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Ekeland's variational principle in locally p-convex spaces and related results

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Abstract. In the framework of locally p-convex spaces, two versions of Ekeland's variational principle and two versions of Caristi's fixed point theorem are given. It is shown that the four results are mutually equivalent. Moreover, by using the local completeness theory, a p-drop theorem in locally p-convex spaces is proven.

1. Introduction. The variational principle discovered by Ekeland (see [6]) in 1972 is one of the most important results of nonlinear functional analysis and it has significant applications in optimization, control theory, game theory, global analysis and various other fields; see, for example, [1, 7, 8, 18]. It is well known that this principle is equivalent to Caristi's fixed point theorem [2], to Daneš' drop theorem [4, 5], and to the petal theorem [10, 19]. In the past decade, Ekeland's variational principle and some related results were extended from the Banach space setting to the topological vector space setting, in particular, to locally convex spaces (see, for example, [3, 9, 11, 17, 22–24, 28]). By using the local convergence and local completeness theory on locally convex spaces (we refer to [12, p. 225] and [20, Chap. 5]), we presented general versions of Ekeland's variational principle and some related theorems in locally convex spaces (see [23], [24]). Recently [25] we introduced the notion of local completeness of locally p-convex spaces and established a general version of the Borwein–Preiss variational principle in locally p-convex spaces. It is natural to consider extensions of Ekeland's variational principle and the related results to locally complete locally p-convex spaces. This is the main topic of the present paper.

The paper is organized as follows. In Section 2, we recall some basic facts on p-convex sets, absolutely p-convex sets and locally complete locally

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p-convex spaces. In Section 3, using a modified version of the Bishop–Phelps lemma [21] we give two versions of Ekeland's variational principle in locally complete locally p-convex spaces and prove that they are indeed equivalent. In Section 4, by using the two versions of Ekeland's variational principle of Section 3, we deduce two versions of Caristi's fixed point theorem in locally p-convex spaces. It turns that the two versions of fixed point theorems and the two versions of Ekeland's variational principle are all equivalent to each other. Finally, in Section 5, we give a p-drop theorem in locally p-convex spaces, which extends the result in [15]. Our proof is direct and only depends on the local completeness of locally p-convex spaces.

2. Preliminaries and notations. In this paper we need some knowledge of p-convex sets and absolutely p-convex sets (see, for example, [13–16, 26–27]). Moreover, we need the notions of local convergence, local closedness and local completeness in locally p-convex spaces (for details, see [25]), which generalize the corresponding notions in locally convex spaces (see, for example, [20]).

DEFINITION 2.1 (see [13, 14, 16, 27]). Let X be a linear space, $0 < p \leq 1$ and $C \subset X$ be nonempty. If $\alpha x + \beta y \in C$ for any $x, y \in C$ and any $\alpha, \beta \geq 0$ with $\alpha^p + \beta^p = 1$, then C is called *p-convex.* If C is in addition circled then it is called *absolutely p-convex.* Obviously C is absolutely p-convex if and only if $\alpha x + \beta y \in C$ for any $x, y \in C$ and any scalars α, β such that $|\alpha|^p + |\beta|^p \leq 1$.

By induction, C is p-convex (resp. absolutely p-convex) if and only if $\sum_{i=1}^n \alpha_i A \subset A$ whenever $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i^p = 1$ (resp. whenever $\sum_{i=1}^n |\alpha_i|^p \leq 1$), with n running through all positive integers. It is easy to verify that arbitrary intersections of p-convex sets (resp. absolutely p-convex sets) are still p-convex (resp. absolutely p-convex). Thus for any nonempty $S \subset X$ there is a smallest p-convex set (resp. absolutely p-convex set) containing S, which is called the *p-convex hull* of S (resp. the *absolutely p-convex hull* of S). We denote the p-convex hull of S by p-co(S) and the absolutely p-convex hull of S by p-aco(S). It is easy to see that

$$p\text{-co}(S) = \Big\{\sum_{i=1}^n \alpha_i x_i : n \in \mathbb{N},\, x_1, \ldots, x_n \in S,\, \alpha_1, \ldots, \alpha_n \geq 0,\, \sum_{i=1}^n \alpha_i^p = 1\Big\},$$

$$p\text{-aco}(S) = \Big\{\sum_{i=1}^n \alpha_i x_i : n \in \mathbb{N},\, x_1, \ldots, x_n \in S,\, \sum_{i=1}^n |\alpha_i|^p \leq 1\Big\}.$$

Let $x \in X$ and $B \subset X$ be nonempty. We denote p-co$(\{x\} \cup B)$ by $D_p(x, B)$, and p-aco$(\{x\} \cup B)$ by $\Gamma_p(x, B)$. If B is p-convex, we call $D_p(x, B)$ the *p-drop* determined by x and B.

Remark 2.1. Assume that $0 < p < 1$ and C is a p-convex set in a linear space X. Then $\alpha x \in C$ for any $x \in C$ and any α such that $2^{1-1/p} \le \alpha \le 1$. Hence $\alpha x \in C$ for any $x \in C$ and any $0 < \alpha \le 1$. Thus either 0 belongs to C or it is an algebraic boundary point of C (see [14, p. 177]). If the p-convex set C is algebraic closed or locally closed, then clearly $0 \in C$. Hence for any nonempty subset S of X, p-co(S) can also be written as

$$p\text{-co}(S) = \Big\{\sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N},\ x_1, \ldots, x_n \in S,\ \alpha_1, \ldots, \alpha_n \ge 0,\ 0 < \sum_{i=1}^{n} \alpha_i^p \le 1\Big\}.$$

Besides, we remark that if C is a convex set (i.e. $p = 1$), we cannot deduce $\alpha x \in C$ from $x \in C$ and $0 < \alpha < 1$.

Remark 2.2. From Remark 2.1 we know that for any p-convex set C ($0 < p < 1$), we can define its *gauge* as follows:

$$q_C(x) = \begin{cases} \inf\{t > 0 : x \in t^{1/p} C\} & \text{if there exists } t > 0 \text{ such that } x \in t^{1/p} C, \\ \infty & \text{else.} \end{cases}$$

Clearly, for any $x, y \in X$ and any $\alpha > 0$, we have $q_C(x + y) \le q_C(x) + q_C(y)$ and $q_C(\alpha x) = \alpha^p q_C(x)$. Moreover, if C is absorbing, then $q_C(0) = 0$ and $0 \le q_C(x) < \infty$ for any $x \in X$.

