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Product of operators and numerical range preserving maps

by

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To Professor Miroslav Fiedler on the occasion of his 80th birthday

Abstract. Let $\mathbf{V}$ be the C^*-algebra $B(H)$ of bounded linear operators acting on the Hilbert space H, or the Jordan algebra $S(H)$ of self-adjoint operators in $B(H)$. For a fixed sequence $(i_1, \ldots, i_m)$ with $i_1, \ldots, i_m \in \{1, \ldots, k\}$, define a product of $A_1, \ldots, A_k \in \mathbf{V}$ by $A_1 * \cdots * A_k = A_{i_1} \cdots A_{i_m}$. This includes the usual product $A_1 * \cdots * A_k = A_1 \cdots A_k$ and the Jordan triple product $A * B = ABA$ as special cases. Denote the numerical range of $A \in \mathbf{V}$ by $W(A) = \{(Ax, x) : x \in H, (x, x) = 1\}$. If there is a unitary operator U and a scalar μ satisfying $\mu^m = 1$ such that $\phi : \mathbf{V} \to \mathbf{V}$ has the form

$$A \mapsto \mu U^* A U \quad \text{or} \quad A \mapsto \mu U^* A^t U,$$

then ϕ is surjective and satisfies

$$W(A_1 * \cdots * A_k) = W(\phi(A_1) * \cdots * \phi(A_k)) \quad \text{ for all } A_1, \ldots, A_k \in \mathbf{V}.$$

It is shown that the converse is true under the assumption that one of the terms in $(i_1, \ldots, i_m)$ is different from all other terms. In the finite-dimensional case, the converse can be proved without the surjectivity assumption on ϕ. An example is given to show that the assumption on $(i_1, \ldots, i_m)$ is necessary.

1. Introduction. Let H be a Hilbert space having dimension at least 2. Denote by $B(H)$ the C^*-algebra of bounded linear operators acting on H, and $S(H)$ the Jordan algebra of self-adjoint operators in $B(H)$. If H has dimension $n < \infty$, then $B(H)$ is identified with the algebra M_n of $n \times n$ complex matrices and $S(H)$ is identified with the set S_n of $n \times n$ complex Hermitian matrices. Define the *numerical range* of $A \in B(H)$ by

$$W(A) = \{(Ax, x) : x \in H, (x, x) = 1\}.$$

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Let $U \in B(H)$ be a unitary operator, and define a mapping ϕ on $B(H)$ or $S(H)$ by

$$A \mapsto U^*AU \quad \text{or} \quad A \mapsto U^*A^tU,$$

where A^t is the transpose of A with respect to a fixed orthonormal basis. (We will always use this interpretation of A^t in our discussion.) Then ϕ is a bijective linear map preserving the numerical range, i.e., $W(\phi(A)) = W(A)$ for all A.

There has been considerable interest in studying the converse of the above statement. Pellegrini [8] obtained an interesting result on numerical range preserving maps on a general C^*-algebra, which implies that a surjective linear map $\phi : B(H) \to B(H)$ preserving the numerical range must be of the above form. Furthermore, by the result in [7], the same conclusion also holds for linear maps ϕ acting on $S(H)$. In [6], the author showed that additive preservers of the numerical range of matrices must be linear and have the standard form $A \mapsto U^*AU$ or $A \mapsto U^*A^tU$. In [2], it was shown that a multiplicative map $\phi : M_n \to M_n$ satisfies $W(\phi(A)) = W(A)$ for all $A \in M_n$ if and only if ϕ has the form $A \mapsto U^*AU$ for some unitary matrix $U \in M_n$. In [5], the authors replaced the condition that "ϕ is multiplicative and preserves the numerical range" on the surjective map $\phi : B(H) \to B(H)$ by the condition that "$W(AB) = W(\phi(A)\phi(B))$ for all A, B", and showed that such a map has the form $A \mapsto \pm U^*AU$ for some unitary operator $U \in B(H)$. They also showed that a surjective map $\phi : B(H) \to B(H)$ satisfies $W(ABA) = W(\phi(A)\phi(B)\phi(A))$ for all $A, B \in B(H)$ if and only if ϕ has the form $A \mapsto \mu U^*AU$ or $A \mapsto \mu U^*A^tU$ for some unitary operator $U \in B(H)$ and $\mu \in \mathbb{C}$ with $\mu^3 = 1$. Similar results for mappings on $S(H)$ were also obtained. Recently, Gau and Li [3] obtained a similar result for surjective maps $\phi : \mathbf{V} \to \mathbf{V}$, where $\mathbf{V} = B(H)$ or $S(H)$, preserving the numerical range of the Jordan product, i.e., $W(AB + BA) = W(\phi(A)\phi(B) + \phi(B)\phi(A))$ for all $A, B \in \mathbf{V}$. Specifically, they showed that such a map must be of the form $A \mapsto \pm U^*AU$ or $A \mapsto \pm U^*A^tU$ for some unitary operator $U \in B(H)$. Moreover, the surjectivity assumption can be removed in the finite-dimensional case.

It is interesting that all the results mentioned in the preceding paragraph illustrate that under some mild assumptions, a numerical range preserving map ϕ is a C^*-isomorphism on $B(H)$ or a Jordan isomorphism on $S(H)$ up to a scalar multiple. Following this line of study, we consider a product of matrices involving k matrices with $k \geq 2$ which includes the usual product $A_1 * \cdots * A_k = A_1 \cdots A_k$, and the Jordan triple product $A * B = ABA$. We prove the following result.

