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Linear maps preserving elements annihilated by the polynomial $XY - YX^\dagger$

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Abstract. Let H and K be complex complete indefinite inner product spaces, and $\mathcal{B}(H,K)$ ($\mathcal{B}(H)$ if $K = H$) the set of all bounded linear operators from H into K. For every $T \in \mathcal{B}(H,K)$, denote by $T^\dagger$ the indefinite conjugate of T. Suppose that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a bijective linear map. We prove that Φ satisfies $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)^\dagger$ for all $A, B \in \mathcal{B}(H)$ with $AB = BA^\dagger$ if and only if there exist a nonzero real number c and a generalized indefinite unitary operator $U \in \mathcal{B}(H,K)$ such that $\Phi(A) = cUAU^\dagger$ for all $A \in \mathcal{B}(H)$.

1. Introduction and main results. Roughly speaking, linear preserver problems concern characterizing linear maps between operator algebras that leave certain properties of elements invariant. Over the past decades a lot of work has been done on linear preserver problems on matrix algebras. Recently, interest in similar questions on operator algebras over infinite-dimensional spaces has also been growing.

Here, we would like to mention a kind of linear preserver problems concerning zeros of polynomials in several elements. Particularly, the most extensive study was done for commutativity preserving linear maps, that is, linear maps preserving zeros of the polynomial $p(X,Y) = XY - YX$ (see [3], [5]–[7] and the reference therein). As to some other polynomials, for example, for $p(X,Y) = XY$, the reader is referred to [12]–[13], [22] and [26], for $p(X,Y) = XY + YX$, to [19], [34], and for any polynomial $p(X)$ in one element, to papers [2], [18], [20] and [30], and so on. Linear maps preserving zeros of $*$-polynomials have also been studied by many authors. For instance, the studies of linear maps preserving normal elements ([8] and

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[23]), preserving unitary elements ([28] and [29]) and preserving the orthogonality of operators ([1] and [27]) belong to this type of topics. Indeed, these maps preserve the zeros of the *-polynomial $p(X,X^*) = X^*X - XX^*$, of the commutative *-polynomial $p(X,X^*) = X^*X - I$, and of the *-polynomial $p(X,Y^*) = XY^*$, respectively. It seems that the problem of characterizing linear maps preserving zeros of $*$-polynomials in several variables is much more difficult.

As a kind of new product in a $*$-ring, the operation $XY - YX^*$ was discussed in [6]. This product $XY - YX^*$ is found to play a more and more important role in some research topics. For example, it is closely related to Jordan $*$-derivations. Let $\mathcal{A}$ be a $*$-ring and $X \in \mathcal{A}$ be fixed. Define an additive map $\delta : \mathcal{A} \to \mathcal{A}$ by $\delta(A) = AX - XA^*$ for all $A \in \mathcal{A}$. Then it is easily checked that $\delta(A^2) = A\delta(A) + \delta(A)A^*$ for all $A \in \mathcal{A}$, that is, δ is a Jordan $*$-derivation. For more results concerning Jordan $*$-derivations, the reader is referred to [9]. Recently, M. A. Chebotar *et al.* [10] have characterized the bijective linear maps preserving the zeros of the $*$-polynomial $p(X,X^*,Y) = XY - YX^*$ on $M_n(\mathbb{F})$, where $\mathbb{F}$ is a field with an involution $*$ and $n \geq 20$. They proved that any such map ϕ is of the form $\phi(x) = \lambda uxu^{-1}$ for all $x \in M_n(\mathbb{F})$, where λ is a nonzero symmetric scalar and u is a normal matrix such that uu^* is a nonzero scalar.

As indefinite inner product spaces are useful both for the discussion of physical problems and for some mathematical questions (see the introduction in [4]), motivated by the work of Molnár [26], some preserver problems were studied and solved for operator algebras on such spaces (see, for example, [14]–[15], [24], [26] and the references therein). In particular, it is an interesting question to characterize linear maps preserving zeros of the $*$-polynomial $p(X,X^*,Y) = XY - YX^*$ in indefinite inner product space setting.

Denote by $\mathbb{F}$ the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Recall that an *indefinite inner product space* means a linear space H over $\mathbb{F}$ equipped with a nondegenerate sesquilinear Hermite functional $[\cdot,\cdot]$. Let $(H,[\cdot,\cdot])$ be such a space. If there are subspaces H_+ and H_- such that

$$H = H_+ \oplus H_-, \tag{1.1}$$

and both $(H_+,[\cdot,\cdot])$ and $(H_-,-[\cdot,\cdot])$ are Hilbert spacs, then H is called a *complete indefinite inner product space*. The decomposition (1.1) is called a *regular decomposition* of H (see [33]). In the following we always assume that the indefinite inner product spaces considered are complete. If $H = H_+ \oplus H_-$ is a regular decomposition, then any $x, y \in H$ can be uniquely represented as $x = x_+ + x_-$ and $y = y_+ + y_-$, where $x_\pm$, $y_\pm \in H_\pm$. Define an inner product on H by

$$\langle x, y\rangle = [x_+, y_+] - [x_-, y_-].$$

Then it is obvious that $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space. We call $\langle\cdot,\cdot\rangle$ the *inner product induced by the regular decomposition* $H = H_+ \oplus H_-$.

A linear operator T from an indefinite inner product space H into an indefinite inner product space K is said to be *bounded* if T is bounded with respect to the inner products of H and K induced by some regular decompositions. The boundedness of T does not depend on the choice of the regular decompositions. We denote by $\mathcal{B}(H,K)$ ($\mathcal{B}(H)$ if $K = H$) the set of all bounded linear operators from H into K. For any $T \in \mathcal{B}(H,K)$, the *indefinite conjugate* of T with respect to the indefinite inner product $[\cdot,\cdot]$ is an operator $T^\dagger \in \mathcal{B}(K,H)$ defined by the equation $[Tx,y] = [x, T^\dagger y]$ for all $x \in H$ and $y \in K$ (similarly, for a bounded conjugate-linear operator $T : H \to K$, its indefinite conjugate operator $T^\dagger : K \to H$ is defined by $[Tx,y] = [T^\dagger y, x]$ for all $x \in H$ and $y \in K$). For a linear operator T, if both $T^\dagger T$ and $TT^\dagger$ are the identity (resp. a nonzero real scalar multiple of the identity), we say that T is an *indefinite unitary operator* (resp. a *generalized indefinite unitary operator*); in the case that T is conjugate linear, we say that T is an *indefinite anti-unitary operator* (resp. a *generalized indefinite anti-unitary operator*).

Now we are in a position to state the main result of this paper.

