STUDIA MATHEMATICA 186 (3) (2008)

On linear extension for interpolating sequences

by

ERIC AMAR (Bordeaux)

Abstract. Let A be a uniform algebra on X and σ a probability measure on X. We define the Hardy spaces $H^p(\sigma)$ and the $H^p(\sigma)$ interpolating sequences S in the p-spectrum $\mathcal{M}_p$ of σ. We prove, under some structural hypotheses on A and σ, that if S is a "dual bounded" Carleson sequence, then S is $H^s(\sigma)$-interpolating with a linear extension operator for $s < p$, provided that either $p = \infty$ or $p \le 2$.

In the case of the unit ball of $\mathbb{C}^n$ we find, for instance, that if S is dual bounded in $H^\infty(\mathbb{B})$ then S is $H^p(\mathbb{B})$-interpolating with a linear extension operator for any $1 \le p < \infty$. Already in this case this is a new result.

1. Introduction. Let $\mathbb{B}$ be the unit ball of $\mathbb{C}^n$; in this case we take the algebra of holomorphic functions in $\mathbb{B}$ continuous on $\overline{\mathbb{B}}$ for A and for σ the normalized Lebesgue measure on $\partial\mathbb{B}$ and, as usual, we denote by $H^p(\mathbb{B})$ the Hardy space of holomorphic functions in $\mathbb{B}$, i.e. the closure in $L^p(\sigma)$ of A if $p < \infty$ and the algebra of bounded holomorphic functions in $\mathbb{B}$ if $p = \infty$.

Let S be a sequence of points in $\mathbb{B}$ and $1 \le p \le \infty$; we say that S is $H^p(\mathbb{B})$-*interpolating*, $S \in IH^p$ for short, if

$$\forall \lambda \in \ell^p(S),\ \exists f \in H^p(\mathbb{B}),\ \forall a \in S, \qquad f(a) = \lambda_a (1 - |a|^2)^{n/p};$$

for $p = \infty$, we set, as usual, $(1 - |a|^2)^{n/p} = 1$ for all $a \in \mathbb{B}$.

It is a well known consequence of Baire's theorem that if $S \in IH^p(\mathbb{B})$ we can choose an f interpolating the sequence $\lambda \in \ell^p$ such that $\|f\|_p \le C\|\lambda\|_p$ with a constant $C > 0$ independent of λ.

Let $a \in \mathbb{B}$ and $k_a(z) := 1/(1 - \overline{a} \cdot z)^n$ be the reproducing kernel for a (the Cauchy kernel), i.e.

$$\forall f \in H^1(\mathbb{B}), \qquad f(a) = \langle f, k_a \rangle := \int_{\partial\mathbb{B}} f(\zeta)\overline{k}_a(\zeta)\, d\sigma(\zeta).$$

Let $k_{p,a} := k_a/\|k_a\|_p$, the normalized reproducing kernel for a in $H^p(\mathbb{B})$.

Because $\|k_a\|_{p'} \simeq (1 - |a|^2)^{-n/p}$, with p' the conjugate exponent for p, $S \in IH^p(\mathbb{B})$ is equivalent to

2000 *Mathematics Subject Classification*: 42B30, 46J15, 32A35.

Key words and phrases: interpolating sequences, Carleson measure, Hardy spaces.

$$\forall \lambda \in \ell^p,\ \exists f \in H^p(\mathbb{B}),\ \forall a \in S, \qquad \langle f, k_{p',a} \rangle = \lambda_a.$$

Now if S is $H^p(\mathbb{B})$-interpolating, then by interpolating the basic sequences of ℓ^p we have

$$\exists C > 0,\ \forall a \in S,\ \exists \varrho_a \in H^p(\mathbb{B}), \qquad \|\varrho_a\|_p \le C, \quad \langle \varrho_a, k_{p',b} \rangle = \delta_{ab}.$$

Hence the system $\{\varrho_a\}_{a\in S}$ is dual to $\{k_{p',a}\}_{a\in S}$ and bounded in $H^p(\mathbb{B})$. This leads to the definition:

Definition 1.1. We shall say that S is *dual bounded* (or *uniformly minimal* [15]) in $H^p(\mathbb{B})$ if:

$$\exists C > 0,\ \forall a \in S,\ \exists \varrho_a \in H^p(\mathbb{B}), \qquad \|\varrho_a\|_p \le C, \quad \langle \varrho_a, k_{p',b} \rangle = \delta_{ab}.$$

Hence if S is $H^p(\mathbb{B})$-interpolating, then it is dual bounded in $H^p(\mathbb{B})$.

Definition 1.2. We say that the $H^p(\mathbb{B})$-interpolating sequence S has the *linear extension property* (L.E.P.) if there is a bounded linear operator $E : \ell^p \to H^p(\mathbb{B})$ such that for every $\lambda \in \ell^p$, $E\lambda$ interpolates the sequence λ in $H^p(\mathbb{B})$ on S, i.e.

$$\forall \lambda \in \ell^p,\ \forall a \in S, \qquad E\lambda(a) = \lambda_a \|k_a\|_{p'}.$$

Natural questions are the following:

- If S is dual bounded in $H^p(\mathbb{B})$, is $S \in IH^p(\mathbb{B})$?
- If $S \in IH^p(\mathbb{B})$, does S automatically have the L.E.P.?

This is true in the classical case of the Hardy spaces of the unit disc $\mathbb{D}$:

For $p = \infty$ the positive answer to the first question is the famous characterization of H^∞-interpolating sequences by L. Carleson [8].

The positive answer to the second question was given by P. Beurling [7].

For $p \in [1, \infty[$ the positive answer to the first question was established by H. Shapiro and A. Shields [18], who also proved that the interpolating sequences are the same for all $p \in [1, \infty]$.

The positive answer to the second question was obtained explicitly by $\overline{\partial}$ methods in [2].

For the Bergman classes $A^p(\mathbb{D})$, it is no longer true that the interpolating sequences are the same for $A^p(\mathbb{D})$ and $A^q(\mathbb{D})$, $q \ne p$. But A. P. Schuster and K. Seip [17], [16] proved that S dual bounded in $A^p(\mathbb{D})$ implies that S is $A^p(\mathbb{D})$-interpolating with the L.E.P.

The first question is still open, even in the ball $\mathbb{B}$ of $\mathbb{C}^n$, $n \ge 2$, for $H^p(\mathbb{B})$, the usual Hardy spaces of the ball and in the polydisc $\mathbb{D}^n$ of $\mathbb{C}^n$, $n \ge 2$, for the usual Hardy spaces.