Theorem 1.1. *Let* $(\mathbb{F}, \mathbf{V}) = (\mathbb{C}, B(H))$ *or* $(\mathbb{R}, S(H))$. *Fix a positive integer* k *and a finite sequence* $(i_1, \ldots, i_m)$ *such that* $\{i_1, \ldots, i_m\} = \{1, \ldots, k\}$

and there is an i_r not equal to i_s for all other s. For $A_1, \ldots, A_k \in \mathbf{V}$, let

$$A_1 * \cdots * A_k = A_{i_1} \cdots A_{i_m}.$$

A surjective map $\phi : \mathbf{V} \to \mathbf{V}$ satisfies

$$W(\phi(A_1) * \cdots * \phi(A_k)) = W(A_1 * \cdots * A_k) \quad \text{for all } A_1, \ldots, A_k \in \mathbf{V} \tag{1.1}$$

if and only if there exist a unitary operator $U \in B(H)$ and a scalar $\mu \in \mathbb{F}$ with $\mu^m = 1$ such that one of the following conditions holds:

(a) *ϕ has the form $A \mapsto \mu U^* A U$.*
(b) *$r = (m+1)/2$, $(i_1, \ldots, i_m) = (i_m, \ldots, i_1)$, and ϕ has the form $A \mapsto \mu U^* A^t U$.*
(c) *$\mathbf{V} = S_2$, $(i_{r+1}, \ldots, i_m, i_1, \ldots, i_{r-1}) = (i_{r-1}, \ldots, i_1, i_m, \ldots, i_{r+1})$ and ϕ has the form $A \mapsto \mu U^* A^t U$.*

Here A^t denotes the transpose of A with respect to a certain orthonormal basis of H. Furthermore, if the dimension of H is finite, then the surjectivity assumption on ϕ can be removed.

Note that the assumption that there is $i_r \notin \{i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_m\}$ is necessary. For example, if $A * B = ABBA$, then mappings ϕ satisfying $W(\phi(A)*\phi(B)) = W(A*B)$ may not have nice structure. For instance, ϕ can send all involutions, i.e., those operators $X \in B(H)$ such that $X^2 = I_H$, to a fixed involution, and $\phi(X) = X$ for other X.

For the usual product $A_1 * \cdots * A_k = A_1 \cdots A_k$ and the Jordan triple product $A*B = ABA$, Hou and Di [5] have also obtained the result for $B(H)$ in Theorem 1.1 with the surjectivity assumption. Our result is stronger when H is finite-dimensional.

It turns out that Theorem 1.1 can be deduced from the following special case.

Theorem 1.2. *Let $(\mathbb{F}, \mathbf{V}) = (\mathbb{C}, B(H))$ or $(\mathbb{R}, S(H))$. Suppose r, s and m are nonnegative integers such that $m - 1 = r + s > 0$. A surjective map $\phi : \mathbf{V} \to \mathbf{V}$ satisfies*

$$W(\phi(A)^r \phi(B) \phi(A)^s) = W(A^r B A^s) \quad \text{for all } A, B \in \mathbf{V} \tag{1.2}$$

if and only if there exist a unitary operator $U \in B(H)$ and a scalar $\mu \in \mathbb{F}$ with $\mu^m = 1$ such that one of the following conditions holds:

(a) *ϕ has the form $A \mapsto \mu U^* A U$.*
(b) *$r = s$ and ϕ has the form $A \mapsto \mu U^* A^t U$.*
(c) *$\mathbf{V} = S_2$ and ϕ has the form $A \mapsto \mu U^* A^t U$.*

Here A^t denotes the transpose of A with respect to a certain orthonormal basis of H. Furthermore, if the dimension of H is finite, then the surjectivity assumption on ϕ can be removed.

We present some auxiliary results in Section 2, and the proofs of the theorems in Section 3.

2. Auxiliary results. For any $x, y \in H$, denote by xy^* the rank one operator $(xy^*)z = (z, y)x$ for all $z \in H$. Then for any operator $A \in B(H)$ with finite rank, A can be written as $x_1y_1^* + \cdots + x_ky_k^*$ for some $x_i, y_i \in H$. Define the *trace* of A by

$$\mathrm{tr}(A) = (x_1, y_1) + \cdots + (x_k, y_k).$$

If H is finite-dimensional, $\mathrm{tr}(A)$ is equivalent to the usual matrix trace, i.e., the sum of all diagonal entries of the matrix A. For each positive integer m, let

$$\mathcal{R}^m = \{\mu xx^* : \mu \in \mathbb{F} \text{ and } x \in H \text{ with } (x, x) = 1 = \mu^m\}.$$

Note that $\mathcal{R}^1$ is the set of Hermitian rank one idempotents, and for all $m > 1$, $\mathcal{R}^1 \subseteq \mathcal{R}^m$.

PROPOSITION 2.1. *Let* $\mathbf{V} = B(H)$ *or* $S(H)$ *and* $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{R}$ *accordingly. Suppose* m *is a positive integer with* $m > 1$, *and* $\phi : \mathbf{V} \to \mathbf{V}$ *is a map satisfying*

$$\mathrm{tr}(\phi(A)^{m-1}\phi(B)) = \mathrm{tr}(A^{m-1}B) \quad \text{for all } A \in \mathcal{R}^m \text{ and } B \in \mathbf{V}. \tag{2.1}$$

If H *is finite-dimensional, then* ϕ *is an invertible* $\mathbb{F}$*-linear map. If* H *is infinite-dimensional and* $\phi(\mathcal{R}^m) = \mathcal{R}^m$, *then* ϕ *is* $\mathbb{F}$*-linear.*

Proof. Suppose H is finite-dimensional. We use an argument similar to that in the proof of Proposition 1.1 in [1]. Let $\mathbf{V} = M_n$ or S_n. For every $X = (x_{ij}) \in \mathbf{V}$, let R_X be the n^2 row vector

$$R_X = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{n1}, \ldots, x_{nn}),$$

and C_X the n^2 column vector

$$C_X = (x_{11}, x_{21} \ldots, x_{n1}, x_{12}, \ldots, x_{n2}, \ldots, x_{1n}, \ldots, x_{nn})^t.$$

Then we deduce from (2.1) that for all $A \in \mathcal{R}^m$ and $B \in \mathbf{V}$,

$$R_{\phi(A)^{m-1}}C_{\phi(B)} = \mathrm{tr}(\phi(A)^{m-1}\phi(B)) = \mathrm{tr}(A^{m-1}B) = R_{A^{m-1}}C_B. \tag{2.2}$$