Theorem 1. *Let H and K be complex complete indefinite inner product spaces. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective linear map. Then Φ satisfies the condition that $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)^\dagger$ for all $A, B \in \mathcal{B}(H)$ with $AB = BA^\dagger$ if and only if there exist a nonzero real number c and a generalized indefinite unitary operator $U \in \mathcal{B}(H,K)$ such that $\Phi(A) = cUAU^\dagger$ for all $A \in \mathcal{B}(H)$.*

Assume that H is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and $J \in \mathcal{B}(H)$ is an invertible self-adjoint operator (i.e., $J^* = J$). Then $(H, [\cdot,\cdot]_J)$ is a complete indefinite inner product space with the indefinite inner product $[\cdot,\cdot]_J = \langle J(\cdot),\cdot\rangle$ induced by J. It is clear that, with respect to $[\cdot,\cdot]_J$, the indefinite conjugate $T^\dagger$ of an operator $T \in \mathcal{B}(H)$ is of the form $T^\dagger = J^{-1}T^*J$, where T^* stands for the usual conjugate of T relative to the inner product $\langle\cdot,\cdot\rangle$. If K is another Hilbert space and $L \in \mathcal{B}(K)$ is an invertible self-adjoint operator, then, with respect to the indefinite inner products $[\cdot,\cdot]_J$ and $[\cdot,\cdot]_L$, we have $S^\dagger = J^{-1}S^*L$ for every $S \in \mathcal{B}(H,K)$. Thus, in terms of definite inner products, Theorem 1 may be restated as follows.

Theorem 1′. *Let H and K be complex Hilbert spaces. Let $J \in \mathcal{B}(H)$ and $L \in \mathcal{B}(K)$ be given invertible self-adjoint operators. Suppose that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a linear bijective map. Then $\Phi(A)\Phi(B) = \Phi(B)L^{-1}\Phi(A)^*L$ for all $A, B \in \mathcal{B}(H)$ with $AB = BJ^{-1}A^*J$ if and only if there exist a nonzero real number c and an invertible operator $U \in \mathcal{B}(H,K)$ satisfying $J^{-1}U^*LU = aI$ on H, $UJ^{-1}U^*L = aI$ on K for some nonzero real number a, such that $\Phi(A) = cUAU^{-1}$ for all $A \in \mathcal{B}(H)$.*

In particular, when both J and L are the identity, we have

COROLLARY 2. *Let H and K be complex Hilbert spaces. Suppose that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a linear bijective map. Then $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)^*$ for all $A, B \in \mathcal{B}(H)$ with $AB = BA^*$ if and only if there exist a nonzero scalar $c \in \mathbb{R}$ and a unitary operator $U \in \mathcal{B}(H, K)$ such that $\Phi(A) = cUAU^*$ for all $A \in \mathcal{B}(H)$.*

We should mention here that, by a different approach, Theorem $1'$ was obtained in [11] under the additional assumptions that $\dim H \geq 3$, $K = H$ and Φ is weakly continuous.

2. Proof of the main result. The proof of Theorem 1 is based on the following result which was proved in [5].

LEMMA 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be centrally closed prime algebras over a field $\mathbb{F}$ of characteristic different from 2 and 3. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective linear map satisfying $\Phi(A^2)\Phi(A) = \Phi(A)\Phi(A^2)$ for all $A \in \mathcal{A}$. If neither $\mathcal{A}$ nor $\mathcal{B}$ satisfies S_4, the standard polynomial identity of degree 4, then*

$$\Phi(A) = c\phi(A) + q(A)$$

for all $A \in \mathcal{A}$, where $c \in \mathbb{F}$ is nonzero, ϕ is an isomorphism or an anti-isomorphism of $\mathcal{A}$ onto $\mathcal{B}$, and q is a linear map from $\mathcal{A}$ into the center of $\mathcal{B}$.

We also need the following simple lemmas.

LEMMA 2.2. *Let H be a complex complete indefinite inner product space with $\dim H \geq 3$. For any rank-one operator $B \in \mathcal{B}(H)$, there exists a nonzero rank-one operator $A \in \mathcal{B}(H)$ such that $AB = BA^\dagger = 0$, $BA = A^\dagger B = 0$ and $A - A^\dagger$ is of rank one.*

Proof. Let $H = H_- \oplus H_+$ be a regular decomposition of H and $P_\pm$ the corresponding projections from H onto $H_\pm$, and let $J = P_+ - P_-$. Then $\langle \cdot, \cdot \rangle = [J(\cdot), \cdot]$ is an inner product on H induced by the regular decomposition. Write $B = u \otimes v$ with $\|u\| = 1$. A rank-one operator $A = x \otimes y$ satisfies $AB = BA^\dagger$ if and only if

$$\langle u, y \rangle x \otimes v = \langle J^{-1}y, v \rangle u \otimes Jx.$$

Since $\dim H \geq 3$, there must be $\dim J^{-1}([u]^\perp) \geq 2$. Here $[u]$ denotes the linear span of u and $[u]^\perp = H \ominus [u]$, that is, the orthogonal complement of $[u]$ in H with respect to the inner product $\langle \cdot, \cdot \rangle$. Thus there exists a nonzero $y \in [u]^\perp$ such that $J^{-1}y \perp v$. Let $x = \alpha J^{-1}y$, where α is any complex number with $\overline{\alpha} \neq \alpha$. Then it is easily seen that $AB = BA^\dagger = 0$, $BA = A^\dagger B = 0$ and $A - A^\dagger = (1 - \overline{\alpha}/\alpha)A$ is of rank one. ∎

Lemma 2.3. *Let H be a complex complete indefinite inner product space and $T \in \mathcal{B}(H)$. If $0 \in \sigma_{\mathrm{p}}(T)$ (the point spectrum of T), then there exists a rank-one operator B such that $TB = BT^{\dagger}$.*

Proof. Since $0 \in \sigma_{\mathrm{p}}(T)$, there exists a nonzero $x \in H$ so that $Tx = 0$. Let J be as in the proof of Lemma 2.2 and $B = x \otimes Jx$. Then B is as claimed. ∎

Now we are ready to prove our result.

Proof of Theorem 1. We need only check the "only if" part.

Assume that $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)^{\dagger}$ whenever $AB = BA^{\dagger}$. It follows from $I \cdot A = A \cdot I^{\dagger}$ for all $A \in \mathcal{B}(H)$ that $\Phi(I)\Phi(A) = \Phi(A)\Phi(I)^{\dagger}$. Since Φ is surjective, we have

$$\Phi(I)B = B\Phi(I)^{\dagger} \quad \text{for all } B \in \mathcal{B}(H),$$

and therefore $\Phi(I) = \Phi(I)^{\dagger}$. Thus the above equality entails that $\Phi(I)$ belongs to the center of $\mathcal{B}(H)$ which is $\mathbb{C}I$ and hence $\Phi(I) = aI$ for some $a \in \mathbb{R}$. It follows from the injectivity of Φ that $a \neq 0$. For any †-Hermitian operator $S \in \mathcal{B}(H)$ (that is, $S^{\dagger} = S$), we have $\Phi(S)\Phi(I) = \Phi(I)\Phi(S)^{\dagger}$, and hence $\Phi(S) = \Phi(S)^{\dagger}$. This implies obviously that $\Phi(A^{\dagger}) = \Phi(A)^{\dagger}$ for all $A \in \mathcal{B}(H)$. Also, for all †-Hermitian operators $S \in \mathcal{B}(H)$, we have

$$\Phi(S^2)\Phi(S) = \Phi(S)\Phi(S^2). \tag{2.1}$$

Let $S, T \in \mathcal{B}(H)$ be arbitrary †-Hermitian. Replacing S in (2.1) by $S + T$, and letting $[S, T]$ denote the commutator $ST - TS$, we get

$$\begin{aligned}([\Phi(ST + TS), \Phi(S)] + [\Phi(S^2), \Phi(T)])& \\ + ([\Phi(T^2), \Phi(S)] + [\Phi(ST + TS), \Phi(T)]) &= 0.\end{aligned}$$

