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Diameter, extreme points and topology

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Abstract. We study the extremal structure of Banach spaces of continuous functions with the diameter norm.

1. Introduction. Let X be a Banach space. The closed unit ball of X and the set (maybe empty) of its extreme points will be denoted by B_X and E_X, respectively. As usual, we denote by $\mathrm{co}(E_X)$ the convex hull of E_X and by $\overline{\mathrm{co}}(E_X)$ its closure, that is, the closed convex hull of E_X.

From now on, K will stand for a compact Hausdorff space and $C(K)$ will be the space of continuous functions from K into $\mathbb{R}$. For every $f \in C(K)$, $\varrho(f)$ will denote the diameter of $f(K)$:

$$\varrho(f) = \max\{|f(t) - f(t')| : t, t' \in K\}.$$

It is clear that ϱ is a seminorm on $C(K)$ and, given $f \in C(K)$, $\varrho(f) = 0$ if, and only if, f is a constant function. The quotient of $C(K)$ by the constant functions becomes a Banach space with respect to the diameter by defining $\varrho([f]) = \varrho(f)$ for every $f \in C(K)$.

Alternatively, we can fix a point t_0 in K and consider the following subspace of $C(K)$:

$$X = \{f \in C(K) : f(t_0) = 0\}.$$

Below we will work with this subspace, which is—for the diameter norm—a Banach space isometric to the above-mentioned quotient. In order to avoid the trivial case we will suppose that K has at least two points. The diameter norm on X is equivalent to the uniform norm. In fact,

$$\|f\|_\infty \le \varrho(f) \le 2\|f\|_\infty \quad \text{for every } f \in X.$$

In recent years, several authors have been considering the diameter to get Banach–Stone type theorems. In this connection, we mention the first work on diameter preserving linear bijections by M. Győry and L. Molnár [2].

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However, our purpose is different. We will obtain Krein–Milman type theorems in spaces of continuous functions with the diameter norm.

2. The results. We start with a useful description of extreme points of the unit ball of X.

LEMMA 1. *The extreme points of B_X are the functions $e \in X$ such that $e(K) = \{0,1\}$ or $e(K) = \{-1,0\}$.*

Proof. Let e be in X such that $e(K) = \{0,1\}$ or $e(K) = \{-1,0\}$ and take $f, g \in X$ with $\varrho(f) \le 1$, $\varrho(g) \le 1$ and $e = (f+g)/2$. Given $t \in K$ with $|e(t)| = 1$, it is obvious that $e(t) = f(t) = g(t)$. Suppose that $e(t) = 0$ and fix a point $t' \in K$ such that $|e(t')| = 1$. Then $e(t') = f(t') = g(t')$ and

$$e(t') = e(t') - e(t) = \frac{f(t') + g(t')}{2} - \frac{f(t) + g(t)}{2} = \frac{f(t') - f(t)}{2} + \frac{g(t') - g(t)}{2}.$$

Taking into account that $|e(t')| = 1$ and $|f(t') - f(t)| \le 1$, $|g(t') - g(t)| \le 1$, it follows that

$$e(t') = f(t') - f(t) = g(t') - g(t),$$

and consequently $e(t) = 0 = f(t) = g(t)$.

Conversely, let e be an extreme point of the unit ball of X. For each t in K define

$$\alpha(t) = \max\{|e(t) - e(t')| : t' \in K\}$$

and choose a point $t_1 \in K$. Suppose, to obtain a contradiction, that $\alpha(t_1) < 1$ and let ε be in $]0, (1 - \alpha(t_1))/2[$. Let us consider a continuous function $\varphi : K \to [0,1]$ such that

$$\varphi(t) = \begin{cases} 1 & \text{if } |e(t) - e(t_1)| \le \varepsilon, \\ 0 & \text{if } |e(t) - e(t_1)| \ge 1 - \alpha(t_1)/2, \end{cases}$$

and let t, t' be in K. First suppose that $|e(t) - e(t_1)| \le (1 - \alpha(t_1))/2$. Then