The second one is known only in the case $p = \infty$, and is positive (see [3] and the references therein).

Nevertheless in the case of the unit ball of $\mathbb{C}^n$, B. Berndtsson [5] proved that if the product of the Gleason distances of the points of S is bounded below away from 0, then S is $H^\infty(\mathbb{B})$-interpolating. He also proved that this condition is not necessary for $n > 1$, in contrast to the case of $n = 1$.

B. Berndtsson, S-Y. A. Chang and K.-C. Lin [6] proved the same theorem in the polydisc of $\mathbb{C}^n$.

In this paper we shall prove that S dual bounded in $H^p(\mathbb{B})$ implies that for all $s < p$, $S \in IH^s(\mathbb{B})$ with the L.E.P., provided that $1 < p \le 2$ or $p = \infty$. In particular:

THEOREM 1.3. *If $S \subset \mathbb{B}$ is dual bounded in $H^p(\mathbb{B})$, then it is $H^s(\mathbb{B})$-interpolating for any $1 \le s < p$, provided that either $p \in]1, 2]$ or $p = \infty$. Moreover, in these cases, S has the linear extension property.*

In [3] a generalization of the interpolating Blaschke products to the ball is studied via $\overline{\partial}$ methods, and a condition is given on a sequence S of points in $\mathbb{B}$ to ensure that S is $H^p(\mathbb{B})$-interpolating for all $p \in [1, \infty[$. This condition implies that S is dual bounded in $H^\infty(\mathbb{B})$, hence the result here is stronger than the one in [3].

I want to thank the referee for all his suggestions and his very pertinent questions.

The methods we use being purely functional-analytic, our results extend to the setting of uniform algebras.

The paper is organized this way:

- Section 2: we recall facts related to uniform algebras and reproducing kernels.
- Section 3: we define and study Carleson and weak Carleson sequences.
- Section 4: we state the structural hypotheses we shall need.
- Section 5: we define the H^p-interpolating sequences and the dual boundedness in this abstract context.
- Section 6: we state our main results. The reader may go directly to this section to get an idea of the results and to have some comments about them.
- Section 7: we apply the main results in the special cases of the ball and of the polydisc.

2. Uniform algebras and reproducing kernels. Let A be a uniform algebra on the compact space X, i.e. A is a subalgebra of $\mathcal{C}(X)$, the continuous functions on X, which separates the points of X and contains 1.

Let σ be a probability measure on X. For $1 \le p < \infty$ we define as usual the Hardy space $H^p(\sigma)$ as the closure of A in $L^p(\sigma)$; $H^\infty(\sigma)$ will be the weak-* closure of A in $L^\infty(\sigma)$.

Let $\mathcal{M}$ be the *Gelfand spectrum* of A, i.e. the non-zero multiplicative elements of A', the dual space of A. We denote an element of A and its

Gelfand transform using the same notation:

$$\forall a \in \mathcal{M} \subset A', \forall f \in A, \qquad f(a) := \hat{f}(a) = a(f).$$

We shall use the following notions, already introduced in [4].

DEFINITION 2.1. Let $\mathcal{M}$ be the spectrum of A and $a \in \mathcal{M}$. We call $k_a \in H^p(\sigma)$ a *p-reproducing kernel* for the point a if

$$\forall f \in A, \qquad f(a) = \int_X f(\zeta)\overline{k}_a(\zeta)\, d\sigma(\zeta).$$

We define the *p-spectrum* of σ as the subset $\mathcal{M}_p$ of those elements of $\mathcal{M}$ that have a p'-reproducing kernel, with p' the conjugate exponent for p, $1/p + 1/p' = 1$.

The reproducing kernel for $a \in \mathcal{M}$, if it exists, is unique. Indeed, suppose there are two, say $k_a \in H^p(\sigma)$ and $l_a \in H^q(\sigma)$. Then

$$\forall f \in A, \qquad 0 = f(a) - f(a) = \int_X f(\overline{k}_a - \overline{l}_a)\, ds, \quad \text{so} \quad k_a = l_a \ \sigma\text{-a.e.},$$

because, by definition, A is dense in $H^r(\sigma)$ with $r := \min(p, q)$. Hence it is correct to denote it by k_a without reference to the $H^p(\sigma)$ where it belongs.

Let $a \in \mathcal{M}_p$. Then $k_a \in H^{p'}(\sigma)$; if $p < q$ then $q' < p'$, hence $k_a \in H^{q'}(\sigma)$ because σ is a probability measure, so $a \in \mathcal{M}_q$. Thus $p < q \Rightarrow \mathcal{M}_p \subset \mathcal{M}_q$.

To simplify the notation we shall write $\langle f, g\rangle := \int_X f\overline{g}\, d\sigma$, whenever this is meaningful.

3. Carleson sequences.

We denote by $k_{q,a} := k_a/\|k_a\|_q$ the normalized reproducing kernel in $H^q(\sigma)$.

DEFINITION 3.1. Let $1 \le q < \infty$. We say that a sequence $S \subset \mathcal{M}_{q'}$ is a *q-Carleson sequence* if

$$\exists D_q > 0, \forall \mu \in \ell^q, \qquad \Big\| \sum_{a \in S} \mu_a k_{q,a} \Big\|_q \le D_q \|\mu\|_q.$$

Let $2 \le q < \infty$. We say that the sequence $S \subset \mathcal{M}_{q'}$ is a *weakly q-Carleson sequence* if

$$\exists D_q > 0, \forall \mu \in \ell^q, \qquad \Big\|\sum_{a \in S} |\mu_a|^2 |k_{q,a}|^2 \Big\|_{q/2} \le D_q \|\mu\|_q^2.$$

We call the second condition "weakly" Carleson because:

LEMMA 3.2. *If $2 \le q < \infty$ and S is q-Carleson then it is weakly q-Carleson.*

Proof. With every sequence S we associate a sequence $\{\varepsilon_a\}_{a\in S}$ of independent random variables with the same law $P(\varepsilon_a = 1) = P(\varepsilon_a = -1) = 1/2$. We shall denote by $\mathbb{E}$ the associated expectation.

In the following, $a \lesssim b$ means that there exists a constant C, independent of a and b, such that $a \le Cb$; $a \simeq b$ means that $a \lesssim b$ and $b \lesssim a$.