Replacing T with $-T$ in the above equality, one obtains

$$\begin{aligned}-([\Phi(ST + TS), \Phi(S)] + [\Phi(S^2), \Phi(T)])& \\ + ([\Phi(T^2), \Phi(S)] + [\Phi(ST + TS), \Phi(T)]) &= 0.\end{aligned}$$

Comparing the above two equalities, we see that for all †-Hermitian operators $T, S \in \mathcal{B}(H)$, we have

$$[\Phi(ST + TS), \Phi(S)] + [\Phi(S^2), \Phi(T)] = 0 \tag{2.2}$$

and

$$[\Phi(T^2), \Phi(S)] + [\Phi(ST + TS), \Phi(T)] = 0. \tag{2.3}$$

For any $A \in \mathcal{B}(H)$, let $S = (A + A^{\dagger})/2$ and $T = (A - A^{\dagger})/2i$; then S and T are †-Hermitian and $A = S + iT$. A straightforward computation shows that

$$\begin{aligned}[\Phi(A^2), \Phi(A)] = -([\Phi(ST + TS), \Phi(T)] + [\Phi(T^2), \Phi(S)])& \\ + i([\Phi(S^2), \Phi(T)] + [\Phi(ST + TS), \Phi(S)]),&\end{aligned}$$

which, together with (2.2) and (2.3), ensures that $[\Phi(A^2), \Phi(A)] = 0$. Thus, in summary, we have showed that

$$\Phi(A^2)\Phi(A) = \Phi(A)\Phi(A^2), \tag{2.4}$$

$\Phi(A^\dagger) = \Phi(A)^\dagger$ for all $A \in \mathcal{B}(H)$, and $\Phi(I) = aI$ for some nonzero $a \in \mathbb{R}$.

Since the case $\dim H = 1$ is trivial, in the following we will assume that $\dim H \geq 2$ and complete the proof by considering the cases $\dim H \geq 3$ and $\dim H = 2$ separately.

Case 1: $\dim H \geq 3$. It is well known that $\mathcal{B}(H)$ is a prime algebra, that is, for $A, B \in \mathcal{B}(H)$, $A\mathcal{B}(H)B = 0$ implies $A = 0$ or $B = 0$. Moreover, $\mathcal{B}(H)$ is centrally closed over the field of complex numbers [25]. By standard PI theory [21], a prime ring $\mathcal{R}$ satisfies S_4 if and only if $\mathcal{R}$ is commutative or $\mathcal{R}$ embeds in $M_2(\mathbb{F})$ for some field $\mathbb{F}$. If $\dim H \geq 3$, then $\dim K \geq 3$, so both algebras $\mathcal{B}(H)$ and $\mathcal{B}(K)$ satisfy the assumptions in Lemma 2.1. Thus, it follows from (2.4) that Φ satisfies all assumptions of Lemma 2.1. Also note that every isomorphism and anti-isomorphism between $\mathcal{B}(H)$ and $\mathcal{B}(K)$ is spatial. Hence there exist a nonzero complex number c and a linear functional f on $\mathcal{B}(H)$ such that either

(i) Φ has the form

$$\Phi(A) = cVAV^{-1} + f(A)I \quad \text{for every } A \in \mathcal{B}(H), \tag{2.5}$$

where $V \in \mathcal{B}(H, K)$ is an invertible operator; or

(ii) Φ has the form

$$\Phi(A) = cVA^\dagger V^{-1} + f(A)I \quad \text{for every } A \in \mathcal{B}(H), \tag{2.6}$$

where $V : H \to K$ is a bounded bijective conjugate linear operator.

Since $\Phi(A^\dagger) = \Phi(A)^\dagger$ for all $A \in \mathcal{B}(H)$, for any rank-one operator $F \in \mathcal{B}(H)$, it follows from (2.5) that

$$(f(F^\dagger) - \overline{f(F)})I = \overline{c}(V^{-1})^\dagger F^\dagger V^\dagger - cVF^\dagger V^{-1}$$

is an operator of rank at most two. With $\dim H \geq 3$ in mind, this would imply that $f(F^\dagger) = \overline{f(F)}$ for all rank-one operators $F \in \mathcal{B}(H)$. It is clear that the same is true for f in (2.6).

Assume that Φ has the form (i), that is, (2.5) holds for all A. We claim that $c \in \mathbb{R}$, V is a generalized indefinite unitary in $\mathcal{B}(H, K)$, and $f(A) = 0$ for every $A \in \mathcal{B}(H)$.

Let J be as in the proof of Lemma 2.2 and L be an invertible self-adjoint operator determined by some regular decomposition of K. For convenience, we shall denote the corresponding inner products on H and K by the same symbol $\langle \cdot, \cdot \rangle$. Note that $x \otimes Jx$ is $\dagger$-Hermitian for every $x \in H$. It follows that $\Phi(x \otimes Jx) = \Phi(x \otimes Jx)^\dagger$ and $f(x \otimes Jx) = \overline{f(x \otimes Jx)}$ for every $x \in H$.

Hence

$$cVx \otimes (V^*)^{-1}Jx = \bar{c}L^{-1}(V^*)^{-1}Jx \otimes LVx$$

for every $x \in H$, which implies that LVx is linearly dependent of $(V^*)^{-1}Jx$. This entails that $LV = \lambda(V^*)^{-1}J$, or equivalently, $V^\dagger V = \lambda I$, for some nonzero $\lambda \in \mathbb{R}$ (see [33]). Now it is clear that $c \in \mathbb{R}$ and V is a generalized indefinite unitary operator. So, if we can show that $f = 0$, then Φ has the form stated in Theorem 1.

To prove $f = 0$, let $\Psi(\cdot) = c^{-1}V^{-1}\Phi(\cdot)V$ and $g = c^{-1}f$. Then $\Psi(A) = A + g(A)I$ for all $A \in \mathcal{B}(H)$ and $\Psi(A)\Psi(B) = \Psi(B)\Psi(A)^\dagger$ whenever $AB = BA^\dagger$. We claim that $g(F) = 0$ for every $F \in \mathcal{F}(H)$, the set of all finite rank operators in $\mathcal{B}(H)$. For any $A, B \in \mathcal{B}(H)$ with $AB = BA^\dagger$, we have

$$(A + g(A)I)(B + g(B)I) = (B + g(B)I)(A^\dagger + \overline{g(A)}I).$$

Hence

$$(g(A) - \overline{g(A)})B + g(B)(A - A^\dagger) = (\overline{g(A)} - g(A))g(B)I \tag{2.7}$$

for all $A, B \in \mathcal{B}(H)$ satisfying $AB = BA^\dagger$.