For any convex set C containing 0, we can define its gauge as follows:

$$q_C(x) = \begin{cases} \inf\{t > 0 : x \in tC\} & \text{if there exists } t > 0 \text{ such that } x \in tC, \\ \infty & \text{else.} \end{cases}$$

Clearly, for any $x, y \in C$ and any $\alpha \ge 0$, we have $q_C(x + y) \le q_C(x) + q_C(y)$ and $q_C(\alpha x) = \alpha q_C(x)$. Moreover, if C is absorbing, then $0 \le q_C(x) < \infty$ for any $x \in X$.

In general, a function $q : X \to \mathbb{R} \cup \{+\infty\}$ is called *subadditive* if $q(x + y) \le q(x) + q(y)$ for any $x, y \in X$, and *positive p-homogeneous* if $q(\alpha x) = \alpha^p q(x)$ for any $x \in X$ and $\alpha \ge 0$ (when $p = 1$, we call it briefly positive homogeneous). Obviously, the gauge of a p-convex set ($0 < p \le 1$) containing 0 is a subadditive, positive p-homogeneous function.

Now we turn to the discussion of locally p-convex spaces. A Hausdorff topological linear space (briefly, a topological linear space) is called *locally p-convex* ($0 < p \le 1$) if there is a basis $\mathcal{U}$ of 0-neighborhoods consisting of absolutely p-convex sets. It is easy to see that a topological linear space X is locally p-convex if and only if there is a family $\{\|\ \|_\lambda\}_{\lambda \in \Lambda}$ of p-homogeneous F-pseudonorms which determines the topology of X (see, for example, [27]). In particular, a locally 1-convex space is called *locally convex.*

Definition 2.2 (see [25]). Let X be a locally p-convex space. A subset B of X is called a *p-disc* if it is bounded and absolutely p-convex. Let E_B

denote the linear span of B, span$[B]$, endowed with the topology determined by the gauge q_B. If E_B is complete, then the p-disc B is called *self-complete.* A sequence (x_n) in X is called a *locally Cauchy sequence* if there is a p-disc B in X such that (x_n) is a Cauchy sequence in E_B. A sequence (x_n) in X is said to be *locally convergent* to a point x_0 if there is a p-disc B such that $x_n \to x_0$ in E_B. Clearly, (x_n) is locally convergent to x_0 if and only if $(x_n - x_0)$ is locally convergent to 0. A point x_0 is called a *locally limit point* of a set A in X if there is a sequence (x_n) in A locally convergent to x_0. The set A is called *locally closed* if every locally limit point of A belongs to A.

LEMMA 2.1 (see [25]). *Let X be a locally p-convex space. Then the following three statements are equivalent*:

(i) *For each bounded set A in X there is a self-complete p-disc B such that $A \subset B$.*
(ii) *Every locally Cauchy sequence in X is locally convergent.*
(iii) *Every closed p-disc is self-complete.*

DEFINITION 2.3 (see [25]). A locally p-convex space X is called *locally complete* if one of the three equivalent statements in Lemma 2.1 is satisfied. A subset A of a locally p-convex space is called *locally complete* if every locally Cauchy sequence in A is locally convergent to a point in A.

Obviously, every locally complete set is locally closed and every locally closed subset of a locally complete space is locally complete.

Let X be a locally p-convex space and let $f : X \to \mathbb{R} \cup \{+\infty\}$. The *effective domain* of f is $\operatorname{dom} f = \{x \in X : f(x) < \infty\}$. If $\operatorname{dom} f \neq \emptyset$, we say that f is *proper.* As in [23], we introduce the notion of locally lower semicontinuous function as follows.

DEFINITION 2.4. Let X be a locally p-convex space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. If the set $\{x \in X : f(x) \le r\}$ is locally closed in X for every $r \in \mathbb{R}$, then f is called *locally lower semicontinuous.*

Concerning the relationship between locally lower semicontinuous functions, sequentially lower semicontinuous functions and lower semicontinuous functions, we refer to [23]. In the next section we shall consider some extensions of Ekeland's variational principle concerning locally lower semicontinuous functions in the setting of locally p-convex spaces.

3. Ekeland's variational principle in locally p-convex spaces.

First we give a modified version of the Bishop–Phelps lemma.

LEMMA 3.1. *Let X be a locally complete locally p-convex space and $q : X \to \mathbb{R}^+ \cup \{+\infty\}$ be a locally lower semicontinuous*, *positive p-homogeneous*, *subadditive proper function such that $B := \{x \in X : q(x) \le 1\}$ is bounded.*

If a nonempty set $A \subset X \times \mathbb{R}$ is locally closed and $\inf\{r : (x,r) \in A\} = 0$, *then for any $\beta > 0$ and any $(x_0, r_0) \in A$, there exists*

$$(x,r) \in A \cap (K_\beta + (x_0, r_0))$$

such that

$$\{(x,r)\} = A \cap (K_\beta + (x,r)),$$

where $K_\beta := \{(x,r) \in X \times R : \beta q(x) \le -r\}$.

Proof. It is easy to verify that $(0,0) \in K_\beta$, $K_\beta + K_\beta \subset K_\beta$ and K_β is locally closed in $X \times \mathbb{R}$. Define $h : X \times \mathbb{R} \to \mathbb{R}$ as follows:

$$h(x,r) = r, \quad \forall (x,r) \in X \times \mathbb{R}.$$

Put $A_0 := A \cap (K_\beta + (x_0, r_0))$. Then

$$0 = \inf h(A) \le \inf h(A_0) \le r_0 < \infty.$$

There exists $(x_1, r_1) \in A_0$ such that

$$r_1 < \inf h(A_0) + 1.$$

Put $A_1 := A \cap (K_\beta + (x_1, r_1))$. Then

$$A_1 \subset A \cap (K_\beta + K_\beta + (x_0, r_0)) \subset A \cap (K_\beta + (x_0, r_0)) = A_0.$$