$$\begin{aligned} &\left| \left(e(t) \pm \frac{1 - \alpha(t_1)}{2} \varphi(t) \right) - \left(e(t') \pm \frac{1 - \alpha(t_1)}{2} \varphi(t') \right) \right| \\ &\quad = \left| e(t) - e(t') \pm \frac{1 - \alpha(t_1)}{2} (\varphi(t) - \varphi(t')) \right| \\ &\quad \le |e(t) - e(t')| + \frac{1 - \alpha(t_1)}{2} \le |e(t) - e(t_1)| + |e(t_1) - e(t')| + \frac{1 - \alpha(t_1)}{2} \\ &\quad \le \frac{1 - \alpha(t_1)}{2} + \alpha(t_1) + \frac{1 - \alpha(t_1)}{2} = 1. \end{aligned}$$

Of course, the same holds if $|e(t') - e(t_1)| \le (1 - \alpha(t_1))/2$. On the other hand, if $|e(t) - e(t_1)| \ge (1 - \alpha(t_1))/2$ and $|e(t') - e(t_1)| \ge (1 - \alpha(t_1))/2$, we have

$$\left|\left(e(t) \pm \frac{1-\alpha(t_1)}{2}\,\varphi(t)\right) - \left(e(t') \pm \frac{1-\alpha(t_1)}{2}\,\varphi(t')\right)\right| = |e(t) - e(t')| \le 1.$$

This proves that $\varrho\big(e \pm \frac{1-\alpha(t_1)}{2}\varphi\big) \le 1$ and contradicts, since $\varphi \neq 0$, the fact that e is an extreme point of the unit ball of X.

Consequently, since t_1 was arbitrary in K, $\alpha(t) = 1$ for every $t \in K$. In particular, $\alpha(t_0) = 1$, and thus $e(K) \cap \{-1, 1\} \neq \emptyset$. If $1 \in e(K)$ it is clear that $0 \le e(t) \le 1$ for each $t \in K$ and, according to the above, there is not a point t in K such that $0 < e(t) < 1$. Therefore, $e(K) = \{0, 1\}$. A similar argument proves that $e(K) = \{-1, 0\}$ if $-1 \in e(K)$. ∎

As an immediate consequence, we have

Corollary 2. *$E_X \neq \emptyset$ if, and only if, K is not connected.*

It is worth noting that, in general, the geometry of X with the diameter norm is different from its geometry with the uniform norm. For instance, if K does not have isolated points and it is not connected, then E_X is nonempty by the previous lemma but the unit ball of X with the uniform norm does not contain extreme points.

On the other hand, the preceding corollary emphasizes that the extremal structure of X depends on the degree of disconnectedness of K.

Let us recall that a topological space is *totally disconnected* if every element has a neighbourhood base of closed and open sets.

Theorem 3. *The following assertions are equivalent*:

(i) *For each $x \in B_X$ there is a sequence $\{\lambda_n\}$ in $[0,1]$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and a sequence $\{e_n\}$ of extreme points of the unit ball of X such that $x = \sum_{n=1}^{\infty} \lambda_n e_n$.*

(ii) $B_X = \overline{\mathrm{co}}(E_X)$.

(iii) *K is totally disconnected.*

Proof. It is clear that (i) implies (ii).

In order to prove that (ii) implies (iii), let t_1 be an element of K with $t_1 \neq t_0$. Let U and V be open and disjoint neighbourhoods of t_0 and t_1, respectively. Select two continuous functions $x, y : K \to [0,1]$ such that $x(t_0) = 0$ and $x(t) = 1$ for every $t \in K \backslash U$, and $y(t_1) = 1$ and $y(t) = 0$ for each $t \in K \backslash V$. Obviously $x, y \in B_X$ and, by (ii), there are $f, g \in \mathrm{co}(E_X)$ such that $\varrho(f - x) < 1/2$ and $\varrho(g - y) < 1/2$. Taking into account that $f(K)$ and $g(K)$ are finite, the sets $U_1 = \{t \in K : f(t) \le 1/2\}$ and $V_1 = \{t \in K : g(t) \ge 1/2\}$ are open and closed. Furthermore, $t_0 \in U_1 \subset U$ and $t_1 \in V_1 \subset V$. Thus, every point in K has a neighbourhood base of closed and open sets, so K is totally disconnected.

