ then

$$\mathbb{C}^n \ni z \leftrightarrow (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n}.$$

2) Let $M_{\mathbb{C}^n}$ be the matrix of a linear transformation of \mathbb{C}^n . If $M_{\mathbb{C}^n} = \Re M_{\mathbb{C}^n} + i\Im M_{\mathbb{C}^n} = A + iB$ then

$$M_{\mathbb{R}^{2n}} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

is the matrix of the corresponding transformation of \mathbb{R}^{2n} .

Length of a tangent vector. Now we want to look at the unit ball in \mathbb{C}^n with its Hermitian metric as a subset of \mathbb{R}^{2n} with the Riemannian metric. By the above, the Riemannian metric g_{ij} , $i, j = 1, \dots, 2n$, in the unit ball of \mathbb{R}^{2n} is given by the matrix $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$, where

$$a_{ij} = \frac{(1 - |z|^2)\delta_{ij} + \Re(\bar{z}_i z_j)}{(1 - |z|^2)^2}, \quad i, j = 1, \dots, n,$$

$$b_{ij} = \frac{\Im(\bar{z}_i z_j)}{(1 - |z|^2)^2}, \quad i, j = 1, \dots, n.$$

Let $T_{(x_1, \dots, x_n, y_1, \dots, y_n)}$ denote the tangent space at the point $(x_1, \dots, x_n, y_1, \dots, y_n)$ and let $v \in T_{(x_1, \dots, x_n, y_1, \dots, y_n)}$. Then the form determining the length of $v = (v_1, \dots, v_{2n})$ is given by the formula

$$ds^2 = [v_1 \ \dots \ v_{2n}] \begin{bmatrix} A & B \\ -B & A \end{bmatrix} [v_1 \ \dots \ v_{2n}]^T.$$

In particular we get (after a long computation)

$$\begin{aligned}
 ds^2 &= [x_1 \ \dots \ x_n \ y_1 \ \dots \ y_n] \begin{bmatrix} A & B \\ -B & A \end{bmatrix} [x_1 \ \dots \ x_n \ y_1 \ \dots \ y_n]^T \\
 &= \frac{x_1^2 + \dots + x_n^2 + y_1^2 + \dots + y_n^2}{(1 - x_1^2 - \dots - x_n^2 - y_1^2 - \dots - y_n^2)^2},
 \end{aligned}$$

so that

$$|(x_1, \dots, x_n, y_1, \dots, y_n)|_R = \frac{|(x_1, \dots, x_n, y_1, \dots, y_n)|_E}{1 - |(x_1, \dots, x_n, y_1, \dots, y_n)|_E^2},$$

where $|v|_E$ denotes the Euclidean and $|v|_R$ the Riemannian length of a vector v .

The same result can be obtained without computations: the Bergman metric is invariant with respect to unitary transformations of the unit ball and the group $U(n)$ acts transitively on the spheres S_r , $0 < r < 1$. Let $z \in \mathbb{C}^n$ with $|z|_E = r < 1$ and choose $A_z \in U(n)$ such that $A_z(z) = (r, 0, \dots, 0) \in \mathbb{C}^n$. This implies that the length of any vector v in $T_z = T_{(x_1, \dots, x_n, y_1, \dots, y_n)}$ is the same as the length of $A_z(v)$ in $T_{(r, 0, \dots, 0)}$. In particular the length of $z = (x_1, \dots, x_n, y_1, \dots, y_n)$ is the same as the length of $(r, 0, \dots, 0)$ in $T_{(r, 0, \dots, 0)}$. But we can identify $(r, 0, \dots, 0) \in \mathbb{C}^n$ with $r \in \mathbb{C}^1$. The form of the Bergman metric $h_{i,j}$ implies that both vectors have the same length and in the case $n = 1$ the Bergman metric in the unit disc of \mathbb{C} is the same as the Poincaré metric, so that

$$|(r, 0, \dots, 0)|_R = \frac{|(r, 0, \dots, 0)|_E}{1 - r^2}.$$

Derivative in the normal direction. Let $0 < r < 1$ and fix a point $(x_1, \dots, x_n, y_1, \dots, y_n)$ with Euclidean norm r , i.e. a point from the sphere S_r . For a differentiable function f we want to compute the directional derivative $\partial f / \partial \mathbf{n}$, where \mathbf{n} is the normal vector at $(x_1, \dots, x_n, y_1, \dots, y_n)$ and has Euclidean length 1. In order to use the formula

