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On the power boundedness of certain Volterra operator pencils

by

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Abstract. Let V be the classical Volterra operator on $L^2(0,1)$, and let z be a complex number. We prove that $I - zV$ is power bounded if and only if $\operatorname{Re} z \ge 0$ and $\operatorname{Im} z = 0$, while $I - zV^2$ is power bounded if and only if $z = 0$. The first result yields

$$\|(I-V)^n - (I-V)^{n+1}\| = O(n^{-1/2}) \quad \text{as } n \to \infty,$$

an improvement of [Py]. We also study some other related operator pencils.

1. Preliminaries. We say that an operator A is *power-bounded* if $\sup_{n\ge 0} \|A^n\| < \infty$. We denote by V the classical *Volterra operator*

$$(Vf)(x) = \int_0^x f(s)\,ds, \quad 0 < x < 1, \quad \text{on } L^p(0,1),\ 1 \le p \le \infty.$$

We recall the well-known formula

$$(V^n f)(x) = \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} f(s)\,ds \quad \text{for } n \in \mathbb{N}.$$

A generalization of this formula is the definition of the *Riemann–Liouville integral operator* of any fractional order $\alpha > 0$,

$$(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)\,ds$$

(Γ is the Euler gamma function) on $L^p(0,1)$, $1 \le p \le \infty$. In particular, $V = J^1$.

Recall that the *Ritt condition* for the resolvent $R(\lambda, A) = (A - \lambda I)^{-1}$ of a bounded linear operator A on a Banach space is

$$\|R(\lambda, A)\| \le \frac{\text{const}}{|\lambda - 1|}, \quad |\lambda| > 1,$$

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which is equivalent to a geometric condition much stronger than the power boundedness of A [NaZe], [Ne2]. If the operator A is merely power-bounded, then the weaker *Kreiss condition*

$$\|R(\lambda, A)\| \le \frac{\text{const}}{|\lambda| - 1}, \quad |\lambda| > 1,$$

holds, but not conversely in general.

2. Introduction. In 1997, Allan [Al] recorded the observation made by T. V. Pedersen that $I - V$ is similar to $(I + V)^{-1}$, namely

$$S^{-1}(I - V)S = (I + V)^{-1} \tag{1}$$

where $(Sf)(t) = e^t f(t)$, $f \in L^p(0,1)$, $1 \le p \le \infty$. By [Ha, Problem 150], we know that $\|(I+V)^{-1}\| = 1$ on $L^2(0,1)$. Hence $I - V$ is a power-bounded operator on $L^2(0,1)$.

In 1987, Pytlik [Py], basing on an upper estimate for the Fejér expression for Laguerre polynomials (see [Sz, p. 198]), proved

$$\|(I - V)^n - (I - V)^{n+1}\| = O(n^{-1/4}) \tag{2}$$

as $n \to +\infty$ on $L^2(0,1)$. The same argument gives the same result also on $L^1(0,1)$, in which case it is sharp [ToZe]. By this method, one is unable to distinguish the delicate properties of the L^p-norms. We shall show, by an algebraic argument, the power boundedness of $I - tV$ for $t > 0$, on $L^2(0,1)$, which will improve Pytlik's estimate to $O(n^{-1/2})$. Our method, however, does not apply to $L^1(0,1)$, because $I - V$ is not power-bounded there (see [Hi, p. 247]), and (2) actually cannot be improved on $L^1(0,1)$ as mentioned above [ToZe]. We also study some other related operator pencils. The details of some calculations as well as alternative proofs of some cases are given in [Ts].

3. The results

PROPOSITION 1. *Let A and B be two commuting power-bounded operators on a Banach space, and $0 \le t \le 1$. Then the convex combination $tA + (1-t)B$ is a power-bounded operator.*

Proof. By the binomial formula,

$$\begin{aligned}\|(tA + (1-t)B)^n\| &\le \sum_{k=0}^{n} \binom{n}{k} t^k \|A^k\| (1-t)^{n-k} \|B^{n-k}\| \\ &\le \text{const} \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} = \text{const}(t + (1-t))^n = \text{const}. \ \blacksquare\end{aligned}$$

THEOREM 1. *The operator* $I - zV$ *is power-bounded on* $L^2(0,1)$ *if and only if* $\operatorname{Re} z \geq 0$ *and* $\operatorname{Im} z = 0$.

Proof. (*If*) It follows from Proposition 1 and the power boundedness of $I - V$ (explained above) that $I - tV = (1-t)I + t(I - V)$ is power-bounded for $0 \leq t \leq 1$ on $L^2(0,1)$.