Next, we prove that (iii) implies (i). Let f be in X with $f(K) \subset [0,1]$. Given t in K, we can choose an open and closed neighbourhood V_t of t satisfying

$$\begin{aligned} V_t \cap f^{-1}(\{1\}) = \emptyset \quad & \text{if } f(t) \neq 1, \\ V_t \cap f^{-1}(\{0\}) = \emptyset \quad & \text{if } f(t) = 1. \end{aligned}$$

Since $\{V_t : t \in K\}$ is an open (and closed) covering of K there are $t_1, \ldots, t_n$ in K such that $K = \bigcup_{j=1}^{n} V_{t_j}$. The sets

$$U_{t_1} = V_{t_1}, \quad U_{t_2} = V_{t_2} \setminus V_{t_1}, \; \ldots, \; U_{t_n} = V_{t_n} \setminus \bigcup_{j=1}^{n-1} V_{t_j}$$

are open and closed, pairwise disjoint, and they are also a covering of K.

Define $J = \{j \in \{1, \ldots, n\} : U_{t_j} \cap f^{-1}(\{0\}) \neq \emptyset\}$ and $U = \bigcup_{j \in J} U_{t_j}$. Obviously U is open and closed. Now, let us pick arbitrarily a point $t \in K$ and let j in $\{1, \ldots, n\}$ be such that $t \in U_{t_j}$. If $f(t) = 0$ then $U_{t_j} \cap f^{-1}(\{0\}) \neq \emptyset$, and so $j \in J$. Therefore $t \in U$. If $f(t) = 1$, then $V_{t_j} \cap f^{-1}(\{1\}) \neq \emptyset$ and necessarily $V_{t_j} \cap f^{-1}(\{0\}) = \emptyset$ (since every V_t is disjoint from $f^{-1}(\{1\})$ or $f^{-1}(\{0\})$). Thus $j \notin J$ and consequently $t \in K \setminus U$. This proves that $f^{-1}(\{0\}) \subset U$ and $f^{-1}(\{1\}) \subset K \setminus U$. Since U is open and closed, the function $e : K \to \{0,1\}$ defined by

$$e(t) = \begin{cases} 0 & \text{if } t \in U, \\ 1 & \text{if } t \in K \setminus U, \end{cases}$$

is continuous. In fact, $e \in X$. Furthermore, given $t \in K$,

$$f(t) = 0 \;\Rightarrow\; e(t) = 0, \qquad f(t) = 1 \;\Rightarrow\; e(t) = 1.$$

Let y in X satisfy $\varrho(y) \leq 1$ and suppose for the time being that $y(t) \geq 0$ for each $t \in K$. Let us select three real numbers $\lambda, \varepsilon_1, \varepsilon_2$ such that $0 < \lambda < \varepsilon_1 < \varepsilon_2 < 1/2$ and a continuous function $\varphi : [0,1] \to [0,1]$ satisfying

$$\varphi(s) = \begin{cases} 0 & \text{if } s \leq \varepsilon_1, \\ 1 & \text{if } s \geq \varepsilon_2. \end{cases}$$

Then it is clear that the function $f : K \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 0 & \text{if } y(t) \leq \varepsilon_1, \\ \varphi(y(t))y(t)/\varepsilon_2 & \text{if } \varepsilon_1 \leq y(t) \leq \varepsilon_2, \\ 1 & \text{if } y(t) \geq \varepsilon_2, \end{cases}$$

is continuous. Actually, $f \in X$ and $f(K) \subset [0,1]$. According to the above argument there is a continuous mapping $e : K \to \{0,1\}$ such that $e(t) = 0$ if $f(t) = 0$ and $e(t) = 1$ if $f(t) = 1$. Now, we define $h = (y - \lambda e)/(1 - \lambda)$.