$$\frac{\partial f}{\partial \mathbf{n}} = \langle \mathbf{grad} f, \mathbf{n} \rangle,$$

we have to know $\mathbf{grad} f$ and the length of \mathbf{n} (which we have already computed). Now we compute the gradient of f . Let $[g^{ij}]$ be the matrix inverse to $[g_{ij}]$ and let $(U, (x_1, \dots, x_{2n}))$ be a local coordinate system. Then by the well-known formula for Riemannian manifolds ([KN], [Ch])

$$\mathbf{grad} f = \sum_{i,j=1}^{2n} \left(g^{ij} \frac{\partial f}{\partial x_j} \right) \frac{\partial}{\partial x_i},$$

where $(\partial / \partial x_i)_{i=1, \dots, 2n}$ is a basic vector field. In the case we consider, the

inverse matrix $[g^{ij}]$ at $(x_1, \dots, x_n, y_1, \dots, y_n)$ is

$$[g^{ij}]_{i,j=1,\dots,2n} = \begin{bmatrix} C & D \\ -D & C \end{bmatrix},$$

where

$$\begin{aligned} c^{ij} &= (1 - |z|^2)(\delta_{ij} - \Re(\bar{z}_i z_j)), & i, j &= 1, \dots, n, \\ d^{ij} &= (1 - |z|^2)(-\Im(\bar{z}_i z_j)), & i, j &= 1, \dots, n. \end{aligned}$$

Putting $z_k = x_k + iy_k$, $k = 1, \dots, n$, with $|z|^2 = \sum_{i=1}^n (x_i^2 + y_i^2) = r^2$, we get, for $i, j = 1, \dots, n$,

$$\begin{aligned} c^{ij} &= \left(1 - \sum_{k=1}^n x_k^2 - \sum_{k=1}^n y_k^2\right)(\delta_{ij} - x_i x_j - y_i y_j) = (1 - r^2)(\delta_{ij} - x_i x_j - y_i y_j), \\ d^{ij} &= \left(1 - \sum_{k=1}^n x_k^2 - \sum_{k=1}^n y_k^2\right)(-x_i y_j + x_j y_i) = (1 - r^2)(-x_i y_j + x_j y_i). \end{aligned}$$

Let \mathbf{n} be the normal vector at $(x_1, \dots, x_n, y_1, \dots, y_n) \in S_r$. Then $\mathbf{n} = (x_1, \dots, x_n, y_1, \dots, y_n)$. By the previous computation its length is equal to $|\mathbf{n}|_R = |\mathbf{n}|_E / (1 - r^2)$. Thus

$$\begin{aligned} \langle \mathbf{grad} f, \mathbf{n} \rangle &= \frac{1}{1 - r^2} \langle \mathbf{grad} f, (x_1, \dots, x_n, y_1, \dots, y_n) \rangle \\ &= \frac{1}{1 - r^2} \left\langle \left(\sum_{j=1}^n g^{ij} \frac{\partial f}{\partial x_j} + \sum_{j=1}^n g^{i n+j} \frac{\partial f}{\partial y_j} \right)_{i=1}^{2n}, (x_1, \dots, x_n, y_1, \dots, y_n) \right\rangle \\ &= \sum_{i=1}^n C_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^n D_i \frac{\partial f}{\partial y_i}. \end{aligned}$$

Now we compute the coefficients C_i and D_i , using the form of the matrix $[g^{ij}] = \begin{bmatrix} C & D \\ -D & C \end{bmatrix}$. For $i = 1, \dots, n$ we have

$$\begin{aligned} C_i &= \sum_{j=1}^n (\delta_{ji} - x_j x_i - y_j y_i) x_j + \sum_{j=1}^n (x_j y_i - x_i y_j) y_j \\ &= x_i \left(1 - \sum_{j=1}^n x_j^2 - \sum_{j=1}^n y_j^2\right) = (1 - r^2) x_i, \\ D_i &= \sum_{j=1}^n (-x_j y_i + x_i y_j) x_j + \sum_{j=1}^n (\delta_{ji} - x_j x_i - y_j y_i) y_j \\ &= y_i \left(1 - \sum_{j=1}^n x_j^2 - \sum_{j=1}^n y_j^2\right) = (1 - r^2) y_i. \end{aligned}$$