Let S be a q-Carleson sequence. For the associated $\{\varepsilon_a\}_{a\in S}$ we have

$$\Big\|\sum_{a\in S} \mu_a \varepsilon_a k_{q,a}\Big\|_q^q \lesssim \|\mu\|_q^q$$

because $|\varepsilon_a| = 1$. Taking expectation on both sides leads to

$$\Big\|\mathbb{E}\Big[\Big|\sum_{a\in S} \mu_a \varepsilon_a k_{q,a}\Big|^q\Big]\Big\|_1 = \mathbb{E}\Big[\Big\|\sum_{a\in S} \mu_a \varepsilon_a k_{q,a}\Big\|_q^q\Big] \lesssim \|\mu\|_q^q.$$

Now using Khinchin's inequalities for the left expression,

$$\Big\|\mathbb{E}\Big[\Big|\sum_{a\in S} \mu_a \varepsilon_a k_{q,a}\Big|^q\Big]\Big\|_1 \simeq \Big\|\sum_{a\in S} |\mu_a|^2 |k_{q,a}|^2\Big\|_{q/2}^{q/2},$$

we get

$$\Big\|\sum_{a\in S} |\mu_a|^2 |k_{q,a}|^2\Big\|_{q/2}^{q/2} \lesssim \mathbb{E}\Big[\Big\|\sum_{a\in S} \mu_a \varepsilon_a k_{q,a}\Big\|_q^q\Big] \lesssim \|\mu\|_q^q,$$

and the lemma. ■

Now if S is weakly q-Carleson, is S weakly p-Carleson for other p? Notice that any sequence S is weakly 2-Carleson:

$$\forall \nu \in \ell^1, \qquad \Big\|\sum_{a\in S} \nu_a |k_{2,a}|^2\Big\|_1 \le \sum_{a\in S} |\nu_a| \, \||k_{2,a}|^2\|_1 \le \|\nu\|_1,$$

because $\|k_{2,a}\|_2 = \| \, |k_{2,a}|^2\|_1 = 1$. Hence if S is weakly q-Carleson with $q > 2$ we can try to use interpolation of linear operators.

By a theorem of E. Stein and G. Weiss [19] we know that if a linear operator U is bounded from $\ell^q(\gamma_q)$ to $L^q(\sigma)$ and from $\ell^1(\gamma_1)$ to $L^1(\sigma)$ then U is bounded from $\ell^p(\gamma_p)$ to $L^p(\sigma)$ with $1 \le p \le q$ provided that the weight satisfies the condition

$$(*) \qquad \frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{q} \ \Rightarrow \ \forall a \in S, \ \gamma_p(a) = \gamma_1(a)^{p(1-\theta)} \gamma_q(a)^{p\theta/q}.$$

Here the weighted ℓ^q space $\ell^q(\gamma_q)$ is defined by

$$\ell^q(\gamma_q) := \Big\{\lambda : \|\lambda\|_{\ell^q(\gamma_q)}^q := \sum_{a\in S} |\lambda_a|^q \gamma_q(a) < \infty\Big\}.$$

The hypothesis means that

$$\forall \lambda \in \ell^q(\gamma_q), \qquad \|U\lambda\|_{L^q(\sigma)} \le M_q \|\lambda\|_{\ell^q(\gamma_q)},$$

and

$$\forall \lambda \in \ell^1(\gamma_q), \qquad \|U\lambda\|_{L^1(\sigma)} \le M_1 \|\lambda\|_{\ell^1(\gamma_q)},$$

and the conclusion is, provided that $(*)$ is true:

$$\forall p \in [1,q],\ \exists M_p > 0,\ \forall \lambda \in \ell^p(\gamma_q), \qquad \|U\lambda\|_{L^p(\sigma)} \le M_p \|\lambda\|_{\ell^p(\gamma_p)}.$$

In order to use this result, we need

LEMMA 3.3. *Let $q \ge 1$ and $1/p = (1-\theta)/1 + \theta/q$ with $0 < \theta < 1$. Then*

$$\|k_a\|_{2p} \le \|k_a\|_2^{1-\theta} \|k_a\|_{2q}^{\theta}.$$

Proof. Let

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{q} = \frac{1}{s} + \frac{1}{r} \quad \text{with} \quad s = \frac{1}{1-\theta} \quad \text{and} \quad r = \frac{q}{\theta}.$$

Hölder's inequality gives, for $f \in L^s(\sigma),\ g \in L^r(\sigma)$,

$$\Big(\int_X |fg|^p \, d\sigma\Big)^{1/p} \le \Big(\int_X |f|^s \, d\sigma\Big)^{1/s} \Big(\int_X |g|^r \, d\sigma\Big)^{1/r}.$$

Setting $f = |k_a|^{2(1-\theta)}$ and $g := |k_a|^{2\theta}$ we get

$$\Big(\int_X |k_a|^{2p} \, d\sigma\Big)^{1/p} \le \Big(\int_X |k_a|^{2(1-\theta)s} \, d\sigma\Big)^{1/s} \Big(\int_X |k_a|^{2\theta r} \, d\sigma\Big)^{1/r},$$

hence replacing s, r gives

$$\Big(\int_X |k_a|^{2p} \, d\sigma\Big)^{1/p} \le \Big(\int_X |k_a|^{2} \, d\sigma\Big)^{1-\theta} \Big(\int_X |k_a|^{2q} \, d\sigma\Big)^{\theta/q}. \ \blacksquare$$

For $p \in [1,q]$ define

$$T : \ell^p(\omega_p) \to L^p(\sigma), \qquad T\lambda := \sum_{a \in S} \lambda_a |k_a|^2,$$

with $\omega_p(a) := \|k_a\|_{2p}^{-2p}$.

To say that T is bounded is the same as saying that the sequence S is weakly p-Carleson.