Clearly,

$$0 \le \inf h(A_0) \le \inf h(A_1) \le r_1 < \infty.$$

Hence there exists $(x_2, r_2) \in A_1$ such that

$$r_2 < \inf h(A_1) + 1/2.$$

Repeating this process we obtain a sequence $\{(x_n, r_n)\}_{n \in \mathbb{N}}$ such that

$$(x_{n+1}, r_{n+1}) \in A_n = A \cap (K_\beta + (x_n, r_n)) \quad \text{and} \quad r_{n+1} < \inf h(A_n) + \frac{1}{n+1}.$$

Clearly, for any $m > n > 1$,

$$\begin{aligned} \beta q(x_m - x_n) &\le r_n - r_m < \inf h(A_{n-1}) + \frac{1}{n} - r_m \\ &\le \inf h(A_{m-1}) + \frac{1}{n} - r_m \le \frac{1}{n}, \end{aligned}$$

so $\{x_n\}_{n \in \mathbb{N}}$ is a locally Cauchy sequence in X, and hence is locally convergent to some $x \in X$. Also, $\{r_n\}$ is convergent to some $r \in \mathbb{R}$. Since $A_n = A \cap (K_\beta + (x_n, r_n))$ is locally closed, $(x_m, r_m) \in A_n$ for every $m > n$ and $\{(x_m, r_m)\}_{m > n}$ is locally convergent to (x,r), we have $(x,r) \in A_n$ and hence $(x,r) \in \bigcap_{n=0}^{\infty} A_n$. In particular,

$$(x,r) \in A_0 = A \cap (K_\beta + (x_0, r_0)).$$

Next we show that $\{(x,r)\} = A \cap (K_\beta + (x,r))$. Indeed, if $(y,s) \in A \cap (K_\beta + (x,r))$, then for every n,

$$(y,s) \in A \cap (K_\beta + K_\beta + (x_n, r_n)) \subset A \cap (K_\beta + (x_n, r_n)) = A_n.$$

Thus $(y - x_n, s - r_n) \in K_\beta$ and hence

$$\beta q(y - x_n) \le r_n - s < \inf h(A_{n-1}) + \frac{1}{n} - s \le \inf h(A_n) + \frac{1}{n} - s \le \frac{1}{n}.$$

Therefore $\{x_n\}$ is locally convergent to y and $\{r_n\}$ is convergent to s. By the uniqueness of limits, we have $y = x$ and $s = r$, as desired. ∎

THEOREM 3.1. *Let X be a locally complete locally p-convex space and $q : X \to \mathbb{R}^+ \cup \{+\infty\}$ be a locally lower semicontinuous, positive p-homogeneous, subadditive proper function such that $B := \{x \in X : q(x) \le 1\}$ is bounded. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a locally lower semicontinuous, bounded from below, proper function and let $x_0 \in \operatorname{dom} f$. Then for any $\beta > 0$, there exists $z \in X$ such that*

(i) $f(z) + \beta q(z - x_0) \le f(x_0)$;
(ii) *for any* $x \ne z$, $f(z) < f(x) + \beta q(x - z)$.

Proof. We may assume that $\inf f(X) = 0$. Put $A := \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}$. Then $\inf\{r : (x, r) \in A\} = 0$. Clearly $(x_0, f(x_0)) \in A$. By Lemma 3.1, there exists

$$(z, r) \in A \cap (K_\beta + (x_0, f(x_0))) \tag{1}$$

such that

$$\{(z, r)\} = A \cap (K_\beta + (z, r)). \tag{2}$$

From (1), we have $f(z) \le r$ and

$$\beta q(z - x_0) \le f(x_0) - r \le f(x_0) - f(z),$$

proving (i).

Next we show that $f(z) = r$. If not, we have $f(z) < r$. Clearly $(z, f(z)) \in A$ and by (2),

$$(z, f(z)) \notin K_\beta + (z, r),$$

which implies that

$$(0, f(z) - r) \notin K_\beta, \quad \text{i.e.} \quad 0 = \beta q(0) > r - f(z),$$

contradicting $f(z) < r$. Thus $r = f(z)$ and hence (2) becomes

$$\{(z, f(z))\} = A \cap (K_\beta + (z, f(z))). \tag{3}$$

If $x \in X$ and $f(x) = \infty$, then certainly (ii) holds. Let $x \in \operatorname{dom} f$ and $x \ne z$. Then by (3) we have

$$(x, f(x)) \notin K_\beta + (z, f(z)),$$

that is, $(x - z, f(x) - f(z)) \notin K_\beta$, or equivalently, $\beta q(x - z) > f(z) - f(x)$. ∎

LEMMA 3.2. *Let X be a locally p-convex space with the topology generated by a family $\{\|\,\|_\lambda\}_{\lambda \in \Lambda}$ of p-homogeneous F-pseudonorms and $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ be a family of positive real numbers. Then $B := \bigcap_{\lambda \in \Lambda} \{x \in X : \alpha_\lambda \|x\|_\lambda \le 1\}$ is a* *closed, bounded, absolutely p-convex set in X, and $q_B(x) = \sup_{\lambda\in\Lambda} \alpha_\lambda \|x\|_\lambda$ for any $x \in X$, where*

$$q_B(x) = \begin{cases} \inf\{t > 0 : x \in t^{1/p}B\} & \text{if there exists } t > 0 \text{ such that } x \in t^{1/p}B, \\ \infty & \text{else.} \end{cases}$$

Proof. It is easy to verify that B is closed, bounded, and absolutely p-convex. For any $x \in X$ and $\varepsilon > 0$, by the definition of q_B we have

$$\frac{x}{(q_B(x)+\varepsilon)^{1/p}} \in B,$$

which means that

$$\alpha_\lambda \left\| \frac{x}{(q_B(x)+\varepsilon)^{1/p}} \right\|_\lambda \le 1, \quad \forall \lambda \in \Lambda.$$

Hence

$$\frac{\alpha_\lambda}{q_B(x)+\varepsilon} \|x\|_\lambda \le 1, \quad \forall \lambda \in \Lambda.$$

This implies that

$$\sup_{\lambda\in\Lambda} \alpha_\lambda \|x\|_\lambda \le q_B(x).$$

Suppose that there exist $x_0 \in X$ and $\varepsilon > 0$ such that

$$\sup_{\lambda\in\Lambda} \alpha_\lambda \|x_0\|_\lambda < q_B(x_0) - \varepsilon.$$