Let m be a natural number. Note the following extension of formula (1):

$$S^{-1}(I - mV)S = (I - (m-1)V)(I + V)^{-1} \tag{3}$$

where $(Sf)(t) = e^t f(t)$, $f \in L^p(0,1)$. We shall verify it by induction. If $m = 1$, we have (1). Suppose that (3) holds for some m. Then

$$\begin{aligned} S^{-1}(I - (m+1)V)S & \\ &= I - S^{-1}(mV)S - S^{-1}VS \\ &= (I - (m-1)V)(I+V)^{-1} + (I+V)^{-1} - I \\ &= (I - (m-1)V)(I+V)^{-1} + (I+V)^{-1} - (I+V)(I+V)^{-1} \\ &= (I - (m-1)V)(I+V)^{-1} + (I - (I+V))(I+V)^{-1} \\ &= (I - mV)(I+V)^{-1}. \end{aligned}$$

This proves (3) and yields the power boundedness of $I - mV$ for all $m \in \mathbb{N}$.

Then the convex combination $(1-t)(I - mV) + t(I - (m+1)V) = I - (m+t)V$ is power-bounded for $0 \leq t \leq 1$ and $m \in \mathbb{N}$.

(*Only if*) We shall show that the operator $I - zV$ does not satisfy the Kreiss condition on $L^2(0,1)$ for $\operatorname{Im} z \neq 0$. Thus $I - zV$ is not power-bounded on this space for those z. Indeed, using the well-known formula for the resolvent of V (see e.g. [Ne1, p. 27]), we obtain

$$(R(\lambda, I - zV)f)(x) = -\frac{f(x)}{\lambda - 1} + \frac{z}{(\lambda-1)^2}\int_0^x e^{-(x-s)z/(\lambda-1)} f(s)\, ds, \qquad \lambda \neq 1.$$

We have

$$\limsup_{n\to\infty} (|1 + i/n| - 1)\|R(1 + i/n, I - zV)e^{in\cdot}\| = \infty \quad \text{ for } \operatorname{Im} z < 0,$$
$$\limsup_{n\to\infty} (|1 - i/n| - 1)\|R(1 - i/n, I - zV)e^{in\cdot}\| = \infty \quad \text{ for } \operatorname{Im} z > 0.$$

Of course, $I - zV$ is not power-bounded for $\operatorname{Re} z < 0$ and $\operatorname{Im} z = 0$, because for $f \equiv 1$, we have $\limsup_{n\to\infty} \|(I - zV)^n 1\| = \infty$. ∎

COROLLARY 1. *On* $L^2(0,1)$, *we have*

$$\|(I - V)^n - (I - V)^{n+1}\| = O(n^{-1/2}) \quad \textit{as } n \to \infty.$$

Proof. Set $L = I - \mu V$ for $\mu > 1$, which is power-bounded by Theorem 1. Then $L_\omega = (1-\omega)I + \omega L = (1-\omega)I + \omega(I - \mu V) = I - \omega\mu V$ is power-bounded for $0 < \omega < 1$ by Proposition 1. Now, Nevanlinna's theorem [Ne1, Theorem 4.5.3] yields

$$\limsup_{n\to\infty} n^{1/2} \|L_\omega^n (L_\omega - I)\| \le \text{const} \left(\frac{\omega}{2\pi(1-\omega)} \right)^{1/2}.$$

So, for $\omega = 1/\mu$ we get the claim. ∎

REMARK 1. Corollary 1 does not follow from Nevanlinna's paper [Ne2, p. 121] because his resolvent assumption (1.35) is not satisfied for any positive $\alpha < 1$.

REMARK 2. Alternatively, one can also use [FoWe, Lemma 2.1] instead of [Ne1, Theorem 4.5.3]; observe that the proof in [FoWe, Lemma 2.1] works also for power-bounded commuting pairs, or use [BoDu, Theorem 4.1].

REMARK 3. It would be interesting to know if the above estimate $O(n^{-1/2})$ is already sharp, and if it extends, together with Theorem 1, to $L^p(0,1)$, $1 < p < \infty$. The above proof of Theorem 1 extends to these spaces as soon as we know that $I - V$ is power-bounded there. Perhaps the Riesz–Thorin convexity theorem [BeSh, p. 196] could be applied.

REMARK 4. It has been pointed out by Yuri Tomilov that Corollary 1 also follows from [Sa] and (1), by using [FoWe] as above. However, this approach does not seem to give Theorem 1. On the other hand, our Theorem 1 yields the corresponding information about the power boundedness of the Sarason operator pencil.

REMARK 5. Consider the matrix

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $I - z\mathbb{A}$, $z \in \mathbb{C}$, is power-bounded if and only if $z = 0$.