Assume that there exists a rank-one operator B such that $g(B) \neq 0$. Then, by (2.7),

$$A - A^\dagger = g(B)^{-1}(\overline{g(A)} - g(A))(g(B)I + B) \tag{2.8}$$

for all $A \in \mathcal{B}(H)$ with $AB = BA^\dagger$. By Lemma 2.2, for the rank-one operator B, there exists a rank-one operator A with $AB = BA^\dagger$ and $A - A^\dagger$ being rank one such that (2.8) holds. But this is impossible since the rank of $g(B)^{-1}(\overline{g(A)} - g(A))(g(B)I + B)$ is always greater than one, a contradiction. So we have proved that $g(B) = 0$ for all rank-one operators B, and consequently, $g(F) = 0$ for every $F \in \mathcal{F}(H)$. Hence

$$\Psi(F) = F \quad \text{for all } F \in \mathcal{F}(H).$$

Next we prove that g is the zero functional on $\mathcal{B}(H)$. For any $A \in \mathcal{B}(H)$, let $x \in H$ be a unit vector and $\xi \in \mathbb{C}$ be such that $\xi \neq \bar{\xi}$. Let $F = (A + \xi I)x \otimes x$. Then $0 \in \sigma_{\mathrm{p}}(A + \xi I - F)$ and, by Lemma 2.3, there exists a rank-one operator B such that

$$(A + \xi I - F)B = B(A + \xi I - F)^\dagger.$$

Write $g(I) = d$ with d a real number. Note that $g(B) = 0$ and $g(F) = 0$. Replacing A by $A + \xi I - F$ in (2.7), we obtain, for the rank-one operator B,

$$(g(A) + d\xi - \overline{g(A)} - d\bar{\xi})B = 0$$

for all $\xi \in \mathbb{C}$ with $\xi \neq \bar{\xi}$. This implies that $d = 0$ and $g(A) = \overline{g(A)}$ for all $A \in \mathcal{B}(H)$. Now it is clear that $g = 0$ since a real-valued linear functional on a complex vector space is the zero functional.

To complete the proof of Case 1, we have to show that Φ can never take the form (ii). On the contrary, assume that (2.6) holds for all A. Similar to the discussion of the case where Φ has the form (i), it is easy to check that $c \in \mathbb{R}$ and $V^\dagger V = \lambda I$ for some nonzero $\lambda \in \mathbb{R}$. Replacing Φ by $\Psi(\cdot) = c^{-1}V^{-1}\Phi(\cdot)V$, we find that Ψ is conjugate linear and preserves the zeros of the polynomial $p(A, A^\dagger, B) = AB - BA^\dagger$. So without loss of generality we might as well assume that Φ maps $\mathcal{B}(H)$ onto itself and has the form

$$\Phi(A) = A^\dagger + f(A)I \quad \text{for all } A \in \mathcal{B}(H),$$

where f is a conjugate linear functional.

If $A, B \in \mathcal{B}(H)$ satisfy $AB = BA^\dagger$, then $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)^\dagger$. Thus

$$(BA - A^\dagger B)^\dagger + f(B)(A^\dagger - A) = (\overline{f(A)} - f(A))(f(B)I + B^\dagger) \tag{2.9}$$

for all $A, B \in \mathcal{B}(H)$ satisfying $AB = BA^\dagger$. It follows from Lemma 2.2 that for any rank-one operator B, there exists a rank-one operator A so that $AB = BA^\dagger = 0$, $BA = A^\dagger B = 0$ and $A^\dagger - A$ is of rank one. So (2.9) implies that

$$f(B)(A^\dagger - A) = (\overline{f(A)} - f(A))(f(B)I + B^\dagger).$$

If there exists a rank-one operator B such that $f(B) \neq 0$, then the left side of the above equality is a rank-one operator, while the operator on the right side is always of rank greater than one, a contradiction. Hence we get $f(F) = 0$ for all $F \in \mathcal{F}(H)$, and consequently

$$\Phi(F) = F^\dagger \qquad \text{for all } F \in \mathcal{F}(H).$$

However, it is easily seen that there exist $A, B \in \mathcal{F}(H)$ such that $AB = BA^\dagger$ but $A^\dagger B^\dagger \neq B^\dagger A$ (for instance, let $A = x \otimes y$ and $B = x \otimes x$ with $y \in [x, J^{-1}x]^\perp$) contrary to the assumption $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)^\dagger$. Therefore, Φ cannot take the form (ii), completing the proof of Case 1.

Case 2: $\dim H = 2$. In this case we must also have $\dim K = 2$. Fix a regular decomposition $H = H_- \oplus H_+$ and denote by $P_\pm$ the projections from H onto $H_\pm$. Let $J = P_+ - P_-$. Then $J^2 = I$ on H and $\langle \cdot, \cdot \rangle = [J(\cdot), \cdot]$ is an inner product on H induced by the regular decomposition. In the same way, assume that L is an invertible self-adjoint operator in $\mathcal{B}(K)$ determined by some regular decomposition of K; then $L^2 = I$ on K. We also use the symbol $\langle \cdot, \cdot \rangle$ to denote the inner product $[L(\cdot), \cdot]$ on K.

For any $A \in \mathcal{B}(H)$, define $\Delta(A, A^\dagger) = \{B \in \mathcal{B}(H) \mid AB = BA^\dagger\}$. It is easily seen that $\Delta(A, A^\dagger)$ is a linear subspace of $\mathcal{B}(H)$. We prove the following assertion.

Assertion. $\dim \Delta(A, A^\dagger) = 1$ *if and only if* $\sigma(A) = \{a_1, a_2\}$ *with* a_1 *real and* a_2 *nonreal*; $\dim \Delta(A, A^\dagger) = 2$ *if and only if* $\sigma(A) \subset \mathbb{R}$ *and* $A \notin \mathbb{C}I$ *or* $a_1 = \overline{a}_2 \notin \mathbb{R}$; $\dim \Delta(A, A^\dagger) = 4$ *if and only if* $A \in \mathbb{R}I$; $\dim \Delta(A, A^\dagger) = 3$ *does not occur. Here* $\sigma(A)$ *denotes the spectrum of an operator* A.