Then

$$\sup_{\lambda\in\Lambda} \alpha_\lambda \left\| \frac{x_0}{(q_B(x_0)-\varepsilon)^{1/p}} \right\|_\lambda = \sup_{\lambda\in\Lambda} \frac{\alpha_\lambda \|x_0\|_\lambda}{q_B(x_0)-\varepsilon} < 1.$$

Hence

$$\frac{x_0}{(q_B(x_0)-\varepsilon)^{1/p}} \in B, \quad \text{i.e.,} \quad x_0 \in (q_B(x_0)-\varepsilon)^{1/p}B.$$

But this contradicts the definition of q_B. Therefore

$$q_B(x) = \sup_{\lambda\in\Lambda} \alpha_\lambda \|x\|_\lambda, \quad \forall x \in X. \ \blacksquare$$

THEOREM 3.2. *Let X be a locally complete locally p-convex space with the topology generated by a family $\{\|\ \|_\lambda\}_{\lambda\in\Lambda}$ of p-homogeneous F-pseudonorms and $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ be a family of positive real numbers. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a locally lower semicontinuous, bounded from below, proper function and let $x_0 \in \operatorname{dom} f$. Then for any $\beta > 0$, there exists $z \in X$ such that*

(i) $f(z) + \beta \sup_{\lambda\in\Lambda} \alpha_\lambda \|z - x_0\|_\lambda \le f(x_0)$;
(ii) *for any $x \ne z$, $f(z) < f(x) + \beta \sup_{\lambda\in\Lambda} \alpha_\lambda \|x - z\|_\lambda$.*

Proof. Put $B := \bigcap_{\lambda\in\Lambda} \{x \in X : \alpha_\lambda \|x\|_\lambda \le 1\}$. By Lemma 3.2,

$$q_B(x) = \sup_{\lambda\in\Lambda} \alpha_\lambda \|x\|_\lambda, \quad \forall x \in X. \tag{4}$$

Combining (4) and Theorem 3.1 completes the proof. $\blacksquare$

Since a complete p-norm space is a locally complete locally p-convex space whose topology is generated by a single p-norm, from Theorem 3.2 we have the following:

Corollary 3.1. *Let* $(X, \| \|)$ *be a complete* p*-normed space*, $f : (X, \| \|) \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous*, *bounded from below*, *proper function and* $x_0 \in \operatorname{dom} f$. *Then for any* $\beta > 0$, *there exists* $z \in X$ *such that*

(i) $f(z) + \beta \|z - x_0\| \le f(x_0)$;
(ii) *for any* $x \neq z$, $f(z) < f(x) + \beta \|x - z\|$.

Theorem 3.3. *Theorems* 3.1 *and* 3.2 *are mutually equivalent.*

Proof. We only need to prove that Theorem 3.2 implies Theorem 3.1. In fact, we shall prove that Corollary 3.1 implies Theorem 3.1. Let X, q, B and f be as in Theorem 3.1 and let $x_0 \in \operatorname{dom} f$. We denote by T the closure of $\Gamma_p(x_0, B)$. Since X is a locally complete locally p-convex space, by Lemma 2.1 and Definition 2.3 we know that T is a self-complete p-disc, i.e., $(X_T, \| \|_T)$ is a complete p-normed space. Here $X_T := \operatorname{span}[T]$ and for any $x \in X_T$, $\|x\|_T := \inf\{t > 0 : x \in t^{1/p} T\}$. Put

$$C := \{x \in X_T : f(x) + \beta q(x - x_0) \le f(x_0)\}.$$

Then C is closed in $(X_T, \| \|_T)$ since f and q are locally lower semicontinuous. Define a function g on X_T as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in C, \\ \infty & \text{if } x \in X_T \setminus C. \end{cases}$$

Then g is a lower semicontinuous, bounded from below, proper function on $(X_T, \| \|_T)$ and $x_0 \in \operatorname{dom} g$. Now applying Corollary 3.1 to the complete p-normed space $(X_T, \| \|_T)$ and the function g, we conclude that there exists $z \in X_T$ such that

(i) $g(z) + \beta \|z - x_0\|_T \le g(x_0) = f(x_0)$;
(ii) for any $x \in X_T$ and $x \neq z$,

$$g(z) < g(x) + \beta \|x - z\|_T. \tag{5}$$

From (i), we know that $g(z) < \infty$ and hence $z \in C$, that is,

$$f(z) + \beta q(z - x_0) \le f(x_0). \tag{6}$$

Thus (i) in Theorem 3.1 holds. Next we show that (ii) in Theorem 3.1 holds in each of the following three possible cases.

Case 1. Let $x \neq z$ and $x \in C$. Then by (5) we have

$$f(z) < f(x) + \beta \|x - z\|_T. \tag{7}$$

Since $T \supset B$ and $\| \|_T$ is the gauge of T, and clearly q is the gauge of B, we have $\| \cdot \|_T \le q(\cdot)$. Combining this with (7), we conclude that $f(z) < f(x) + \beta q(x - z)$.

CASE 2. Let $x \neq z$ and $x \in X_T \setminus C$. From the definition of C, we have

$$f(x) + \beta q(x - x_0) > f(x_0).$$

Combining this with (6), we have

$$\begin{aligned} f(z) + \beta q(z - x_0) &\le f(x_0) < f(x) + \beta q(x - x_0) \\ &\le f(x) + \beta q(x - z) + \beta q(z - x_0). \end{aligned} \tag{8}$$

By (6), we know that $\beta q(z - x_0) < \infty$, so (8) yields $f(z) < f(x) + \beta q(x - z)$.

CASE 3. Let $x \neq z$ and $x \notin X_T$. Since $z \in X_T$, we have $x - z \notin X_T$. Thus $x - z \notin t^{1/p}B$ for any $t > 0$, so $q(x - z) = \infty$ and obviously $f(z) < f(x) + \beta q(x - z)$. ∎

Obviously Theorem 3.1 is an extension of [11, Theorem 2] and [24, Corollary 3.2]. And Theorem 3.2 is an extension of [11, Theorem 3] and [24, Corollary 3.1]. In fact, by slightly modifying Lemma 3.1 we can obtain the following extensions of [24, Theorems 3.2 and 3.1].

