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Algebraic reflexivity of $C(X,E)$ and Cambern's theorem

by

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Abstract. The algebraic and topological reflexivity of $C(X)$ and $C(X,E)$ are investigated by using representations for the into isometries due to Holsztyński and Cambern.

1. Introduction. In [6], Holsztyński established the following representation for into isometries between spaces of continuous functions $C(X)$ and $C(Y)$, with X and Y compact Hausdorff spaces.

THEOREM 1.1. *If X and Y are compact Hausdorff spaces and $T : C(X) \to C(Y)$ is a linear isometry, then there exist a closed subset Y_0 of Y, a surjective continuous map $\varphi : Y_0 \to X$, and $\alpha \in C(Y)$ with $\|\alpha\|_\infty = 1$ and $|\alpha(y)| = 1$ for every $y \in Y_0$, such that*

$$T(f)(y) = \alpha(y)\, f(\varphi(y)) \quad \text{for } f \in C(X),\ y \in Y_0. \tag{1.1}$$

Holsztyński's representation for into isometries of $C(X)$ has applications to the algebraic reflexivity problem for $C(X)$. We show that the isometry group of the space $C(X)$ of continuous real-valued functions is algebraically reflexive under mild conditions on X. Our proofs are different from those presented by Molnár and Zalar in [10]. The fact that Holsztyński's representation works for the space of real-valued functions is an essential step in our argument. We observe that Molnár and Zalar [10] used the Russo–Dye theorem to derive the algebraic reflexivity of the isometry group of $C(X,\mathbb{C})$, the Banach space of all complex-valued continuous functions on X. We note that the Russo–Dye theorem is not available in the real case.

Holsztyński's theorem was extended to vector-valued spaces of continuous functions by Cambern. A characterization of into isometries for the vector-valued function setting is done in [2], provided the range space is strictly convex. Cambern's result can be generalized to complex strictly

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convex range spaces. This generalization enables us to prove some new results about the algebraic reflexivity of the isometry group of $C(X,E)$, also under this new condition on E. We mention that our hypotheses on E are weaker than those considered by Jarosz and Rao in [8].

2. Remarks on Holsztyński's theorem and the algebraic reflexivity of $C(X)$. Holsztyński's proof is given for complex-valued functions, however it is mentioned in [6], as a footnote, that the same characterization is also valid for real-valued functions. For completeness of exposition we provide the minor modification to Holsztyński's proof for the real-valued case. We first recall some essential notation from [6]:

$$\begin{aligned} S_x &= \{f \in C(X) : \|f\| = 1 \text{ and } |f(x)| = 1\}, && x \in X,\\ R_y &= \{g \in C(Y) : \|g\| = 1 \text{ and } |g(y)| = 1\}, && y \in Y,\\ Q_x &= \{y \in Y : T(S_x) \subset R_y\}, && x \in X. \end{aligned}$$

If $C(X)$ refers to real-valued continuous functions all the six steps (i–vi) in Holsztyński's proof are valid with a minor modification necessary to show step (i). This first step asserts that if $f \in C(X)$ vanishes at $x \in X$, then $T(f)(y) = 0$ for every $y \in Q_x$. Indeed, suppose that there exists $f \in C(X)$ so that $f(x) = 0$ and $T(f)(y) \neq 0$ for some $y \in Q_x$. We may assume that f has norm 1. We set $g = \min\{1 + f, 1, 1 - f\}$. Then $g(x) = 1$ and $\|g\|_\infty = 1$. This implies that g and $g - f$ are in S_x. Hence $|T(g)(y)| = 1$ and $|T(g - f)(y)| = 1$. This implies that $T(f)(y) = 0$, contradicting our initial assumption.

Holsztyński's characterization of isometries allows us to establish the algebraic reflexivity of the isometry group of $C(X)$, for both the real and complex cases, provided that X satisfies the first countability axiom and an additional topological property. We first review the definition of algebraic reflexivity for this particular case.

Definition 2.1. An isometry T of $C(X)$ is said to be *locally surjective* if for every $f \in C(X)$ there exists a surjective isometry T_f so that $T(f) = T_f(f)$. The space $C(X)$ is *algebraically reflexive* if every locally surjective isometry is surjective.

The following example shows that not every isometry is locally surjective.