Note that we can choose $A_1, \ldots, A_{n^2}$ in $\mathcal{R}^m$ such that $\{A_1^{m-1}, \ldots, A_{n^2}^{m-1}\}$ forms a basis for $\mathbf{V}$. Let Δ and Δ_ϕ be $n^2 \times n^2$ matrices having rows $R_{A_1^{m-1}}, \ldots, R_{A_{n^2}^{m-1}}$ and $R_{\phi(A_1)^{m-1}}, \ldots, R_{\phi(A_{n^2})^{m-1}}$, respectively. By (2.2),

$$\Delta_\phi C_{\phi(B)} = \Delta C_B \quad \text{for all } B \in \mathbf{V}.$$

Now take a basis $\{B_1, \ldots, B_{n^2}\}$ in $\mathbf{V}$ and let Ω and Ω_ϕ be the $n^2 \times n^2$ matrices having columns $C_{B_1}, \ldots, C_{B_{n^2}}$ and $C_{\phi(B_1)}, \ldots, C_{\phi(B_{n^2})}$, respectively. Then $\Delta_\phi\Omega_\phi = \Delta\Omega$. Note that both Δ and Ω are invertible, hence so is Δ_ϕ.

Therefore, for any $B \in \mathbf{V}$,

$$C_{\phi(B)} = \Delta_\phi^{-1} \Delta C_B.$$

Hence, ϕ is invertible and $\mathbb{F}$-linear.

Next, suppose H is infinite-dimensional and $\phi(\mathcal{R}^m) = \mathcal{R}^m$. Take any $X, Y \in \mathbf{V}$. For any $x \in H$ with $(x,x) = 1$, since $\mathcal{R}^1 \subseteq \mathcal{R}^m = \phi(\mathcal{R}^m)$, there is $A \in \mathcal{R}^m$ such that $\phi(A) = xx^*$. Then $\phi(A)^{m-1} = xx^*$ and

$$\begin{aligned}(\phi(X+Y)x, x) &= \operatorname{tr}(xx^*\phi(X+Y)) = \operatorname{tr}(\phi(A)^{m-1}\phi(X+Y)) \\ &= \operatorname{tr}(A^{m-1}(X+Y)) = \operatorname{tr}(A^{m-1}X) + \operatorname{tr}(A^{m-1}Y) \\ &= \operatorname{tr}(\phi(A)^{m-1}\phi(X)) + \operatorname{tr}(\phi(A)^{m-1}\phi(Y)) \\ &= (\phi(X)x, x) + (\phi(Y)x, x).\end{aligned}$$

Since this is true for all unit vectors $x \in H$, it follows that $\phi(X+Y) = \phi(X) + \phi(Y)$. Similarly, we can show that $\phi(\lambda X) = \lambda\phi(X)$ for all $\lambda \in \mathbb{F}$ and $X \in \mathbf{V}$. ∎

It is well known that if $A \in M_2$ then $W(A)$ is an elliptical disk with the eigenvalues of A as foci. Moreover, if $A \in B(H)$ is unitarily similar to $A_1 \oplus A_2$ then $W(A)$ is the convex hull of $W(A_1) \cup W(A_2)$. In particular, if A has rank one, then A is unitarily similar to $C \oplus 0$, where C has a matrix representation $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$; hence $W(A) = W(C)$ is an elliptical disk with 0 as a focus. These facts are used in the proof of the following lemma, which is an extension of a result in [5].

LEMMA 2.2. *Let r and s be two nonnegative integers with $r + s > 0$. For any $B \in B(H)$, B has rank one if and only if for all $A \in B(H)$, $W(A^rBA^s)$ is an elliptical disk with zero as one of the foci.*

Proof. Let $B \in B(H)$. If B is rank one, then so is A^rBA^s. Therefore $W(A^rBA^s)$ is an elliptical disk with 0 as a focus by the discussion before the lemma.

Conversely, suppose B has rank at least 2. Then there exist $x, y \in H$ such that $\{Bx, By\}$ is an orthonormal set. Let $C = x(Bx)^* - y(By)^*$. Then $BC = Bx(Bx)^* - By(By)^*$ has numerical range $[-1, 1]$. Suppose $r = 0$. Since C has rank two, it has an operator matrix of the form $C_1 \oplus 0$, where $C_1 \in M_k$ with $2 \le k \le 4$, with respect to an orthonormal basis of H. Let D have operator matrix $\operatorname{diag}(1, \ldots, k) \oplus 0$ with respect to the same basis. Then $C + \nu D$ has operator matrix $(C_1 + \nu D_1) \oplus 0$, where $D_1 = \operatorname{diag}(1, \ldots, k)$. Except for finitely many $\nu \in \mathbb{R}$, $C_1 + \nu D_1$ has distinct eigenvalues so that there is A_ν satisfying $A_\nu^s = C + \nu D$, and $W(BA_\nu^s) = W(BC + \nu BD)$. By [4, Problem 220], the mapping $\nu \mapsto \operatorname{Closure}(W(BC + \nu BD))$ is continuous. Since $W(BC) = [-1, 1]$, there is a sufficiently small $\nu > 0$ such that $W(BA_\nu^s)$ is not an elliptical disk with 0 as a focus. If $s = 0$, we can fix an orthonormal basis of H, and apply the above argument to B^t to show that there exists

A such that $W(A^rB) = W(B^t(A^t)^r)$ is not an elliptical disk with 0 as a focus.