Note that $AB = BA^{\dagger}$ if and only if $ABJ = BJA^{*}$. So, to prove our assertion we need only deal with the case that $J = I$. It is clear that there exists $B \neq 0$ such that $AB = BA^{*}$ if and only if there exists a complex number a such that $\{a, \overline{a}\} \subseteq \sigma(A)$. The set $\mathcal{C}$ of all such A is the union of four disjoint subsets:

$$\begin{aligned}
&\mathcal{C}_1 = \mathbb{R}I,\\
&\mathcal{C}_2 = \{A \mid \sigma(A) = \{a_1, a_2\},\ a_1 \in \mathbb{R},\ a_2 \notin \mathbb{R}\},\\
&\mathcal{C}_3 = \{A \mid \sigma(A) = \{a, \overline{a}\},\ a \notin \mathbb{R}\},\\
&\mathcal{C}_4 = \{A \mid \sigma(A) \subset \mathbb{R},\ A \notin \mathbb{R}I\}.
\end{aligned}$$

So,

$$\Delta(A, A^{*}) \neq \{0\} \ \Leftrightarrow\ A \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4. \tag{2.10}$$

Obviously, $\dim \Delta(A, A^{*}) = 4$ if and only if $A \in \mathbb{R}I$. For $A \in \mathcal{C}_2 \cup \mathcal{C}_3$, let $\sigma(A) = \{a_1, a_2\}$ with a_2 nonreal. Pick unit vectors $x_1, x_2 \in H$ such that $x_1 \in \ker(a_1 I - A)$ and $\langle x_1, x_2\rangle = 0$. An elementary 2×2 matrix argument shows that $\dim \Delta(A, A^{*}) = 1$ if and only if $A \in \mathcal{C}_2$, and in this case, there exists a rank-one projection P such that $\Delta(A, A^{*}) = \mathbb{C}P$. In fact, if $A \in \mathcal{C}_2$, then a_1 is real and $\Delta(A, A^{*}) = \mathbb{C}(x_1 \otimes x_1)$. If $A \in \mathcal{C}_3$, then $a_2 = \overline{a}_1$ and

$$\begin{aligned}
&\Delta(A, A^{*})\\
&\quad = \left\{ b_1 x_1 \otimes x_2 + \left(\frac{\langle x_1, Ax_2\rangle b_1 - \langle Ax_2, x_1\rangle b_2}{a_1 - \overline{a}_1} x_1 + b_2 x_2 \right) \otimes x_1 \,\middle|\, b_1, b_2 \in \mathbb{C} \right\}.
\end{aligned}$$

Hence, for $A \in \mathcal{C}_3$, we have $\dim \Delta(A, A^{*}) = 2$. It is also easy to check that, for the case of $\sigma(A) \subset \mathbb{R}$ and $A \notin \mathbb{C}I$, we have $\dim \Delta(A, A^{*}) = 2$, and hence $\dim \Delta(A, A^{*}) \neq 3$ for each A, finishing the proof of the Assertion.

Now assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a bijective linear map such that $AB = BA^{\dagger} \Rightarrow \Phi(A)\Phi(B) = \Phi(B)\Phi(A)^{\dagger}$. Then $\Phi(A^{\dagger}) = \Phi(A)^{\dagger}$ for all A and $\Phi(I) \in \mathbb{R}I$. Without loss of generality, assume that $\Phi(I) = I$. Note that

$$\Phi(\Delta(A, A^{\dagger})) \subseteq \Delta(\Phi(A), \Phi(A)^{\dagger}) \tag{2.11}$$

for all $A \in \mathcal{B}(H)$. Thus, for every A, we have

$$\dim(\Delta(A, A^{\dagger})) \leq \dim(\Delta(\Phi(A), \Phi(A)^{\dagger})). \tag{2.12}$$

Since $\Phi(\mathbb{R}I) = \mathbb{R}I$, using the above assertion, we obtain

$$\dim \Delta(A, A^{\dagger}) = 2 \ \Rightarrow\ \dim \Delta(\Phi(A), \Phi(A)^{\dagger}) = 2. \tag{2.13}$$

Choose an orthonormal basis $\{e_1, e_2\}$ of H so that J is diagonal. Set $E_{ij} = e_i \otimes e_j$ and $\Phi(E_{ij}) = T_{ij}$, $i, j = 1, 2$. It is clear that $T_{11} + T_{22} = I$ and $\{T_{ij} \mid i, j = 1, 2\}$ is a basis of $\mathcal{B}(K)$. It is easily seen that

$$\Delta(E_{12}, E_{12}^{\dagger}) = \{B = b_{11}E_{11} + b_{12}E_{12} + b_{12}E_{12}^{\dagger} \mid b_{11}, b_{12} \in \mathbb{C}\},$$

so, by (2.11), we obtain

$$b_{11}T_{11}+b_{12}T_{12}+b_{12}T_{12}^{\dagger}=\Phi(B)\in\Delta(T_{12},T_{12}^{\dagger}).$$

Thus,

$$b_{11}T_{12}T_{11}+b_{12}T_{12}^2+b_{12}T_{12}T_{12}^{\dagger}=b_{11}T_{11}T_{12}^{\dagger}+b_{12}T_{12}T_{12}^{\dagger}+b_{12}(T_{12}^{\dagger})^2$$

for all complex numbers b_{11},b_{12}. It follows that

$$\begin{cases} T_{12}T_{11}=T_{11}T_{12}^{\dagger}, \\ T_{12}^2=(T_{12}^{\dagger})^2. \end{cases} \tag{2.14}$$

We complete the proof of Case 2 by considering four subcases.

SUBCASE 1: Both J and L are linearly independent of the identity. We choose an orthonormal basis $\{e_1,e_2\}$ of H such that $J=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ with respect to this basis. By the Assertion and (2.13), we get $\dim\Delta(T_{11},T_{11}^{\dagger})=\dim\Delta(E_{11},E_{11}^{\dagger})=2$, and therefore, either $T_{11}\in\mathcal{C}_3$ or $T_{11}\in\mathcal{C}_4$. Assume that the former occurs. Then there is a complex number $t_1\notin\mathbb{R}$ such that $\sigma(T_{11})=\{t_1,\bar{t}_1\}$. Take an orthonormal basis $\{u_1,u_2\}$ of K so that T_{11} has the matrix representation

$$T_{11}=\begin{pmatrix} t_1 & t_{12} \\ 0 & \bar{t}_1 \end{pmatrix}.$$

For any $\alpha\notin\mathbb{R}$, we have

$$T_{11}+\alpha T_{22}=\begin{pmatrix} t_1+(1-t_1)\alpha & t_{12}(1-\alpha) \\ 0 & \bar{t}_1+(1-\bar{t}_1)\alpha \end{pmatrix}.$$