Finally, the above formulae give

$$\langle \mathbf{grad} f, \mathbf{n} \rangle = (1 - r^2) \langle \nabla f, (x_1, \dots, x_n, y_1, \dots, y_n) \rangle.$$

In the space \mathbb{R}^{2n} we have (λ denotes here the Lebesgue measure):

$$\lambda(B(0, r)) = \int_0^r \omega_{2n-1} \varrho^{2n-1} d\varrho = \frac{\omega_{2n-1}}{2n} r^{2n},$$

whence the measure of the sphere S_r , which is the derivative of $\lambda(B(0, r))$, is $\omega_{2n-1} r^{2n-1}$.

In the unit ball of \mathbb{R}^{2n} equipped with the Riemann metric (inherited from the Bergman metric in the unit ball of \mathbb{C}^n) we have the following Riemannian volume V of the ball (cf. [R, Thm. 2.2.6(ii)]):

$$V(B(0, r)) = \int_0^r \omega_{2n-1} \frac{\varrho^{2n-1}}{(1 - \varrho^2)^{n+1}} d\varrho.$$

This implies that the Riemannian measure of the sphere S_r is equal to

$$(50) \quad A(r) = \lim_{\varepsilon \rightarrow 0^+} \frac{V(B(0, r + \varepsilon)) - V(B(0, r))}{\varepsilon / (1 - r^2)} = \omega_{2n-1} \frac{r^{2n-1}}{(1 - r^2)^n},$$

because the interval (in the unit ball of \mathbb{C}^n) with endpoints $(r, 0, \dots, 0)$ and $(r + \varepsilon, 0, \dots, 0)$ has length $\varepsilon / (1 - r^2) + o(\varepsilon)$. Thus the measure on S_r , induced by the Riemannian measure, is equal to the Euclidean measure divided by $(1 - r^2)^n$.

LEMMA 11. Consider B_1 , the unit ball in \mathbb{C}^n , equipped with the Bergman metric and let Δ_{LB} be the Laplace–Beltrami operator on this Riemannian manifold. Fix $r \in (0, 1)$ and denote by $P_r(x, y)$ the Poisson kernel, and by $G_r(x, y)$ the Green function of B_r with respect to Δ_{LB} . Then, in the above notation,

$$-\frac{d}{dR} \Big|_{R=r} (G_r(x, Ry))_E = (1 - r^2)^{n-1} P_r(x, ry),$$

where the subscript E in $(G_r(x, Ry))_E$ denotes that this quantity is computed with respect to the Euclidean metric.

Proof. By Theorem 8 of [Ch, p. 174], for u harmonic in B_r and $u \in C^1(\overline{B}_r)$,

$$-u(x) = \int_{\partial B_r} \left(\frac{\partial G_r}{\partial v_w} \right)_R (x, w) u(w) dA(w),$$

where now the subscript R denotes that the derivative is computed with respect to the Riemannian metric. We want to use the Euclidean metric and then, by (50), for $w \in S_r$ we have

$$dA(w) = \frac{d\sigma_r(w)}{(1 - r^2)^n},$$

and the Riemannian directional derivative is $1 - r^2$ times the Euclidean one, hence

$$-u(x) = \frac{1}{(1 - r^2)^{n-1}} \int_{\partial B_r} \left(\frac{\partial G}{\partial v_w} \right)_E(x, w) u(w) d\sigma_r(w).$$

Because the function $\partial G / \partial v_w$ reproduces harmonic functions, it has to be the Poisson kernel. This gives

$$-\frac{d}{dR} \Big|_{R=r} (G_D(x, Ry))_E = (1 - r^2)^{n-1} P_r(x, ry). \blacksquare$$

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Institute of Mathematics and Computer Science
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: Tomasz.Zak@pwr.wroc.pl

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