THEOREM 3.1′. *Let X be a locally p-convex space and $q : X \to \mathbb{R}^+ \cup \{+\infty\}$ be a locally lower semicontinuous, positive p-homogeneous, subadditive proper function such that $B := \{x \in X : q(x) \le 1\}$ is bounded. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a locally lower semicontinuous, bounded from below, proper function and let $x_0 \in \operatorname{dom} f$. If $\{x \in X : f(x) \le f(x_0)\}$ or B is locally complete, then for any $\beta > 0$, there exists $z \in X$ such that*

(i) $f(z) + \beta q(z - x_0) \le f(x_0)$;
(ii) *for any* $x \neq z$, $f(z) < f(x) + \beta q(x - z)$.

THEOREM 3.2′. *Let X be a locally p-convex space with the topology generated by a family $\{\| \|_\lambda\}_{\lambda \in \Lambda}$ of p-homogeneous F-pseudonorms and $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ be a family of positive real numbers. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a locally lower semicontinuous, bounded from below, proper function and let $x_0 \in \operatorname{dom} f$. If $\{x \in X : f(x) \le f(x_0)\}$ or $\bigcap_{\lambda \in \Lambda} \{x \in X : \alpha_\lambda \|x\|_\lambda \le 1\}$ is locally complete, then for any $\beta > 0$, there exists $z \in X$ such that*

(i) $f(z) + \beta \sup_{\lambda \in \Lambda} \alpha_\lambda \|z - x_0\|_\lambda \le f(x_0)$;
(ii) *for any* $x \neq z$, $f(z) < f(x) + \beta \sup_{\lambda \in \Lambda} \alpha_\lambda \|x - z\|_\lambda$.

4. Equivalence of Ekeland's variational principle and Caristi's fixed point theorem. By using Theorems 3.1 and 3.2, we can deduce the following two versions of Caristi's fixed point theorem in locally p-convex spaces.

THEOREM 4.1 (Extended Caristi's fixed point theorem). *Let X be a locally complete locally p-convex space and $q : X \to \mathbb{R}^+ \cup \{+\infty\}$ be a locally lower semicontinuous, positive p-homogeneous, subadditive, proper function such that $\{x \in X : q(x) \le 1\}$ is bounded. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a locally* *lower semicontinuous, bounded from below, proper function. If* $T : X \to 2^X$ *has the property that for any* $x \in X$ *and any* $y \in Tx$,

$$\beta q(y-x) + f(y) \leq f(x), \quad \text{where } \beta > 0 \text{ is a constant},$$

then for any $x_0 \in \mathrm{dom} f$, *there exists* $z \in (Tx_0)^{\sim}$ *such that* $Tz = \{z\}$, *where* $(Tx_0)^{\sim} := \{y \in X : \beta q(y - x_0) + f(y) \leq f(x_0)\}$.

Proof. By Theorem 3.1, there exists $z \in X$ such that

(i) $f(z) + \beta q(z - x_0) \leq f(x_0)$;
(ii) for any $x \neq z$, $f(z) < f(x) + \beta q(x - z)$.

From (i) we know that $z \in (Tx_0)^{\sim}$. We show that $Tz = \{z\}$. If not, there exists $y \in Tz$ and $y \neq z$. From (ii), we have

$$f(z) < f(y) + \beta q(y - z). \tag{9}$$

On the other hand, as $y \in Tz$, by the hypothesis on T we have $\beta q(y-z)+f(y) \leq f(z)$, which contradicts (9). ∎

Similarly we can deduce the following version of Caristi's fixed point theorem from Theorem 3.2.

THEOREM 4.2 (Extended Caristi's fixed point theorem). *Let* X *be a locally complete locally* p*-convex space with the topology generated by a family* $\{\| \, \|_\lambda\}_{\lambda \in \Lambda}$ *of* p*-homogeneous* F*-pseudonorms*, $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ *be a family of positive real numbers and let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a locally lower semicontinuous*, *bounded from below*, *proper function. If* $T : X \to 2^X$ *has the property that for any* $x \in X$ *and any* $y \in Tx$,

$$\alpha_\lambda \|x - y\|_\lambda + f(y) \leq f(x), \quad \forall \lambda \in \Lambda,$$

then for any $x_0 \in \mathrm{dom}\, f$, *there exists* $z \in (Tx_0)^{\sim}$ *such that* $Tz = \{z\}$, *where* $(Tx_0)^{\sim} := \{y \in X : \alpha_\lambda \|x_0 - y\|_\lambda + f(y) \leq f(x_0), \forall \lambda \in \Lambda\}$.

In fact, the two versions of Caristi's fixed point theorem and the two versions of Ekeland's variational principle of Section 3 are equivalent to one another.

THEOREM 4.3. *Theorems* 3.1, 3.2, 4.1 *and* 4.2 *are equivalent to one another.*

Proof. We only need to prove that Theorem 4.1 implies Theorem 3.1 and that Theorem 4.2 implies Theorem 3.2. Here we only give the proof of the former implication; the other proof is similar. Let X, q, B and f be as in Theorem 3.1 and let $x_0 \in \mathrm{dom}\, f$. Define $T : X \to 2^X$ by

$$Tx := \{y \in X : \beta q(y - x) + f(y) \leq f(x)\}.$$

Obviously $x \in Tx$ and T satisfies the hypothesis of Theorem 4.1. Hence there exists $z \in (Tx_0)^{\sim} = \{y \in X : \beta q(y - x_0) + f(y) \leq f(x_0)\}$ such that $Tz = \{z\}$. Thus $\beta q(z - x_0) + f(z) \le f(x_0)$, i.e., Theorem 3.1(i) holds. Moreover, for any $x \ne z$, we have $x \notin Tz$, i.e.,

$$\beta q(x - z) + f(x) > f(z).$$

Thus Theorem 3.1(ii) holds. ■

5. A p-drop theorem in locally p-convex spaces $(0 < p < 1)$**.** In order to obtain a general p-drop theorem in locally p-convex spaces, we need the following lemmas.