Now, suppose that $rs > 0$. Let H_0 be the subspace of H spanned by $\{x, y, Bx, By\}$, which has dimension $p \in \{2, 3, 4\}$. Suppose $B_0 \in M_p$ is the compression of B on H_0. Then $B_0 = PU$ for some positive semidefinite $P \in S_p$ with rank at least 2, and a unitary matrix $U \in M_p$. Let $V \in M_p$ be a unitary matrix such that V^*UV is in diagonal form. Then V^*PV is positive semidefinite with rank at least 2. Note that the 2×2 principal minors of V^*PV are nonnegative, and their sum is the 2-elementary symmetric function of the eigenvalues of V^*PV, which is positive. So, at least one 2×2 principal minor of V^*PV is nonzero. Since V^*B_0V is the product of V^*PV and the diagonal unitary matrix V^*UV, the 2×2 principal minors of V^*B_0V are unit multiples of those of V^*PV. It follows that at least one 2×2 principal minor of V^*B_0V is nonzero. Hence, there exists a two-dimensional subspace H_1 of H_0 such that the compression B_1 of B on H_1 is invertible. Suppose $\{u, v\}$ is an orthonormal basis of H_1 such that $B_1 = auu^* + buv^* + cvv^*$. Then $\det(B_1) = ac \neq 0$. Let $A = \alpha uu^* + \beta vv^*$ so that $\alpha^{r+s}a = 1$ and $\beta^{r+s}c = -1$. Then $A^rBA^s = uu^* - vv^* + \alpha^r\beta^s buv^*$ and $W(A^rBA^s)$ is an elliptical disk with foci $1, -1$. ∎

Note that the analog of the above result for $\mathbf{V} = S(H)$ does not hold if H has dimension at least 3. For example, if $A * B = ABA$ and $B = uu^* + vv^*$ for some orthonormal set $\{u, v\}$ in H, then $W(ABA)$ is always a line segment with 0 as an end point. To prove our main theorems, we need a characterization of elements in $\mathcal{R}^m$ when $\mathbf{V} = S(H)$.

Lemma 2.3. *Let r, s and m be nonnegative integers such that $m - 1 = r + s > 0$. Suppose $X \in S(H)$ is such that $W(X^m) = [0, 1]$. Then $X \in \mathcal{R}^m$ if and only if the following holds:*

(†) *For any $Y \in S(H)$ satisfying $W(Y^m) = [0, 1] = W(X^rYX^s)$, we have*

$$\{Z \in S(H) : W(Z^m) = [0, 1],\ Y^rZY^s = 0_H\} \subseteq \{Z \in S(H) : W(Z^m) = [0, 1],\ X^rZX^s = 0_H\}.$$

Proof. Since $W(X^m) = [0, 1]$, X has an eigenvalue μ satisfying $\mu^m = 1$ with a unit eigenvector u. Assume that $X \neq \mu uu^*$. Then $X = [\mu] \oplus X_2$ on $H = \mathrm{span}\{u\} \oplus \{u\}^\perp$, where X_2 is nonzero. Let $Y = [\mu] \oplus 0_{\{u\}^\perp}$. Then $W(Y^m) = [0, 1] = W(X^rYX^s)$. Note that the operator $Z = [0] \oplus I_{\{u\}^\perp}$ satisfies $W(Z^m) = [0, 1]$ and $Y^rZY^s = 0_H$ but $X^rZX^s = [0] \oplus X_2^{m-1} \neq 0_H$.

Conversely, suppose $X = \mu uu^*$ on $H = \mathrm{span}\{u\} \oplus \{u\}^\perp$. For any $Y \in S(H)$ satisfying $W(Y^m) = [0, 1] = W(X^rYX^s)$, we have $Y = [\mu] \oplus Y_1$ and

$W(Y_1^m) \subseteq [0,1]$. Suppose

$$Z = \begin{pmatrix} \alpha & z_1^* \\ z_1 & Z_2 \end{pmatrix}$$

on $\operatorname{span}\{u\} \oplus \{u\}^\perp$ satisfying $W(Z^m) = [0,1]$ and $Y^r Z Y^s = 0_H$. If $rs > 0$ then $\alpha = 0$; if $rs = 0$ then $\alpha = 0$ and $z_1 = 0$. In both cases, we see that $X^r Z X^s = 0_H$. ∎

3. Proofs of the main theorems

3.1. *Proof of Theorem 1.2.* We need the following lemma.

LEMMA 3.1. *Let* $\mathbf{V} = M_n$ *or* S_n, *and let* $\phi : \mathbf{V} \to \mathbf{V}$ *be a map satisfying* (1.2). *Then*

$$\phi(\mathcal{R}^m) \subseteq \mathcal{R}^m. \tag{3.1}$$

Proof. Each matrix $A \in \mathcal{R}^m$ can be written as $\mu U^* E_{11} U$ for some unitary matrix U and $\mu \in \mathbb{F}$ with $\mu^m = 1$. It suffices to prove that $\phi(E_{11}) \in \mathcal{R}^m$. For the other cases, we may replace the map ϕ by the map $A \mapsto \phi(\mu U^* A U)$.

We first consider the case when $\mathbf{V} = S_n$. For $i = 1, \ldots, n$, let $F_i = \phi(E_{ii})$. Since $E_{ii}^r E_{jj} E_{ii}^s = 0_n$ for all $i \neq j$, we have

$$W(F_i^r F_j F_i^s) = W(E_{ii}^r E_{jj} E_{ii}^s) = W(0_n) = \{0\}.$$

It follows that $F_i^r F_j F_i^s = 0_n$ for all $i \neq j$.

We claim that $F_i F_j = F_j F_i = 0_n$ for all $i \neq j$. If the claim holds, then there are $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and a unitary matrix V such that $F_i = \alpha_i V^* E_{ii} V$. Furthermore, as $W(F_i^m) = W(E_{ii}^m) = [0,1]$, $\alpha_i^m = 1$. Therefore, $\phi(E_{11}) = F_1 = \alpha_1 V^* E_{11} V \in \mathcal{R}^m$ and the result follows.

When m is odd, as $W(\phi(I_n)^m) = W(I_n^m) = \{1\}$, we have $\phi(I_n) = I_n$. Then for any $i = 1, \ldots, n$,

$$W(F_i) = W(\phi(I_n)^r \phi(E_{ii}) \phi(I_n)^s) = W(I_n^r E_{ii} I_n^s) = W(E_{ii}) = [0,1].$$

Thus, F_i is positive semidefinite. Now for any $i \neq j$, as $F_i^r F_j F_i^s = 0_n$, we deduce that $F_i F_j = F_j F_i = 0_n$.