LEMMA 3.4. *If S is weakly s-Carleson with $s > 2$, then it is weakly r-Carleson for any $2 \le r \le s$.*

Proof. Let $q := s/2$. We know that T is bounded for $p = 1$ and for $p = q$. Applying the Stein–Weiss result to $U := T$ with $\gamma_1 = \omega_1,\ \gamma_q = \omega_q$, we get, for any $1 \le p \le q$,

$$\|T\lambda\|_{L^p(\sigma)} \le M_p \|\lambda\|_{\ell^p(\gamma_p)},$$

with the weight γ_p defined as:

$$\text{if} \quad \frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{q}, \quad \text{then} \quad \gamma_p(a) := \gamma_1(a)^{p(1-\theta)} \gamma_q(a)^{p\theta/q} \text{ for } a \in S.$$

Replacing $\omega_1,\ \omega_q$ by their values, this means
$$\gamma_p(a) = \|k_a\|_2^{-2p(1-\theta)}\|k_a\|_{2q}^{-2p\theta}.$$
Hence
$$\exists M_p > 0, \quad \|T\lambda\|_p^p \le M_p\|\lambda\|^p_{\ell^p(\gamma_p)} = M_p \sum_{a\in S} |\lambda_a|^p \gamma_p(a).$$
But Lemma 3.3 gives $\|k_a\|_{2p} \le \|k_a\|_2^{1-\theta}\|k_a\|_{2q}^{\theta}$, which precisely says that $\gamma_p(a) \le \omega_p(a)$ for all $a \in S$, hence
$$\|T\lambda\|_p^p \le M_p \sum_{a\in S} |\lambda_a|^p \omega_p(a),$$
and T is indeed bounded from $l^p(\omega_p)$ to $L^p(\sigma)$, which proves the lemma with $r = 2p$. ■

We also notice that any sequence S is 1-Carleson:
$$\forall \mu \in \ell^1, \quad \Big\|\sum_{a\in S} \mu_a k_{1,a}\Big\|_1 \le \sum_{a\in S} |\mu_a|\, \|k_{1,a}\|_1 \le \|\mu\|_1,$$
and the same proof as above also gives

Lemma 3.5. *If S is q-Carleson with $q > 1$, then S is p-Carleson for any $1 \le p \le q$.*

4. Structural hypotheses. We shall need some structural hypotheses on σ relative to the reproducing kernels.

Definition 4.1. Let $q \in]1,\infty[$. Then we say that the measure σ satisfies the *structural hypothesis $SH(q)$* if, with q' the conjugate of q:

$$\exists \alpha = \alpha_q > 0, \forall a \in \mathcal{M}_q \cap \mathcal{M}_{q'} \subset \mathcal{M}_2, \quad \|k_a\|_2^2 \ge \alpha\|k_a\|_q\|k_a\|_{q'}. \tag{4.1}$$

This goes the opposite way to Hölder inequalities. Because $a \in \mathcal{M}_q \cap \mathcal{M}_{q'} \subset \mathcal{M}_2$, we have $k_a(a) = \int_X k_a(\zeta)\overline{k}_a(\zeta)\,d\sigma = \|k_a\|_2^2$ and the condition above is the same as
$$\|k_a\|_q\|k_a\|_{q'} \le \alpha_q^{-1}k_a(a).$$

Definition 4.2. Let $p, s \in [1,\infty]$ and q be such that $1/s = 1/p + 1/q$. We say that the measure σ satisfies the *structural hypothesis $SH(p,s)$* if

$$\exists \beta = \beta_{p,q} > 0, \forall a \in \mathcal{M}_s, \quad \|k_a\|_{s'} \le \beta\|k_a\|_{p'}\|k_a\|_{q'}. \tag{4.2}$$

This is meaningful because $s < p,\ s < q$, and hence $\mathcal{M}_s \subset \mathcal{M}_p \cap \mathcal{M}_q$.

We proved in [4] that $SH(q)$ and $SH(p,s)$ are true for *all values* of $q,\ p$ and s in the case of the unit ball of $\mathbb{C}^n$ with σ the Lebesgue mesure on $\partial\mathbb{B}$ and the algebra $A(\mathbb{B})$ of holomorphic functions in $\mathbb{B}$, continuous in $\overline{\mathbb{B}}$.

The same is true [4] in the case of the unit polydisc $\mathbb{D}^n$ of $\mathbb{C}^n$, with σ the Lebesgue mesure on $\mathbb{T}^n$ and the algebra $A(\mathbb{D}^n)$ of holomorphic functions in $\mathbb{D}^n$, continuous in $\overline{\mathbb{D}}^n$, still for *all values* of $q,\ p$ and s.

5. Interpolating sequences

Definition 5.1. We say that $S \subset \mathcal{M}_p$ is $H^p(\sigma)$-*interpolating* for $1 \leq p < \infty$, abbreviated as $S \in IH^p(\sigma)$, if

$$\forall \lambda \in \ell^p,\ \exists f \in H^p(\sigma),\ \forall a \in S, \quad f(a) = \lambda_a \|k_a\|_{p'}.$$

We say that $S \subset \mathcal{M}_\infty$ is $H^\infty(\sigma)$-*interpolating*, written $S \in IH^\infty(\sigma)$, if

$$\forall \lambda \in \ell^\infty,\ \exists f \in H^\infty(\sigma),\ \forall a \in S, \quad f(a) = \lambda_a.$$

Remark 5.2. If S is $H^p(\sigma)$-interpolating then there is a constant C_I, the interpolating constant, such that [4]

$$\forall \lambda \in \ell^p,\ \exists f \in H^p(\sigma), \quad \|f\|_p \leq C_I \|\lambda\|_p, \quad f(a) = \lambda_a \|k_a\|_{p'},\ \forall a \in S.$$

Definition 5.3. We say that the $H^p(\sigma)$-interpolating sequence S has the *linear extension property* (L.E.P.) if there is a bounded linear operator $E\ \ell^p \to H^p(\sigma)$ such that for every $\lambda \in \ell^p$, $E\lambda$ interpolates the sequence λ in $H^p(\sigma)$ on S, i.e.

$$\forall \lambda \in \ell^p,\ \forall a \in S, \quad E\lambda(a) = \lambda_a \|k_a\|_{p'}.$$

Let $S \subset \mathcal{M}_p$, so $k_{p',a} := k_a/\|k_a\|_{p'}$, the normalized reproducing kernel, exists for any $a \in S$; consider a dual system $\{\varrho_a\}_{a\in S} \subset H^p(\sigma)$, i.e. $\langle \varrho_a, k_{p',b}\rangle = \delta_{a,b}$ for all $a, b \in S$, when it exists.