Example 2.2. An isometry T of $C(X)$ determines a surjective continuous map φ, defined on a subset X_0, as stated in Theorem 1.1. For instance, let $T : C([0, 1], \mathbb{R}) \to C([0,1], \mathbb{R})$ be defined by

$$T(f)(z) = \begin{cases} f(2z) & \text{if } 0 \le z \le 1/2,\\ -4f(1)(z - 3/4) & \text{if } 1/2 \le z \le 3/4,\\ 0 & \text{if } 3/4 \le z \le 1. \end{cases}$$

In this case, $X_0 = [0, 1/2]$ and $\varphi : X_0 \to X$ is given by $\varphi(z) = 2z$. The isometry T is not locally surjective. Indeed, the support of $T(\mathbf{1})$ is equal to $[0, 3/4]$, where $\mathbf{1}$ is the constant function equal to 1. However, any surjective isometry maps $\mathbf{1}$ to a modulus 1 continuous function.

The next proposition characterizes locally surjective isometries of $C(X,\mathbb{R})$ and $C(X,\mathbb{C})$. Throughout the rest of this paper, $C(X)$ represents either $C(X,\mathbb{R})$ or $C(X,\mathbb{C})$.

PROPOSITION 2.3. *If X is a Hausdorff, compact, first countable topological space and T is a locally surjective isometry on $C(X)$, then there exist a closed subset X_0 of X, a homeomorphism $\varphi : X_0 \to X$, and $\alpha \in C(X)$ with $\|\alpha\|_\infty = 1$ and $|\alpha(z)| = 1$ for every $z \in X_0$, such that*

$$T(f)(z) = \alpha(z) f(\varphi(z)) \quad \text{for } f \in C(X),\ z \in X_0. \tag{2.1}$$

Proof. Let T be a locally surjective isometry on $C(X)$. Holsztyński's proof in [6] asserts that, for each $x \in X$, Q_x is nonempty and $\bigcup_{x\in X} Q_x$ is a closed subset of X; denote it by X_0. Furthermore, for every $z \in X_0$,

$$T(g)(z) = \alpha(z) g(\varphi(z))$$

with α and φ as described in Theorem 1.1.

We start by proving that Q_x is a singleton. Let y_1 and y_2 be in Q_x and $y_1 \neq y_2$. Since X is first countable and compact, there exists $f \in C(X)$, with values in $[0, 1]$, such that $f(x) = \|f\|_\infty = 1$ and $|f(y)| < 1$ for all $y \neq x$ (see [4]). Since T is a locally surjective isometry, there exists a surjective isometry T_f such that $T(f) = T_f(f)$. The Banach–Stone theorem asserts that $T_f(g)(z) = \alpha_f(z) g(\tau_f(z))$, with τ_f a homeomorphism of X and α_f a scalar-valued, modulus 1 continuous map defined on X. In particular, this implies that

$$T_f(f)(y_i) = \alpha_f(y_i) f(\tau_f(y_i)) = T(f)(y_i) = \alpha(y_i) f(\varphi(y_i)) = \alpha(y_i) f(x)$$

for $i = 1, 2$ and

$$f(x) = \alpha(y_1)^{-1} \alpha_f(y_1) f(\tau_f(y_1)) = \alpha_2(y_2)^{-1} \alpha_f(y_2) f(\tau_f(y_2)).$$

Consequently,

$$|f(x)| = 1 = |f(\tau_f(y_1))| = |f(\tau_f(y_2))|$$

and $\tau_f(y_1) = \tau_f(y_2) = x$. Since τ_f is a homeomorphism, this leads to a contradiction. Therefore Q_x consists of at most a single point. Since Q_x is nonempty, it must be a single point.

In addition, every function f that attains its norm $\|f\|_\infty$ at a single point (say $x \in X$) determines a surjective isometry and a homeomorphism τ_f that satisfies

$$\varphi(Q_x) = \tau_f(Q_x).$$

The previous considerations also imply that φ is injective. Moreover, since X_0 is a closed subset of X, it follows that φ is a homeomorphism between X_0 and X. ∎

The next theorem asserts the algebraic reflexivity of the isometry group of $C(X)$, under some topological constraints on X.

THEOREM 2.4. *If X is a Hausdorff, compact, first countable topological space such that either*

(1) *there exists an injective and continuous real-valued function on X, or*
(2) *X is a connected n-dimensional manifold without boundary,*

then $C(X)$ is algebraically reflexive.

Proof. (1) Let T be a locally surjective isometry of $C(X)$. Without loss of generality we may choose an injective function f with values in the interval $[0,1]$. Theorem 1.1 implies the existence of a closed subset X_0 of X, a surjective continuous map $\varphi : X_0 \to X$ and a modulus 1 complex-valued continuous function such that

$$T(f)(z) = \alpha(z) f(\varphi(z)) \quad \text{for every } z \in X_0.$$

The Banach–Stone theorem states that

$$T(f)(x) = \alpha_f(x) f(\tau_f(x)) \quad \text{for every } x \in X,$$

where τ_f is a homeomorphism on X and α_f a complex-valued, modulus 1 continuous function on X. Therefore, for every $z \in X_0$ we have $f(\varphi(z)) = f(\tau_f(z))$. The injectivity of f implies that $\varphi(z) = \tau_f(z)$, and the surjectivity of φ implies that $X = X_0$. This proves the first statement.