THEOREM 2. *The operator $I - zV$, $z \in \mathbb{C}$, is power-bounded on $L^1(0,1)$ if and only if $z = 0$.*

Proof. We consider the following three cases:

Case $t < 0$. The operator $I - tV$ is not power-bounded on $L^1(0,1)$ for $t < 0$ since as before, from the binomial formula it is clear that

$$\limsup_{n\to\infty} \|(I - tV)^n 1\| = \infty.$$

Case $t > 0$. As in [Py, p. 292] we can write

$$((I - tV)^n f)(x) - ((I - tV)^{n+1} f)(x) = (tV(I - tV)^n f)(x)$$
$$= t\Big(\sum_{k=0}^{n} \binom{n}{k} (-1)^k t^k V^{k+1} f\Big)(x) = t\int_0^x \sum_{k=0}^{n} \binom{n}{k} (-1)^k t^k \frac{(x-s)^k}{k!} f(s)\,ds$$
$$= t\int_0^x L_n^{(0)}(t(x-s)) f(s)\,ds$$

where

$$L_n^{(0)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{x^k}{k!}, \qquad n \ge 1,$$

are the Laguerre polynomials with parameter 0. By summing these formulas and using [Sz, p. 102, formula (5.1.13)], we get

$$((I - tV)^{n+1} f)(x) = f(x) - t\int_0^x L_n^{(1)}(t(x-s)) f(s)\,ds$$

where $L_n^{(1)}(x)$ are the Laguerre polynomials with parameter 1.

Using the classical estimates for Laguerre polynomials [Sz, p. 177 and 198] and the formula for the norm of an integral operator on $L^1(0,1)$ given in [ToZe, Lemma 4.5], we deduce as in [ToZe, Example 4.6] that

$$\limsup_{n\to\infty} \|(I - tV)^n\| = \infty.$$

Case $z \in \mathbb{C} \setminus \mathbb{R}$. We show that the operator $I - zV$ does not satisfy the Kreiss condition on $L^1(0,1)$ for $\operatorname{Im} z \ne 0$. Thus $I - zV$ is not power-bounded on $L^1(0,1)$ for those z. Indeed, on $L^1(0,1)$, we have

$$\limsup_{n\to\infty} (|1 + i/n| - 1)\|R(1 + i/n, I - zV)e^{in\cdot}\| = \infty \quad \text{for } \operatorname{Im} z < 0,$$
$$\limsup_{n\to\infty} (|1 - i/n| - 1)\|R(1 - i/n, I - zV)e^{in\cdot}\| = \infty \quad \text{for } \operatorname{Im} z > 0. \ \blacksquare$$

Remark 6. By duality, the same characterization holds on $L^\infty(0,1)$.

Proposition 2. *Let* $\sigma(Q) = \{0\}$. *If* $I - Q$ *satisfies the Ritt condition, then so does* $I - tQ$ *for* $t > 0$.

Proof. We can write

$$R(\lambda, I - tQ) = (I - tQ - \lambda I)^{-1} = \frac{1}{t}\Big(\frac{1-\lambda}{t} I - Q\Big)^{-1}$$
$$= \frac{1}{t}\Big[(I - Q) - I + \frac{1-\lambda}{t} I\Big]^{-1}$$
$$= \frac{1}{t}\Big[I - Q - \Big(1 - \frac{1-\lambda}{t}\Big) I\Big]^{-1}.$$

Whenever $|t-1+\lambda| > t$, i.e. $|\lambda-(1-t)| > t$, which certainly holds for $\operatorname{Re}\lambda > 1$, we have

$$\|R(\lambda, I-tQ)\| \le \frac{1}{t}\,\frac{\mathrm{const}}{\left|\frac{t-1+\lambda}{t}-1\right|} = \frac{\mathrm{const}}{|\lambda-1|},$$

and this yields the Ritt condition by [NaZe, Lemma, p. 146] because $\sigma(Q) = \{0\}$. ■

REMARK 7. The operator $I - J^\alpha$ satisfies the Ritt condition for $0 < \alpha < 1$ on $L^p(0,1), 1 \le p \le \infty$, by [Ly2, p. 137], hence $I - tJ^\alpha$ satisfies the Ritt condition for all $t > 0$ on $L^p(0,1), 1 \le p \le \infty$, by Proposition 2. Hence these operators are power-bounded by [Ly1, Theorem 1, p. 154] or [NaZe, Theorem, p. 147].

This observation does not seem to follow by the method used above in the case $\alpha = 1$, because there is no analogy of (1) and (3) for $\alpha \ne 1$.

We know from Theorem 1 that $I-tV$ is power-bounded on $L^2(0,1)$, while $I+tV$ is not for $t > 0$ (for $t = 1$ the latter also follows from the Gelfand Theorem [Ge]). This leads to the natural question whether the product $(I-tV)(I+tV) = I - t^2V^2$ is power-bounded. The answer is negative.

THEOREM 3. *The operator* $I-zV^2$, $z \in \mathbb{C}$, *is power-bounded on* $L^p(0,1)$, $1 \le p \le \infty$, *if and only if* $z = 0$.