If $y(t) \leq \varepsilon_1$ then $f(t) = 0$ and so $e(t) = 0$. It follows that $h(t) = y(t)/(1 - \lambda) \in [0,1]$.

If $\varepsilon_1 \le y(t) \le \varepsilon_2$ we have

$$0 \le \frac{\varepsilon_1 - \lambda e(t)}{1-\lambda} \le \frac{y(t) - \lambda e(t)}{1-\lambda} \le \frac{y(t)}{1-\lambda} \le \frac{\varepsilon_2}{1-\lambda} < 1.$$

Therefore, in this case also $h(t) \in [0,1]$.

Finally, if $y(t) \ge \varepsilon_2$, then $f(t) = 1$ and so $e(t) = 1$. We thus have

$$h(t) = \frac{y(t) - \lambda}{1-\lambda} \in [0,1].$$

In this way, it is clear that h belongs to X and $\varrho(h) \le 1$. Moreover, from the definition of h it follows that

$$y = \lambda e + (1-\lambda)h.$$

Note that e is an extreme point of the unit ball of X and that, in case $y(t) \le 0$, for every t in K, the previous reasoning applied to $-y$ provides the same representation of y by means of an extreme point e and a point h of the unit ball of X such that $e(K) = \{0,-1\}$ and $h(K) \subset [-1,0]$.

It remains to treat the case where there are two real numbers a and b, with $a < 0 < b$ and $b - a = 1$, such that $a \le y(t) \le b$ for every t in K.

Under this assumption let us consider the functions $y_1, y_2 : K \to \mathbb{R}$ given by

$$y_1(t) = \begin{cases} y(t)/b & \text{if } y(t) \ge 0, \\ 0 & \text{if } y(t) \le 0, \end{cases} \qquad y_2(t) = \begin{cases} 0 & \text{if } y(t) \ge 0, \\ -y(t)/a & \text{if } y(t) \le 0, \end{cases}$$

which are elements of X with $0 \le y_1(t) \le 1$ and $-1 \le y_2(t) \le 0$ for every t in K. Moreover, it is easy to see that $y = by_1 - ay_2$. By the above, given $\lambda \in \,]0,1/2[$ there exist $e \in E_X$ with $e(K) = \{0,1\}$ and $h \in B_X$ with $h(K) \subset [0,1]$ such that $y_1 = \lambda e + (1-\lambda)h$. Thus

$$\begin{aligned} y &= b(\lambda e + (1-\lambda)h) - ay_2 = b\lambda e + b(1-\lambda)h - ay_2 \\ &= b\lambda e + (1-b\lambda)\,\frac{b(1-\lambda)h - ay_2}{1-b\lambda}. \end{aligned}$$

Set

$$g = \frac{b(1-\lambda)h - ay_2}{1-b\lambda}$$

and let t be in K. If $y(t) \ge 0$,

$$\frac{a}{1-b\lambda} \le 0 \le \frac{b(1-\lambda)h(t)}{1-b\lambda} = g(t) \le \frac{b(1-\lambda)}{1-b\lambda}.$$

On the other hand, if $y(t) \le 0$, we have (using the fact that $y(t) \ge a$)

$$\begin{aligned} \frac{a}{1-b\lambda} \le \frac{y(t)}{1-b\lambda} &\le \frac{b(1-\lambda)h(t) + y(t)}{1-b\lambda} \\ &= \frac{b(1-\lambda)h(t) - ay_2(t)}{1-b\lambda} = g(t) \le \frac{b(1-\lambda)}{1-b\lambda}. \end{aligned}$$

Therefore, $g(K) \subset \left[\frac{a}{1-b\lambda}, \frac{b(1-\lambda)}{1-b\lambda}\right]$, and hence

$$\varrho(g) \leq \frac{b(1-\lambda)}{1-b\lambda} - \frac{a}{1-b\lambda} = \frac{1-b\lambda}{1-b\lambda} = 1.$$

Of course, the previous argument may be performed in an analogous way with the use of a representation of y_2 (instead of y_1) in the form $\lambda e+(1-\lambda)h$, with $e(K) = \{-1, 0\}$ and $h(K) \subset [-1, 0]$. Taking into account that λ can be chosen in the interval $]0, 1/2[$ and $\max\{-a, b\} \geq 1/2$, we have proved in fact that for every α in $]0, 1/4[$ and y in B_X, there are an extreme point e and an element g in the unit ball of X such that $y = \alpha e + (1-\alpha)g$.