Thus there exists a scalar α so that $T_{11}+\alpha T_{22}\notin\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3\cup\mathcal{C}_4$, and hence, by the Assertion and the inequality (2.12), one sees that

$$1=\dim\Delta(E_{11}+\alpha E_{22},(E_{11}+\alpha E_{22})^{\dagger})\le\dim\Delta(T_{11}+\alpha T_{22},(T_{11}+\alpha T_{22})^{\dagger})=0,$$

a contradiction. So we must have $T_{11}\in\mathcal{C}_4$, and

$$T_{11}=\begin{pmatrix} t_1 & t_{12} \\ 0 & t_2 \end{pmatrix}$$

with respect to the basis $\{u_1,u_2\}$ of K, where $t_1,t_2\in\mathbb{R}$ and $t_{12}\neq 0$ if $t_2=t_1$. Notice further that if $(t_1,t_2)\neq(1,0)$ or $(0,1)$, then there exists some $\alpha\in\mathbb{C}$ with $\alpha\neq\overline{\alpha}$ so that either $T=T_{11}+\alpha T_{22}$ or $T=\alpha T_{11}+T_{22}$ is not in the class $\mathcal{C}$, i.e., $\Delta(T,T^{\dagger})=\{0\}$. However, this contradicts the fact that $\dim\Delta(T,T^{\dagger})\ge\dim\Delta(A,A^{\dagger})=1$, where $A=E_{11}+\alpha E_{22}$ or $A=\alpha E_{11}+E_{22}$ respectively. Hence $(t_1,t_2)=(1,0)$ or $(0,1)$. Without loss of generality, we assume that $t_1=1$ and $t_2=0$. Observe also that, with respect to this basis,

$$L = \begin{pmatrix} l & \sqrt{1-l^2}\, e^{i\theta} \\ \sqrt{1-l^2}\, e^{-i\theta} & -l \end{pmatrix}$$

with $l \in \mathbb{R}$ and $|l| \le 1$. Since $T_{11}^\dagger = T_{11}$, there must be $l^2 + l\sqrt{1-l^2}\, e^{i\theta}\overline{t}_{12} = 1$. Hence $l \neq 0$ and

$$T_{11} = \begin{pmatrix} 1 & \frac{\sqrt{1-l^2}}{l}\, e^{i\theta} \\ 0 & 0 \end{pmatrix}, \quad T_{22} = \begin{pmatrix} 0 & -\frac{\sqrt{1-l^2}}{l}\, e^{i\theta} \\ 0 & 1 \end{pmatrix}. \tag{2.15}$$

Also, fixing the bases $\{e_1, e_2\}$ of H and $\{u_1, u_2\}$ of K, we can view Φ as a linear map from $M_2(\mathbb{C})$ onto itself.

Write

$$T_{12} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.$$

Then

$$-T_{21} = T_{12}^\dagger = LT_{12}^* L = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix},$$

where

$$\begin{aligned}
r_{11} &= l^2\overline{s}_{11} + l\sqrt{1-l^2}\, e^{i\theta}\overline{s}_{12} + l\sqrt{1-l^2}\, e^{-i\theta}\overline{s}_{21} + (1-l^2)\overline{s}_{22},\\
r_{21} &= l\sqrt{1-l^2}\, e^{-i\theta}\overline{s}_{11} - l^2\overline{s}_{12} + (1-l^2)e^{-2i\theta}\overline{s}_{21} - l\sqrt{1-l^2}\, e^{-i\theta}\overline{s}_{22},\\
r_{12} &= l\sqrt{1-l^2}\, e^{i\theta}\overline{s}_{11} + (1-l^2)e^{2i\theta}\overline{s}_{12} - l^2\overline{s}_{21} - l\sqrt{1-l^2}\, e^{i\theta}\overline{s}_{22},\\
r_{22} &= (1-l^2)\overline{s}_{11} - l\sqrt{1-l^2}\, e^{i\theta}\overline{s}_{12} - l\sqrt{1-l^2}\, e^{-i\theta}\overline{s}_{21} + l^2\overline{s}_{22}.
\end{aligned}$$

Substituting (2.15) and the above representation of $T_{12}^\dagger$ into the equation $T_{12}T_{11} = T_{11}T_{12}^\dagger$ in (2.14), we get $s_{21} = 0$ and $s_{11} \in \mathbb{R}$. Since $\dim \Delta(T_{12}, T_{12}^\dagger) = 2$, we see that also $s_{22} \in \mathbb{R}$. Assume that either $s_{11} \neq 0$ or $s_{22} \neq 0$. Let $A = tE_{11} + \alpha E_{22} + \beta E_{12}$, where $t \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \overline{\alpha}$. Then $\sigma(\Phi(A)) = \{t + \beta s_{11}, \alpha + \beta s_{22}\}$. It is clear that one can choose suitable scalars t, α, β so that $\Phi(A) \notin \mathcal{C}$. Thus we get a contradiction that $1 = \dim \Delta(A, A^\dagger) \le \dim \Delta(\Phi(A), \Phi(A)^\dagger) = 0$. So there must be $s_{11} = s_{22} = 0$, and consequently,

$$T_{12} = \begin{pmatrix} 0 & s_{12} \\ 0 & 0 \end{pmatrix}$$

and

$$-T_{21} = T_{12}^\dagger = \begin{pmatrix} l\sqrt{1-l^2}\, e^{i\theta}\overline{s}_{12} & (1-l^2)\, e^{2i\theta}\overline{s}_{12} \\ -l^2\overline{s}_{12} & -l\sqrt{1-l^2} e^{i\theta}\overline{s}_{12} \end{pmatrix}. \tag{2.16}$$

For any $\alpha, \beta \in \mathbb{C}$ with $\overline{\alpha} \neq \alpha$, since $A = \alpha E_{11} + \overline{\alpha} E_{22} + \beta E_{12} \in \mathcal{C}_3$, it follows that $\dim \Delta(A, A^\dagger) = 2$ and

$$\Delta(A, A^\dagger) = \left\{ B = \frac{\overline{\beta} b_{12} + \beta b_{21}}{\overline{\alpha} - \alpha} E_{11} + b_{12} E_{12} + b_{21} E_{21} \,\middle|\, b_{12}, b_{21} \in \mathbb{C} \right\}.$$

By (2.11), $\Phi(\Delta(A, A^\dagger)) \subseteq \Delta(\Phi(A), \Phi(A)^\dagger)$, so

$$\begin{aligned}(\alpha T_{11} + \overline{\alpha} T_{22} + \beta T_{12}) &\left(\frac{\overline{\beta} b_{12} + \beta b_{21}}{\overline{\alpha} - \alpha} T_{11} + b_{12} T_{12} + b_{21} T_{21} \right) \\ &= \left(\frac{\overline{\beta} b_{12} + \beta b_{21}}{\overline{\alpha} - \alpha} T_{11} + b_{12} T_{12} + b_{21} T_{21} \right) (\alpha T_{11} + \overline{\alpha} T_{22} + \beta T_{12})^\dagger\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\begin{pmatrix} \alpha & \frac{\sqrt{1-l^2}}{l} e^{i\theta}(\alpha - \overline{\alpha}) + \beta s_{12} \\ 0 & \overline{\alpha} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \\ &\qquad = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \\ &\qquad\quad \cdot \begin{pmatrix} \overline{\alpha} + l\sqrt{1-l^2}\, e^{i\theta} \overline{s}_{12} \overline{\beta} & -\frac{\sqrt{1-l^2}}{l} e^{i\theta}(\alpha - \overline{\alpha}) + (1-l^2) e^{2i\theta} \overline{s}_{12} \overline{\beta} \\ -l^2 \overline{s}_{12} \overline{\beta} & \alpha - l\sqrt{1-l^2}\, e^{i\theta} \overline{s}_{12} \overline{\beta} \end{pmatrix},\end{aligned}$$