When m is even, since $W(\phi(I_n)^m) = \{1\}$, the eigenvalues of $\phi(I_n)$ can be either 1 or -1 only. Write $\phi(I_n) = V^*(I_p \oplus -I_q)V$ for some unitary matrix V and nonnegative integers p and q such that $p + q = n$. Then for any $i = 1, \ldots, n$,

$$W(\phi(I_n)^r \phi(E_{ii}) \phi(I_n)^s) = W(I_n^r E_{ii} I_n^s) = W(E_{ii}) = [0,1].$$

Since one of r and s is odd while the other one must be even, either $\phi(I_n) F_i$ or $F_i \phi(I_n)$ is positive semidefinite. In both cases, we conclude that $F_i = V^*(P_i \oplus -Q_i)V$ for some positive semidefinite matrices $P_i \in H_p$ and $Q_i \in H_q$. Since $F_i^r F_j F_i^s = 0_n$, we have $P_i^r P_j P_i^s = 0_p$ and $Q_i^r Q_j Q_i^s = 0_q$ for all $i \neq j$. Then we conclude that $P_iP_j = P_jP_i = 0_p$ and $Q_iQ_j = Q_jQ_i = 0_q$ and hence $F_iF_j = F_jF_i = 0_n$.

So, our claim is proved and the lemma follows if $\mathbf{V} = S_n$.

Next, we turn to the case when $\mathbf{V} = M_n$. We divide the proof into a sequence of assertions.

Assertion 1. *Let* $D = \mathrm{diag}(0, e^{i\theta_2}, \ldots, e^{i\theta_n})$ *with* $0 < \theta_2 < \cdots < \theta_n < \pi/m$. *Then*

$$\phi(D) = V^*([0] \oplus T)V$$

for some unitary matrix $V \in M_n$ *and invertible upper triangular matrix* $T \in M_{n-1}$.

Proof. Note that D^m has n distinct eigenvalues and $W(D^m)$ is a polygon with n vertices including zero. Since $W(\phi(D)^m) = W(D^m)$, it follows that $\phi(D)^m$ has n distinct eigenvalues, including one zero eigenvalue. Then so has $\phi(D)$. Therefore, we may write

$$\phi(D) = V^* \begin{pmatrix} 0 & x^* \\ 0 & T \end{pmatrix} V$$

for some $x \in \mathbb{C}^{n-1}$, a unitary matrix V and an upper triangular matrix $T \in M_{n-1}$ such that all eigenvalues of T are nonzero. Then T is invertible. Since $W(\phi(D)^m)$ is a polygon with n vertices, $\phi(D)^m$ is a normal matrix. Note that an upper triangular matrix is normal if and only if it is diagonal. Observe that

$$\phi(D)^m = V^* \begin{pmatrix} 0 & x^*T^{m-1} \\ 0 & T^m \end{pmatrix} V.$$

It follows that $x = 0$ as T is invertible, i.e., $\phi(D) = V^*([0] \oplus T)V$. The proof of the assertion is complete.

Assertion 2. *The inclusion* (3.1) *holds if* $rs = 0$.

Proof. Suppose $r = 0$. Then, as $E_{11}D^s = 0_n$, where D is the matrix defined in Assertion 1, $\phi(E_{11})\phi(D)^s = 0_n$. It follows that only the first column of $V^*\phi(E_{11})V$ is nonzero, where V is the unitary matrix defined in Assertion 1. Hence, $\phi(E_{11})$ is a rank one matrix. Note that $W(\phi(E_{11})^m) = W(E_{11}^m) = [0, 1]$, and since a rank one matrix $A \in M_n$ satisfies $W(A^m) = [0, 1]$ if and only if $A \in \mathcal{R}^m$, we conclude that $\phi(E_{11}) \in \mathcal{R}^m$. The proof for $s = 0$ is similar. Thus, our assertion is true.

Assertion 3. *Suppose* $rs > 0$. *For any nonzero*

$$A = \begin{pmatrix} a & w^* \\ z & 0_{n-1} \end{pmatrix} \in M_n,$$

we have

$$\phi\left(\begin{pmatrix} a & w^* \\ z & 0_{n-1} \end{pmatrix}\right) = V^*\begin{pmatrix} \alpha & x^* \\ y & 0_{n-1} \end{pmatrix}V$$

for some $\alpha \in \mathbb{C}$ and $x, y \in \mathbb{C}^{n-1}$, where V is the unitary matrix defined in Assertion 1. Furthermore, if $A^m \neq 0_n$ is Hermitian, then $x = \beta y$ for some nonzero $\beta \in \mathbb{C}$.

Proof. Let D be the matrix defined in Assertion 1. Since $D^r A D^s = 0_n$, it follows that $\phi(D)^r\phi(A)\phi(D)^s = 0_n$. Thus

$$\phi(A) = V^*\begin{pmatrix} \alpha & x^* \\ y & 0_{n-1} \end{pmatrix}V$$

for some $\alpha \in \mathbb{C}$ and $x, y \in \mathbb{C}^{n-1}$, where V is defined in Assertion 1. If $A^m \neq 0$ is Hermitian, $W(\phi(A)^m) = W(A^m) \subseteq \mathbb{R}$. Hence, $\phi(A)^m$ is Hermitian as well. Clearly, if x or y is the zero vector, say $x = 0$, then $\alpha \neq 0$ as $A^m \neq 0_n$. Therefore, y must also be zero. Thus the assertion holds.

Now we assume that both x and y are nonzero vectors. By induction, we have

$$\phi(A)^k = V^*\begin{pmatrix} a_{k+1} & a_k x^* \\ a_k y & a_{k-1} y x^* \end{pmatrix}V \quad \text{for all } k = 1, 2, \ldots,$$

where the sequence $\{a_k\}$ satisfies $a_{k+1} = \alpha a_k + x^* y a_{k-1}$ with $a_0 = 0$, $a_1 = 1$ and $a_2 = \alpha$.