Definition 5.4. We say that $S \subset \mathcal{M}_p$ is *dual bounded* in $H^p(\sigma)$ if a dual system $\{\varrho_a\}_{a\in S} \subset H^p(\sigma)$ exists and is bounded in $H^p(\sigma)$, i.e.

$$\exists C > 0,\ \forall a \in S,\ \exists \varrho_a \in H^p(\sigma), \quad \|\varrho_a\|_p \leq C, \quad \langle \varrho_a, k_{p',b}\rangle = \delta_{ab}.$$

We shall need the following facts proved in [4]:

Theorem 5.5. *Let $p > 1$. If $S \subset \mathcal{M}_p$ and $S \in IH^\infty(\sigma)$, and σ satisfies $SH(p)$, then $S \in IH^p(\sigma)$ with the L.E.P.*

Theorem 5.6. *If $S \subset \mathcal{M}_1$ and S is dual bounded in $H^p(\sigma)$ for some $p > 1$, then $S \in IH^1(\sigma)$.*

We shall also need to truncate S to its first N elements, written S_N. Clearly if $S \in IH^p(\sigma)$ then $S_N \in IH^p(\sigma)$ with a smaller interpolating constant. Let $I^p_{S_N} := \{f \in H^p(\sigma) : f_{|S_N} = 0\}$ be the module over A of functions that are zero on S_N. For $\lambda \in \ell^p$, with $\{\varrho_a\}_{a\in S}$ a bounded dual sequence, the function $f_N := \sum_{a\in S_N} \lambda_a \varrho_a$ interpolates λ on S_N and $\|f_N\|_{H^p(\sigma)/I^p_{S_N}} \leq C_I \|\lambda\|_p$.

We also have the converse for $1 < p \leq \infty$, which is all that we need [4]:

Lemma 5.7. *If all truncations S_N of S are in $IH^p(\sigma)$ for some $p > 1$, with a uniform constant C_I, then $S \in IH^p(\sigma)$ with the same constant.*

6. Main results. Now we are in a position to state and comment on our main results.

THEOREM 6.1. *Let $1 < p \leq 2$, $1 \leq s < p$ and q be such that $1/s = 1/p+1/q$. Suppose that $S \subset \mathcal{M}_s \cap \mathcal{M}_{q'}$ is dual bounded in $H^p(\sigma)$ and weakly q-Carleson, and σ satisfies the structural hypotheses $SH(q)$ and $SH(p,s)$. Then S is $H^s(\sigma)$-interpolating and has the L.E.P. in $H^s(\sigma)$.*

The passage from $p = 2$ to $p \leq 2$ in the case of the ball is due to F. Bayart (oral communication): he uses Khinchin's inequalities which prove to be very well fitted to this problem.

Using this time the fact that Khinchin's inequalities also provide a way to put absolute values inside sums, we get the other extremity of the range of p's:

THEOREM 6.2. *Let $1 \leq s < \infty$ and $p > 2s$. Suppose that $S \subset \mathcal{M}_s \cap \mathcal{M}_{s'}$ is dual bounded in $H^\infty(\sigma)$ and weakly p-Carleson, and σ satisfies the structural hypotheses $SH(p,s)$ and $SH(q)$ for q such that $1/s = 1/p + 1/q$. Then S is $H^s(\sigma)$-interpolating with the L.E.P.*

This theorem is the best possible in this generality. There is no hope to show that dual boundedness in H^∞ implies H^∞-interpolation as L. Carleson proved for the unit disc:

In [11] and [13] the authors proved that in the spectrum $\mathcal{M}$ of the uniform algebra $H^\infty(\mathbb{D})$ there are sequences S of points, $S \subset \mathcal{M}\backslash\mathbb{D}$, such that the product of the Gleason distances is bounded below away from 0, which implies that S is dual bounded in $H^\infty(\mathbb{D})$, but S is *not* H^∞-interpolating.

The above theorems will be consequences of the next lemma.

As above, to a sequence S of points in $\mathcal{M}$, we associate a sequence $\{\varepsilon_a\}_{a\in S}$ of independent Bernoulli variables.

LEMMA 6.3. *Let $S \subset \mathcal{M}_p$ be a sequence of points such that a dual system $\{\varrho_{p,a}\}_{a\in S}$ exists in $H^p(\sigma)$. Let $1 \leq s < p$ and q be such that $1/s = 1/p+1/q$. Suppose that*

$$\forall\lambda \in \ell^p(S), \qquad \mathbb{E}\Big[\Big\|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big\|_p^p\Big] \lesssim \|\lambda\|_{\ell^p}^p,$$

S is weakly q-Carleson (if $q > 2$), and σ satisfies $SH(q)$ and $SH(p,s)$. Then S is $H^s(\sigma)$-interpolating and moreover S has the L.E.P.

REMARK 6.4. If $q \leq 2$ we do not assume any Carleson condition on S.

Proof. If $p = 1$ we set

$$\forall\lambda \in \ell^1, \qquad T(\lambda) := \sum_{a\in S}\lambda_a\varrho_{1,a};$$

the sum converges in $H^1(\sigma)$ because the functions $\varrho_{1,a}$ are uniformly bounded in $H^1(\sigma)$. Then $T(\lambda)$ interpolates the sequence λ, and clearly the operator T is linear. Moreover, $\|T(\lambda)\|_1 \le \sup_{a\in S} \|\varrho_{1,a}\| \, \|\lambda\|_{\ell^1}$, so T is bounded.

If $p > 1$, we may suppose that $1 < s < p$ because if $S \in IH^s(\sigma)$ then by Theorem 5.6, for $S \subset \mathcal{M}_1$ we also have $S \in IH^1(\sigma)$.

First we truncate the sequence to the first N elements and get estimates independent of N, i.e. for $s \in [1,p[$ and $\nu \in \ell^s_N$ we shall build a function $h \in H^s(\sigma)$ such that

$$\forall j = 0,\dots,N-1, \qquad h(a_j) = \nu_j \|k_{a_j}\|_{s'} \quad \text{and} \quad \|h\|_{H^s} \le C\|\nu\|_{\ell^s_N},$$

with the constant C independent of N. The conclusion then follows using Lemma 5.7.

Since $1/s = 1/p + 1/q$, we have $q \in]p', \infty[$. We set $\nu_j = \lambda_j \mu_j$ with $\mu_j := |\nu_j|^{s/q} \in \ell^q$ and $\lambda_j := (\nu_j/|\nu_j|)|\nu_j|^{s/p} \in \ell^p$. Then $\|\nu\|_s = \|\lambda\|_p \|\mu\|_q$. Let

$$c_a := \frac{\|k_a\|_{s'}}{\|k_a\|_{p'} k_{q,a}(a)} = \frac{\|k_a\|_{s'} \|k_a\|_q}{\|k_a\|_{p'} k_a(a)}.$$

By $SH(q)$ we have $k_a(a) \ge \alpha \|k_a\|_q \|k_a\|_{q'}$, hence

$$c_a \le \frac{\|k_a\|_{s'}}{\alpha \|k_a\|_{p'} \|k_a\|_{q'}} \le \frac{\beta}{\alpha}$$

by $SH(p,s)$.