LEMMA 5.1 (see [15]). *Let X be a linear space and let $C \subset X$ be p-convex. Then*

$$D_p(x, C) = \{\alpha x + \beta y : y \in C,\ \alpha, \beta \in [0, 1],\ \alpha^p + \beta^p = 1\}.$$

Proof. Fix $z \in D_p(x, C)$. Then $z = \alpha_0 x + \sum_{i=1}^n \alpha_i y_i$, where $y_1, \ldots, y_n \in C$ and $\alpha_0, \alpha_1, \ldots, \alpha_n \in [0, 1]$ with $\sum_{i=0}^n \alpha_i^p = 1$.

If $\alpha_0 = 1$, then clearly $z = x \in \{\alpha x + \beta y : y \in C,\ \alpha, \beta \in [0, 1],\ \alpha^p + \beta^p = 1\}$.

If $0 \le \alpha_0 < 1$, then

$$z = \alpha_0 x + \Big(\sum_{i=1}^n \alpha_i^p\Big)^{1/p} \sum_{i=1}^n \frac{\alpha_i}{(\sum_{j=1}^n \alpha_j^p)^{1/p}}\, y_i.$$

Observe that

$$\sum_{i=1}^n \bigg(\frac{\alpha_i}{(\sum_{j=1}^n \alpha_j^p)^{1/p}}\bigg)^p = 1, \qquad y_i \in C \quad \text{and} \quad C \text{ is } p\text{-convex},$$

so we have

$$\sum_{i=1}^n \frac{\alpha_i}{(\sum_{j=1}^n \alpha_j^p)^{1/p}}\, y_i \in C.$$

Since $\alpha_0^p + [(\sum_{i=1}^n \alpha_i^p)^{1/p}]^p = 1$, the proof is complete. ■

LEMMA 5.2. *Let C be a p-convex set and $a \in D_p(x, C)$. Then $D_p(a, C) \subset D_p(x, C)$.*

Proof. For any $z \in D_p(a, C)$, by Lemma 5.1 we can write

$$z = \alpha a + \beta y, \quad \text{where} \quad y \in C,\ \alpha, \beta \in [0, 1],\ \alpha^p + \beta^p = 1.$$

Since $a \in D_p(x, C)$, we have

$$a = \lambda x + \mu y', \quad \text{where} \quad y' \in C,\ \lambda, \mu \in [0, 1],\ \lambda^p + \mu^p = 1.$$

Thus

$$\begin{aligned} z &= \alpha a + \beta y = \alpha(\lambda x + \mu y') + \beta y = \alpha\lambda x + \alpha\mu y' + \beta y \\ &= \alpha\lambda x + ((\alpha\mu)^p + \beta^p)^{1/p} \cdot \frac{\alpha\mu y' + \beta y}{((\alpha\mu)^p + \beta^p)^{1/p}}. \end{aligned}$$

Since C is p-convex and $y, y' \in C$, we have

$$\frac{\alpha\mu y' + \beta y}{((\alpha\mu)^p + \beta^p)^{1/p}} \in C.$$

To complete the proof, it suffices to observe that

$$(\alpha\lambda)^p + (\alpha\mu)^p + \beta^p = \alpha^p(\lambda^p + \mu^p) + \beta^p = \alpha^p + \beta^p = 1. \ \blacksquare$$

In the following we always assume that X is a locally p-convex space.

LEMMA 5.3. *Let $B \subset X$ be a locally closed, bounded, p-convex set, and let $x_0 \in X$. Then $D_p(x_0, B)$ is also locally closed and bounded.*

Proof. Since X has a 0-neighborhood base consisting of absolutely p-convex sets, the absolutely p-convex hull of the bounded set $\{x_0\} \cup B$, i.e. $\Gamma_p(x_0, B)$, is bounded, and hence so is $D_p(x_0, B) \subset \Gamma_p(x_0, B)$.

Now we prove that $D_p(x_0, B)$ is locally closed. Let $(a_i) \subset D_p(x_0, B)$ be locally convergent to a_0, that is, there is a bounded p-disc A such that $a_i \to a_0$ in E_A. We may assume that $A \supset \Gamma_p(x_0, B) \supset D_p(x_0, B)$. Since $a_i \in D_p(x_0, B)$, by Lemma 5.1 we have

$$a_i = \lambda_i x_0 + (1 - \lambda_i^p)^{1/p} b_i, \quad \text{where} \quad b_i \in B,\ 0 \le \lambda_i \le 1.$$

Since $(\lambda_i) \subset [0, 1]$, there exists a subsequence (λ_{i_j}) convergent to some $\lambda_0 \in [0, 1]$.

If $\lambda_0 = 1$, then $(1 - \lambda_{i_j}^p)^{1/p} b_{i_j} \to 0$ and $\lambda_{i_j} x_0 \to \lambda_0 x_0 = x_0$. Hence $a_{i_j} = \lambda_{i_j} x_0 + (1 - \lambda_{i_j}^p)^{1/p} b_{i_j} \to x_0$. Thus $a_0 = x_0 \in D_p(x_0, B)$.

If $0 \le \lambda_0 < 1$, then in E_A,

$$\lambda_{i_j} x_0 \to \lambda_0 x_0 \quad \text{and} \quad \lambda_{i_j} x_0 + (1 - \lambda_{i_j}^p)^{1/p} b_{i_j} \to a_0.$$

Hence in E_A,

$$(1 - \lambda_{i_j}^p)^{1/p} b_{i_j} \to a_0 - \lambda_0 x_0 \quad \text{and} \quad b_{i_j} \to \frac{a_0 - \lambda_0 x_0}{(1 - \lambda_0^p)^{1/p}}.$$

Put

$$b_0 := \frac{a_0 - \lambda_0 x_0}{(1 - \lambda_0^p)^{1/p}}.$$

Then $b_0 \in B$ since B is locally closed. Thus

$$a_0 = \lambda_0 x_0 + (1 - \lambda_0^p)^{1/p} b_0 \in D_p(x_0, B). \ \blacksquare$$

LEMMA 5.4 (see [13, p. 101] or [27, p. 94]). *Let $S \subset X$ be an absolutely p-convex set. Then for any scalars $\alpha_1, \ldots, \alpha_n$, we have $\alpha_1 S + \cdots + \alpha_n S \subset \gamma S$ provided that $\gamma \ge (\sum_{i=1}^n |\alpha_i|^p)^{1/p}$.*