It is impossible to have both a_m and a_{m-1} equal to zero, since then $a_{m+1} = 0$, and hence $\phi(A)^m = 0_n$. Then $W(A^m) = W(\phi(A)^m) = \{0\}$, which contradicts our assumption that $A^m \neq 0_n$. Thus, a_m or a_{m-1} must be nonzero. In both cases, as A^m is Hermitian, we must have $x = \beta y$ for some nonzero $\beta \in \mathbb{C}$. The proof of our assertion is complete.

ASSERTION 4. *The inclusion* (3.1) *holds if* $rs > 0$.

Proof. For $i = 1, \ldots, n$, let $H_i = \frac{1}{2}(E_{1i} + E_{i1})$. Then H_i^m is Hermitian and $H_i^m \neq 0_n$. By Assertion 3, we write

$$\phi(H_i) = V^*\begin{pmatrix} \alpha_i & \beta_i z_i^* \\ z_i & 0_{n-1} \end{pmatrix}V$$

for some $\alpha_i, \beta_i \in \mathbb{C}$ and $z_i \in \mathbb{C}^{n-1}$ with $\beta_i \neq 0$. Denote by Z_i the $n \times 2$ matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & z_i \end{smallmatrix}\right)$ and by K_i the 2×2 matrix $\left(\begin{smallmatrix} \alpha_i & \beta_i \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$\phi(H_i) = V^*\left[\begin{pmatrix} 1 & 0 \\ 0 & z_i \end{pmatrix}\begin{pmatrix} \alpha_i & \beta_i \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z_i^* \end{pmatrix}\right]V = V^* Z_i K_i Z_i^* V.$$

Observe that for any distinct $i < j$, $H_i^r H_j H_i^s = 0_n$. Setting $R_{ij} = Z_i^* Z_j$, we have

$$(3.2) \qquad \begin{aligned} 0_n &= \phi(H_i)^r \phi(H_j) \phi(H_i)^s \\ &= V^* Z_i\,(K_i R_{ii})^{r-1} K_i\,[R_{ij}\,K_j\,R_{ij}^*]\,K_i\,(R_{ii} K_i)^{s-1}\,Z_i^* V. \end{aligned}$$

Next we claim that for any $1 \le i < j \le n$,

$$z_i^* z_j = \alpha_j = 0 \quad \text{and} \quad z_j \neq 0 \quad \text{whenever } z_i \neq 0.$$

To see this, suppose $z_i \neq 0$. Then the $n \times 2$ matrix Z_i has rank 2 and hence the 2×2 matrix $Z_i^* Z_i$ is invertible. Also both K_i and K_j are invertible. Then (3.2) holds only when

$$\begin{pmatrix} 1 & 0 \\ 0 & z_i^* z_j \end{pmatrix} \begin{pmatrix} \alpha_j & \beta_j \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z_j^* z_i \end{pmatrix} = (Z_i^* Z_j) K_j (Z_j^* Z_i) = 0_2.$$

Thus, $\beta_j z_j^* z_i = z_i^* z_j = \alpha_j = 0$. Finally, since $W(\phi(H_j)^m) = W(H_j^m) \neq \{0\}$, we have $z_j \neq 0$.

Now we must have $z_1 = 0$. Otherwise, $\alpha_j = z_1^* z_j = 0$ and $z_j \neq 0$ for all $j = 2, \ldots, n$. We can then further deduce that $z_i^* z_j = 0$ for all $i \neq j$. Thus, we have n nonzero orthogonal vectors $z_1, \ldots, z_n$ in $\mathbb{C}^{n-1}$, which is impossible. Therefore, $z_1 = 0$ and hence $\alpha_1 \neq 0$. Finally, as $W(\phi(H_1)^m) = W(H_1^m) = [0,1]$, we have $\alpha_1^m = 1$. So $\phi(E_{11}) = \phi(H_1) = \alpha_1 V^* E_{11} V \in \mathcal{R}^m$ and the result follows. The proof of our assertion is complete.

Combining the assertions, we get the result for $\mathbf{V} = M_n$. ■

Proof of Theorem 1.2. First, consider the sufficiency part. If (a) or (b) holds, then clearly ϕ satisfies (1.2). Suppose (c) holds. Then for any $A, B \in S_2$, there is a unitary $V \in M_2$ such that $V^* A V = D$ is a real diagonal matrix, and $V^* B V = C$ is a real symmetric matrix. Thus,

$$\begin{aligned} \phi(A^r B A^s) &= W(D^r C D^s) = W(\overline{D^r C D^s}) \\ &= W((D^t)^r C^t (D^t)^s) = W(\phi(A)^r \phi(B) \phi(A)^s). \end{aligned}$$

Next we turn to the necessity. Suppose $\mathbf{V} = B(H)$ or $S(H)$. Assume that $\phi : \mathbf{V} \to \mathbf{V}$ satisfies (1.2), and that ϕ is surjective if H is infinite-dimensional. We divide the proof into several steps.

Step 1. We show that $\phi(\mathcal{R}^m) = \mathcal{R}^m$ and ϕ is linear. Suppose H is finite-dimensional with no surjectivity assumption on ϕ. By Lemma 3.1, $\phi(\mathcal{R}^m) \subseteq \mathcal{R}^m$. Suppose H is infinite-dimensional. For $\mathbf{V} = S(H)$, we have $\phi(\mathcal{R}^m) = \mathcal{R}^m$ by Lemma 2.3 and the surjectivity of ϕ. For $\mathbf{V} = B(H)$, by Lemma 2.2 and the surjectivity of ϕ, we see that ϕ maps the set of rank one operators onto itself; since a rank one operator $A \in B(H)$ satisfies $W(A^m) = [0,1]$ if and only if $A \in \mathcal{R}^m$, we also have $\phi(\mathcal{R}^m) = \mathcal{R}^m$.