Now set $h(z) = \sum_{a\in S} \nu_a c_a \varrho_a k_{q,a}$. Then

$$\forall a \in S, \qquad h(a) = \nu_a \|k_a\|_{s'},$$

because $\varrho_a(b) = \delta_{ab} \|k_a\|_{p'}$. This means that h interpolates ν and clearly h is linear in ν. To estimate the $H^s(\sigma)$ norm of h, set

$$f(\varepsilon, z) := \sum_{a\in S} \lambda_a c_a \varepsilon_a \varrho_a(z), \qquad g(\varepsilon, z) := \sum_{a\in S} \mu_a \varepsilon_a k_{q,a}(z).$$

Then $h(z) = \mathbb{E}(f(\varepsilon,z) g(\varepsilon,z))$ because $\mathbb{E}(\varepsilon_j \varepsilon_k) = \delta_{jk}$. So we get

$$|h(z)|^s = |\mathbb{E}(fg)|^s \le (\mathbb{E}(|fg|))^s \le \mathbb{E}(|fg|^s),$$

hence

$$\|h\|_s = \Big(\int_X |h(z)|^s \, d\sigma(z)\Big)^{1/s} \le \Big(\int_X \mathbb{E}(|fg|^s) \, d\sigma(z)\Big)^{1/s}.$$

Using Hölder's inequality, we obtain

$$\begin{aligned} (6.1) \qquad \int_X \mathbb{E}(|fg|^s) \, d\sigma(z) &= \mathbb{E}\Big[\int_X |fg|^s \, d\sigma(z)\Big] \\ &\le \Big(\mathbb{E}\Big[\int_X |f|^p \, d\sigma\Big]\Big)^{s/p} \Big(\mathbb{E}\Big[\int_X |g|^q \, d\sigma\Big]\Big)^{s/q}. \end{aligned}$$

Let $\widetilde{\lambda}_a := c_a\lambda_a$ for $a \in S$. Then $\|\widetilde{\lambda}\|_p \le \alpha^{-1}\beta\|\lambda\|_p$ and the first factor above is controlled by hypothesis:

$$\mathbb{E}\Big[\int_X |f|^p\,d\sigma\Big] = \mathbb{E}\Big[\Big\|\sum_{a\in S}\lambda_a c_a\varepsilon_a\varrho_{p,a}\Big\|_p^p\Big] \lesssim \|\widetilde{\lambda}\|_p^p \lesssim \|\lambda\|_{\ell^p}^p. \tag{6.2}$$

Fubini's theorem gives, for the second factor in (6.1),

$$\mathbb{E}\Big[\int_X |g|^q\,d\sigma\Big] = \int_X \mathbb{E}[|g|^q]\,d\sigma.$$

Khinchin's inequalities yield

$$\mathbb{E}[|g|^q] \simeq \Big(\sum_{a\in S}|\mu_a|^2|k_{q,a}|^2\Big)^{q/2}$$

Now, if $q > 2$, S weakly q-Carleson implies

$$\int_X \mathbb{E}[|g|^q]\,d\sigma \lesssim \int_X \Big(\sum_{a\in S}|\mu_a|^2|k_{q,a}|^2\Big)^{q/2} d\sigma \lesssim \|\mu\|_{\ell^q}^q. \tag{6.3}$$

If $q \le 2$ then $(\sum_{a\in S}|\mu_a|^2|k_{q,a}|^2)^{q/2} \le \sum_{a\in S}|\mu_a|^q|k_{q,a}|^q$, hence integrating over X we get

$$\int_X \mathbb{E}[|g|^q]\,d\sigma \le \int_X \Big(\sum_{a\in S}|\mu_a|^q|k_{q,a}|^q\Big) d\sigma \le \sum_{a\in S}|\mu_a|^q \int_X |k_{q,a}|^q\,d\sigma = \|\mu\|_{\ell^q}. \tag{6.4}$$

So putting (6.2) and (6.3) or (6.4) in (6.1) we get the lemma. ∎

6.1. *Proof of Theorem 6.1.* We state and prove a more precise version of Theorem 6.1:

THEOREM 6.5. *Let $1 < p \le 2$, $1 \le s < p$ and q be such that $1/s = 1/p+1/q$. Suppose that $S \subset \mathcal{M}_s\cap\mathcal{M}_{q'}$, $\{\varrho_{p,a}\}_{a\in S}$ is a norm bounded sequence in $H^p(\sigma)$, S is weakly q-Carleson, and σ satisfies the structural hypotheses $SH(q)$ and $SH(p,s)$. Then S is $H^s(\sigma)$-interpolating with the L.E.P.*

Proof. It remains to prove that the hypotheses of the theorem imply those of Lemma 6.3.

We have to prove that

$$\mathbb{E}\Big[\Big\|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big\|_p^p\Big] \lesssim \|\lambda\|_{\ell^p}^p,$$

knowing that $\{\varrho_{p,a}\}_{a\in S}$ is bounded in $H^p(\sigma)$, i.e. $\sup_{a\in S}\|\varrho_{p,a}\|_p \le C$. By Fubini's theorem,

$$\mathbb{E}\Big[\Big\|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big\|_p^p\Big] = \int_X \mathbb{E}\Big[\Big|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big|^p\Big]\,d\sigma,$$

and by Khinchin's inequalities,

$$\mathbb{E}\Big[\Big|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big|^p\Big]\simeq\Big(\sum_{a\in S}|\lambda_a|^2|\varrho_{p,a}|^2\Big)^{p/2}.$$

Now $p\le 2$, so $(\sum_{a\in S}|\lambda_a|^2|\varrho_{p,a}|^2)^{1/2}\le(\sum_{a\in S}|\lambda_a|^p|\varrho_{p,a}|^p)^{1/p}$, hence

$$\int_X\mathbb{E}\Big[\Big|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big|^p\Big]\,d\sigma\le\int_X\Big(\sum_{a\in S}|\lambda_a|^p|\varrho_{p,a}|^p\Big)\,d\sigma=\sum_{a\in S}|\lambda_a|^p\|\varrho_{p,a}\|_p^p.$$

So, finally,

$$\mathbb{E}\Big[\Big\|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big\|_p^p\Big]\lesssim\sup_{a\in S}\ \|\varrho_{p,a}\|_p^p\|\lambda\|_p^p,$$

and the assertion holds. ∎

An alternative proof was suggested to me by F. Lust-Piquard. Recall the definition of the *type* of a Banach space ([14, p. 162]) for the reader's convenience:

Definition 6.6. The Banach space X is of *type* p, $1\le p\le 2$, if there is a constant $C\ge 1$ such that

$$\forall n\in\mathbb{N},\ \forall x_1,\dots,x_n\in X,\qquad\Big(\mathbb{E}\Big(\Big\|\sum_{j=1}^n\varepsilon_jx_j\Big\|_X^2\Big)\Big)^{1/2}\le C\Big(\sum_{j=1}^n\|x_j\|_X^p\Big)^{1/p}.$$

Now since any subspace of $L^p(\sigma)$ is of type p for $1\le p\le 2$ ([14, Th. III.9, p. 169]), for instance $H^p(\sigma)$, we get

$$\Big(\mathbb{E}\Big[\Big\|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big\|_p^2\Big]\Big)^{1/2}\le C\Big(\sum_{a\in S}|\lambda_a|^p\|\varrho_{p,a}\|_p^p\Big)^{1/p},$$

hence

$$\Big(\mathbb{E}\Big[\Big\|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big\|_p^2\Big]\Big)^{1/2}\le C\sup_{a\in S}\|\varrho_{p,a}\|_p\Big(\sum_{a\in S}|\lambda_a|^p\Big)^{1/p}=C\sup_{a\in S}\|\varrho_{p,a}\|_p\|\lambda\|_p.$$

On the other hand, because $p\le 2$, we have

$$\Big(\mathbb{E}\Big[\Big\|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big\|_p^p\Big]\Big)^{1/p}\le\Big(\mathbb{E}\Big[\Big\|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big\|_p^2\Big]\Big)^{1/2}.$$

Finally,

$$\mathbb{E}\Big[\Big\|\sum_{a\in S}\lambda_a\varepsilon_a\varrho_{p,a}\Big\|_p^p\Big]\le C^p(\sup_{a\in S}\ \|\varrho_{p,a}\|_p^p)\|\lambda\|_{\ell^p}^p\lesssim\|\lambda\|_{\ell^p}^p,$$

and again the conclusion follows.

6.2. *Proof of Theorem 6.2.* Again we state a more precise version of Theorem 6.2:

THEOREM 6.7. *Let $1 \leq s < \infty$ and $p > 2s$. Suppose that $S \subset \mathcal{M}_s \cap \mathcal{M}_{s'}$, $\{\varrho_a\}_{a\in S}$ is a norm bounded sequence in $H^\infty(\sigma)$, S is weakly p-Carleson, and σ satisfies the structural hypotheses $SH(p,s)$ and $SH(q)$ for q such that $1/s = 1/p + 1/q$. Then S is $H^s(\sigma)$-interpolating with the L.E.P.*

Proof. The idea is still to use Lemma 6.3, but in two steps.

Set $\varrho_{p,a} := \varrho_a k_{p,a}$ for $a \in S$. Then $\|\varrho_{p,a}\|_p \leq \|\varrho_a\|_\infty \|k_{p,a}\|_p = \|\varrho_a\|_\infty \leq C$ by hypothesis.

By Khinchin's inequalities we have

$$\mathbb{E}\Big[\Big|\sum_{a\in S} \lambda_a \varepsilon_a \varrho_{p,a}\Big|^p\Big] \simeq \Big(\sum_{a\in S} |\lambda_a|^2 |\varrho_{p,a}|^2\Big)^{p/2},$$

but this time we use the fact that $|\varrho_{p,a}| \leq \|\varrho_{\infty,a}\|\,|k_{a,p}| \leq C|k_{a,p}|$, hence

$$\mathbb{E}\Big[\Big|\sum_{a\in S} \lambda_a \varepsilon_a \varrho_{p,a}\Big|^p\Big] \lesssim C^p\Big(\sum_{a\in S} |\lambda_a|^2 |k_{a,p}|^2\Big)^{p/2}.$$

Since S is weakly p-Carleson, we get

$$\Big\|\sum_{a\in S} |\lambda_a|^2 |k_{a,p}|^2\Big\|_{p/2}^{p/2} \leq D\|\lambda\|_p^p,$$

hence

$$\mathbb{E}\Big[\Big\|\sum_{a\in S} \lambda_a \varepsilon_a \varrho_{p,a}\Big\|_p^p\Big] \lesssim \int_X \Big(\sum_{a\in S} |\lambda_a|^2 |k_{a,p}|^2\Big)\,d\sigma \lesssim \|\lambda\|_{\ell^p}^p,$$

and Lemma 6.3 shows that S is $H^s(\sigma)$-interpolating with the L.E.P. provided that S is weakly q-Carleson if $q > 2$. But because $p > 2s$ we have $q = sp/(p-s) \leq p$, hence if $q > 2$, then because $q \leq p$, S weakly p-Carleson implies S weakly q-Carleson by Lemma 3.4, and the theorem is proved. ∎

7. Application to the polydisc and to the ball. In [4] it is proved that the structural hypotheses hold in the polydisc. In that case the Carleson sequences are characterized geometrically and they are the same for all $p \in]1,\infty[$ (see [9], [10]), i.e. if S is a p-Carleson sequence for some $p > 1$, then it is r-Carleson for any $r \in [1,\infty]$. So in that case we just say that S is a Carleson sequence if it is p-Carleson for some $p > 1$.

Moreover, still in that case, p-Carleson is the same as weakly p-Carleson, so it is enough to say "Carleson sequence" in the theorem:

THEOREM 7.1. *Let $S \subset \mathbb{D}^n$ be a Carleson sequence that is dual bounded in $H^p(\mathbb{D}^n)$ with either $p = \infty$ or $p \leq 2$. Then S is $H^s(\mathbb{D}^n)$-interpolating for any $1 \leq s < p$ with the L.E.P.*

Still in [4] it is proved that the structural hypotheses hold in the ball. Again the Carleson measures, hence the Carleson sequences, are characterized geometrically and they are the same for all $p \in]1,\infty[$ (see [12]); also, S Carleson is the same as S weakly Carleson.