Proof. Since $\sum_{i=1}^n |\alpha_i/\gamma|^p \le 1$, for any $x_1, \ldots, x_n \in S$, we have $\sum_{i=1}^n (\alpha_i/\gamma) x_i \in S$. From this, $\sum_{i=1}^n \alpha_i x_i \in \gamma S$, and hence $\alpha_1 S + \cdots + \alpha_n S \subset \gamma S$. $\blacksquare$

LEMMA 5.5. *Let $B \subset X$ be p-convex and $A \subset X$ be nonempty. If there exists a p-convex absorbing set W such that $W \cap (A - B) = \emptyset$, then for any $x_0 \in A$ and any ε, $0 < \varepsilon < 1$, there exists $a \in D_p(x_0, B) \cap A$ such that*

$$D_p(a, B) \cap A \subset \{\lambda a + (1 - \lambda^p)^{1/p} b : (1 - \varepsilon)^{1/p} < \lambda \leq 1,\, b \in B\}.$$

Proof. Let q_W be the gauge of W. Since $W \cap (A - B) = \emptyset$ and W is absorbing, we have

$$1 \leq q_W(x - y) < \infty, \qquad \forall x \in A,\, y \in B.$$

Clearly,

$$1 \leq \alpha := \inf\{q_W(x - y) : x \in D_p(x_0, B) \cap A,\, y \in B\} < \infty. \tag{10}$$

For any ε such that $0 < \varepsilon < 1$, there exist $a \in D_p(x_0, B) \cap A$ and $b_1 \in B$ such that

$$q_W(a - b_1) < \alpha(1 + \varepsilon). \tag{11}$$

Take any $z \in D_p(a, B) \cap A$. By Lemma 5.1 we may assume that $z = \lambda a + \mu b_2$, where $\lambda, \mu \geq 0$, $\lambda^p + \mu^p = 1$ and $b_2 \in B$. Since $b_1, b_2 \in B$ and B is p-convex, we have

$$\lambda b_1 + \mu b_2 \in B. \tag{12}$$

By Lemma 5.2 we know that $D_p(a, B) \subset D_p(x_0, B)$, hence

$$z \in D_p(x_0, B) \cap A. \tag{13}$$

From (10), (12) and (13), we have

$$\begin{aligned} \alpha &\leq q_W(z - \lambda b_1 - \mu b_2) = q_W(\lambda a + \mu b_2 - \lambda b_1 - \mu b_2) \\ &= q_W(\lambda(a - b_1)) = \lambda^p q_W(a - b_1). \end{aligned}$$

Now using (11), we obtain

$$\alpha \leq \lambda^p q_W(a - b_1) < \lambda^p \alpha(1 + \varepsilon).$$

Since $0 < 1 \leq \alpha < \infty$, we have $1 < \lambda^p(1 + \varepsilon)$. From this,

$$1 - \varepsilon < \frac{1}{1 + \varepsilon} < \lambda^p, \quad \text{and so} \quad (1 - \varepsilon)^{1/p} < \lambda.$$

Thus we have shown that

$$\begin{aligned} D_p(a, B) \cap A &\subset \{\lambda a + \mu b : (1 - \varepsilon)^{1/p} < \lambda \leq 1,\, \mu \geq 0,\, \lambda^p + \mu^p = 1,\, b \in B\} \\ &= \{\lambda a + (1 - \lambda^p)^{1/p} b : (1 - \varepsilon)^{1/p} < \lambda \leq 1,\, b \in B\}. \ \blacksquare \end{aligned}$$

Now we can give a p-drop theorem in locally p-convex spaces under a very weak assumption.

THEOREM 5.1. *Let X be a locally complete locally p-convex space, $A \subset X$ be locally closed and $B \subset X$ be locally closed, bounded and p-convex. If there exists a p-convex absorbing set W such that $W \cap (A - B) = \emptyset$, then for any $x_0 \in A$ there exists $a \in D_p(x_0, B) \cap A$ such that $D_p(a, B) \cap A = \{a\}$.*

Proof. Take a sequence (ε_n) of positive real numbers such that

$$0 < \varepsilon_n < 1 - (1 - 1/2^n)^p < 1/2^n, \qquad n = 1, 2, \ldots.$$

By Lemma 5.5 there exists $a_1 \in D_p(x_0, B) \cap A$ such that

$$D_p(a_1, B) \cap A \subset \{\lambda a_1 + (1 - \lambda^p)^{1/p} b : (1 - \varepsilon_1)^{1/p} < \lambda \le 1,\ b \in B\}.$$

Similarly, there exists $a_2 \in D_p(a_1, B) \cap A$ such that

$$D_p(a_2, B) \cap A \subset \{\lambda a_2 + (1 - \lambda^p)^{1/p} b : (1 - \varepsilon_2)^{1/p} < \lambda \le 1,\ b \in B\}.$$

Continuing, we obtain a sequence (a_n) with

$$\begin{aligned}
a_1 &= \lambda_0 x_0 + (1 - \lambda_0^p)^{1/p} b_0, \quad \text{where} \quad 0 \le \lambda_0 \le 1,\ b_0 \in B,\\
a_2 &= \lambda_1 a_1 + (1 - \lambda_1^p)^{1/p} b_1, \quad \text{where} \quad (1 - \varepsilon_1)^{1/p} < \lambda_1 \le 1,\ b_1 \in B,\\
&\ \vdots\\
a_{n+1} &= \lambda_n a_n + (1 - \lambda_n^p)^{1/p} b_n, \quad \text{where} \quad (1 - \varepsilon_n)^{1/p} < \lambda_n \le 1,\ b_n \in B,\\
&\ \vdots
\end{aligned}$$

Hence

$$\begin{aligned}
a_{n+2} - a_{n+1} &= (1 - \lambda_{n+1}^p)^{1/p} b_{n+1} - (1 - \lambda_{n+1}) a_{n+1},\\
&\ \vdots\\
a_{n+k+1} - a_{n+k} &= (1 - \lambda_{n+k}^p)^{1/p} b_{n+k} - (1 - \lambda_{n+k}) a_{n+k}.
\end{aligned}$$