Now, for any $A \in \mathcal{R}^m$ and $B \in \mathbf{V}$, both A^rBA^s and $\phi(A)^r\phi(B)\phi(A)^s$ have rank at most one. As a result, $W(A^rBA^s)$ is an elliptical disk with foci $\mathrm{tr}(A^rBA^s)$ and 0, and $W(\phi(A)^r\phi(B)\phi(A)^s)$ is an elliptical disk with foci $\mathrm{tr}(\phi(A)^r\phi(B)\phi(A)^s)$ and 0. Since $W(A^rBA^s) = W(\phi(A)^r\phi(B)\phi(A)^s)$, we conclude that

$$\mathrm{tr}(A^{r+s}B) = \mathrm{tr}(A^rBA^s) = \mathrm{tr}(\phi(A)^r\phi(B)\phi(A)^s) = \mathrm{tr}(\phi(A)^{r+s}\phi(B)) \tag{3.3}$$

for all $A \in \mathcal{R}^m$ and $B \in \mathbf{V}$. By Proposition 2.1, ϕ is linear. Moreover, if H is finite-dimensional, ϕ is invertible. Indeed, ϕ^{-1} also satisfies (1.2), and hence (3.1) and (3.3). So, $\phi(\mathcal{R}^m) = \mathcal{R}^m$.

Step 2. We show that $\phi(I_H) = \mu I_H$ with $\mu^m = 1$. For any $x \in H$ with $(x, x) = 1$, there are $y \in H$ and $\mu \in \mathbb{F}$ with $(y, y) = \mu^m = 1$ such that $\phi(\mu yy^*) = xx^*$. Then by (3.3),

$$\begin{aligned}(\phi(I_H)x, x) &= \mathrm{tr}(xx^*\phi(I_H)) = \mathrm{tr}((xx^*)^{m-1}\phi(I_H)) = \mathrm{tr}(\phi(\mu yy^*)^{m-1}\phi(I_H))\\ &= \mathrm{tr}((\mu yy^*)^{m-1}I_H) = \mu^{m-1}(y, y) = \mu^{-1}.\end{aligned}$$

It follows that $W(\phi(I_H)) \subseteq \{\mu^{-1} : \mu^m = 1\} = \{\mu : \mu^m = 1\}$. By the convexity of numerical range, $W(\phi(I_H))$ is a singleton set. Thus, $\phi(I_H) = \mu I_H$ for some $\mu^m = 1$.

Step 3. We show that ϕ has the asserted form. Using the result in Step 2, and replacing ϕ by the map $A \mapsto \mu^{-1}\phi(A)$, we have $\phi(I_H) = I_H$. Furthermore,

$$W(\phi(A)) = W(\phi(I_H)^r\phi(A)\phi(I_H)^s) = W(I_H^rAI_H^s) = W(A) \quad \text{for all } A \in \mathbf{V}.$$

Since ϕ is linear, by the results in [7, 8] the map ϕ has the form

$$A \mapsto U^*AU \quad \text{or} \quad A \mapsto U^*A^tU$$

for some unitary operator $W \in B(H)$.

Step 4. It remains to show that $r = s$ when $\mathbf{V} \neq S_2$ and ϕ has the form $A \mapsto U^*A^tU$. For any $A, B \in \mathbf{V}$,

$$\begin{aligned}W(A^sBA^r) &= W((A^t)^rB^t(A^t)^s) = W(U^*(A^t)^rB^t(A^t)^sU)\\ &= W(\phi(A)^r\phi(B)\phi(A)^s) = W(A^rBA^s).\end{aligned}$$

For $\mathbf{V} = B(H)$, let $\{u, v\}$ be an orthonormal set in H, $A = uu^* + uv^* + vv^*$ and $B = vv^*$. Then

$$W(suv^* + vv^*) = W(A^sBA^r) = W(A^rBA^s) = W(ruv^* + vv^*).$$

Thus, $r = s$ and the result follows.

Now consider $\mathbf{V} = S(H)$, where H has dimension at least 3. Suppose $r \neq s$. Without loss of generality, we assume that $r > s$. Let $A, B \in S(H)$ be such that

$$A^{r-s} = D \oplus 0 \quad \text{and} \quad A^sBA^s = E \oplus 0,$$

where

$$D = \operatorname{diag}(3,2,1) \quad \text{and} \quad E = \begin{pmatrix} 1 & 1 & i \\ 1 & 0 & 1 \\ -i & 1 & 0 \end{pmatrix}$$

with respect to a suitable orthonormal basis. Then

$$\begin{aligned} W(DE \oplus 0) &= W(A^r B A^s) = W(A^s B A^r) = W(ED \oplus 0) \\ &= W(\overline{DE \oplus 0}) = \overline{W(DE \oplus 0)}. \end{aligned}$$

Therefore, $W(DE \oplus 0)$ is symmetric about the real axis. But this is impossible as the eigenvalues of $DE - ED$ are $2i$, $(\sqrt{3}-1)i/2$ and $(-\sqrt{3}-1)i/2$. Hence $\{\operatorname{Im} z : z \in W(DE \oplus 0_{n-3})\} = [(-\sqrt{3}-1)/2, 2]$ so that the two horizontal support lines of $W(DE \oplus 0)$ are $\{z : \operatorname{Im} z = 2\}$ and $\{z : \operatorname{Im} z = (-\sqrt{3}-1)/2\}$, which is a contradiction. Therefore, we must have $r = s$.

The proof of our theorem is complete. ■

3.2. *Proof of Theorem 1.1.* If (a) holds then ϕ clearly satisfies (1.1). Suppose (b) holds. Then for any $A_1, \ldots, A_k \in \mathbf{V}$, we have

$$\begin{aligned} &W(\phi(A_1) * \cdots * \phi(A_k)) \\ &\quad = W(\phi(A_{i_1}) \cdots \phi(A_{i_m})) = W(U^* A_{i_1}^t \cdots A_{i_m}^t U) = W((A_{i_m} \cdots A_{i_1})^t) \\ &\quad = W(A_{i_m} \cdots A_{i_1}) = W(A_{i_1} \cdots A_{i_m}) = W(A_1 * \cdots * A_k). \end{aligned}$$

Suppose (c) holds. Note that $X, Y \in M_2$ have the same numerical range if and only if the two matrices have the same eigenvalues and the same Frobenius norm, equivalently, $\operatorname{tr}(X) = \operatorname{tr}(Y)$, $\det(X) = \det(Y)$ and $\operatorname{tr}(XX^*) = \operatorname{tr}(YY^*)$. One readily checks that these conditions are satisfied for $X = A_1 * \cdots * A_k$ and $Y = \phi(A_1) * \cdots * \phi(A_k)$ for any $A_1, \ldots, A_k \in S_2$ if (c) holds. So, condition (1.1) follows.