Proof. We consider the following three cases:

Case $t < 0$. The operator $I - tV^2$ is not power-bounded on $L^p(0,1)$, $1 \le p \le \infty$, for $t < 0$ because, as before, from the binomial formula it is clear that

$$\limsup_{n\to\infty} \|(I-tV^2)^n 1\| = \infty.$$

Case $t > 0$. The resolvent formula for V^2 is

$$(R(\lambda, I-V^2)f)(x) = -\frac{f(x)}{\lambda-1} + \frac{1}{(\lambda-1)^{3/2}} \int_0^x \sinh\frac{x-s}{(\lambda-1)^{1/2}}\, f(s)\,ds \quad \text{for } \lambda \ne 1$$

(see [Hi, p. 260] or [Ne1, p. 130]). Therefore the resolvent formula for $I-tV^2$ is

$$(R(\lambda, I-tV^2)f)(x) = \frac{1}{t}\left(R\left(1-\frac{1-\lambda}{t}, I-V^2\right)f\right)(x)$$
$$= -\frac{f(x)}{\lambda-1} + \frac{t^{1/2}}{(\lambda-1)^{3/2}} \int_0^x \sinh\frac{(x-s)t^{1/2}}{(\lambda-1)^{1/2}}\, f(s)\,ds.$$

We choose $f \equiv 1$. Then

$$(R(\lambda, I - tV^2)1)(x) = -\frac{1}{\lambda - 1} + \frac{t^{1/2}}{(\lambda-1)^{3/2}} \int_0^x \sinh \frac{(x-s)t^{1/2}}{(\lambda-1)^{1/2}}\, ds$$
$$= -\frac{2}{\lambda - 1} + \frac{1}{\lambda - 1} \cosh \frac{t^{1/2}x}{(\lambda-1)^{1/2}}.$$

We note that

$$\int_0^x \sinh \frac{(x-s)t^{1/2}}{(\lambda-1)^{1/2}}\, ds$$
$$= -\frac{(\lambda-1)^{1/2}}{t^{1/2}} + \frac{1}{2}\,\frac{(\lambda-1)^{1/2}}{t^{1/2}}[e^{t^{1/2}x/(\lambda-1)^{1/2}} + e^{-t^{1/2}x/(\lambda-1)^{1/2}}].$$

Hence, for $\lambda_n = 1 + 1/n$, we get

$$(R(\lambda_n, I - tV^2)1)(x) = -2n + n \cosh \sqrt{nt}x = n(\cosh \sqrt{nt}x - 2),$$

and an easy calculation shows that

$$\limsup_{n\to\infty} (|\lambda_n| - 1)\|(R(\lambda_n, I - tV^2)1)(x)\| = \infty.$$

Therefore, $R(\lambda, I - tV^2)$ does not satisfy the Kreiss condition for $t > 0$.

Case $z \in \mathbb{C} \setminus \mathbb{R}$. We show that the operator $I - zV^2$ does not satisfy the Kreiss condition on $L^p(0,1)$, $1 \le p \le \infty$, for $\operatorname{Im} z \ne 0$. Indeed, we can write $z = (\alpha + i\beta)^2$ with $\alpha, \beta \in \mathbb{R}$, where $\alpha \ne 0$. In the resolvent formula for $I - zV^2$,

$$(R(\lambda, I - zV^2)f)(x) = -\frac{f(x)}{\lambda - 1} + \frac{z^{1/2}}{(\lambda-1)^{3/2}} \int_0^x \sinh \frac{(x-s)z^{1/2}}{(\lambda-1)^{1/2}} f(s)\, ds$$

we set $\lambda_n = 1 + 1/n^2$. Then

$$(R(1 + 1/n^2, I - z^2V^2)e^{in\cdot})(x)$$
$$= -n^2 e^{inx} + n^3(\alpha + i\beta) \int_0^x \sinh[n(x-s)(\alpha + i\beta)] e^{ins}\, ds.$$

We note that

$$\int_0^x \sinh[n(x-s)(\alpha+i\beta)]e^{ins} ds = -\frac{1}{2}\,\frac{e^{inx}}{n(\alpha + i(\beta-1))} + \frac{e^{n(\alpha+i\beta)x}}{2n(\alpha+i(\beta-1))}$$
$$-\frac{1}{2}\,\frac{e^{inx}}{n(\alpha + i(\beta+1))} + \frac{e^{-n(\alpha+i\beta)x}}{2n(\alpha+i(\beta+1))}.$$

We get

$$\limsup_{n\to\infty} \frac{1}{n^2}\|(R(\lambda_n, I - zV^2)e^{in\cdot})(x)\| = \infty.$$

Therefore, $R(\lambda, I - zV^2)$ does not satisfy the Kreiss condition. ∎

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