for all $\alpha, \beta, b_{12}, b_{21} \in \mathbb{C}$ with $\alpha \neq \overline{\alpha}$, where

$$\begin{aligned} w_{11} &= \frac{\overline{\beta} b_{12} + \beta b_{21}}{\overline{\alpha} - \alpha} - l\sqrt{1-l^2}\, e^{i\theta} \overline{s}_{12} b_{21}, \\ w_{12} &= \frac{\sqrt{1-l^2}}{l} e^{i\theta} \frac{\overline{\beta} b_{12} + \beta b_{21}}{\overline{\alpha} - \alpha} + s_{12} b_{12} - (1-l^2) e^{2i\theta} \overline{s}_{12} b_{21}, \\ w_{21} &= l^2 \overline{s}_{12} b_{21}, \\ w_{22} &= l\sqrt{1-l^2}\, e^{i\theta} \overline{s}_{12} b_{21}. \end{aligned}$$

Thus, we obtain

$$l^2 |s_{12}|^2 = 1. \tag{2.17}$$

Note that

$$\begin{aligned} T_{21} &= -\begin{pmatrix} l\sqrt{1-l^2}\, e^{i\theta} \overline{s}_{12} & (1-l^2) e^{2i\theta} \overline{s}_{12} \\ -l^2 \overline{s}_{12} & -l\sqrt{1-l^2}\, e^{i\theta} \overline{s}_{12} \end{pmatrix} \\ &= -l^2 \overline{s}_{12} \begin{pmatrix} \frac{\sqrt{1-l^2}}{l} e^{i\theta} & \frac{1-l^2}{l^2} e^{2i\theta} \\ -1 & -\frac{\sqrt{1-l^2}}{l} e^{i\theta} \end{pmatrix} \end{aligned}$$

is of rank one and $T_{21}^2 = 0$. For any $c \in \mathbb{C}$, let

$$E_c = \begin{pmatrix} c & c^2 \\ -1 & -c \end{pmatrix}.$$

Then E_c is a rank-one nilpotent matrix, and it can be easily checked by (2.17) that

$$\Phi(E_c) = l^2\overline{s}_{12}E_{(\frac{c}{l^2\overline{s}_{12}}+\frac{\sqrt{1-l^2}}{l}e^{i\theta})},$$

which is also rank-one nilpotent. Conversely, for any $c \in \mathbb{C}$, we have

$$\Phi((l^2\overline{s}_{12})^{-1}E_{(l^2\overline{s}_{12}c-l\sqrt{1-l^2}e^{i\theta}\overline{s}_{12})}) = E_c.$$

Note also that $\mathcal{L}$ is a maximal additive subgroup of rank-one nilpotent matrices in $M_2(\mathbb{C})$ if and only if $\mathcal{L}$ has one of the following forms:

(i) $\mathcal{L} = \mathbb{C}E_{12}$;
(ii) $\mathcal{L} = \mathbb{C}E_{21} = \mathbb{C}E_0$;
(iii) there is a nonzero number $c \in \mathbb{C}$ such that $\mathcal{L} = \mathbb{C}E_c$.

Therefore the map $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ preserves rank-one nilpotent matrices in both directions. By [17, Theorem 2.4], there exist a nonzero scalar c, an invertible matrix $V \in M_2(\mathbb{C})$ and a linear map $\varphi : \mathbb{C}I \to M_2(\mathbb{C})$ such that

$$\Phi(A) = cVAV^{-1} + \varphi(\mathrm{tr}(A)I)$$

for all $A \in M_2(\mathbb{C})$, where $\mathrm{tr}(A)$ denotes the trace of a matrix A. Since $\Phi(I) = I$, we see that $\varphi(I) \in \mathbb{C}I$. So, there is a linear functional f on $M_2(\mathbb{C})$ such that

$$\Phi(A) = cVAV^{-1} + f(A)I$$

for all $A \in M_2(\mathbb{C})$.

We claim that $f(A^\dagger) = \overline{f(A)}$ for all A. Since $\Phi(A^\dagger) = \Phi(A)^\dagger$ for all A, we see that

$$\overline{c}(V^{-1})^\dagger A^\dagger V^\dagger - cVA^\dagger V^{-1} = (f(A^\dagger) - \overline{f(A)})I$$

for every A. Let $S = V^\dagger V$, $a = \overline{c}/c$ and $h(A) = c^{-1}(f(A) - \overline{f(A^\dagger)})$. Then it follows that

$$aS^{-1}AS = h(A)I + A \quad \text{for all } A \in M_2(\mathbb{C}).$$

Considering the spectrum one observes that $\sigma(A) = \{0\}$ implies $h(A) = 0$. Regarding h as a linear functional on $M_2(\mathbb{C})$, the above fact shows that the three-dimensional linear subspace $\mathrm{sl}_2(\mathbb{C})$ spanned by nilpotent matrices is contained in the kernel of h. Assume that $h \neq 0$; then $h(A) \neq 0$ whenever $\mathrm{tr}(A) \neq 0$. For any operator A_0 such that $\sigma(A_0) = \{0, 1\}$, we have $\{0, a\} = \{h(A_0), h(A_0) + 1\}$. As $h(A_0) \neq 0$, one sees that $a = h(A_0) = -1$. Hence

$h(A) = -\operatorname{tr}(A)$ and

$$S^{-1}AS + A = \operatorname{tr}(A)I \quad \text{for all } A.$$

In particular, $AS = -SA$ for all A with $\operatorname{tr}(A) = 0$, but this would imply that $S = 0$, a contradiction. Hence, $h = 0$ and $f(A^\dagger) = \overline{f(A)}$ for every A.