Moreover, an easy corollary [4] of a theorem of P. Thomas [20] gives that S dual bounded in $H^p(\mathbb{B})$ implies S Carleson, hence:

Theorem 7.2. *Let $S \subset \mathbb{B}$ be dual bounded in $H^p(\mathbb{B})$ with either $p = \infty$ or $p \le 2$. Then S is $H^s(\mathbb{B})$-interpolating for any $1 \le s < p$ with the L.E.P.*

We have for free the same result for the Bergman classes of the ball by the "subordination lemma" [1]: to a function $f(z)$ defined for $z = (z_1, \dots, z_n) \in \mathbb{B}_n \subset \mathbb{C}^n$ associate the function $\widetilde{f}(z,w) := f(z)$ defined for $(z,w) = (z_1, \dots, z_n, w) \in \mathbb{B}_{n+1} \subset \mathbb{C}^{n+1}$. Then $f \in A^p(\mathbb{B}_n) \Leftrightarrow \widetilde{f} \in H^p(\mathbb{B}_{n+1})$ with the same norm. Moreover, if $F \in H^p(\mathbb{B}_{n+1})$ then $f(z) := F(z,0) \in A^p(\mathbb{B}_n)$ with $\|f\|_{A^p(\mathbb{B}_n)} \le \|F\|_{H^p(\mathbb{B}_{n+1})}$.

Suppose that $S \subset \mathbb{B}_n$ is dual bounded in $A^p(\mathbb{B}_n)$. This means that

$$\exists \{\varrho_a\}_{a\in S},\ \forall a \in S, \qquad \|\varrho_a\|_{A^p(\mathbb{B}_n)} \le C \text{ and } \varrho_a(b) = \delta_{ab}(1-|a|^2)^{-(n+1)/p},$$

because the normalized reproducing kernel for $A^p(\mathbb{B}_n)$ is

$$b_a(z) := \frac{(1-|a|^2)^{(n+1)/p'}}{(1-\overline{a}\cdot z)^{n+1}}.$$

Embed S in $\mathbb{B}_{n+1}$ as $\widetilde{S} := \{(a,0) : a \in S\}$, as in [1]. Then the sequence $\{\widetilde{\varrho}_a\}_{a\in S}$ is precisely a bounded dual sequence for $\widetilde{S} \subset \mathbb{B}_{n+1}$ in $H^p(\mathbb{B}_{n+1})$, hence we can apply the previous theorem: if $p = \infty$ or $p \le 2$ and $s < p$, then $\widetilde{S}$ is $H^s(\mathbb{B}_{n+1})$-interpolating with the L.E.P. If T is the operator realizing the extension,

$$\lambda \in \ell^s \to T\lambda \in H^s(\mathbb{B}_{n+1}),$$
$$(T\lambda)(a,0) = \lambda_a \|k_{(a,0)}\|_{H^{s'}(\mathbb{B}_{n+1})}, \qquad \|T\lambda\|_{H^s(\mathbb{B}_{n+1})} \le C_I \|\lambda\|_s,$$

then the operator $(U\lambda)(z) := (T\lambda)(z,0)$ is a bounded linear operator from ℓ^s to $A^s(\mathbb{B}_n)$ realizing the extension because $\|k_{(a,0)}\|_{H^{s'}(\mathbb{B}_{n+1})} = \|b_a\|_{A^{s'}(\mathbb{B}_n)}$, where k is the kernel for $H^s(\mathbb{B}_{n+1})$ and b is the kernel for $A^s(\mathbb{B}_n)$. Hence we proved

Corollary 7.3. *Let $S \subset \mathbb{B}$ be dual bounded in $A^p(\mathbb{B})$ with either $p = \infty$ or $p \le 2$. Then S is $A^s(\mathbb{B})$-interpolating for any $1 \le s < p$ with the L.E.P.*

We also get the same result for the Bergman spaces with weight of the form $(1-|z|^2)^k$, $k \in \mathbb{N}$, just by the same method, but considering $H^p(\mathbb{B}_{n+k+1})$ instead of $H^p(\mathbb{B}_{n+1})$.

References

[1] E. Amar, *Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de $\mathbb{C}^n$*, Canad. J. Math. 30 (1978), 711–737.

[2] —, *Extension de fonctions holomorphes et courants*, Bull. Sci. Math. 107 (1983), 24–48.

[3] —, *Interpolating sequences in the ball of $\mathbb{C}^n$*, Ark. Mat. 38 (2000), 1–20.

[4] —, *On interpolation of interpolating sequences*, Indag. Math. (N.S.) 18 (2007), 177–187.

[5] B. Berndtsson, *Interpolating sequences for H^∞ in the ball*, Proc. Kon. Ned. Akad. Wetensch. 88 (1985), 1–10.

[6] B. Berndtsson, S.-Y. A. Chang, and K.-C. Lin, *Interpolating sequences in the polydisc*, Trans. Amer. Math. Soc. 302 (1987), 161–169.

[7] P. Beurling and L. Carleson, *Research on interpolation problems*, preprint, Uppsala, 1962.

[8] L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. 80 (1958), 921–930.

[9] S.-Y. A. Chang, *Carleson measure in the bi-disc*, Ann. of Math. 109 (1979), 613–620.

[10] S.-Y. A. Chang and R. A. Fefferman, *Some recent developments in Fourier analysis and H^p-theory on product domains*, Bull. Amer. Math. Soc. 12 (1985), 1–43.

[11] P. Gorkin, R. Mortini, and H. M. Lingenberg, *Homeomorphic disks in the spectrum of H^∞*, Indiana Univ. Math. J. 39 (1990), 961–983.

[12] L. Hörmander, *L^p estimates for (pluri-) subharmonic functions*, Math. Scand. 20 (1967), 65–78.

[13] K. Izuchi, *Interpolating sequences in the maximal ideal space of H^∞*, J. Math. Soc. Japan 43 (1991), 721–731.

[14] D. Li et H. Queffélec, *Introduction à l'étude des espaces de Banach*, Cours spécialisés 12, Soc. Math. France, 2004.

[15] N. K. Nikolskii, *Treatise on the Shift Operator*, Grundlehren Math. Wiss. 273, Springer, 1986.

[16] A. P. Schuster and K. Seip, *Weak conditions for interpolation in holomorphic spaces*, Publ. Mat. Barc. 44 (2000), 277–293.

[17] —, —, *A Carleson type condition for interpolation in Bergman spaces*, J. Reine Angew. Math. 497 (1998), 223–233.

[18] H. Shapiro and A. L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. 83 (1961), 513–522.

[19] E. M. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. 87 (1958), 159–172.

[20] P. J. Thomas, *Hardy space interpolation in the unit ball*, Indag. Math. 90 (1987), 325–351.

Université Bordeaux I
351 Cours de la Libération
33405 Talence, France
E-mail: Eric.Amar@math.u-bordeaux1.fr

Received October 27, 2006
Revised version March 14, 2008 (6029)