To conclude, let α be in $]0, 1/4[$ and x in B_X. According to the above, there are $e_1 \in E_X$ and $g_1 \in B_X$ such that $x = \alpha e_1 + (1-\alpha)g_1$. For the same reason, $g_1 = \alpha e_2 + (1-\alpha)g_2$ for certain $e_2 \in E_X$ and $g_2 \in B_X$. Proceeding in this manner, we obtain a sequence $\{e_n\}$ of extreme points and a sequence $\{g_n\}$ of elements of the unit ball of X such that $g_n = \alpha e_{n+1} + (1-\alpha)g_{n+1}$ for every n in $\mathbb{N}$. Consequently,

$$x = \alpha e_1 + (1-\alpha)\alpha e_2 + (1-\alpha)^2\alpha e_3 + \cdots + (1-\alpha)^n\alpha e_{n+1} + (1-\alpha)^{n+1}g_{n+1}$$

for each $n \in \mathbb{N}$. From this, it follows immediately that

$$x = \sum_{n=1}^{\infty}(1-\alpha)^{n-1}\alpha e_n.$$

Since $\sum_{n=1}^{\infty}(1-\alpha)^{n-1}\alpha = 1$, it is sufficient to define $\lambda_n = (1-\alpha)^{n-1}\alpha$ for every $n \in \mathbb{N}$. ∎

In [1] R. M. Aron and R. H. Lohman introduced the following interesting concepts:

A Banach space X is said to have the *λ-property* if, for every $y \in B_X$, there are $\lambda \in]0, 1]$, $e \in E_X$ and $z \in B_X$ such that $y = \lambda e + (1-\lambda)z$. If it is possible to find a common λ for all y in B_X, then Y is said to have the *uniform λ-property.*

In view of the preceding proof, it is clear that the space X we have considered has the uniform λ-property if, and only if, K is totally disconnected. In the last paragraph of the above proof it has been proved that the uniform λ-property implies the statement (i). This fact is valid for every Banach space (see [1]) and it has been included for the sake of completeness.

To conclude, we note that the previous theorem also holds for spaces of type $C_0(L)$, where L is a noncompact, locally compact Hausdorff space. Under these conditions the diameter is a norm (equivalent to the uniform norm) and if $K = L \cup \{\infty\}$ is the one-point compactification of L, it suffices to define $t_0 = \infty$ to observe that $C_0(L)$, with the diameter norm, is nothing other than the space X defined in the introduction, for such a selection of t_0. On the other hand, L is totally disconnected if, and only if, so is its one-point compactification. It is perhaps worth remarking that the local compactness of L is important for this last fact. The existence of totally disconnected (not locally compact) spaces whose one-point compactification is connected is well known (see [3]).

Taking into account the above results and comments, we have:

Corollary 4. *Let L be a noncompact locally compact Hausdorff space and $C_0(L)$ the space of real continuous functions vanishing at infinity*, *equipped with the diameter norm. The following statements are equivalent*:

(i) *For every x in the unit ball of $C_0(L)$ there is a sequence $\{\lambda_n\}$ in $[0,1]$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and a sequence $\{e_n\}$ of extreme points of $B_{C_0(L)}$ such that $x = \sum_{n=1}^{\infty} \lambda_n e_n$.*

(ii) $B_{C_0(L)} = \overline{\mathrm{co}}(E_{C_0(L)})$.

(iii) *L is totally disconnected.*

In particular, let us observe that c_0, provided with the diameter norm, has the properties (i) and (ii) of this corollary.

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