Now it is obvious that c is real and $V^\dagger = \alpha V^{-1}$ for some real number α. We claim further that $f = 0$. Considering $\Psi(\cdot) = c^{-1}V^{-1}\Phi(\cdot)V$, we may assume that Φ has the form $\Phi(A) = A + g(A)I$ for all A, where $g = c^{-1}f$. To show $g = 0$, let $g(E_{ij}) = \alpha_{ij}$ and $\Phi(I) = dI$. Then α_{ii} is real for $i = 1, 2$, $\alpha_{22} = d - 1 - \alpha_{11}$, and $\alpha_{21} = -\overline{\alpha}_{12}$. Since g is linear, we need only show that $\alpha_{11} = \alpha_{12} = 0$ and $d = 1$. For any $\alpha, \beta, b_{12}, b_{21} \in \mathbb{C}$ with $\alpha \neq \overline{\alpha}$, let

$$A = \alpha E_{11} + \overline{\alpha}E_{22} + \beta E_{12}, \qquad B = \frac{\overline{\beta}b_{12} + \beta b_{21}}{\overline{\alpha} - \alpha}\, E_{11} + b_{12}E_{12} + b_{21}E_{21}.$$

Then $AB = BA^\dagger$. Hence $(g(A) - \overline{g(A)})B + g(B)(A - A^\dagger) = (\overline{g(A)} - g(A))g(B)I$. It follows that

$$(g(A) - \overline{g(A)})b_{12} + g(B)\beta = 0 \tag{2.18}$$

for all $\alpha, \beta, b_{12}, b_{21} \in \mathbb{C}$. Notice that

$$g(B) = \frac{\overline{\beta}b_{12} + \beta b_{21}}{\overline{\alpha} - \alpha}\, \alpha_{11} + b_{12}\alpha_{12} - b_{21}\overline{\alpha}_{12}.$$

Let $b_{12} = 0$; from (2.18) one gets

$$g(B)\beta = \left(\frac{\beta^2}{\overline{\alpha} - \alpha}\, \alpha_{11} - \beta\overline{\alpha}_{12}\right) b_{21} = 0$$

for all $\alpha, \beta, b_{21} \in \mathbb{C}$. Thus we must have $\alpha_{11} = \alpha_{12} = 0$. So $g(B) = 0$ and by (2.18) again we get $(g(A) - \overline{g(A)})b_{12} = 0$ for all $\alpha, b_{21} \in \mathbb{C}$ with $\alpha \neq \overline{\alpha}$. Since $g(A) - \overline{g(A)} = (d-1)(\overline{\alpha} - \alpha)$, it follows that $(1-d)(\overline{\alpha} - \alpha)b_{12} = 0$ for all scalars $\alpha, b_{21} \in \mathbb{C}$ with $\alpha \neq \overline{\alpha}$, and consequently, $d = 1$. Therefore, $g = 0$ and Φ has the form stated in Theorem 1 as desired, completing the proof of Subcase 1.

Remark. To show that the map Φ has the structure stated in Theorem 1 in this subcase, we mention another approach by using a result in the geometry of 2×2 matrices [32]. We give a sketch of proof as follows. By (2.17), we can get $\det A = 0 \Leftrightarrow \det \Phi(A) = 0$. Thus Φ preserves rank-one matrices in both directions. Note that $\Phi(I) = I$ and $\Phi(A^\dagger) = \Phi(A)^\dagger$ for every A. Hence, by a result in the geometry of 2×2 matrices [32], either there exist a real scalar c and a generalized indefinite unitary operator $U : H \to K$ such that $\Phi(A) = cUAU^\dagger$ for all A; or there exist a real scalar c and a generalized indefinite anti-unitary operator $U : H \to K$ such that $\Phi(A) = cUA^\dagger U^\dagger$ for all A. The last form cannot occur because there are operators A, B such that $AB = BA^\dagger$ but $A^\dagger B^\dagger \neq B^\dagger A$ (to see this, for example, take $A = e_1 \otimes e_2$,

$B = e_1 \otimes e_1$, where e_1 and e_2 are unit vectors such that $Je_1 = e_1$ and $Je_2 = -e_2$ with J as in the proof of Subcase 1). Therefore, Φ has the form stated in Theorem 1.

SUBCASE 2: Both J and L are linearly dependent of the identity. In this case, the assumption becomes $AB = BA^* \Rightarrow \Phi(A)\Phi(B) = \Phi(B)\Phi(A)^*$ and we may assume that $J = I$ and $L = I$. It is easily checked that there exist a real number c and a unitary operator $U : H \to K$ such that $\Phi(A) = cUAU^*$ for all A, and Φ has the form stated in Theorem 1.

SUBCASE 3: $J \notin \mathbb{R}I$ and $L \in \mathbb{R}I$. We shall show that this subcase does not occur, that is, the assumption $AB = BA^\dagger \Rightarrow \Phi(A)\Phi(B) = \Phi(B)\Phi(A)^*$ for all A cannot be satisfied for any bijective linear map Φ. Without loss of generality we may assume that $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Taking the same symbols as in the proof of Subcase 1, it is easily checked that $T_{11} = \Phi(E_{11}) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $T_{22} = \Phi(E_{22}) = I - T_{11}$. By (2.14), we have $T_{12}T_{11} = T_{11}T_{12}^*$ and $T_{12}^2 = (T_{12}^*)^2$. It follows from a similar argument to the proof of Subcase 1 that $T_{12} = \left(\begin{smallmatrix} 0 & s_{12} \\ 0 & 0 \end{smallmatrix}\right)$ and $T_{21} = T_{12}^*$. Moreover,

$$B = \frac{\overline{\beta} b_{12} + \beta b_{21}}{\overline{\alpha} - \alpha} E_{11} + b_{12}E_{12} + b_{21}E_{21} \in \Delta(A, A^\dagger)$$

with $A = \alpha E_{11} + \overline{\alpha} E_{22} + \beta E_{12}$ implies that

$$\frac{\overline{\beta} b_{12} + \beta b_{21}}{\overline{\alpha} - \alpha} T_{11} + b_{12}T_{12} + b_{21}T_{21} \in \Delta(\Phi(A), \Phi(A)^*)$$

for all $\alpha, \beta, b_{12}, b_{21} \in \mathbb{C}$ with $\alpha \neq \overline{\alpha}$. However, this will lead to a contradiction $|s_{12}|^2 = -1$.

SUBCASE 4: $J \in \mathbb{R}I$ and $L \notin \mathbb{R}I$. This cannot occur either. To see this, without loss of generality, we assume that $J = I$. Taking the same symbols as in Subcase 1, we see that (2.15) is still true. A similar argument to Subcase 1 shows that $T_{12} = \left(\begin{smallmatrix} 0 & s_{12} \\ 0 & 0 \end{smallmatrix}\right)$ and $T_{12}^\dagger$ has the form (2.16). Now, for any $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \overline{\alpha}$ and $\beta \neq 0$, let $A = \alpha E_{11} + \overline{\alpha} E_{22} + \beta E_{12}$. Since, for all $b_{12}, b_{21} \in \mathbb{C}$ with $b_{12} \neq 0$, we have

$$\frac{\overline{\beta} b_{12} - \beta b_{21}}{\alpha - \overline{\alpha}} E_{11} + b_{12}E_{12} + b_{21}E_{21} \in \Delta(A, A^*),$$

it follows that

$$\frac{\overline{\beta} b_{12} - \beta b_{21}}{\alpha - \overline{\alpha}} T_{11} + b_{12}T_{12} + b_{21}T_{21} \in \Delta(\Phi(A), \Phi(A)^\dagger).$$

However, by taking $b_{21} = 0$, this would lead to $l^2|s_{12}|^2 = -1$, a contradiction.

Thus, in the case $\dim H = 2$, we have also proved that the map Φ has the form stated in Theorem 1. Now the proof of Theorem 1 is complete. ∎

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