(2) Proposition 2.3 asserts the existence of a subset X_0 of X that is homeomorphic to X. Therefore X_0 must be a compact n-manifold. This implies that the boundary of X_0 in X is empty, so X_0 is both open and closed in X. Since X is connected we have $X = X_0$, which concludes the proof. ∎

EXAMPLE 2.5. Examples of topological spaces satisfying condition (1) of Theorem 2.4 are Cantor sets, compact totally disconnected metric spaces, and one-dimensional manifolds.

DEFINITION 2.6. A Banach space is said to be *topologically reflexive* provided that every isometry that is the strong limit of a sequence of surjective isometries is also a surjective isometry.

REMARK 2.7. We observe that $C([0,\,1],\mathbb{R})$ is not topologically reflexive. Let T be defined by $T(f)(x) = f(\tau(x))$ where $\tau(x) = 0$ if $0 \le x \le 1/2$, and $\tau(x) = 2x - 1$ if $1/2 \le x \le 1$. The isometry T is the strong limit of the sequence of surjective isometries $T_n(f)(x) = f(\tau_n(x))$ with $\tau_n(x) = \frac{2}{n+1}x$ if $0 \le x \le (n+1)/2n$, and $\tau_n(x) = 2x - 1$ if $(n+1)/2n \le x \le 1$.

Similar constructions exist for topological spaces containing a point with a locally Euclidean neighborhood, i.e. homeomorphic to a Euclidean space.

3. Spaces of continuous vector-valued functions. We consider the characterization of isometries due to Cambern (see [2]) between two spaces of vector-valued continuous functions, $C(X, E)$ and $C(Y, E_1)$, with X and Y compact topological spaces, E and E_1 Banach spaces, and E_1 strictly convex. These spaces are equipped with the standard norm $\|\cdot\|_\infty$. We recall Cambern's characterization of isometries on spaces of vector-valued continuous functions, which generalizes a pioneering theorem on surjective isometries due to Jerison.

THEOREM 3.1. (1) (Jerison, [9]) *If A is an isometry from $C(X, E)$ onto $C(Y, E)$, with E strictly convex, then there exists a homeomorphism τ of Y onto X and a continuous map $y \mapsto A_y$ from Y into the space of bounded operators on E, equipped with the strong operator topology, such that for all $y \in Y$, A_y is an isometry of E and*

$$A(F)(y) = A_y(F)(\tau(y)) \quad \text{for } F \in C(X, E),\ y \in Y.$$

(2) (Cambern, [2]) *Let E and E_1 be Banach spaces with E_1 strictly convex and A an isometry from $C(X, E)$ into $C(Y, E_1)$. Then there exists a subset $B(A) \subset Y$, a continuous function $\phi : Y \to \mathcal{B}(E, E_1)$ such that $\phi(y) = A_y$ ($\mathcal{B}(E, E_1)$ denotes all bounded operators from E into E_1 equipped with the strong operator topology) with $\|A_y\| \le 1$ for all $y \in Y$ and $\|A_y\| = 1$ for all $y \in B(A)$, and there exists a continuous map τ from $B(A)$ onto X such that*

$$A(F)(y) = A_y(F)(\tau(y)) \quad \text{for } F \in C(X, E),\ y \in B(A).$$

If E is finite-dimensional then $B(A)$ is a closed subset of Y.

Cambern's proof follows Holsztyński's approach for the scalar case. We can show that Cambern's characterization also holds for E_1 complex strictly convex. We recall that a Banach space E is said to be *complex strictly convex* if whenever $x, y \in E$ satisfy $\|x\| = \|e^{i\theta}y + x\| = 1$ for every $\theta \in \mathbb{R}$, then $y = 0$. Equivalently, if $x, y \in E$ and $\|x\| = \|\pm iy + x\| = 1$, then $y = 0$ (cf. [14]). A Banach space E is said to be *strictly convex* if whenever $x, y \in E$ are of norm 1 and $\|(x + y)/2\| = 1$, then $x = y$. The space $L^1(\mu)$ is complex strictly convex but not strictly convex. For many other examples of complex strictly convex spaces we refer the reader to [7].