By adding the above k equalities, we obtain

$$\begin{aligned}
(14)\qquad a_{n+k+1} - a_{n+1} = (1 - \lambda_{n+1}^p)^{1/p} b_{n+1} + \cdots + (1 - \lambda_{n+k}^p)^{1/p} b_{n+k}\\
- (1 - \lambda_{n+1}) a_{n+1} - \cdots - (1 - \lambda_{n+k}) a_{n+k}.
\end{aligned}$$

Obviously for every i,

$$(15)\qquad b_{n+i} \in B \subset \Gamma_p(x_0, B).$$

Also, by Lemma 5.2,

$$a_{n+i} \in D_p(a_{n+i-1}, B) \subset \cdots \subset D_p(a_n, B) \subset \cdots \subset D_p(x_0, B) \subset \Gamma_p(x_0, B),$$

where $\Gamma_p(x_0, B)$ is circled, hence

$$(16)\qquad -a_{n+i} \in \Gamma_p(x_0, B).$$

Combining (14)–(16) and applying Lemma 5.4, we have

$$\begin{aligned}
(17)\quad a_{n+k+1} - a_{n+1} &\in (1 - \lambda_{n+1}^p)^{1/p} \Gamma_p(x_0, B) + \cdots + (1 - \lambda_{n+k}^p)^{1/p} \Gamma_p(x_0, B)\\
&\quad + (1 - \lambda_{n+1}) \Gamma_p(x_0, B) + \cdots + (1 - \lambda_{n+k}) \Gamma_p(x_0, B)\\
&\subset [\varepsilon_{n+1} + \cdots + \varepsilon_{n+k} + (1 - (1 - \varepsilon_{n+1})^{1/p})^p + \cdots\\
&\quad + (1 - (1 - \varepsilon_{n+k})^{1/p})^p]^{1/p} \Gamma_p(x_0, B).
\end{aligned}$$

By the choice of (ε_n) we know that $(1-\varepsilon_n)^{1/p} > 1-1/2^n$, so $1-(1-\varepsilon_n)^{1/p} < 1/2^n$ and hence

$$(1-(1-\varepsilon_n)^{1/p})^p < 1/2^{np},$$

so from (17) we have

$$\begin{aligned} a_{n+k+1} - a_{n+1} &\in \left[\frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+k}} + \frac{1}{2^{(n+1)p}} + \cdots + \frac{1}{2^{(n+k)p}}\right]^{1/p} \Gamma_p(x_0, B) \\ &\subset \left(\frac{1}{2^n} + \frac{1}{2^p-1}\cdot\frac{1}{2^{np}}\right)\Gamma_p(x_0, B). \end{aligned}$$

Thus (a_n) is a locally Cauchy sequence, and hence is locally convergent to some point a. Next we consider two cases.

Case 1. Assume that there exists a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that $a_{n_i} = x_0$ for all i. Then for any $z \in D_p(x_0, B) \cap A$, we have $z \in D_p(a_{n_i}, B) \cap A$ for $i = 1, 2, \ldots$, and z can be written as

$$z = \lambda'_{n_i} a_{n_i} + (1-\lambda'_{n_i})^p)^{1/p} b'_i, \quad \text{where} \quad (1-\varepsilon_{n_i})^{1/p} < \lambda'_{n_i} \le 1 \text{ and } b'_i \in B.$$

Letting $i \to \infty$, we obtain

$$z = \lim_{i\to\infty}(\lambda'_{n_i} a_{n_i} + (1-(\lambda'_{n_i})^p)^{1/p} b'_i) = \lim_{i\to\infty}(\lambda'_{n_i} x_0 + (1-(\lambda'_{n_i})^p)^{1/p} b'_i) = x_0.$$

Hence $D_p(x_0, B) \cap A = \{x_0\}$.

Case 2. Assume that there exists some $m \in \mathbb{N}$ such that $a_n \ne x_0$ for every $n > m$. We may assume that $a_n \ne x_0$ for all n. Since B is a locally closed bounded p-convex set, by Lemma 5.3, $D_p(a_n, B)$ is locally closed and hence $D_p(a_n, B) \cap A$ is locally closed. As $(a_{n+k})_{k\in\mathbb{N}} \subset D_p(a_n, B) \cap A$ and $(a_{n+k})_{k\in\mathbb{N}}$ is locally convergent to a, we infer that $a \in D_p(a_n, B) \cap A$. This holds for every n, so $a \in \bigcap_{n=1}^{\infty} D_p(a_n, B) \cap A$. We assert that $D_p(a, B) \cap A = \{a\}$. In fact, take any $z \in D_p(a, B) \cap A$. Then $z \in D_p(a_n, B) \cap A$ for every n. Thus we may assume that

$$z = \mu_n a_n + (1-{\mu_n}^p)^{1/p} y_n, \quad \text{where} \quad (1-\varepsilon_n)^{1/p} < \mu_n \le 1,\ y_n \in B.$$

Hence

$$\begin{aligned} z - a_n &= (1-\mu_n^p)^{1/p} y_n + \mu_n a_n - a_n \\ &\in (1-\mu_n^p)^{1/p}\Gamma_p(x_0, B) + (1-\mu_n)\Gamma_p(x_0, B) \\ &\subset [1-\mu_n^p + (1-\mu_n)^p]^{1/p}\Gamma_p(x_0, B) \\ &\subset [\varepsilon_n + (1-(1-\varepsilon_n)^{1/p})^p]^{1/p}\Gamma_p(x_0, B) \\ &\subset \left(\frac{1}{2^n} + \frac{1}{2^{np}}\right)^{1/p}\Gamma_p(x_0, B) \subset \left(\frac{1}{2^{np}} + \frac{1}{2^{np}}\right)^{1/p}\Gamma_p(x_0, B) \\ &\subset \frac{2^{1/p}}{2^n}\,\Gamma_p(x_0, B). \end{aligned}$$

Thus (a_n) is locally convergent to z. At the same time, it is locally convergent to a. Therefore $z = a$, i.e., $D_p(a, B) \cap A = \{a\}$. ∎

Remark 5.1. From the proof of Theorem 5.1 we observe that every $a_n \in D_p(x_0, B) \cap A$. If A or B is locally complete, we can deduce that $D_p(x_0, B) \cap A$ is still locally complete. Thus we see that in Theorem 5.1 the condition that X is locally complete can be replaced by the condition that B or A is locally complete.

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