Next, we turn to the necessity. Applying Theorem 1.2 with $A_{i_r} = B$ and $A_{i_s} = A$ for all other $s \neq r$, we conclude that there exist a unitary operator $U \in B(H)$ and a scalar $\mu \in \mathbb{F}$ with $\mu^m = 1$ such that one of the following holds:

(a) $A \mapsto \mu U^* A U$ for all $A \in \mathbf{V}$.
(b) $r = (m+1)/2$ and ϕ has the form $A \mapsto \mu U^* A^t U$.
(c) $\mathbf{V} = S_2$ and ϕ has the form $A \mapsto \mu U^* A^t U$.

It remains to prove $(i_{r+1}, \ldots, i_m, i_1, \ldots, i_{r-1}) = (i_{r-1}, \ldots, i_1, i_m, \ldots, i_{r+1})$ if (b) or (c) holds.

Evidently, the result holds for $k = 2$ as we must have $i_1 = \cdots = i_{r-1} = i_{r+1} = \cdots = i_m$ in this case. Now we assume that $k \geq 3$. Then we have

$$\begin{aligned} W(A_{i_1} \cdots A_{i_m}) &= W(\phi(A_{i_1}) \cdots \phi(A_{i_m})) = W(U^* A_{i_1}^t \cdots A_{i_m}^t U) \\ &= W(A_{i_1}^t \cdots A_{i_m}^t) = W(A_{i_m} \cdots A_{i_1}). \end{aligned}$$

By taking $A_{i_r} = R$, where R is a Hermitian rank one idempotent, and considering the foci of the elliptical disks for the above numerical ranges, we conclude that

$$\begin{aligned}\operatorname{tr}(A_{i_{r+1}}\cdots A_{i_m}A_{i_1}\cdots A_{i_{r-1}}R) &= \operatorname{tr}(A_{i_1}\cdots A_{i_{r-1}}RA_{i_{r+1}}\cdots A_{i_m})\\ &= \operatorname{tr}(A_{i_m}\cdots A_{i_{r+1}}RA_{i_{r-1}}\cdots A_{i_1})\\ &= \operatorname{tr}(A_{i_{r-1}}\cdots A_{i_1}A_{i_m}\cdots A_{i_{r+1}}R).\end{aligned}$$

Since R can be an arbitrary Hermitian rank one idempotent, by the fact that X and Y are equal if $\operatorname{tr}(XR) = \operatorname{tr}(YR)$ for all Hermitian rank one idempotent R, we deduce that

$$A_{i_{r+1}}\cdots A_{i_m}A_{i_1}\cdots A_{i_{r-1}} = A_{i_{r-1}}\cdots A_{i_1}A_{i_m}\cdots A_{i_{r+1}} \tag{3.4}$$

for all choices of $A_1,\ldots,A_k$.

We now use a similar argument to the one in the proof of [1, Theorem 2.1]. We give the details for the sake of completeness. For simplicity, we rename $(i_{r+1},\ldots,i_m,i_1,\ldots,i_{r-1})$ as $(j_1,\ldots,j_{m-1})$ and we have to show that $(j_1,\ldots,j_{m-1}) = (j_{m-1},\ldots,j_1)$. Suppose $(j_1,\ldots,j_{m-1}) \neq (j_{m-1},\ldots,j_1)$. Let $1 \le p \le m/2$ be the smallest integer such that $j_p \neq j_{m-p}$. For any $\lambda > 0$, let $D = \operatorname{diag}(\lambda,1)$ and S be some 2×2 symmetric matrix with positive entries. Fix a two-dimensional subspace H_1 in H and take $A_{j_p} = D\oplus I_{H_1^\perp}$ and $A_{j_t} = S\oplus I_{H_1^\perp}$ for all other $j_t \neq j_p$ on $H = H_1\oplus H_1^\perp$. Then

$$A_{j_p}\cdots A_{j_{m-p}} = (D^{d_1}S^{s_1}D^{d_2}S^{s_2}\cdots D^{d_q}S^{s_q})\oplus I_{H_1^\perp}$$

for positive integers d_i, s_i. Note that

$$D^{d_i}S^{s_i} = \begin{pmatrix}\lambda^{d_i}e_i & \lambda^{d_i}f_i\\ g_i & h_i\end{pmatrix} \quad\text{and}\quad S^{s_i}D^{d_i} = \begin{pmatrix}\lambda^{d_i}e_i & f_i\\ \lambda^{d_i}g_i & h_i\end{pmatrix}$$

for some positive numbers e_i, f_i, g_i, h_i. We check that the $(1,2)$ entry of $D^{d_1}S^{s_1}\cdots D^{d_q}S^{s_q}$ is a polynomial of degree $d_1+\cdots+d_q$ in λ, while the $(1,2)$ entry of $S^{s_q}D^{d_q}\cdots S^{s_1}D^{d_1}$ is a polynomial of degree $d_2+\cdots+d_q$. So, there is $\lambda > 0$ such that

$$\begin{aligned}A_{j_p}\cdots A_{j_{m-p}} &= (D^{d_1}S^{s_1}\cdots D^{d_q}S^{s_q})\oplus I_{H_1^\perp}\\ &\neq (S^{s_q}D^{d_q}\cdots S^{s_1}D^{d_1})\oplus I_{H_1^\perp} = A_{j_{m-p}}\cdots A_{j_p}.\end{aligned}$$

It follows that $A_{j_1}\cdots A_{j_{m-1}} \neq A_{j_{m-1}}\cdots A_{j_1}$, which is a contradiction. Hence, $(j_1,\ldots,j_{m-1}) = (j_{m-1},\ldots,j_1)$ as asserted. ∎

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