The notation to be used in the remainder of this section follows Cambern's paper [2]. The operator $A : C(X,E) \to C(X,E_1)$ denotes an isometry so that

$$A(F)(y) = A_y(F)(\tau(y)).$$

The operators A_y are given by $A_y(e) = A(\mathbb{E})(y)$ with $\mathbb{E}(x) = e$ the constant function in $C(X,E)$. We also set

$$\begin{aligned}\mathcal{F}_{e,x} &= \{F \in C(X,E) : F(x) = \|F\|_\infty \cdot e\},\\ B(e,x) &= \{y \in Y : \|(A(F))(y)\| = \|F\|_\infty \text{ for all } F \in \mathcal{F}_{e,x}\},\\ B(x) &= \bigcup_{\{e:\,\|e\|=1\}} B(e,x), \qquad B(A) = \bigcup_{x\in X} B(x).\end{aligned}$$

It requires a fairly straightforward modification of Cambern's arguments to prove the following result.

Corollary 3.2. *Let E and E_1 be Banach spaces with E_1* ***complex strictly convex*** *and A an isometry from $C(X,E)$ into $C(Y,E_1)$. Then there exists a subset $B(A) \subset Y$, a continuous function $\phi : Y \to \mathcal{B}(E, E_1)$ such that $\phi(y) = A_y$ with $\|A_y\| \le 1$ for all $y \in Y$ and $\|A_y\| = 1$ for all $y \in B(A)$, and a continuous map τ from $B(A)$ onto X such that*

$$A(F)(y) = A_y(F)(\tau(y)) \quad \text{for } F \in C(X,E),\ y \in B(A).$$

If E is finite-dimensional then $B(A)$ is a closed subset of Y.

We recall that $\tau(y) = x$ for $y \in B(x)$.

We now have enough machinery to address the algebraic reflexivity of $C(X,E)$ whenever E is assumed to be strictly convex (or complex strictly convex). This theorem extends the results of Jarosz and Rao [8].

Theorem 3.3. *If X is a compact connected n-manifold without boundary, and E is algebraically reflexive and strictly convex or complex strictly convex, then $C(X,E)$ is algebraically reflexive.*

Proof. If A denotes a locally surjective isometry on $C(X,E)$ then A has the representation stated in Theorem 3.1(2). Given $F \in C(X,E)$ there exist a homeomorphism φ_F of X and a bounded operator I_F defined on X and with values in the surjective isometries on E, i.e. $I_F(x) = I_{(F,x)}$ is a surjective isometry on E, such that

$$A(F)(\xi) = I_{(F,\xi)}(F(\varphi_F(\xi))).$$

If we assume that there exist distinct points x_0 and x_1 in $B(A)$ with $\tau(x_0) = \tau(x_1) = x$, then, given $e \in E$ (of norm 1), and $F = \mathbb{E} \in C(X,E)$ ($F(\xi) = e$ for every $\xi \in X$) we must have

$$A_{x_0}(e) = A(F)(x_0) = I_{(F,x_0)}(F(\varphi_F(x_0))) = I_{(F,x_0)}(e)$$

and

$$A_{x_1}(e) = A(F)(x_1) = I_{(F,x_1)}(e).$$

It follows that $\|A_{x_0}(e)\| = \|I_{(F,x_0)}(e)\| = \|e\| = 1 = \|I_{(F,x_1)}(e)\| = \|A_{x_1}(e)\|$. Since X is first countable, we select a continuous function β on X and with values in $[0,1]$ such that $\beta(x) = 1$ and $\beta(y) < 1$ for all $y \neq x$. We set $F(\xi) = \beta(\xi) \cdot e$. We have $A(F)(x_0) = A_{x_0}(e) = I_{(F,x_0)}(F(\varphi_F(x_0)))$ and $A(F)(x_1) = A_{x_1}(e) = I_{(F,x_1)}(F(\varphi_F(x_1)))$. Therefore $\varphi_F(x_0) = \varphi_F(x_1) = x$, since

$$1 = \|A_{x_0}(e)\| = \|I_{(F,x_0)}(F(\varphi_F(x_0)))\| = \|F(\varphi_F(x_0))\| = \|F(\varphi_F(x_1))\|.$$

This contradiction shows that $B(x)$ reduces to a single point and τ is injective. As shown in [2], the set $B = \{(x,y) : \tau(y) = x\}$ is closed in $X \times X$, hence compact, as also is its projection on the second component. This implies that $B(A)$ is compact and τ is a homeomorphism between $B(A)$ and X. Therefore $X = B(A)$. It remains to show that A_x is a surjective isometry for every $x \in X$. Given $e \in E$ of norm 1, we have

$$A_x(e) = A(\mathbb{E})(x) = I_{(\mathbb{E},x)}(\mathbb{E}(\varphi_{\mathbb{E}}(x))) = I_{(\mathbb{E},x)}(e),$$

which implies that A_x is a locally surjective isometry. Since E is algebraically reflexive, A_x is onto. ∎

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